The space of $D\!\!\!\!/\,$-minimal Riemannian metrics

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Spinors and Dirac operators

Let M be a closed manifold with fixed spin structure
$n = \dim M$.
$\mathcal{R}(M) := \{\text{Riemannian metrics on } M\}$

For $g \in \mathcal{R}(M)$ we have the *spinor bundle* $\Sigma^g M \to M$.

In this talk: the *real* spinor bundle. It carries: a fiberwise scalar product, a connection and Clifford multiplication.

Dirac operator $D\!\!\!/^g : \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$. Locally:

$$D\!\!\!/^g \varphi := \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \varphi \,.$$

$D\!\!\!/^g$ is a self-adjoint elliptic operator of first order.
$\Longrightarrow$ real and discrete spectrum, eigenspaces finite-dimensional, Fredholm

$D\!\!\!/^g$ strongly depends on g.

$\mathbb{K}$-structure

The real spinor bundle $\Sigma^g M$ often has additionally the structure of a $\mathbb{K}$-bundle.

$n \mod 8$	1	2	3	4	5	6	7	0
$\mathbb{K}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$

The Dirac operator $D\!\!\!/^g : \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$ is $\mathbb{K}$-linear,

$$D\!\!\!/^g \in \mathrm{Fred}_{\mathbb{K}}^{\mathrm{sym},\infty} .$$

Index $\alpha(M) \in KO_n(*)$

$n \mod 8$	1	2	3	4	5	6	7	0
$KO_n(*)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

A simple definition of the index $\alpha(M) \in \mathrm{KO}_n(*)$

$$\omega_n := e_1 \cdot e_2 \cdot \ldots \cdot e_n$$

For $n = 4k$: $\omega_{4k} \cdot \omega_{4k} = 1$ and thus

$$\Sigma^g M = \underbrace{\Sigma^g_+ M}_{\omega_{4k} \equiv +1} \oplus \underbrace{\Sigma^g_- M}_{\omega_{4k} \equiv -1}$$

$$D\!\!\!/^g = D\!\!\!/^g_+ + D\!\!\!/^g_-, \quad D\!\!\!/^g_\pm : \Gamma(\Sigma^g_\pm M) \to \Gamma(\Sigma^g_\mp M)$$

For $n \in \mathbb{N}$

$$\alpha(M) = \begin{cases} \dim_{\mathbb{R}} \ker D\!\!\!/^g_+ - \dim_{\mathbb{R}} \ker D\!\!\!/^g_- \in \mathbb{Z} & \text{if } n \equiv 0 \mod 8, \\ \dim_{\mathbb{C}} \ker D\!\!\!/^g \mod 2 \in \mathbb{Z}/2 & \text{if } n \equiv 1 \mod 8, \\ \dim_{\mathbb{H}} \ker D\!\!\!/^g \mod 2 \in \mathbb{Z}/2 & \text{if } n \equiv 2 \mod 8, \\ \dim_{\mathbb{H}} \ker D\!\!\!/^g_+ - \dim_{\mathbb{H}} \ker D\!\!\!/^g_- \in \mathbb{Z} & \text{if } n \equiv 4 \mod 8, \\ 0 & \text{if } n \equiv 3, 5, 6, 7 \mod 8. \end{cases}$$

Lower bound on the kernel

Thus: $\dim_{\mathbb{K}} \ker \not{D}^g \geq |\alpha(M)|$ where

$$|\alpha(M)| := \begin{cases} |\alpha(M)| & \text{if } n \equiv 0 \mod 8 \\ 1 & \text{if } n \equiv 1,2 \mod 8 \text{ and } \alpha(M) \neq 0 \\ 0 & \text{if } n \equiv 1,2 \mod 8 \text{ and } \alpha(M) = 0 \\ |\alpha(M)| & \text{if } n \equiv 4 \mod 8 \\ 0 & \text{if } n \equiv 3,5,6,7, \end{cases}$$

$$\mathcal{R}_{\min}(M) := \left\{ g \in \mathcal{R}(M) \mid \dim_{\mathbb{K}} \ker \not{D}^g = |\alpha(M)| \right\}$$

$\not{D}$-minimal metrics

Theorem 1 (Ammann–Dahl–Humbert 2009).
For connected M, the space $\mathcal{R}_{\min}(M)$ is C^1-open and C^∞-dense in $\mathcal{R}(M)$.

