THE SUPREMUM OF FIRST EIGENVALUES OF CONFORMALLY COVARIANT OPERATORS IN A CONFORMAL CLASS

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ABSTRACT. Let (M,g) be a compact Riemannian manifold of dimension ≥ 3. We show that there is a metric $\tilde{g}$ conformal to g and of volume 1 such that the first positive eigenvalue of the conformal Laplacian with respect to $\tilde{g}$ is arbitrarily large. A similar statement is proven for the first positive eigenvalue of the Dirac operator on a spin manifold of dimension ≥ 2.

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1. INTRODUCTION

The goal of this article is to prove the following theorems.

Theorem 1.1. *Let (M,g_0,χ) be compact Riemannian spin manifold of dimension $n\geq 2$. For any metric g in the conformal class $[g_0]$, we denote the first positive eigenvalue of the Dirac operator on (M,g,χ) by $\lambda_1^+(D_g)$. Then*

$$\sup_{g\in[g_0]} \lambda_1^+(D_g)\mathrm{Vol}(M,g)^{1/n}=\infty.$$

Theorem 1.2. *Let (M,g_0,χ) be compact Riemannian manifold of dimension $n\geq 3$. For any metric g in the conformal class $[g_0]$, we denote the first positive eigenvalue of the conformal Laplacian $L_g := \Delta_g + \frac{n-2}{4(n-1)}\mathrm{Scal}_g$ (also called Yamabe operator) on (M,g,χ) by $\lambda_1^+(L_g)$. Then*

$$\sup_{g\in[g_0]} \lambda_1^+(L_g)\mathrm{Vol}(M,g)^{2/n}=\infty.$$

The Dirac operator and the conformal Laplacian belong to a large family of operators, defined in details in subsection 2.3. These operators are called conformally covariant elliptic operators of order k and of bidegree $((n-k)/2,(n+k)/2)$, acting on manifolds (M,g) of dimension $n>k$. In our definition we also claim formal self-adjointness.

All such conformally covariant elliptic operators of order k and of bidegree $((n-k)/2,(n+k)/2)$ share several analytical properties, in particular they are associated to the non-compact embedding $H^{k/2}\to$

$L^{2n/(n-k)}$. Often they have interpretations in conformal geometry. To give an example, we define for a compact Riemannian manifold (M, g_0)

$$Y(M,[g_0]) := \inf_{g\in[g_0]} \lambda_1(L_g)\mathrm{Vol}(M,g)^{2/n},$$

where $\lambda_1(L_g)$ is the lowest eigenvalue of L_g. If $Y(M,[g_0]) > 0$, then the solution of the Yamabe problem [29] tells us that the infimum is attained and the minimizer is a metric of constant scalar curvature. This famous problem was finally solved by Schoen and Yau using the positive mass theorem.

In a similar way, for $n = 2$ the Dirac operator is associated to constant-mean-curvature conformal immersions of the universal covering into $\mathbb{R}^3$. If a Dirac-operator-analogue of the positive mass theorem holds for a given manifold (M, g_0), then the infimum

$$\inf_{g\in[g_0]} \lambda_1^+(D_g)\mathrm{Vol}(M,g)^{1/n}$$

is attained [3]. However, it is still unclear whether such a Dirac-operator-analogue of the positive mass theorem holds in general.

The Yamabe problem and its Dirac operator analogue, as well as the analogues for other conformally covariant operators are typically solved by minimizing an associated variational problem. As the Sobolev embedding $H^{k/2} \to L^{2n/(n-k)}$ is non-compact, the direct method of the calculus of variation fails, but perturbation techniques and conformal blow-up techniques typically work. Hence all these operators share many properties.

However, only few statements can be proven simultaneously for all conformally covariant elliptic operators of order k and of bidegree $((n-k)/2,(n+k)/2)$. Some of the operators are bounded from below (e.g. the Yamabe and the Paneitz operator), whereas others are not (e.g. the Dirac operator). Some of them admit a maximum principle, others do not. Some of them act on functions, others on sections of vector bundles. The associated Sobolev space $H^{k/2}$ has non-integer order if k is odd, hence it is not the natural domain of a differential operator. For Dirac operators, the spin structure has to be considered in order to derive a statement as Theorem 1.1 for $n = 2$. Because of these differences, most analytical properties have to be proven for each operator separately.

