Topology of spaces of metrics generalizing positive scalar curvature

Overview over joint work with Klaus Kröncke, Hartmut Weiß, Frederik Witt, Olaf Müller and Jonathan Glöckle

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Plan

I. Riemannian metrics of psc and nnsc

II. Strict DEC on Lorentzian initial data

III. DEC and the kernel of the Dirac–Witten operator

Recap: The Cauchy problem for Lorentzian manifolds with parallel spinors

DEC – Conclusions

Slides available on
`http://www.mathematik.uni-regensburg.de/ammann/talks/2021A-Fri-Ka-handout.pdf` or
`http://www.berndammann.de/talks`.

I. Riemannian metrics of psc and nnsc

The subtle difference between $\mathrm{scal} > 0$ and $\mathrm{scal} \geq 0$

Let M be a compact spin manifold.
For any Riemannian metric g we have a Dirac operator $D\!\!\!/ : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$. It is an elliptic self-adjoint operator, and thus a Fredholm operator.

Schrödinger-Lichnerowicz formula:

$$D\!\!\!/^2 \varphi = \nabla^* \nabla \varphi + \frac{\mathrm{scal}}{4} \varphi.$$

If $\mathrm{scal} > 0$, then $D\!\!\!/$ is invertible.

This allows many applications: obstructions to positive scalar curvature, information about the moduli space of psc metrics.

Questions for our talk:

- What about $\mathrm{scal} \geq 0$?
- Why are $\mathrm{scal} > 0$ and $\mathrm{scal} \geq 0$ interesting?
- How far do we get with Lorentzian analogues?

Non-negative scalar curvature

Now let M be a compact connected spin manifold, and let g be a Riemannian metric with $\mathrm{scal}^g \geq 0$.
Assume $D\!\!\!/\varphi = 0$, $\varphi \not\equiv 0$. Then

$$0 = \int_M \langle D\!\!\!/^2 \varphi, \varphi \rangle = \int_M \underbrace{\|\nabla\varphi\|^2}_{\geq 0} + \frac{1}{4} \int_M \underbrace{\mathrm{scal}^g}_{\geq 0} \underbrace{\|\varphi\|^2}_{\geq 0}.$$

Thus we have zero everywhere, e.g. $\nabla\varphi = 0$. (A parallel spinor)

$\Longrightarrow$ Strong implications.

Implications from parallel spinors, Part 1

Let (M, g) be a Riemannian or a Lorentzian connected spin manifold.
Assume that $\varphi \not\equiv 0$ is a parallel spinor.

$\Rightarrow \quad R_{X,Y}\varphi = 0$

$\Rightarrow \quad 0 = \sum \pm e_i \cdot R_{e_i,Y}\varphi \overset{!}{=} \frac{1}{2}\mathrm{Ric}(Y) \cdot \varphi$

$\Rightarrow \quad g(\mathrm{Ric}(Y), \mathrm{Ric}(Y))\varphi = -\mathrm{Ric}(Y) \cdot \mathrm{Ric}(Y) \cdot \varphi = 0$

In the Riemannian case: $\mathrm{Ric} = 0$

Theorem

If a connected Riemannian spin manifold (M, g) carries parallel spinor $\varphi \not\equiv 0$, then $\mathrm{Ric}^g = 0$.

Note: in the Lorentzian case we may only conclude that $\mathrm{Ric}(Y)$ is lightlike.

Implications from parallel spinors, Part 1, cont'd

From the Cheeger-Gromoll splitting theorem it follows:
If (M, g) is a compact Ricci-flat manifold, then it has a finite cover

$$(\hat{M}, \hat{h}) = (N, h) \times (\mathbb{R}^k/\Gamma), \quad \pi_1(N) = 1.$$

In particular, $\pi_1(M)$ is virtually abelian
(i.e. it contains an abelian subgroup of finite index).

