Yamabe constants, Yamabe invariants and Gromov-Lawson surgeries

Overview over (joint) work with Emmanuel Humbert, Mattias Dahl, Nadine Große, Nobuhiko Otoba

https://ammann.app.uni-regensburg.de/talks/gromov-lawson.pdf

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Einstein-Hilbert functional

Let M be a compact n-dimensional manifold, $n \geq 3$.
$\mathcal{R}(M) := \{\text{Riem. metrics on } M\}$.
The renormalised Einstein-Hilbert functional is

$$\mathcal{E}_M : \mathcal{R}(M) \to \mathbb{R}, \qquad \mathcal{E}_M(g) := \frac{\int_M \mathrm{scal}^g \, \mathrm{dvol}^g}{\mathrm{vol}(M, g)^{(n-2)/n}}$$

$[g_0] := \{u^{4/(n-2)} g_0 \mid u > 0\}$.

$\{\text{Stationary points of } \mathcal{E}_M : [g_0] \to \mathbb{R}\}$ = $\{$metrics with constant scalar curvature$\}$

$\{\text{Stationary points of } \mathcal{E}_M : \mathcal{R}(M) \to \mathbb{R}\}$ = $\{$Einstein metrics$\}$

(Conformal) Yamabe constant

The (conformal) Yamabe constant is defined as

$$Y_M([g]) := Y(M,[g]) := \inf_{\tilde{g}\in[g]} \mathcal{E}_M(\tilde{g}) > -\infty.$$

If $\mathbb{S}^n$ denote the sphere with the standard structure, then

$$Y_M([g]) \leq Y(\mathbb{S}^n).$$

Yamabe problem $\mathcal{E}_M : [g] \to \mathbb{R}$ attains its infimum.
Minimizers have $\mathrm{scal} = c$.

Proven by Trudinger 1968, Aubin 1976, Schoen (&Yau)) $\approx$ 1984

Remark

$Y_M([g]) > 0$ if and only if $[g]$ contains a metric of positive scalar curvature. Then the space of psc metrics in $[g]$ is contractible.

Reformulation and non-compact manifolds

Let $g = u^{4/(n-2)} g_0$, g_0 a complete metric on M.
Define Yamabe operator $L^{g_0} := 4\frac{n-1}{n-2}\Delta^{g_0} + \mathrm{scal}^{g_0}$.

$$\widetilde{Y}_M(g_0) := \inf \left\{ \frac{\int_M u\, L^{g_0} u \, \mathrm{dvol}^{g_0}}{\|u\|^2_{L^{2n/(n-2)}(M,g_0)}} \;\middle|\; 0 \not\equiv u \in C_c^\infty(M, [0,\infty)) \right\}$$

For compact M we have

$$Y_M([g_0]) = \widetilde{Y}_M(g_0).$$

For non-compact M we use this as a definition.

$\rightsquigarrow$ related work on $Y_M(g)$ for M non-compact by Akutagawa, Große, Ammann&Große, and others.

Obata's theorem

Theorem (Obata, 1971)

Assume:

- *M is connected and compact, $n = \dim M \geq 3$*
- *g_0 is an Einstein metric on M*
- *$g = u^{4/(n-2)} g_0$ with scal^g constant*
- *(M, g_0) not conformal to $\mathbb{S}^n$*

Then u is constant.

Conclusion

$$\mathcal{E}_M(g_0) = Y(M, [g_0])$$

This conclusion also holds on M compact if g_0 is a non-Einstein metric with $\mathrm{scal} = const \leq 0$ (Maximum principle).

So in these two cases, we have determined $Y(M, [g_0])$.

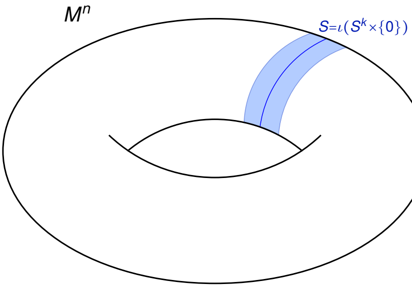
However, in general, it is difficult to get explicit "good" lower bounds for $Y(M, [g_0])$.