We consider it hence as remarkable that the proof of our Theorems 1.1 and 1.2 can be extended to all such operators. Our proof only uses some few properties of the operators, defined axiomatically in 2.3. More exactly we prove the following.

Theorem 1.3. *Let P_g be a conformally covariant elliptic operator of order k, of bidegree $((n-k)/2,(n+k)/2)$ acting on manifolds of dimension $n > k$. We also assume that P_g is invertible on $\mathbb{S}^{n-1}\times\mathbb{R}$ (see Definition 2.4). Let (M, g_0) be compact Riemannian manifold. In the case that P_g depends on the spin structure, we assume that M is oriented and is equipped with a spin structure. For any metric g in the conformal class $[g_0]$, we denote the first positive eigenvalue of P_g by $\lambda_1^+(P_g)$. Then*

$$\sup_{g\in[g_0]} \lambda_1^+(P_g)\mathrm{Vol}(M,g)^{k/n} = \infty.$$

The interest in this result is motivated by three questions. At first, as already mentioned above the infimum

$$\inf_{g\in[g_0]} \lambda_1^+(D_g)\mathrm{Vol}(M,g)^{1/n}$$

reflects a rich geometrical structure [3], [4], [5], [7], [8], similarly for the conformal Laplacian. It seems natural to study the supremum as well.

The second motivation comes from comparing Theorem 1.3 to results about some other differential operators. For the Hodge Laplacian Δ_p^g acting on p-forms, we have $\sup_{g\in[g_0]} \lambda_1(\Delta_p^g)\mathrm{Vol}(M,g)^{2/n} = +\infty$ for $n \geq 4$ and $2 \leq p \leq n-2$ ([19]). On the other hand, for the Laplacian Δ^g acting on functions, we have $\sup_{g\in[g_0]} \lambda_k(\Delta^g)\mathrm{Vol}(M,g)^{2/n} < +\infty$ (the case $k = 1$ is proven in [20] and the general case in [27]). See [25] for a synthetic presentation of this subject.

The essential idea in our proof is to construct metrics with longer and longer cylindrical parts. We will call this an *asymptotically cylindrical blowup*. Such metrics are also called *Pinocchio metrics* in [2, 6]. In [2, 6] the behavior of Dirac eigenvalues on such metrics has already been studied partially, but the present article has much stronger results. To extend these existing results provides the third motivation.

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2. Preliminaries

2.1. **Notations.** In this article $B_y(r)$ denotes the ball of radius r around y, $S_y(r) = \partial B_y(r)$ its boundary. The standard sphere $S_0(1) \subset \mathbb{R}^n$ in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$, its volume is ω_{n-1}. For the volume element of (M, g) we use the notation dv^g. In our article, $\Gamma(V)$ (resp. $\Gamma_c(V)$) always denotes the set of all smooth sections (resp. all compactly supported smooth sections) of the vector bundle $V \to M$.

For sections u of $V \to M$ over a Riemannian manifold (M, g) the Sobolev norms L^2 and H^s, $s \in \mathbb{N}$, are defined as

$$\begin{aligned} \|u\|^2_{L^2(M,g)} &:= \int_M |u|^2\, dv^g \\ \|u\|^2_{H^s(M,g)} &:= \|u\|^2_{L^2(M,g)} + \|\nabla u\|^2_{L^2(M,g)} + \ldots + \|\nabla^s u\|^2_{L^2(M,g)}. \end{aligned}$$

The vector bundle V will be suppressed in the notation. If M and g are clear from the context, we write just L^2 and H^s. The completion of $\{u \in \Gamma(V) \,|\, \|u\|_{H^s(M,g)} < \infty\}$ with respect to the $H^s(M, g)$-norm is denoted by $\Gamma_{H^s(M,g)}(V)$, or if (M, g) or V is clear from the context, we alternatively write $\Gamma_{H^s}(V)$ or $H^s(M, g)$ for $\Gamma_{H^s(M,g)}(V)$. The same definitions are used for L^2 instead of H^s. And similarly $\Gamma_{C^k(M,g)}(V) = \Gamma_{C^k}(V) = C^k(M, g)$ is the set of all C^k-sections, $k \in \mathbb{N} \cup \{\infty\}$.