Conclusion

If M is a compact spin manifold, and $\pi_1(M)$ is not virtually abelian, then we obtain

$$\begin{array}{rcl} \mathcal{R}^{\geq}(M) & \to & \text{Inv-Self-Adj} \\ g & \mapsto & D\!\!\!/^{g} \end{array}$$

$\mathcal{R}^{\geq}(M) := \{g \mid \mathrm{scal}^g \geq 0\}$,
$\text{Inv-Self-Adj} := \{$invert. self-adj. ops. with "some" additional structure$\}$

Implications from parallel spinors, Part 1, cont'd

Using the map

$$\begin{array}{rcl} \mathcal{R}^{\geq}(M) & \to & \text{Inv-Self-Adj} \\ g & \mapsto & D\!\!\!/^{g} \end{array}$$

we get the usual conlusions for psc, e.g.:

- If $0 \neq \operatorname{ind}(M) \in KO_{\dim M}(pt)$, then $\mathcal{R}^{\geq}(M) = \emptyset$.
- One can use the family index theorem to find non-trivial elements in $\pi_k(\mathcal{R}^{\geq}(M))$.
  (work for psc by Hitchin, Crowley–Hanke–Schick–Steimle, Botvinnik–Ebert–Randal-Williams)

Implications from parallel spinors, Part 2

If (M, g) carries a parallel spinor, then it has special holonomy.

$p \in M$:
$\mathrm{Hol}(M, g) := \{\text{Parallel transport along loops } p \rightsquigarrow p\} \subset O(n)$

If there is a parallel spinor, then there is a finite cover $\hat{M} \to M$ such that

$$\hat{M} = N_1 \times \ldots \times N_k,$$

$\mathrm{Hol}(N_i) \in \Big\{\{1\}, \mathrm{SU}(\ell), \mathrm{Sp}(\ell), G_2, \mathrm{Spin}(7)\Big\}$.

$\Rightarrow$ obstructions on Betti-numbers, e.g. (for $\dim M \geq 4$):
$b_4(M) \neq 0$ or $(b_3(\hat{M}) \neq 0$ and $b_6(\hat{M}) \neq 0 \ldots b_{3\dim M/7}(\hat{M}) \neq 0)$

If no metrics with par. spinors exists: conclusions as in Part 1.

Implications from parallel spinors, Part 3

If (M, g) carries a parallel spinor, then it is a **stable** Ricci-flat metric.

g cannot be deformed to a metric of positive scalar curvature

Theorem (Schick–Wraith)

Let M be a closed manifold with a psc metric g_0, and let $\mathcal{R}^{\geq}(M)_{g_0}$ be the path-connected component of g_0 in $\mathcal{R}^{\geq}(M)$. Then we get a map

$$\begin{array}{ccc} \mathcal{R}^{\geq}(M)_{g_0} & \to & \text{Inv-Self-Adj} \\ g & \mapsto & D\!\!\!/^{g} \end{array}$$

Conclusion

Nontrivial homotopy groups $\pi_k\big(\mathcal{R}^{\geq}(M)_{g_0}\big)$, $k \geq 1$.

Implications from parallel spinors, Part 3, cont'd

Important ingredient: good understanding of
$\mathcal{R}_{\parallel}(M) := \{g \in \mathcal{R}(M) \mid g \text{ has a parallel spinor}\}$

- $\mathcal{R}_{\parallel}(M)$ is a Fréchet submanifold of $\Gamma(T^*M \odot T^*M)$
- Smooth, finite-dim. premoduli space $\mathcal{R}_{\parallel}(M)/\operatorname{Diff}_0(M)$
- No psc metric in a neighborhood of $\mathcal{R}_{\parallel}(M)$

The case of irreducible holonomy, $\pi_1(M) = 1$ is well-understood due to work by McK Wang, Tian–Todorov, Joyce, Dai–G. Wang–Wei, Nordstroem,...

Additional effort required for **reducible holonomy** (or $\pi_1(M) \neq 1$)
Kröncke's stability (2015) & A.–Kröncke–Weiß–Witt (2019)

Goals:

- What is the motivation for understanding $\mathrm{scal} \geq 0$ coming from general relativity?
- Are there Lorentzian analogues?