Recap: Surgery

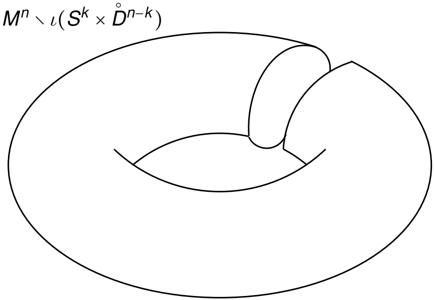
We consider an embedding of $\iota : S^k \times D^{n-k} \hookrightarrow M^n$.

Define $M^\# := \Big(M \setminus \iota(S^k \times \mathring{D}^{n-k}) \Big) \cup_{S^k \times S^{n-k-1}} \Big(D^{k+1} \times S^{n-k-1} \Big)$.

We say: $M^\#$ arises by k-dimensional surgery from M.



Picture for n=2, k=1



Picture for n=2, k=1

Gromov-Lawson surgery for Yamabe constants

Assume that $M^\#$ arises from M by a surgery of dimension $k \le n-3$.

For $\tau \in (0,\infty)$ and $g \in \mathcal{R}(M)$ we define a metric $\mathcal{GL}_\tau(g) \in \mathcal{R}(M^\#)$.

Theorem A (Ammann&Dahl&Humbert (2013))

There is a constant $\Lambda_{n,k} > 0$ *with:*

$$Y_{M^\#}\big(\mathcal{GL}_\tau(g)\big) \ge \min\big\{Y_M(g), \Lambda_{n,k}\big\} - o_\tau(1).$$

- Our metric $\mathcal{GL}_\tau(g)$ is similar to the Gromov-Lawson construction for positive scalar curvature metrics.
- Technical implementation differs.
- Special cases were known, e.g. a version with 0 instead $\Lambda_{n,k} >$ is due to Petean, the $k = 0$-case is due to O. Kobayashi, and the perservation of positivity is the classical Gromov&Lawson/Schoen&Yau result about psc-preserving surgeries.

Technical implementation

We write close to $S := \iota(S^k \times \{0\})$, $r(x) := d(x, S)$

$$g \approx g|_S + dr^2 + r^2 g_{\mathrm{round}}^{n-k-1}$$

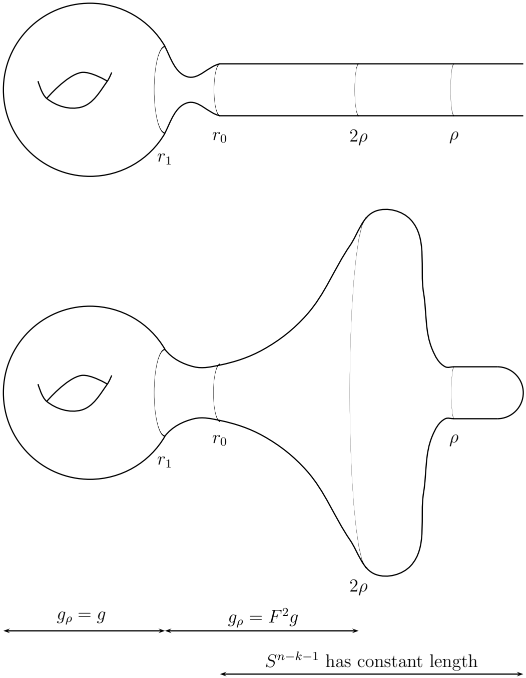
where $g_{\mathrm{round}}^{n-k-1}$ is the round metric on S^{n-k-1}.
$t := -\log r$.

$$\frac{1}{r^2} g \approx e^{2t} g|_S + dt^2 + g_{\mathrm{round}}^{n-k-1}$$

We define a metric

$$\mathcal{GL}_\tau(g) = \begin{cases} g & \text{for } r > r_1 \\ \frac{1}{r^2} g & \text{for } r \in (2\rho, r_0) \\ f^2(t) g|_S + dt^2 + g_{\mathrm{round}}^{n-k-1} & \text{for } r < 2\rho \end{cases}$$

that extends to a metric on $M^\#$.


r_0
2ρ
ρ
r_1
r_0
ρ
r_1
2ρ
$g_\rho = g$
$g_\rho = F^2 g$
S^{n-k-1} has constant length