2.2. **Removal of singularities.** In the proof we will use the following removal of singularities lemma.

Lemma 2.1 (Removal of singularities lemma). *Let Ω be a bounded open subset of $\mathbb{R}^n$ containing 0. Let P be an elliptic differential operator of order k on Ω, $f \in C^\infty(\Omega)$, and let $u \in C^\infty(\Omega \setminus \{0\})$ be a solution of*

$$Pu = f \tag{1}$$

on $\Omega \setminus \{0\}$ with

$$\lim_{\varepsilon\to 0} \int_{B_0(2\varepsilon)-B_0(\varepsilon)} |u| r^{-k} = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \int_{B_0(\varepsilon)} |u| = 0 \tag{2}$$

where r is the distance to 0. Then u is a (strong) solution of (1) on Ω. The same result holds for sections of vector bundles over relatively compact open subset of Riemannian manifolds.

Proof. We show that u is a weak solution of (1) in the distributional sense, and then it follows from standard regularity theory, that it is also a strong solution. This means that we have to show that for any given compactly supported smooth test function $\psi : \Omega \to \mathbb{R}$ we have

$$\int_\Omega u P^*\psi = \int_\Omega f\psi.$$

Let $\eta : \Omega \to [0, 1]$ be a test function that is identically 1 on $B_0(\varepsilon)$, has support in $B_0(2\varepsilon)$, and with $|\nabla^m \eta| \le C_m/\varepsilon^m$. It follows that

$$\sup |P^*(\eta\psi)| \le C(P, \Omega, \psi)\varepsilon^{-k},$$

on $B_0(2\varepsilon) \setminus B_0(\varepsilon)$ and $\sup |P^*(\eta\psi)| \leq C(P, \Omega, \psi)$ on $B_0(\varepsilon)$ and hence

$$\begin{aligned} \left| \int_\Omega uP^*(\eta\psi) \right| &\leq C\varepsilon^{-k} \int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} |u| + C \int_{B_0(\varepsilon)} |u| \\ &\leq C \int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} |u| r^{-k} + C \int_{B_0(\varepsilon)} |u| \to 0. \end{aligned} \tag{3}$$

We conclude

$$\begin{aligned} \int_\Omega uP^*\psi &= \int_\Omega uP^*(\eta\psi) + \int_\Omega uP^*((1-\eta)\psi) \\ &= \underbrace{\int_\Omega uP^*(\eta\psi)}_{\to 0} + \underbrace{\int_\Omega (Pu)(1-\eta)\psi}_{\to \int_\Omega f\psi} \end{aligned} \tag{4}$$

for $\varepsilon \to 0$. Hence the lemma follows. □

Condition (2) is obviously satisfied if $\int_\Omega |u| r^{-k} < \infty$. It is also satisfied if

$$\int_\Omega |u|^2 r^{-k} < \infty \text{ and } k \leq n, \tag{5}$$

as in this case

$$\left(\int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} |u| r^{-k} \right)^2 \leq \int_\Omega |u|^2 r^{-k} \underbrace{\int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} r^{-k}}_{\leq C}.$$

2.3. **Conformally covariant elliptic operators.** In this subsection we present a class of certain conformally covariant elliptic operators. Many important geometric operators are in this class, in particular the conformal Laplacian, the Paneitz operator, the Dirac operator, see also [18, 21, 22] for more examples. Readers who are only interested in the Dirac operator, the Conformal Laplacian or the Paneitz operator, can skip this part and continue with section 3.

Such a conformally covariant operator is not just one single differential operator, but a procedure how to associate to an n-dimensional Riemannian manifold (M, g) (potentially with some additional structure) a differential operator P_g of order k acting on a vector bundle. The important fact is that if $g_2 = f^2 g_1$, then one claims

$$P_{g_2} = f^{-\frac{n+k}{2}} P_{g_1} f^{\frac{n-k}{2}}. \tag{6}$$

One also expresses this by saying that P has bidegree $((n-k)/2, (n+k)/2)$.