We will Lorentzian see analogues for

- $\mathrm{scal} > 0$, $\mathrm{scal} \geq 0$
- Dirac operators
- parallel spinors
- Methods to detect topology in the moduli space
- Analogues to implications for parallel spinors, Part 1 and 2

However, no analogue to stability (yet?).
Obstructions do not seem optimal yet.

II. Strict DEC on Lorentzian initial data

Dominant energy condition

Let h be a Lorentzian metric on N
Energy-momentum tensor or Einstein tensor

$$T^h := \mathrm{Ric}^h - \frac{1}{2}\mathrm{scal}^h h$$

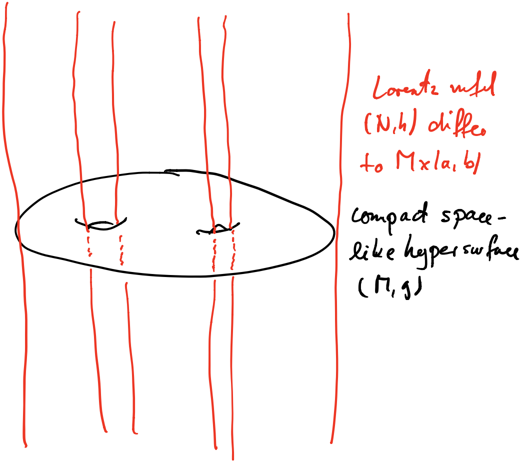
We say that h satisfies the dominant energy condition in $x \in N$ if for all causal future oriented vectors $X, Y \in T_x N$:

$$T(X, Y) \geq 0. \qquad \text{(DEC)}$$

Physical interpretation (Einstein equation):

Non-negative mass density of matter fields.

Assume: (N, h) is time- and space-oriented, globally hyperbolic, spin, compact Cauchy hypersurface



$$\mathcal{I}(M) := \{(g, W) \mid g \text{ Riemannian metric},$$
$$W \in \mathrm{End}(TM) \text{ symmetric}\}.$$

DEC on spacelike hypersurfaces

If M is a space-like hypersurface with induced metric g, and if ν is a future-oriented unit normal, then we define:
Energy density $\rho := T^h(\nu,\nu) = \frac{1}{2}\left(\mathrm{scal}^{\,g} + (\mathrm{tr}\,W)^2 - \mathrm{tr}(W^2)\right)$
Momentum density $j := T^h(\nu,\,\cdot\,)|_{T_xM} = \mathrm{div}\,W - \mathrm{d}\,\mathrm{tr}\,W$
DEC for h implies $\rho \geq |j|$.

Definition

Let g be a Riemannian metric and W a g-symmetric endomorphism section. We say that (g, W) satisfies

- the dominant energy condition if $\rho \geq |j|$ (DEC)

$$\mathcal{I}^{\geq}(M) := \{(g, W) \in \mathcal{I}(M) \text{ satisfying (DEC)}\}.$$

- the strict dominant energy condition if $\rho > |j|$ ($\mathrm{DEC}_>$)

$$\mathcal{I}^{>}(M) := \{(g, W) \in \mathcal{I}(M) \text{ satisfying } (\mathrm{DEC}_>)\}.$$

The inclusion $\mathcal{R}^{\geq}(M) \to \mathcal{I}^{\geq}(M)$

$$\mathcal{R}(M) \hookrightarrow \mathcal{I}(M),\ g \mapsto (g, 0)$$
$$\mathcal{R}^{>}(M) := \{g \in \mathcal{R} \mid \mathrm{scal}^{g} \geq 0\} = \mathcal{R}(M) \cap \mathcal{I}^{>}(M)$$
$$\mathcal{R}^{\geq}(M) := \{g \in \mathcal{R} \mid \mathrm{scal}^{g} > 0\} = \mathcal{R}(M) \cap \mathcal{I}^{\geq}(M)$$

