Parallel spinors on Riemannian and Lorentzian manifolds

Overview over joint work with Klaus Kröncke, Olaf Müller and Jonathan Glöckle

Bernd Ammann[1]

[1] Universität Regensburg, Germany

Einstein Spaces and Special Geometry
Institut Mittag-Leffler
Djursholm, Sweden
July 10, 2023

http://www.berndammann.de/talks/2023Stockholm-handout.pdf

Outline

I. Spinors

I.1. Spinors on semi-Riemannian manifolds

Let (M, g) be a time- and space-oriented semi-Riemannian manifold of signature ℓ.
We assume that we have a fixed spin structure, i.e. a choice of a complex vector bundle $\Sigma^g M$, called the spinor bundle, with

$$\Sigma^g M \otimes_{\mathbb{C}} \Sigma^g M = \bigwedge{}^{\bullet/\mathrm{even}} T^* M \otimes_{\mathbb{R}} \mathbb{C}.$$

This bundle carries (fiberwise over $p \in M$)

- a non-degenerate hermitian product $\langle \cdot, \cdot \rangle$ (positive definite in the Riemannian case)
- a compatible connection
- a compatible Clifford multiplication $\mathrm{cl} : TM \otimes \Sigma^g M \to \Sigma^g M$, $\mathrm{cl}(X \otimes \varphi) =: X \bullet \varphi$ such that

$$X \bullet Y \bullet \varphi + Y \bullet X \bullet \varphi + 2g(X, Y)\,\varphi = 0.$$

$$\langle X \bullet \varphi, \psi \rangle = (-1)^{\ell+1} \langle \varphi, X \bullet \psi \rangle.$$

I.2 Associated vector fields

$$\langle X\cdot\varphi,\psi\rangle = (-1)^{\ell+1}\langle\varphi, X\cdot\psi\rangle.$$

Associated vector field: Quadratic map $\Sigma^g M \to TM$, $\varphi \mapsto V_\varphi$

For all $X \in TM$ we have:

Riemannian $\ell = 0$: $g(V_\varphi, X) = -i\langle X\cdot\varphi,\varphi\rangle$,

Lorentzian $\ell = 1$: $g(V_\varphi, X) = -\langle X\cdot\varphi,\varphi\rangle$

In the Lorentzian case

- V_φ is causal and future oriented,
- $V_\varphi = 0 \Leftrightarrow \varphi = 0$,
- $g(V_\varphi, V_\varphi) = 0 \Leftrightarrow \langle\varphi,\varphi\rangle = 0$,

I.3. Parallel spinors

Let (M, g) be a Riemannian or Lorentzian spin manifold.
Assume that $\varphi \neq 0$ is a parallel spinor,

$\Rightarrow \quad R_{X,Y}\varphi = 0$

$\Rightarrow \quad 0 = \sum \pm e_i \cdot R_{e_i,Y}\varphi \overset{!}{=} \frac{1}{2}\mathrm{Ric}(Y)\cdot\varphi$

$\Rightarrow \quad g(\mathrm{Ric}(Y), \mathrm{Ric}(Y))\varphi = -\mathrm{Ric}(Y)\cdot \mathrm{Ric}(Y)\cdot\varphi = 0$

In the Riemannian case: $\mathrm{Ric} = 0$

In the Lorentzian case:

$\mathrm{Ric}(Y)$ is lightlike for all Y

$\Rightarrow \quad \mathrm{Ric} = f\alpha \otimes \alpha$ for a lightlike 1-form α.

Remarks

(1) *A product $M_1 \times \cdots \times M_k$ of (semi-)Riemannian spin manifolds carries a parallel spinor if and only if each factor carries a parallel spinor.*

(2) *Compared to the Lorentzian case, Riemannian manifolds with parallel spinors are reasonably well understood (special holonomy, non-trivial parallel forms, topological obstructions on closed manifolds)*

I.4. Goals and Motivation

Goal: Better understanding of Lorentzian manifolds with parallel spinors

Motivations

- Parallel spinors are "odd supersymmetries"
- Special holonomy
- Equality case in index-theoretical arguments for Dirac–Witten operators
  (Similar role of parallel spinors and Ricci-flat metrics when studying closed Riemannian manifolds with $\mathrm{scal} \geq 0$.)