The sense of this equation is apparent if P_g is an operator from $C^\infty(M)$ to $C^\infty(M)$. If P_g acts on a vector bundle or if some additional structure (as e.g. spin structure) is used for defining it, then a rigorous and careful definition needs more attention. The language of categories provides a good formal framework [30]. The concept of conformally covariant elliptic operators is already used by many authors, but we do not know of a reference where a formal definition is carried out that fits to our context. (See [26] for a similar categorial approach that includes some of the operators presented here.) Often an intuitive definition is used. The intuitive definition is obviously sufficient if one deals with operators acting on functions, such as the conformal Laplacian or the Paneitz operator. However to properly state Theorem 1.3 we need the following definition.

Let *Riem*^{n} (resp. *Riemspin*^{n}) be the category n-dimensional Riemannian manifolds (resp. n-dimensional Riemannian manifolds with orientation and spin structure). Morphisms from (M_1, g_1) to (M_2, g_2) are conformal embeddings $(M_1, g_1) \hookrightarrow (M_2, g_2)$ (resp. conformal embeddings preserving orientation and spin structure).

Let $\mathit{Laplace}_k^n$ (resp. Dirac_k^n) be the category whose objects are $\{(M,g),V_g,P_g\}$ where (M,g) in an object of Riem^n (resp. $\mathit{Riemspin}^n$), where V_g is a vector bundle with a scalar product on the fibers, where $P_g:\Gamma(V_g)\to\Gamma(V_g)$ is an elliptic formally self-adjoint differential operator of order k.

A morphism (ι,κ) from $\{(M_1,g_1),V_{g_1},P_{g_1}\}$ to $\{(M_2,g_2),V_{g_2},P_{g_2}\}$ consists of a conformal embedding $\iota:(M_1,g_1)\hookrightarrow(M_2,g_2)$ (preserving orientation and spin structure in the case of Dirac_k^n) together with a fiber isomorphism $\kappa:\iota^*V_{g_2}\to V_{g_1}$ preserving fiberwise length, such that P_{g_1} and P_{g_2} satisfy the conformal covariance property (6). For stating this property precisely, let $f>0$ be defined by $\iota^*g_2=f^2g_1$, and let $\kappa_*:\Gamma(V_{g_2})\to\Gamma(V_{g_1})$, $\kappa_*(\varphi)=\kappa\circ\varphi\circ\iota$. Then the conformal covariance property is

$$\kappa_*P_{g_2}=f^{-\frac{n+k}{2}}P_{g_1}f^{\frac{n-k}{2}}\kappa_*. \tag{7}$$

In the following the maps κ and ι will often be evident from the context and then will be omitted. The transformation formula (7) then simplifies to (6).

Definition 2.2. A *conformally covariant elliptic operator of order k and of bidegree* $((n-k)/2,(n+k)/2)$ is a contravariant functor from Riem^n (resp. $\mathit{Riemspin}^n$) to $\mathit{Laplace}_k^n$ (resp. Dirac_k^n), mapping (M,g) to (M,g,V_g,P_g) in such a way that the coefficients are continuous in the C^k-topology of metrics (see below). To shorten notation, we just write P_g or P for this functor.

It remains to explain the C^k-continuity of the coefficients.

For Riemannian metrics g, g_1, g_2 defined on a compact set $K\subset M$ we set

$$d^g_{C^k(K)}(g_1,g_2):=\max_{t=0,\dots,k}\|(\nabla_g)^t(g_1-g_2)\|_{C^0(K)}.$$

For a fixed background metric g, the relation $d^g_{C^k(K)}(\,\cdot\,,\,\cdot\,)$ defines a distance function on the space of metrics on K. The topology induced by d^g is independent of this background metric and it is called the *C^k-topology of metrics on K*.

Definition 2.3. We say that the coefficients of P are *continuous in the C^k-topology of metrics* if for any metric g on a manifold M, and for any compact subset $K\subset M$ there is a neighborhood $\mathcal{U}$ of $g|_K$ in the C^k-topology of metrics on K, such that for all metrics $\tilde g$, $\tilde g|_K\in\mathcal{U}$, there is an isomorphism of vector bundles $\hat\kappa:V_g|_K\to V_{\tilde g}|_K$ over the identity of K with induced map $\hat\kappa_*:\Gamma(V_g|_K)\to\Gamma(V_{\tilde g}|_K)$ with the property that the coefficients of the differential operator

$$P_g-(\hat\kappa_*)^{-1}P_{\tilde g}\hat\kappa_*$$

depend continuously on $\tilde g$ (with respect to the C^k-topology of metrics).