- This is the main reason why Riemannian metrics with $\mathrm{scal} \geq 0$ (or $\mathrm{scal} > 0$) play a central role in general relativity.
- Can we control the topology of $\mathcal{I}^{\geq}(M)$?
- First important step: understand $\mathcal{I}^{>}(M)$ (J. Glöckle, arXiv:1906.00099)

Glöckle's work on $\mathcal{I}^{\geq}(M)$

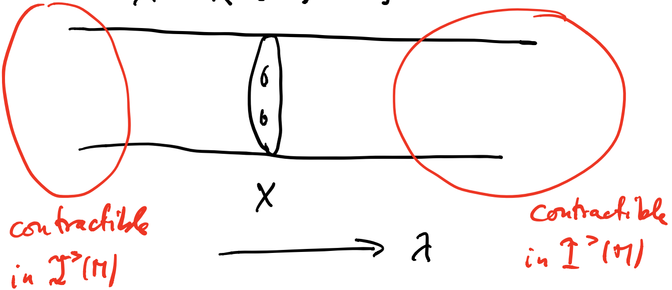
A lot is known about $\mathcal{I}^{>}(M)$.
In particular, we have $(g, \lambda\,\mathrm{Id}) \in \mathcal{I}^{>}(M)$ if

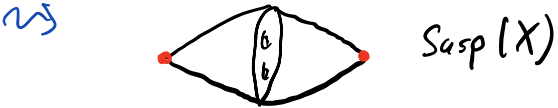
- $g \in \mathcal{R}^{>}(M)$ and $\lambda \in \mathbb{R}$, or
- $g \in \mathcal{R}^{\geq}(M)$ and $\lambda \in \mathbb{R} \setminus \{0\}$, or
- $g \in \mathcal{R}(M)$ and $|\lambda| \gg 0$.

We get a map $\mathrm{Susp}\big(\mathcal{R}^{>}(M)\big) \to \mathcal{I}^{>}(M)$.

$$\mathrm{Susp}\big(\mathcal{R}^{>}(M)\big) = \Big(\mathcal{R}^{>}(M) \times [-1,1]/M \times \{-1\}\Big)/M \times \{1\}.$$

$$X = R^{>}(M) \times \{0\} \subset I^{>}(M)$$





The Lorentzian α-index

For any $\Psi : S^{k+1} \to \mathcal{I}^{>}(M)$ J. Glöckle constructs

$$\alpha_{\mathrm{Lor}}(\Psi) \in \mathrm{KO}_{m+k+1}(\{*\}) = \begin{cases} \mathbb{Z} & \text{if } m+k+1 \in 4\mathbb{N} \\ \mathbb{Z}/2 & \text{if } m+k+1 \in 8\mathbb{N}+1 \\ & \text{or } m+k+1 \in 8\mathbb{N}+2 \\ 0 & \text{else} \end{cases}$$

$m = \dim M$.

Theorem (J. Glöckle 2019)

The diagram

$$\begin{array}{ccccc} \pi_k\big(\mathcal{R}^{>}(M)\big) & \xrightarrow{\mathrm{Susp}} & \pi_{k+1}\big(\mathrm{Susp}(\mathcal{R}^{>}(M)\big) & \longrightarrow & \pi_{k+1}\big(\mathcal{I}^{>}(M)\big) \\ & \searrow^{\alpha_{Riem}} & & \swarrow_{\alpha_{Lor}} & \\ & & KO_{m+k+1}(\{*\}) & & \end{array}$$

commutes.

Key technique in Glöckle's article: The Dirac–Witten operator

Literature: Witten 1981, Parker-Taubes, Hijazi-Zhang, ..., Glöckle 2019.

Restrict the spinor bundle ΣN from (N, h) to (M, g).

As spinor module $\Sigma N|_M$ is one or two copies of ΣM.

However:

scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on ΣN is indefinite (splitt signature),

scalar product $\langle \cdot, \cdot \rangle$ on ΣM positive definite.