Spaces of metrics with Yamabe constants above λ

$$Y_M^{-1}\big((\lambda,\infty)\big) := \{g \in \mathcal{R}(M) \mid Y_M([g]) > \lambda\}$$

$$\begin{aligned} Y_M^{-1}\big((0,\infty)\big) &:= \{g \in \mathcal{R}(M) \mid Y_M([g]) > 0\} \\ &= \{g \in \mathcal{R}(M) \mid [g] \text{ contains a psc metric}\} \\ &\simeq \mathcal{R}_+(M) := \{g \in \mathcal{R}(M) \mid \mathrm{scal}^M > 0\} \end{aligned}$$

Parametrized version of Theorem A

Assume that $M^{\#}$ is obtained by a k-dimensional surgery from M.

$$Y_M^{-1}\big((\lambda,\infty)\big) \xrightarrow{\mathcal{GL}} Y_{M^{\#}}^{-1}\big((\lambda,\infty)\big)$$

A generalized Chernysh-Walsh result follows then with the same analytical tools as in Thm. A = ADH 2013.

Theorem B (A. 2022, in prep.)

The map $\mathcal{GL}: Y_M^{-1}\big((\lambda,\infty)\big) \to Y_{M^{\#}}^{-1}\big((\lambda,\infty)\big)$ *is a (weak) homotopy equivalence for* $2 \leq k \leq n-3$.

The constants $\Lambda_{n,k}$

Obviously $\Lambda_{n,k} > 0$ is **not** unique, the larger the better.
Unless $n = k + 3 \geq 7$, our result holds for

$$\Lambda_{n,k} := \inf_{c \in [0,1]} Y\left(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}\right),$$

where $\mathbb{H}_c^{k+1}$ is the simply connected complete Riemannian manifolds of dimension $k + 1$ with $\sec = -c^2$.

$$\Lambda_n := \min\{\Lambda_{n,0}, \Lambda_{n,1}, \ldots, \Lambda_{n,n-3}\}$$

Examples:
$\Lambda_4 \geq 38.9$, $Y(\mathbb{S}^4) = 61.562\ldots$
$\Lambda_5 \geq 45.1$, $Y(\mathbb{S}^5) = 78.996\ldots$

The constants $\Lambda_{n,k}$ (ct'd)

In most cases we get some explicit values for $\Lambda_{n,k} > 0$:

n	k	known $\Lambda_{n,k}$	conjectured $\Lambda_{n,k}$	$Y(\mathbb{S}^n)$
3	0	43.82323	43.82323	43.82323
4	0	61.56239	61.56239	61.56239
4	1	$\geq$ 38.9	59.40481	61.56239
5	0	78.99686	78.99686	78.99686
5	1	$\geq$ 51.2	78.18644	78.99686
5	2	$\geq$ 45.1	75.39687	78.99686

The blue values rely on special investigations by Petean and Ruiz.

n	k	known $\Lambda_{n,k}$	conjectured $\Lambda_{n,k}$	$Y(^n)$
6	0	96.29728	96.29728	96.29728
6	1	> 0	95.87367	96.29728
6	2	≥ 54.77	94.71444	96.29728
6	3	≥ 49.98	91.68339	96.29728
7	0	113.5272	113.5272	113.5272
7	1	> 0	113.2670	113.5272
7	2	≥ 74.50	112.6214	113.5272
7	3	≥ 74.50	111.2934	113.5272
7	4	> 0	108.1625	113.5272

More values for $\Lambda_{n,k}$ To Conjectures

For $n \geq 7$, there are still problems with the explicit values for $k = 1$ and $k = n - 3$.

(Smooth) Yamabe invariant

For M compact:

$$\sigma(M) := \sup_{[g] \subset \mathcal{R}(M)} Y(M, [g]) \in (-\infty, Y(\mathbb{S}^n)]$$

smooth Yamabe invariant. (Introduced by O. Kobayashi and R. Schoen)

Remark

M caries a psc metric $\Leftrightarrow \sigma(M) > 0$

Supreme Einstein metrics

Following LeBrun, we say a Riemannian Einstein metric g on a closed manifold M is a supreme Einstein metric if

$$\mathcal{E}_M(g) = Y_M([g]) = \sigma(M).$$

The following Riem. manifolds are supreme Einstein:

- Round spheres trivial
- Flat tori (Gromov&Lawson, Schoen&Yau $\approx'$ 83)
  E.g. enlargeable Manifolds
- $\mathbb{R}P^3$ (Bray&Neves '04) Inverse mean curvature flow
- Compact quotients of 3-dim. hyperbolic space (Perelman, M. Anderson '06 (sketch), Kleiner&Lott '08) Ricci flow
- $(\mathbb{C}P^2, g_{\mathrm{FS}})$ (LeBrun) Seiberg-Witten theory, index theory $\rightsquigarrow$ next talk

If our conjectured values for $\Lambda_{n,k}$ hold, then $(\mathbb{C}P^3, g_{\mathrm{FS}})$ is not a supreme Einstein metric.