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For connected M, the space $\mathcal{R}_{\min}(M)$ is C^1-open and C^∞-dense in $\mathcal{R}(M)$.

- As the spectrum depends continuously in C^1-topology, C^1-openness is clear
- For denseness, it suffices to prove $\mathcal{R}_{\min}(M) \neq \emptyset$
- S. Maier 1997: perturbative argument for $n \in \{2, 3, 4\}$
- Bär–Dahl 2002 for $\pi_1(M) = 1$: Gromov-Lawson type surgeries
- Ammann–Dahl–Humbert 2009: surgeries of codimension ≥ 2.

For $\alpha(M) = 0$ we also write $\mathcal{R}_{\mathrm{inv}}(M)$ for $\mathcal{R}_{\min}(M)$.

Our Goal: Better understanding of $\mathcal{R}_{\min}(M)$ and $\mathcal{R}_{\mathrm{inv}}(M)$.

Connections to psc

Schrödinger–Lichnerowicz formula

$$\left(\not{D}^g\right)^2\psi = (\nabla^g)^*\nabla^g\psi + \frac{\mathrm{scal}^{\,g}}{4}\psi$$

$$\mathcal{R}_{\mathrm{psc}}(M) \hookrightarrow \mathcal{R}_{\mathrm{inv}}(M) \xrightarrow{\not{D}^\bullet} \mathrm{Fred}_{\mathbb{K}}^{\mathrm{inv}} \longrightarrow \Omega^{\infty+n+1}\,\mathrm{KO}$$

Thus many statements about psc metrics give information about $\mathcal{R}_{\mathrm{inv}}(M)$.

Existence of non-minimal metrics

Conjecture (Bär–Dahl 2002 for $\alpha(M) = 0$).

On any closed connected spin manifold M of dimension ≥ 3

$$\mathcal{R}_{\text{non-min}}(M) := \mathcal{R}(M) \setminus \mathcal{R}_{\min}(M) \neq \emptyset.$$

Now solved (?) for $\alpha(M) = 0$.

- Hitchin 1974 & Dahl 2008 $n \equiv 7, 0, 1 \mod 8$
- Bär 1996 $n \equiv 3, 7 \mod 8$
- Seeger 2000 S^{2m}, $m \geq 2$
- Conclusion from Crowley-Schick 2018 for $n \geq 6$
- Dahl 2008 S^n, $n \geq 5$
- Ammann-Lockman (in progress), $n = 4$ and 5 (=remaining cases)

$\alpha(M) \neq 0$: Work in progress by Ammann–Bunke–Pilca.
Not the main topic of this talk.

Surfaces, $n = 2$

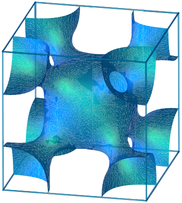
Let M be a connected closed spin surface of genus γ.

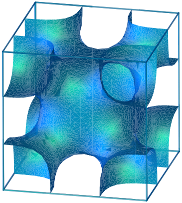
$\mathcal{R}_{\text{non-min}}(M) = \emptyset$ for

- $\alpha(M) = 0$ and $\gamma \leq 2$
- $\alpha(M) = 1$ and $\gamma \leq 4$

$\mathcal{R}_{\text{non-min}}(M) \neq \emptyset$ in all other cases

Example: $\alpha(M) = 0$, $\gamma = 4$, minimal surface in T^3
spinorial Weierstraß representation (Kusner-Schmitt) gives a nontrivial element in $\ker \not{D}$





For $\alpha(M) = 0$, many proofs of $\mathcal{R}_{\text{non-inv}}(M) \neq \emptyset$ determine non-trivial

$$\pi_i(\mathcal{R}_{\text{inv}}(M)).$$

E.g. the proofs of Hitchin 1974, Bär 1996 and Crowley-Schick 2018 study spectral flow for $\not{D}$
$\rightsquigarrow$ relative invariant $\pi_i(\mathcal{R}_{\text{inv}}(M)) \to \mathrm{KO}_{n+i+1}(*)$.

Not the main topic of this talk.

Question.
Does $\mathcal{R}_{\min}(M)$ has topology not detected by index theory?