They are related by

$$\langle \varphi, \psi \rangle = \langle\!\langle \nu \cdot \varphi, \psi \rangle\!\rangle.$$

The connections differ:

$$\nabla^N_X \varphi = \nabla^M_X \varphi - \frac{1}{2} \nu \cdot W(X) \cdot \varphi$$

Dirac–Witten-Operator

$$\not{D}^{(g,W)} \varphi = \sum_{j=1}^{m} e_j \cdot \nabla^N_{e_j} \varphi$$

where $(e_1, \ldots, e_m)$ is a locally defined orthonorm. frame of TM.

$\not{D}^{(g,W)}$ is self-adjoint and Fredholm.

Schrödinger-Lichnerowicz formula:

$$\left(\not{D}^{(g,W)}\right)^2 = (\nabla^N)^* \nabla^N + \frac{1}{2}(\rho - \nu \cdot j^\sharp \cdot),$$

The $*$ is taken on M with respect to $\langle \cdot, \cdot \rangle$.
Recall:
Energy density $\rho := T^h(\nu, \nu) = \frac{1}{2}\left(\mathrm{scal}^g + (\mathrm{tr} W)^2 - \mathrm{tr}(W^2)\right)$
Momentum density $j := T^h(\nu, \cdot)|_{T_x M} = \mathrm{div} W - \mathrm{d}\,\mathrm{tr} W$
DEC for h implies $\rho \geq |j|$.

This implies that $\not{D}^{(g,W)}$ is invertible if $(g, W) \in \mathcal{I}^{>}(M)$.
As a consequence Glöckle can use index theoretical methods.

Understanding $\mathcal{I}^{\geq}(M)$

In the following diagram we assume $k \geq 1$ and that the base point is g_0 resp. $(g_0, 0)$ where g_0 has positive scalar curvature.

$$
\begin{array}{ccc}
\pi_k\big(\mathcal{R}^{>}(M)\big) & \longrightarrow & \pi_{k+1}\big(\mathcal{I}^{>}(M)\big) \\
\big\downarrow & & \big\downarrow \\
\pi_k\big(\mathcal{R}^{\geq}(M)\big) & \longrightarrow & \pi_{k+1}\big(\mathcal{I}^{\geq}(M)\big)
\end{array}
$$

Index theoretically determined non-trivial homotopy groups survive in the upper right and in the lower left corner.
What about the lower right corner?

III. DEC and the kernel of the Dirac–Witten operator

Proposition (Ammann, Glöckle (2021))

Assume that M is a connected closed spin manifold and $(g, W) \in \mathcal{I}^{\geq}(M)$.
We assume that $\varphi \in \ker \not{D}^{(g,W)} \setminus \{0\}$.
Then g, W, φ provides initial data for a Lorentzian manifold with a parallel spinor.

In fact φ: is a parallel section of $\Sigma N|_M \to M$.

Recap: The Cauchy problem for Lorentzian manifolds with parallel spinors

Work by H. Baum, T. Leistner, A. Lischewski

Let (N, h) be a space- and time-oriented Lorentzian spin manifold with a parallel spinor Φ.

The Dirac current of (N, h, Φ) is the vector field V_Φ with

$$h(X, V_\Phi) = -\langle\!\langle X \cdot \Phi, \Phi \rangle\!\rangle \quad \forall X \in TN.$$

As Φ is parallel, the vector field V_Φ is also parallel.

One can show:

- $h(V_\Phi, V_\Phi) \leq 0$, i.e. V_Φ is causal.
- V_Φ is future oriented.
- $\mathrm{Ric}^N \parallel V_\Phi^b \otimes V_\Phi^b$.
- If V_Φ is timelike, then N is stationary and $\mathrm{Ric}^N = 0$.