Manifolds with $0 < \sigma(M) < \Lambda_n$

Are there M with

$$0 < \sigma(M) < \Lambda_n := \min\{\Lambda_{n,0}, \dots, \Lambda_{n,n-3}\}\,?$$

Conjecture (Schoen)

If the finite group $\Gamma \subset \mathrm{SO}(n+1)$ acts freely on S^n, then the round metric g^n_{round} on S^n/Γ is a supreme Einstein metric.

The conjecture would imply

$$\mathcal{E}_{S^n/\Gamma}(g^n_{\mathrm{round}}) = Y(S^n/\Gamma, g^n_{\mathrm{round}}) = \sigma(S^n/\Gamma)$$
$$= n(n-1)\frac{\mathrm{vol}(\mathbb{S}^n)^{2/n}}{(\#\Gamma)^{2/n}} \xrightarrow{\#\Gamma\to\infty} 0.$$

Unfortunately, only known for $\Gamma = \{1\}$ and $\mathbb{R}P^3$.

A Monotonicity formula for surgery

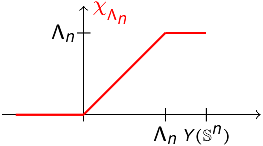
Corollary (ADH, follows from Theo. A)

Let $M^\#$ be obtained from M by k-dimensional surgery, $0 \le k \le n-3$. Then

$$\sigma(M^\#) \ge \min\{\sigma(M), \Lambda_{n,k}\}$$

We define $\Lambda_n := \min\{\Lambda_{n,0}, \ldots, \Lambda_{n,n-3}\}$ and

$$\chi_{\Lambda_n}(t) := \max\{\min\{t, \Lambda_n\}, 0\}.$$



For the truncated Yamabe invariant $\chi_{\Lambda_n}\big(\sigma(M)\big)$ we have

$$\boxed{\chi_{\Lambda_n}\big(\sigma(M^\#)\big) \ge \chi_{\Lambda_n}\big(\sigma(M)\big)}$$

and we have equality for $2 \le k \le n-3$.

Bordism results

Let $n \geq 5$, Γ finitely presented

Bordism techniques (Gromov-Lawson, Stolz,...) and Theorem A yield a well-defined map

$$\begin{aligned} s_\Gamma : \Omega_n^{\mathrm{spin}}(B\Gamma) &\to \mathbb{R} \\ [M, f] &\mapsto \chi_{\Lambda_n}\big(\sigma(M)\big) \end{aligned}$$

where we chose a representative with a connected non-empty M and $f_* : \pi_1(M) \to \Gamma$ bijective.

$$s_\Gamma(a+b) \geq \min\{s_\Gamma(a), s_\Gamma(b)\}$$

We get subgroups $s_\Gamma^{-1}\big((\lambda, \infty)\big) \subset \Omega_n^{\mathrm{spin}}(B\Gamma)$.

Descend to $\mathrm{ko}(B\Gamma)$

Recall from index theory

$$\Omega_n^{\mathrm{spin}}(B\Gamma) \xrightarrow{D} \mathrm{ko}_n(B\Gamma) \xrightarrow{\mathrm{per}} \mathrm{KO}_n(B\Gamma) \xrightarrow{A} \mathrm{KO}_n(C^*\Gamma)$$