Special Case.
Let M be closed connected spin, $n = \dim M$.

If $\mathrm{KO}_{n+1}(\{*\}) \cong 0$, is then $\mathcal{R}_{\min}(M)$ connected?

We have proven: Yes for $n = 2$ and $n = 4$.

Theorem 2 (A.-Dahl 2025).

Let M be a closed connected spin manifold of dimension $n = 2$ or $n = 4$. Then the space $\mathcal{R}_{\text{min}}(M)$ is connected.

If $n = 2$ and $\alpha(M) \neq 0$, then the codimension of $\mathcal{R}_{\text{non-min}}(M)$ is at least 6.
If $n = 4$, then this codimension is at least $2 + 2 \cdot |\alpha(M)|$.

Idea: The space $\mathcal{R}_{\text{non-min}}(M)$ is a "stratified" subset of $\mathcal{R}(M)$ of codimension at least 2, 6, $2 + 2 \cdot |\alpha(M)|$ resp..

Question.

Let $\alpha(M) \neq 0$.
Does $\ker \not{D}^{\bullet} \to \mathcal{R}_{\min}(M)$ extend to a vector bundle $V \to \mathcal{R}(M)$?
I.e. do we have for $g \in \mathcal{R}_{\text{non-min}}(M)$

$$\ker \not{D}^{g} = V_g \oplus C_g$$

in a natural way, where V_g is given by the index and C_g describes the kernel transition?

Here we show: No! The index part and the transition part cannot be separated.

Theorem 3 (A.-Dahl 2025&wip).

Let M be a closed connected spin manifold of dimension $n = 2$ or $n = 4$, $\alpha(M) \neq 0$. Then a splitting as above never exists.

Stratification of $\mathrm{Fred}_{\mathbb{K},z}^{\text{non-min}}$

$\mathrm{Fred}_{\mathbb{K}}^{\text{sym},\infty}$: the space of self-adjoint $\mathbb{K}$-linear Fredholm operators on $\mathcal{H}$ with discrete spectrum and infinitely many positive and negative eigenvalues and further pointwise algebraic structures

$\mathrm{Fred}_{\mathbb{K}}^{\text{sym},\infty} \xrightarrow{\alpha} \mathrm{KO}_n(*)$, $\mathrm{Fred}_{\mathbb{K},z}^{\text{sym},\infty} := \alpha^{-1}(z)$

$\mathrm{Fred}_{\mathbb{K},z}^{\text{min}} \subset \mathrm{Fred}_{\mathbb{K},z}^{\text{sym},\infty}$: the subspace of minimal operators of index z

$\mathrm{Fred}_{\mathbb{K},z}^{\text{non-min}} = \mathrm{Fred}_{\mathbb{K},z}^{\text{sym},\infty} \setminus \mathrm{Fred}_{\mathbb{K},z}^{\text{min}}$: the subspace of non-minimal operators

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$\mathrm{Fred}_{\mathbb{K},z}^{\text{non-min}}$ is a stratified subspace of $\mathrm{Fred}_{\mathbb{K},z}^{\text{sym},\infty}$. (Next slide)

$\mathrm{Fred}^{\text{non-min}}_{\mathbb{K},z}$ is a stratified subspace of $\mathrm{Fred}^{\text{sym},\infty}_{\mathbb{K},z}$

Fix $P_0 \in \mathrm{Fred}^{\text{non-min}}_{\mathbb{K},z}$

Study $P \in \mathrm{Fred}^{\text{sym},\infty}_{\mathbb{K},z}$, $P \approx P_0$

$\pi^g_{(-\epsilon,\epsilon)} P_0 : \mathcal{H} \to \ker P_0$ spectral projection of P_0 for the interval $(-\epsilon, \epsilon)$.

$\iota^g_{(-\epsilon,\epsilon)} P_0 : \ker P_0 \to \mathcal{H}$ its adjoint.