2.4. **Invertibility on $\mathbb{S}^{n-1}\times\mathbb{R}$.** Let P be a conformally covariant elliptic operator of order k and of bidegree $((n-k)/2,(n+k)/2)$. For $(M,g)=\mathbb{S}^{n-1}\times\mathbb{R}$, the operator P_g is a self-adjoint operator $H^k\subset L^2\to L^2$ (see Lemma 3.1 and the comments thereafter).

Definition 2.4. We say that P *is invertible on* $\mathbb{S}^{n-1}\times\mathbb{R}$ if P_g is an invertible operator $H^k\to L^2$ where g is the standard product metric on $\mathbb{S}^{n-1}\times\mathbb{R}$. In order words there is a constant $\sigma>0$ such that the spectrum of $P_g:\Gamma_{H^k}(V_g)\to\Gamma_{L^2}(V_g)$ is contained in $(-\infty,-\sigma]\cup[\sigma,\infty)$ for any $g\in U$. In the following, the largest such σ will be called σ_P.

We conjecture that any conformally covariant elliptic operator of order k and of bidegree $((n-k)/2,(n+k)/2)$ with $k<n$ is invertible on $\mathbb{S}^{n-1}\times\mathbb{R}$.

2.5. **Examples.**
Example 1: The Conformal Laplacian
Let

$$L_g:=\Delta_g+\frac{n-2}{4(n-1)}\mathrm{Scal}_g,$$

be the conformal Laplacian. It acts on functions on a Riemannian manifold (M,g), i.e. V_g is the trivial real line bundle $\underline{\mathbb{R}}$. Let $\iota:(M_1,g_1)\hookrightarrow(M_2,g_2)$ be a conformal embedding. Then we can choose

$\kappa := \mathrm{Id} : \iota^* V_{g_2} \to V_{g_1}$ and formula (7) holds for $k = 2$ (see e.g. [15, Section 1.J]). All coefficients of L_g depend continuously on g in the C^2-topology. Hence L is a conformally covariant elliptic operator of order 2 and of bidegree $((n-2)/2, (n+2)/2)$.

The scalar curvature of $\mathbb{S}^{n-1} \times \mathbb{R}$ is $(n-1)(n-2)$. The spectrum of L_g on $\mathbb{S}^{n-1} \times \mathbb{R}$ of L_g coincides with the essential spectrum of L_g and is $[\sigma_L, \infty)$ with $\sigma_L := (n-2)^2/4$. Hence L is invertible on $S^{n-1} \times \mathbb{R}$ if (and only if) $n > 2$.

Example 2: The Paneitz operator

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 5$. The Paneitz operator P_g is given by

$$P_g u = (\Delta_g)^2 u - \mathrm{div}_g (A_g \, du) + \frac{n-4}{2} Q_g u$$

where

$$A_g := \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \mathrm{Scal}_g g - \frac{4}{n-2} \mathrm{Ric}_g,$$

$$Q_g = \frac{1}{2(n-1)} \Delta_g \mathrm{Scal}_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2 (n-2)^2} \mathrm{Scal}_g^2 - \frac{2}{(n-2)^2} |\mathrm{Ric}_g|^2.$$

This operator was defined by Paneitz [32] in the case $n = 4$, and it was generalized by Branson in [17] to arbitrary dimensions ≥ 4. We also refer to Theorem 1.21 of the overview article [16]. The explicit formula presented above can be found e.g. in [23]. The coefficients of P_g depend continuously on g in the C^4-topology

As in the previous example we can choose for κ the identity, and then the Paneitz operator P_g is a conformally covariant elliptic operator of order 4 and of bidegree $((n-4)/2, (n+4)/2)$.