Thus two cases may arise:

(1) V_Φ timelike

(2) V_Φ lightlike

In both cases $\varphi := \Phi|_M$ is a parallel section of $\Sigma N|_M$.
This is equivalent to the generalized imaginary Killing spinor equation

$$\nabla^M_X \varphi = \frac{i}{2} W(X) \bullet \varphi, \qquad \forall X \in TM \qquad \text{(giKs)}$$

This implies $\not{D}^{(g,W)}(\varphi) = 0$.

Our proposition (A.-Glöckle 2021) implies, conversely:
If M is a closed spin manifold, and if (g, W) satisfies the dominant energy condition, then every $\varphi \in \ker \not{D}^{(g,W)}$ satisfies (ce).

Goal: Determine necessary conditions for (giKs).

The timelike case

Assume V_Φ timelike.
Then (N, h) can be extended such that

$$(\widetilde{N}, \tilde{h}) = (\widetilde{M}, \widetilde{g}_0) \times (\mathbb{R}, -dt^2) \Big/_{\pi_1(M)}$$

where a homomorphism $\pi_1(M) \to \mathbb{R}$ defines the action on $\mathbb{R}$.

Thus M carries a metric g_0 with a parallel spinor.
Same obstructions as in Section I.

The lightlike case: geometric picture

(inspired by Baum, Leistner, and Lischewski)

Assume N to be globally hyperbolic with a parallel lightlike spinor and a compact Cauchy surface M.
Then (N, h) can be extended to be geodesically complete.
$(V_\varphi)^\perp$ is a parallel distribution of codimension 1.
Thus there is a foliation by $(\mathcal{L}_x)$ and if $y \in \mathcal{L}_x$ then

$$V_\varphi|_y = T_y\mathcal{L}_x$$

$V_\varphi \in (V_\varphi)^\perp$, a Killing vector field.
$\mathcal{F}_x := (M \cap \mathcal{L}_x)$ defines a codimension 1 foliation of M.
These leaves carry a metric with a parallel spinor.

Write

$$V_\Phi|_M = -U_\Phi + u_\Phi \nu$$

U_Φ tangential to M
ν future unit normal of M
Then the flow of U_Φ maps leaves to leaves.

Compact leaves

Case 1: One leaf (and thus all leaves) is/are non-compact. Then M is a mapping torus of some spin diffeomorphism $f : Q \to Q$.

$$
\begin{aligned}
M = M_f &= Q \times [0,1] \big/ (x,0) \sim (f(x),1) \\
g &= g_s + \frac{1}{u_\Phi^2} ds^2
\end{aligned}
$$

Here g_s is a family of metrics on Q with a parallel spinor.

We get a smooth path in $\mathcal{R}_\parallel(Q)/\operatorname{Diff}_0(Q)$ and a loop in $\mathcal{R}_\parallel(Q)/\operatorname{Diff}_{\mathrm{spin}}(Q)$.

Questions:

- Why do I not say "a loop in $\mathcal{R}_\parallel(Q)$"?
- Are there examples of unclosed paths in $\mathcal{R}_\parallel(Q)/\operatorname{Diff}_0(Q)$?
- Does every loop in $\mathcal{R}_\parallel(Q)/\operatorname{Diff}_{\mathrm{spin}}(Q)$ give a generalized imaginary Killing spinor (giKs)?

Compact leaves, cont'd

- Why do I not say "a loop in $\mathcal{R}_{\|}(Q)$"?

There is no natural way to identify the leaves. It depends on the choice of the Cauchy hypersurface.

- Are there examples of unclosed paths in $\mathcal{R}_{\|}(Q)/\operatorname{Diff}_0(Q)$?

Yes! $Q = \mathbb{R}/\mathbb{Z}^2$.

Example (Sol geometry, 3-dim)

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \exp(B)$ defines a diffeomorphism $\frac{\mathbb{R}^2}{\mathbb{Z}^2} \xrightarrow{f=A} \frac{\mathbb{R}^2}{\mathbb{Z}^2}$.

$g = g_s + ds^2$ and $g_s := \exp(-sB)^* g_{\mathrm{eucl}}$.