$\Pi(P) := \pi^g_{(-\epsilon,\epsilon)} P_0 \circ P \circ \iota^g_{(-\epsilon,\epsilon)} P_0 : \ker P_0 \to \ker P_0$

After identifying $\ker P \cong \mathbb{K}^k$, we have

$$\mathrm{Fred}^{\text{sym},\infty}_{\mathbb{K},z} \ni P_0 \approx P \mapsto \Pi(P) \in \mathbb{K}^{k \times k}$$

$$\begin{aligned} \mathrm{Fred}^{\text{sym},\infty}_{\mathbb{K},z} \supset \mathcal{U} &\xrightarrow{\Pi} \{A \in \mathbb{K}^{k \times k} \mid A^* = A \text{ and add. struct.}\} \\ P &\longmapsto \Pi(P) \\ \text{stratum of } P_0 &\longrightarrow 0 \\ \text{min. operators} &\longrightarrow \text{max. rank} \end{aligned}$$

Example $n = 3$

$k = \dim_{\mathbb{H}} \ker P_0$

$$\begin{aligned} \mathrm{Fred}_{\mathbb{H}}^{\mathrm{sym},\infty} = \mathrm{Fred}_{\mathbb{H},0}^{\mathrm{sym},\infty} \supset \mathcal{U} &\xrightarrow{\Pi} \{A \in \mathbb{H}^{k\times k} \mid A^* = A\} \\ P &\longmapsto \Pi(P) \\ \mathrm{Fred}_{\mathbb{H}}^{\mathrm{min}} &\longrightarrow \text{invertible matrices} \end{aligned}$$

no further algebraic structure

The top-dimensional strata of $\mathrm{Fred}_{\mathbb{K}}^{\text{non-min}}$ have $k = 1$ and thus codimension 1.

$\mathrm{Fred}_{\mathbb{K}}^{\mathrm{min}}$ is not connected. Connected components distinguished by spectral flow.

$\mathcal{R}_{\mathrm{inv}}(M)$ not connected, $\mathrm{ind\text{-}diff}(g_1, g_2) \in \mathrm{KO}_{m+1}(\{*\}) \cong \mathbb{Z}$ detects countably infinite many connected components

Transversality

Is the map $g \mapsto \not{D}^g$ transversal to the stratification?
Do we have for each stratum S_k of $\mathrm{Fred}_{\mathbb{K},z}^{\text{non-min}}$ and $\not{D}^{g_0} \in S_k$:

$$\mathrm{Image}\left(\frac{d}{dg}\bigg|_{g_0} \not{D}^g\right) + T_{g_0} S_k \overset{?}{=} T_{g_0}\,\mathrm{Fred}_{\mathbb{K},z}^{\text{sym},\infty}?$$

Then

$$\begin{aligned} &\text{codimension of } \mathrm{Fred}_{\mathbb{K},z}^{\text{non-min}} \text{ in } \mathrm{Fred}_{\mathbb{K},z}^{\text{sym},\infty} \\ = &\text{codimension of } \mathcal{R}_{\text{non-min}}(M) \text{ in } \mathcal{R}(M) \end{aligned}$$

Answer: sometimes

Our Tool: Determine $\mathrm{rank}\left(\frac{d}{dg}\Big|_{g_0} \Pi \not{D}^g\right)$

The case $n = 2$

$g_0 \in \mathcal{R}_{\text{non-min}}(M)$, $g \in \mathcal{R}(M)$, $g \approx g_0$

$k := \dim_{\mathbb{H}} \ker \not{D}^{g_0}$

$$\mathbb{K} = \mathbb{H}, \quad (\omega_2)^2 = -1, \quad \omega_2 \not{D}^g = -\not{D}^g \omega_2$$

Decompose into eigenvalues for $i\omega_2$

$$\ker \not{D}^g \mathrel{\widehat{=}} \mathbb{H}^k = \underset{i\omega_2 = 1}{\mathbb{C}^k} \oplus \underset{i\omega_2 = -1}{\mathbb{C}^k}$$

$$\Pi(\not{D}^g) = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad A \in \mathbb{C}^{k \times k}$$

$$J\text{-linearity} \iff A = -A^T$$

$$\left.\begin{array}{l} \text{real codimension is} \leq k(k-1) \\ \operatorname{rank}\left(\frac{d}{dg}\Big|_{g_0} \Pi \not{D}^g \right) \geq k(k-1) \end{array}\right\} \Rightarrow \operatorname{rank} = k(k-1)$$

The case $n = 2$, continued

We have transversality on all strata!
codim of $\mathrm{Fred}^{\text{non-min}}_{\mathbb{H},\alpha(M)}$ = codim of $\mathcal{R}_{\text{non-min}}(M)$ in $\mathcal{R}(M)$

$\alpha(M) = k \mod 2$

If $\alpha(M) = 0$: top-dimensional strata have $k = 2$,
codimension $2 \cdot (2 - 1) = 2$
$\Longrightarrow \mathcal{R}_{\text{inv}}(M)$ connected
Samuel Lockman has showed for some Riemann surface that $\pi_1(\mathcal{R}_{\text{inv}}(M)) \neq 1$.