I.5. Associated vector fields of parallel spinors

Notation

Let $(\overline{M}, \bar{g})$ be a connected (glob. hyp.) Lorentzian spin manifold with spinor bundle $\Sigma\overline{M}$, $\dim \overline{M} = n+1$.

Let $0 \not\equiv \varphi \in \Gamma(\Sigma\overline{M})$ be parallel.

$\Rightarrow$ $V_\varphi \not\equiv 0$ is also parallel.

$\Rightarrow$ $V_\varphi^\perp$ is a parallel collection of hyperplanes of $T\overline{M}$, i.e. a parallel distribution.

This integrates to a foliation by hypersurfaces, i.e. for any $x \in \overline{M}$ there is an injectively immersed hypersurface $\iota_x : \mathcal{F}_x \hookrightarrow \overline{M}$ ($\mathcal{F}_x \triangleq \iota_x(\mathcal{F}_x)$) as follows:

- $x \in \mathcal{F}_x$
- $T_x\mathcal{F}_x = \left(V_\varphi|_x\right)^\perp$

Timelike associated vector field

Let us assume that V_φ is timelike.

Then–at least locally–

$$(\overline{M}, \bar{g}) \cong \big((a,b) \times \mathcal{F}_x, -\mathrm{d}t^2 + g_x^{\mathcal{F}}\big).$$

Then $(\mathcal{F}_x, g_x^{\mathcal{F}})$ is spacelike and has a parallel spinor.

This case reduces to the **Riemannian** case.

$\rightsquigarrow$ "reasonably well understood", not the topic of this talk.

Lightlike associated vector field

From now on consider: $V = V_\varphi$ is lightlike. $\rightsquigarrow$ tangent to $\mathcal{F}_x$.
We assume that the flow $\Phi_t^V \in \mathrm{Diff}(\mathcal{F}_x)$ of V exists for all times $t \in \mathbb{R}$, and this defines an $\mathbb{R}$-action on $\mathcal{F}_x$.

The flow Φ_t^V acts by isometries.

Define the $(n-1)$-dimensional Riemannian manifold

$$\widehat{Q}_x := \mathcal{F}_x \big/_{y \sim \varphi_t^V(y)}, \quad h_x := \text{submersion metric.}$$

The parallel spinor on $(\overline{M}, \bar{g})$ yields a parallel spinor on each $(\widehat{Q}_x, h_x)$.

II. The Lorentzian–Riemannian correspondence

Thus one may expects

$$\left\{\begin{array}{l}\text{(glob. hyp.) connected}\\ (n+1)\text{-dimensional}\\ \text{Lorentzian manifolds } \overline{M}\\ \text{with par. lightl. spinor}\end{array}\right\} \overset{1:1}{\leftrightsquigarrow} \left\{\begin{array}{l}\text{1-parameter families}\\ (n-1)\text{-dimensional}\\ \text{Riemannian manifolds } \widehat{Q}_x\\ \text{with a parallel spinor}\end{array}\right\}$$

This can be partially proven and is one of our goals.

$$\mathcal{R}_{\|}(Q) := \{\text{Riem. metrics with a non-triv. par. spinor}\},$$
$$\mathcal{M}od_{\|}(Q) := \mathcal{R}_{\|}(Q)/\operatorname{Diff}_{\mathrm{Id}}(Q).$$

Reformulate/sharpen:

$$\left\{\begin{array}{l}\text{Lorentzian manifolds } \overline{M}^{n+1}\\ \text{with parallel lightl. spinor}\\ +\text{ extra conditions}\end{array}\right\} \overset{1:1}{\leftrightsquigarrow} \{\text{Curves in } \mathcal{M}od_{\|}(Q^{n-1})\}$$

Assumption CompSpaceHyp

$(\overline{M}, \bar{g})$ is a closed connected Lorentzian spin manifold, $\dim \overline{M} = n + 1$,

- carries a parallel spinor $\varphi \not\equiv 0$ with a lightlike associated vector field $V = V_\varphi$
- carries a compact spacelike hypersurface M^n (without boundary) with induced Riemannian metric g and Weingarten map W

such that

1) the flow Φ_t^V of V exists for all $t \in \mathbb{R}$,
2) the flow lines $t \mapsto \Phi_t^V(x)$ intersect M precisely once.