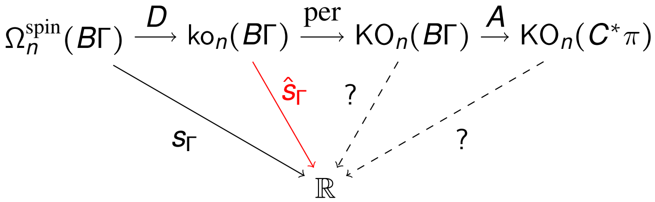
Descend to $\mathrm{ko}(B\Gamma)$

Recall from index theory

$$\begin{array}{ccccccc}
\Omega_n^{\mathrm{spin}}(B\Gamma) & \xrightarrow{D} & \mathrm{ko}_n(B\Gamma) & \xrightarrow{\mathrm{per}} & \mathrm{KO}_n(B\Gamma) & \xrightarrow{A} & \mathrm{KO}_n(C^*\Gamma) \\
 & \searrow^{s_\Gamma} & \downarrow ? & & \swarrow ? & \swarrow ? & \\
 & & & \mathbb{R} & & &
\end{array}$$

Descend to $\mathrm{ko}(B\Gamma)$

Recall from index theory



Theorem C (Ammann&Otoba, in prep.)

For a slightly adapted constant Λ_n, the truncated Yamabe invariant descents to a map $\mathrm{ko}_n(B\Gamma) \to \mathbb{R}$.

Idea of proof

One has to study Yamabe invariants of
$\ker\big(\Omega_n^{\mathrm{Spin}}(B\Gamma) \xrightarrow{D} \mathrm{ko}_n(B\Gamma)\big)$.
Given by Baas-Sullivan singular manifolds, obtained by gluing of multi-$\mathbb{H}P^2$-bundles, see work by Hanke.

Interpretations of Theorem B

Theorem B'

Let $\lambda \in [0, \Lambda_{n,k})$. *The map* $\mathcal{GL} : Y_M^{-1}\big((\lambda, \infty)\big) \to Y_{M^\#}^{-1}\big((\lambda, \infty)\big)$ *is well-defined (up to homotopy) for* $0 \leq k \leq n-3$ *and is a (weak) homotopy equivalences for* $2 \leq k \leq n-3$.

In fact these maps and the associated homotopies are compatible with the inclusion associated to $\lambda \geq \tilde{\lambda}$.
Thus we get a morphism of "filtered topological spaces"

$$\mathcal{GL} : \Big(Y_M^{-1}\big((\lambda, \infty)\big)\Big)_{\lambda \in [0, \Lambda_{n,k})} \to \Big(Y_{M^\#}^{-1}\big((\lambda, \infty)\big)\Big)_{\lambda \in [0, \Lambda_{n,k})},$$

which are "filtered homotopy equivalences" for $2 \leq k \leq n-3$.

Higher Yamabe invariants

Yamabe invariant $\sigma(M) := \sup\left\{\lambda \in \mathbb{R} \mid Y_M^{-1}\big((\lambda, \infty)\big) \neq \emptyset\right\}$

Higher Yamabe invariants

Yamabe invariant $\sigma(M) := \sup\left\{\lambda \in \mathbb{R} \mid Y_M^{-1}\big((\lambda,\infty)\big) \neq \varnothing\right\}$
$\pi_{-1}(\varnothing) = \varnothing, \quad \pi_{-1}(\underbrace{S}_{\neq *}) = \{*\}$

Functor from

$$\begin{array}{ccc} (\mathbb{R}, \geq) & \longrightarrow & \Big(\big\{\varnothing, \{*\}\big\}, \text{maps}\Big) \\ \lambda & \longmapsto & \pi_{-1}\left(Y_M^{-1}\big((\lambda,\infty)\big)\right) \\ \lambda \geq \tilde{\lambda} & \longmapsto & \pi_{-1}(\hookrightarrow) \end{array}$$

So far: nothing than a very complicated way to characterize a real number!

Higher Yamabe invariants

Truncated Yamabe invariant $\chi_{\Lambda_n}(\sigma(M))$
$\pi_{-1}(\varnothing) = \varnothing, \quad \pi_{-1}(\underbrace{S}_{\neq *}) = \{*\}$

Essentially a functor from

$$\begin{array}{rcl} ([0,\Lambda_n),\geq) & \longrightarrow & \left(\{\varnothing,\{*\}\},\text{maps}\right) \\ \lambda & \longmapsto & \pi_{-1}\left(Y_M^{-1}((\lambda,\infty))\right) \\ \lambda \geq \tilde{\lambda} & \longmapsto & \pi_{-1}(\hookrightarrow) \end{array}$$

So far: nothing than a very complicated way to characterize a number in $[0,\Lambda_n]$!