On $\mathbb{S}^{n-1} \times \mathbb{R}$ one calculates

$$A_g = \frac{(n-4)n}{2} \mathrm{Id} + 4\pi_{\mathbb{R}} > 0$$

where $\pi_{\mathbb{R}}$ is the projection to vectors parallel to $\mathbb{R}$.

$$Q_g = \frac{(n-4)n^2}{8}.$$

We conclude

$$\sigma_P = Q = \frac{(n-4)n^2}{8}$$

and P is invertible on $\mathbb{S}^{n-1} \times \mathbb{R}$ if (and only if) $n > 4$.

Examples 3: The Dirac operator.

Let $\tilde{g} = f^2 g$. Let $\Sigma_g M$ resp. $\Sigma_{\tilde{g}} M$ be the spinor bundle of (M, g) resp. $(M, \tilde{g})$. Then there is a fiberwise isomorphism $\beta^g_{\tilde{g}} : \Sigma_g M \to \Sigma_{\tilde{g}} M$, preserving the norm such that

$$D_{\tilde{g}} \circ \beta^g_{\tilde{g}}(\varphi) = f^{-\frac{n+1}{2}} \beta^g_{\tilde{g}} \circ D_g \left(f^{\frac{n-1}{2}} \varphi \right),$$

see [24, 14] for details. Furthermore, the cocycle conditions

$$\beta^g_{\tilde{g}} \circ \beta^{\tilde{g}}_g = \mathrm{Id} \qquad \text{and} \qquad \beta^{\hat{g}}_g \circ \beta^{\tilde{g}}_{\hat{g}} \circ \beta^g_{\tilde{g}} = \mathrm{Id}$$

hold for conformal metrics g, $\tilde{g}$ and $\hat{g}$. We will hence use the map $\beta^g_{\tilde{g}}$ to identify $\Sigma_g M$ with $\Sigma_{\tilde{g}} M$. Hence we simply get

$$D_{\tilde{g}} \varphi = f^{-\frac{n+1}{2}} \circ D_g \left(f^{\frac{n-1}{2}} \varphi \right). \tag{8}$$

All coefficients of D_g depend continuously on g in the C^1-topology. Hence D is a conformally covariant elliptic operator of order 1 and of bidegree $((n-1)/2, (n+1)/2)$.

The Dirac operator on $\mathbb{S}^{n-1}\times\mathbb{R}$ can be decomposed in a part D_{vert} deriving along $\mathbb{S}^{n-1}$ and a part D_{hor} deriving along $\mathbb{R}$, $D_g = D_{\mathrm{vert}} + D_{\mathrm{hor}}$, see [1] or [2]. Locally

$$D_{\mathrm{vert}} = \sum_{i=1}^{n-1} e_i \cdot \nabla_{e_i}$$

for a local frame $(e_1,\ldots,e_{n-1})$ of $\mathbb{S}^{n-1}$. Here $\cdot$ denotes the Clifford multiplication $TM\otimes\Sigma_g M\to\Sigma_g M$. Furthermore $D_{\mathrm{hor}} = \partial_t\cdot\nabla_{\partial_t}$, where $t\in\mathbb{R}$ is the standard coordinate of $\mathbb{R}$. The operators D_{vert} and D_{hor} anticommute. For $n\geq 3$, the spectrum of D_{vert} coincides with the spectrum of the Dirac operator on $\mathbb{S}^{n-1}$, we cite [12] and obtain

$$\mathrm{spec} D_{\mathrm{vert}} = \left\{ \pm\left(\frac{n-1}{2}+k\right) \mid k\in\mathbb{N}_0\right\}.$$

The operator $(D_{\mathrm{hor}})^2$ is the ordinary Laplacian on $\mathbb{R}$ and hence has spectrum $[0,\infty)$. Together this implies that the spectrum of the Dirac operator on $\mathbb{S}^{n-1}\times\mathbb{R}$ is $(-\infty,-\sigma_D]\cup[\sigma_D,\infty)$ with $\sigma_D=\frac{n-1}{2}$.

In the case $n=2$ these statements are only correct if the circle $\mathbb{S}^{n-1}=\mathbb{S}^1$ carries the spin structure induced from the ball. Only this spin structure extends to the conformal compactification that is given by adding one point at infinity for each end. For this reason, we will understand in the whole article that all circles $\mathbb{S}^1$ should be equipped with this bounding spin structure. The extension of the spin structure is essential in order to have a spinor bundle on the compactification. The methods used in our proof use this extension implicitly.