Consequences/Notation

For $x \in M$ define $Q_x := \mathcal{F}_x \cap M \cong \widehat{Q}_x$.

Q_x, $x \in M$, is a foliation of M by hypersurfaces.

III. From Lorentzian manifolds with a parallel spinor to curves in $\mathcal{M}od_{\parallel}(Q)$

$\rightsquigarrow$: Work by H. Baum, T. Leistner, A. Lischewski (2014–19), reinterpreted.

Let $(\overline{M}, \bar{g})$ be a Lorentzian spin manifold with a parallel spinor φ, with Assumption CompSpaceHyp.

$$V_\varphi|_M = -U_\varphi + u_\varphi \nu$$

- ν is the future unit normal of M,
- U_φ is the associated vector field to $\varphi|_M$ on the Riemannian manifold (M, g)

$\alpha := g(U_\varphi, \cdot) \in \Omega^1(M)$ is closed and thus locally $\alpha = \mathrm{d}s$.
The foliation $\{Q_x\}_{x \in M}$ is locally given as $s = \mathrm{const}$.

Q_x is diffeomorphic to Q_y for all $x, y \in M$, and either all of them are closed, or all of them are dense.

Theorem (Baum, Leistner, Lischewski 2014)

If we "restrict" φ to M, then it satisfies the constraint equations

$$\left.\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \bullet \varphi, \ \forall X \in TM, \\ U_\varphi \bullet \varphi &= i u_\varphi \varphi, \end{aligned}\right\} \qquad \text{(CE)}$$

Theorem (Leistner, Lischewski 2019)

If (g, W, φ) is a solutions to (CE) then we have

$$g = \frac{1}{u_\varphi^2} ds^2 + h_s$$

where h_s is a Ricci-flat metric with parallel spinor on the foliation $\{Q_x\}_{x \in M}$.

Thus, this gives a map "$\rightsquigarrow$".

However, the factor $\frac{1}{u_\varphi^2} ds^2$ will **not** be obtained in "$\leftsquigarrow$"

Theorem (Leistner, Lischewski 2019)

If (g, W, φ) is a solutions to (CE) then we have

$$g = \frac{1}{u_\varphi^2}\mathrm{d}s^2 + h_s$$

where h_s is a structured Ricci-flat metric on the foliation $\{Q_x\}_{x \in M}$.

Theorem (A.–Kröncke (in progress))

For some fixed $x \in M$, we assume that $Q \coloneqq Q_x$ is closed. Then we can change the spatial hypersurface M to $M_0 \supset Q$, such that close to Q

$$g = \mathrm{d}s^2 + h_s.$$

Already known locally: Schimming/Galaev–Leistner.
Version above: solution of a Hamilton–Jacobi equation.

IV. From curves in $\mathcal{M}od_{\parallel}(Q)$ to Lorentzian manifolds w.p.s.

IV.1. From (CE) to Lorentzian manifolds w.p.s.

Now we consider $\leftsquigarrow$.

The following theorem was proven by

- H. Baum, T. Leistner, A. Lischewski 2014: real-analytic
- A. Lischewski 2015: smooth
- J. Seipel 2019: simple proof, following an idea by P. Chrusciel

Theorem

If (M, g) is a Riemannian manifold with a non-trivial solution of

$$\left.\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \bullet \varphi, \ \forall X \in TM, \\ U_\varphi \bullet \varphi &= i u_\varphi \varphi, \end{aligned}\right\} \qquad \text{(CE)}$$

then it extends to a Lorentzian metric on $\mathbb{R} \times M$ with a parallel spinor φ with V_φ lightlike.

IV.2. Ansatz for solving the constraint equations (CE)

$$M = (a,b) \times Q, \qquad g = \mathrm{d}s^2 + h_s$$

where Q is a closed spin manifold.
$h_s \in \mathcal{R}_{\parallel}(Q) := \{\text{Riem. metrics with a non-triv. par. spinor}\}$

Proposition (A.–Kröncke–Müller (2021))

If $(h_s)_{a\le s\le b}$ is a divergence-free path of Ricci-flat metrics, then there is a symmetric $W \in \mathrm{End}(TM)$ and $\varphi \in \Gamma(\Sigma M)$, such that (g, W, φ) solves (CE) on $M = (a,b) \times Q$.