A.-Kröncke-Müller (2019 pp): we get a giKs on the mapping torus M_A of A.

$\Rightarrow \; \pi_1(M_A) = \mathbb{Z}^2 \rtimes \mathbb{Z}$ is solvable

Compact leaves, cont'd

Example (Nil geometry, 3-dim)

$$A = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} = \exp\left(\begin{pmatrix} 0 & 0 \\ \ell & 0 \end{pmatrix}\right) = \exp\Bigg(\overbrace{\begin{pmatrix} 0 & \ell/2 \\ \ell/2 & 0 \end{pmatrix}}^{B_{\mathrm{sym}}} + \overbrace{\begin{pmatrix} 0 & -\ell/2 \\ \ell/2 & 0 \end{pmatrix}}^{B_{\mathrm{as}}}\Bigg)$$

defines a diffeomorphism $\in \mathrm{Diff}\left(\frac{\mathbb{R}^2}{\mathbb{Z}\oplus\ell\mathbb{Z}}\right)$; $g_s := \begin{pmatrix} 1 & 0 \\ -s\ell & 1 \end{pmatrix}^* g_{\mathrm{eucl}}$

We get a giKs on $\frac{\mathbb{R}^2}{\mathbb{Z}\oplus\ell\mathbb{Z}} \times \mathbb{R}$, $g = g_s + ds^2$.

Because of the antisymmetric part B_{as}, the spinor will "rotate" and not "close up" in general.

For $L\ell \in 8\pi\mathbb{Z}$ the spinor is L-periodic in s and we get an example on a Heisenberg-manifold.

$1 \to \mathbb{Z} \to \pi_1 \to \mathbb{Z}^2 \to 1$ centrally, $\pi_1 = \mathbb{Z}^2 \rtimes \mathbb{Z}$

Non-compact leaves

Case 2: One leaf (and thus all leaves) is/are non-compact.

All leaves are dense and isometric.
However the flow of U_φ is not isometric, even after an isotopy.
Example: a “tilted” variant of the Nil geometry example.

Theorem (Ammann–Glöckle 2021)

If a closed spin manifold M carries a Riemannian metric g with a non-trivial lightlike giKs and non-compact leaves, then $b_1(M) = 1$, and there is a finite cover

$$\hat{M} = P \times T \xrightarrow{\textit{finite}} M$$

where P is a simply connected, compact manifold with a parallel spinor, and where T is a torus bundle over a closed manifold B. Furthermore B^k is homeomorphic to a torus, and B has a dense codimension-1-foliation by leaves diffeomorphic to $\mathbb{R}^{k-1}$ and a transversal measure.

DEC – Conclusions

If a closed spin manifold M carries a giKs, ...

- then $\pi_1(M)$ is virtually solvable of derived length at most 2, i.e. there is a finite index subgroup $\pi \subset \pi_1(M)$ fitting in the short exact sequence
$$1 \to \mathbb{Z}^\ell \to \pi \to \mathbb{Z}^k \to 1$$
with $\dim M \geq k + \ell$.
- ...and if $\dim M = k + \ell$, then M is finitely covered by torus bundle over a topological torus
- ...and if $\dim M > k + \ell$, then M is finitely covered by some $\hat{M}$ with $b_4(\hat{M}) \neq 0$ or $b_3(\hat{M}) \neq 0$.

...

If M does not satisfy one of these necessary conditions, then Glöckle's Lorentzian α-index yields:

- If $\mathrm{ind}\,(M) \neq 0$ in $\mathrm{KO}_{\dim M}(pt)$ then, $\mathcal{I}^{\geq}(M)$ is not connected: there is no path from a "big bang" to "big crunch".
- If $m = \dim M \geq 6$ and if $m + k \in 4\mathbb{Z} \cup (8\mathbb{Z} + 1) \cup (8\mathbb{Z} + 2)$, then $\pi_k(\mathcal{I}^{\geq}(M)) \to \mathrm{KO}_{m+k}(pt)$ is non-trivial.