If $\alpha(M) = 1$: top-dimensional strata have $k = 3$,
codimension $3 \cdot (3 - 1) = 6$
$\Longrightarrow \mathcal{R}_{\text{min}}(M)$ connected and $\pi_1(\mathcal{R}_{\text{min}}(M)) = \pi_2(\mathcal{R}_{\text{min}}(M)) = \pi_3(\mathcal{R}_{\text{min}}(M)) = \pi_4(\mathcal{R}_{\text{min}}(M)) = \{1\}$.

The case $n = 4$

$$(\omega_4)^2 = 1, \quad \omega_4 D\!\!\!/^g = -D\!\!\!/^g \omega_4$$

$$\Sigma^g M = \Sigma^g_+ M \oplus \Sigma^g_- M$$

$$D\!\!\!/^g_\pm : \Gamma(\Sigma^g_\pm M) \to \Gamma(\Sigma^g_\mp M)$$

$$k_\pm := \dim_{\mathbb{H}} \ker D\!\!\!/^g_\pm$$

$$a := k_+ - k_- = \alpha(M) = \hat{\mathcal{A}}(M)/2.$$

$$\Pi(D\!\!\!/^g) = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad A \in \mathbb{H}^{k_+ \times k_-}$$

No further algebraic structure

Wlog $a \geq 0$.
Then the real dimension of such matrices is $4k_- \cdot (k_- + a)$.

Top-dimensional strata of $\mathrm{Fred}^{\text{non-min}}_{\mathbb{K},a}$ have $k_- = 1$
$\Rightarrow$ real codimension is $4(1 + a)$.

The case $n = 4$

However: no transversality in general!

Lemma 4 (S. Maier 1997).

Let M be a closed connected manifold of dimension 4. Then

$$\mathrm{rank}\left(\frac{d}{dg}\Big|_{g_0}\Pi \not{D}^g\right) = 0 \Leftrightarrow (M, g_0) \text{ is conformal to a flat torus}$$

To the proof: "$\Rightarrow$"

$\ker\left(\frac{d}{dg}\Big|_{g_0}\Pi \not{D}^g\right)^*$ contains an $\mathbb{H}$-line.

Involved calculations show that it is conformal to a flat torus. $\square$

Conformally flat metrics are of infinite codimension.

The case $n = 4$ continued

We improved Maier's Lemma in two ways:

- $\operatorname{rank}\left(\frac{d}{dg}\Big|_{g_0} \Pi D\!\!\!/^g\right) \geq 2$, unless conformal to a flat torus
- For each pair of quaternionic lines $(\Phi_+ \cdot \mathbb{H}, \Phi_- \cdot \mathbb{H})$, $\Phi_\pm \in \ker D\!\!\!/^{g_0}_\pm$ we get a 2-dimensional space of variation, which deforms a harmonic spinor of real multiplicty 8 into an eigenspinor pair to eigenvalues $\pm\lambda$ (of real mult. 4) in $\mathrm{span}_{\mathbb{H}}\{\Phi_+, \Phi_-\}$.

We get on top-dimension strata

$$\operatorname{rank}\left(\frac{d}{dg}\Big|_{g_0} \Pi D\!\!\!/^g\right) \geq 2(1 + a).$$

Thus top-dimension strata of $\mathcal{R}_{\text{non-min}}(M)$ have real codimension $\geq 2(1 + a)$.

Transversality would imply $= 4(1 + a)$.

Full transversality in dimension 4?

$n = 4$: full transversality fails in many other examples, e.g. on products of Riemann surfaces.

Work in progress: metrics without transversality have infinite codimension.

Thank you for your attention!