Higher Yamabe invariants, ct'd

Higher Yamabe invariant $\chi_{\Lambda_n}\left(\sigma^k(M)\right)$, $k \in \mathbb{N} \cup \{0\}$. For $k = 0$ we get a functor from

$$\begin{array}{ccc} ([0,\Lambda_n), \geq) & \xrightarrow{\chi_{\Lambda_n}(\sigma^k)} & \left(\text{sets}, \text{maps}\right) \\ \lambda & \longmapsto & \pi_0\left(Y_M^{-1}((\lambda,\infty))\right) \\ \lambda \geq \tilde{\lambda} & \longmapsto & \pi_0(\hookrightarrow) \end{array}$$

For $k > 1$ we get a functor from

$$\begin{array}{ccc} ([0,\Lambda_n), \geq) & \xrightarrow{\chi_{\Lambda_n}(\sigma^k)} & \left(\text{grps}^{\pi_0}, \text{hom}^{\pi_0}\right) \\ \lambda & \longmapsto & \pi_k\left(Y_M^{-1}((\lambda,\infty))\right) \\ \lambda \geq \tilde{\lambda} & \longmapsto & \pi_k(\hookrightarrow) \end{array}$$

For $k = 1$: similar with conjugacy classes.

Theorem B implies that all higher (truncated) Yamabe invariants are invariant under suitable bordisms, i.e. those that can be decomposed in surgeries of dimension $k \in \{2, 3, \ldots, n-3\}$.

We expect – but we are far from a proof – that the higher Yamabe invariants of $\mathbb{C}P^3$ are non-trivial for $82.986 \leq \lambda \leq 96.297$.

Sketch of Proof for Theorem A

Theorem A (Ammann&Dahl&Humbert (2013))

There is a constant $\Lambda_{n,k} > 0$ *with:*

$$Y_{M^\#}\big(\mathcal{GL}_\tau(g)\big) \geq \min\big\{Y_M(g), \Lambda_{n,k}\big\} - o_\tau(1).$$

Assume we have $\tau_i \to \infty$ with $g_i := \mathcal{GL}_{\tau_i}(g)$ and

$$\lambda_\infty := \lim_{i\to\infty} Y_{M^\#}(g_i) < Y(M,g).$$

Choose Yamabe minimizer $\tilde{g}_i \in [g_i]$.
After passing to a subsequence, then for some $p_i \in M^\#$

$$(M^\#, \tilde{g}_i, p_i) \to (N, h, p_\infty)$$

in the pointed Gromov-Hausdorff-C^∞-sense.

- Either: after removing singularities from (N, h, p_∞) we get (M, g); then $\lambda_\infty \geq Y(M, g)$. ↯
- Or (N, h, p_∞) is in a well-controlled family of model spaces. $\rightsquigarrow \Lambda_{n,k}$

Sketch of Proof for Theorem B

Theorem B (A. 2022, in prep.)

The map $\mathcal{GL} : Y_M^{-1}\big((\lambda,\infty)\big) \to Y_{M\#}^{-1}\big((\lambda,\infty)\big)$ *is a (weak) homotopy equivalence for* $2 \leq k \leq n-3$*.*

Well-definedness

Roughly as the proof of Theorem A, but in families.

Weak homotopy equivalence

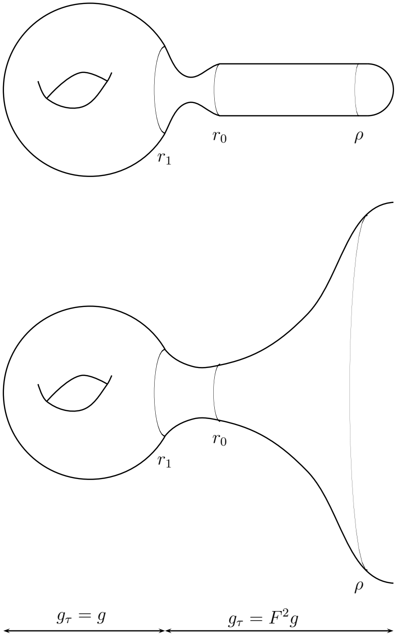
Split the construction in several steps, $\mathcal{R}_{>\lambda}(M) := Y_M^{-1}\big((\lambda,\infty)\big)$

Step no. 1 : $\mathcal{GL}_1$ makes the normal exponential maps coincide for all metrics in compact family, and cuts off the lower order terms.