We say that $(h_s)_{a\le s\le b}$ is divergence-free if

$$\mathrm{div}^{h_s}\left(\frac{d}{ds} h_s\right) = 0.$$

This means that this path of metrics (h_s) is orthogonal to the orbits of $\mathrm{Diff}_{\mathrm{Id}}(Q)$.

So – up to the problem that the normalization for u_φ only works semi-locally – we have:

$$\left\{ \begin{array}{l} \text{Lorentz manifolds } \overline{M}^{n+1} \\ \text{with par. lightl. spinor} \\ \text{twisted with flat line bdl.} \\ + \text{ extra conditions} \\ + \text{ choice of hyp.surf. } M \end{array} \right\} \overset{1:1}{\leftrightsquigarrow} \left\{ \begin{array}{l} \text{curves in the moduli space of} \\ \text{Riemannian metrics on } Q^{n-1} \\ \text{with a parallel spinor} \\ + \text{ scaling function } s \mapsto f(s) \\ + \text{ closing condition} \end{array} \right\}$$

Handout-Comment: Either remove the green or the blue line, to get a 1-to-1 correspondence.

V. Topological obstructions to solutions of (CE)

V.1. Are there topological obstructions to solutions of (CE)?

Motivation: Solutions of (CE) are the ids-analogue of a Riemannian parallel spinor.

$$\exists \nabla\varphi = 0 \;\Rightarrow\; \mathrm{Ric} = 0 \;\Rightarrow\; \pi_1(M) \text{ virtually abelian of rank} \leq n$$

Analyzing the parallel spinors on $\{Q_x\}_{x\in M}$ we get:

Theorem (A.–Glöckle (2023))

If M is a closed Riemannian spin manifold manifold with a solution to (CE), then $\pi_1(M)$ has a subgroup Γ of finite index that fits into the short exact sequence

$$\{1\} \to \mathbb{Z}^k \to \Gamma \to \mathbb{Z}^m \to \{1\},$$

with $k+m \in \{n, n-4, n-6, n-7, n-8, n-10, n-11, n-12, \ldots\}$.
($\pi_1(M)$ is virtually solvable with derived length ≤ 2.)

Further obstructions, e.g. Betti numbers.
If $k+m=n$, then a finite cover of M is homeomorphic to T^n.

V.2. Application of these obstructions

Let M be connected, closed, spin.
An initial data set (ids) on M, is a pair (g, k) of sections of $T^*M \odot T^*M$ such that g is positive definite.

$$2\rho = \mathrm{scal}^{\,g} + (\mathrm{tr}\, k)^2 - \|k\|^2$$
$$j = \mathrm{div} k - \mathrm{d}\,\mathrm{tr}\, k.$$

ids-DEC holds iff $\rho \geq |j|$.
ids-DEC holds strictly iff $\rho \geq |j|$ and at some x we have $\rho(x) > |j(x)|$.

If ids-DEC holds strictly, then the Dirac–Witten operator $\not{D}^W$ is invertible $\rightsquigarrow$ index-theoretical machinery (Jonathan Glöckle).

If ids-DEC holds, then any $\varphi \in \ker \not{D}^W$ is a solution to (CE).

V.3. Application: Example

Theorem (essentially due to J. Glöckle)

Let M be a closed connected spin manifold

(a) *with a "good" index theoretic obstruction to positive scalar curvature,*
e.g. $KO_{\dim M}(\{\bullet\})$, enlargeability,...

(b) *with no solution of (CE)*

*Then there is **no** Lorentzian metric $\bar{g}$ on $\overline{M} := (a,b) \times M$ such that*

- *$(\overline{M}, \bar{g})$ is globally hyperbolic with Cauchy hypersurface $\{t_0\} \times M$,*
- *it evolves from a big bang $\frac{k(X,X)}{g(X,X)} \gg 1$ to a big crunch $\frac{k(X,X)}{g(X,X)} \ll -1$,*
- *$(\overline{M}, \bar{g})$ satisfies the ~~strict~~ dominant energy condition (DEC).*

The theorem also holds for strict DEC without assumption (b).

For $\dim M = 3$:

Theorem applicable to $T^3 \# T^3$, $T^3 \# \mathbb{R}P^3$, hyperbolic pieces,..
Our goal: A Lorentzian metric $\bar{g}$ exists, if, and only if, M is a connected sums of S^3/Γ and $S^2 \times S^1$.