$\mathcal{GL}^1 : \mathcal{R}_{>\lambda}(M) \to \mathcal{R}_{>\lambda}^{S^k,\epsilon}(M)$ is a homotopy inverse to the inclusion $\mathcal{R}_{>\lambda}^{S^k,\epsilon}(M) \hookrightarrow \mathcal{R}_{>\lambda}(M)$

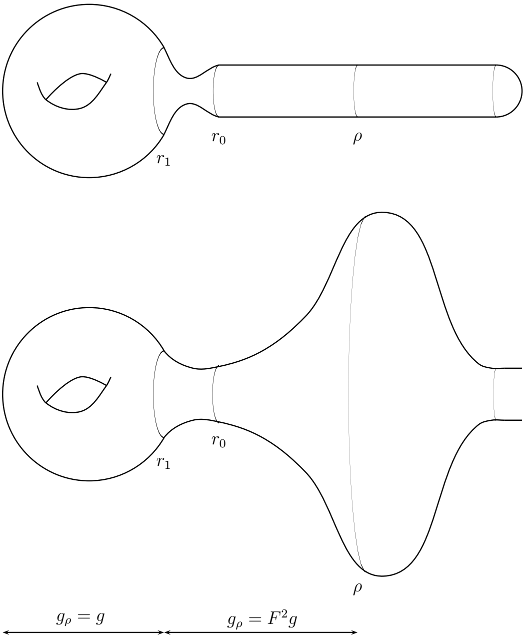
Step no. 2 Make a conformal deformation that makes the metrics in the normal direction of torpedo type. Obviously this step does not affect the Yamabe constant. Thus trivially we have a homotopy equivalence:

$$\mathcal{GL}^2 : \mathcal{R}_{>\lambda}^{S^k,\epsilon}(M) \to \mathcal{R}_{>\lambda}^{S^k,\epsilon,\text{infl-torp}}(M)$$


r_0
ρ
r_1
r_0
r_1
ρ
$g_\tau = g$
$g_\tau = F^2 g$

Step no. 3 Slow down the inflation in tangential direction slowly. One analyses certain blow-up limits in analogy to the proof of Theorem A. Then remove the curvature of the normal bundle of S. Below $\Lambda_{n,k}$ one obtains a homotopy equivalence.

$$\mathcal{GL}^3 : \mathcal{R}_{>\lambda}^{S^k,\epsilon,\text{infl-torp}}(M) \to \mathcal{R}_{>\lambda}^{S^k,\epsilon',\text{torp,prod}}(M)$$



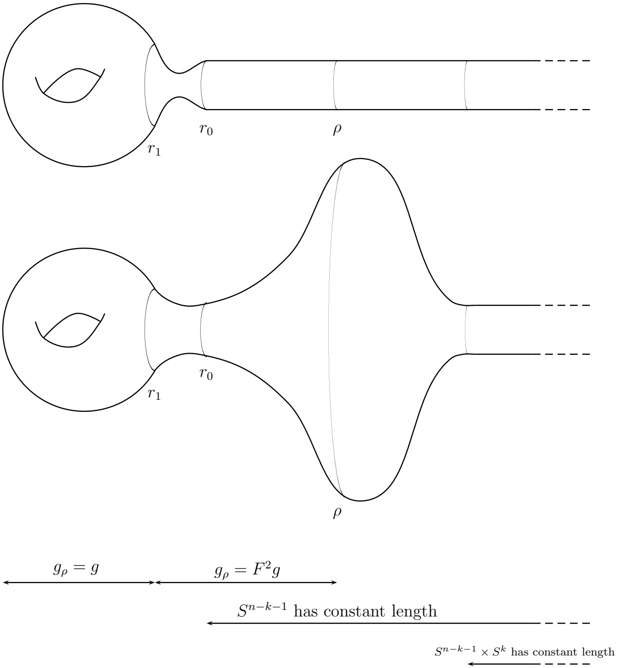
S^{n-k-1} has constant length

Step no. 4 Let the torpedos go to infinity. We get a convergence against manifolds with an end isometric (in a standard way) to

$$\left(S^k \times S^{n-k-1} \times [0,\infty), \mu_1 g^k_{\mathrm{round}} + \mu_2 g^{n-k-1}_{\mathrm{round}} + \mathrm{d}t^2\right).$$

Get a homotopy equivalence

$$\mathcal{GL}^4 : \mathcal{R}^{S^k,\epsilon',\mathrm{torp},\mathrm{prod}}_{>\lambda}(M) \to \mathcal{R}^{S^k \times S^{n-k-1},std}_{>\lambda}\big(M \setminus \iota(S^k \times 0)\big).$$


r_0
ρ
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r_0
r_1
ρ
$g_\rho = g$
$g_\rho = F^2 g$
S^{n-k-1} has constant length
$S^{n-k-1} \times S^k$ has constant length

With

$$M \smallsetminus \iota(S^k \times 0) \cong M^\# \smallsetminus \iota^\#(S^{n-k-1} \times 0)$$

we get

$$\mathcal{R}_{>\lambda}^{S^k \times S^{n-k-1}, std}\big(M \smallsetminus \iota(S^k \times 0)\big) \cong \mathcal{R}_{>\lambda}^{S^{n-k-1} \times S^k, std}\big(M^\# \smallsetminus \iota^\#(S^{n-k-1} \times 0)\big),$$

and this completes the proof. $\square$

Thanks for the attention.

Conjecture #1:

Back

$$Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$$

Conjecture #2:

The infimum in the definition of $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ is attained by an $O(k+1) \times O(n-k)$ invariant function if $0 \leq c < 1$.

$O(n-k)$-invariance is difficult,
$O(k+1)$-invariance follows from standard reflection methods

Comments

If we assume Conjecture #2, then Conjecture #1 reduces to an ODE and $Y(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ can be calculated numerically.
Assuming Conjecture #2, a maple calculation confirmed Conjecture #1 for all tested n, k and c.
The conjecture **would** imply:

$$\sigma(S^2 \times S^2) \geq \Lambda_{4,1} = 59.4\ldots$$

Compare this to

$$Y(\mathbb{S}^4) = 61.5\ldots \qquad Y(\mathbb{S}^2 \times \mathbb{S}^2) = 50.2\ldots \qquad \sigma(\mathbb{C}P^2) = 53.31\ldots$$

More values for $\Lambda_{n,k}$ Back

n	k	$\Lambda_{n,k} \geq$ known	$\Lambda_{n,k} =$ conjectured	$Y(\mathbb{S}^n)$
3	0	43.8	43.8	43.8
4	0	61.5	61.5	61.5
4	1	38.9	59.4	61.5
5	0	78.9	78.9	78.9
5	1	56.6	78.1	78.9
5	2	45.1	75.3	78.9
6	0	96.2	96.2	96.2
6	1	>0	95.8	96.2
6	2	54.7	94.7	96.2
6	3	49.9	91.6	96.2
7	0	113.5	113.5	113.5
7	1	>0	113.2	113.5
7	2	74.5	112.6	113.5
7	3	74.5	111.2	113.5
7	4	>0	108.1	113.5

Back

n	k	$\Lambda_{n,k} \geq$ known	$\Lambda_{n,k} =$ conjectured	$Y(\mathbb{S}^n)$
8	0	130.7	130.7	130.7
8	1	> 0	130.5	130.7
8	2	92.2	130.1	130.7
8	3	95.7	129.3	130.7
8	4	92.2	127.9	130.7
8	5	> 0	124.7	130.7
9	0	147.8	147.8	147.8
9	1	109.2	147.7	147.8
9	2	109.4	147.4	147.8
9	3	114.3	146.9	147.8
9	4	114.3	146.1	147.8
9	5	109.4	144.6	147.8
9	6	> 0	141.4	147.8

Back

n	k	$\Lambda_{n,k} \geq$ known	$\Lambda_{n,k} =$ conjectured	$Y(\mathbb{S}^n)$
10	0	165.0		165.02
10	1	102.6		165.02
10	2	126.4		165.02
10	3	132.0		165.02
10	4	133.3		165.02
10	5	132.0		165.02
10	6	126.4		165.02
10	7	> 0		165.02
11	0	182.1		182.1
11	1	> 0		182.1
11	2	143.3		182.1
11	3	149.4		182.1
11	4	151.3		182.1
11	5	151.3		182.1
11	6	149.4		182.1
11	7	143.3		182.1
11	8	> 0		182.1