Hence D is invertible on $S^{n-1}\times\mathbb{R}$ if (and only if) $n>1$.

Most techniques used in the literature on estimating eigenvalues of the Dirac operators do not use the spin structure and hence these techniques cannot provide a proof in the case $n=2$.

Example 4: The Rarita-Schwinger operator and many other Fegan type operators are conformally covariant elliptic operators of order 1 and of bidegree $((n-1)/2,(n+1)/2)$. See [21] and in the work of T. Branson for more information.

Example 5: Assume that (M,g) is a Riemannian spin manifold that carries a vector bundle $W\to M$ with metric and metric connection. Then there is a natural first order operator $\Gamma(\Sigma_g M\otimes W)\to\Gamma(\Sigma_g M\otimes W)$, the *Dirac operator twisted by* W. This operator has similar properties as conformally covariant elliptic operators of order 1 and of bidegree $((n-1)/2,(n+1)/2)$. The methods of our article can be easily adapted in order to show that Theorem 1.3 is also true for this twisted Dirac operator. However, twisted Dirac operators are not "conformally covariant elliptic operators" in the above sense. They could have been included in this class by replacing the category *Riemspin*^{n} by a category of Riemannian spin manifolds with twisting bundles. In order not to overload the formalism we chose not to present these larger categories.

The same discussion applies to the spin^c-Dirac operator of a spin^c-manifold.

3. Asymptotically cylindrical blowups

3.1. **Convention.** *From now on we suppose that P_g is a conformally covariant elliptic operator of order k, of bidegree $((n-k)/2,(n+k)/2)$, acting on manifolds of dimension n and invertible on $\mathbb{S}^{n-1}\times\mathbb{R}$.*

3.2. **Definition of the metrics.** Let g_0 be a Riemannian metric on a compact manifold M. We can suppose that the injectivity radius in a fixed point $y\in M$ is larger than 1. The geodesic distance from y to x is denoted by $d(x,y)$.

We choose a smooth function $F_\infty : M\setminus\{y\}\to[1,\infty)$ such such that $F_\infty(x)=1$ if $d(x,y)\geq 1$, $F_\infty(x)\leq 2$ if $d(x,y)\geq 1/2$ and such that $F_\infty(x)=d(x,y)^{-1}$ if $d(x,y)\in(0,1/2]$. Then for $L\geq 1$ we define F_L to be a smooth positive function on M, depending only on $d(x,y)$, such that $F_L(x)=F_\infty(x)$ if $d(x,y)\geq e^{-L}$ and $F_L(x)\leq d(x,y)^{-1}=F_\infty(x)$ if $d(x,y)\leq e^{-L}$.

For any $L \geq 1$ or $L = \infty$ set $g_L := F_L^2 g_0$. The metric g_∞ is a complete metric on M_∞.

The family of metrics (g_L) is called an *asymptotically cylindrical blowup*, in the literature it is denoted as a family of *Pinocchio metrics* [6], see also Figure 1.

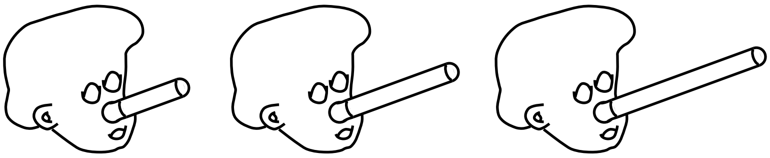

FIGURE 1. Asymptotically cylindrical metrics g_L (alias Pinocchio metrics) with growing nose length L.

3.3. **Eigenvalues and basic properties on** (M, g_L)**.** For the P-operator associated to (M, g_L), $L \in \{0\} \cup [1, \infty)$ (or more exactly its self-adjoint extension) we simply write P_L instead of P_{g_L}. As M is compact the spectrum of P_L is discrete.

We will denote the spectrum of P_L in the following way

$$\ldots \leq \lambda_1^-(P_L) < 0 = 0 \ldots = 0 < \lambda_1^+(P_L) \leq \lambda_2^+(P_L) \leq \ldots,$$

where each eigenvalue appears with the multiplicity corresponding to the dimension of the eigenspace. The zeros might appear on this list or not, depending on whether P_L is invertible or not. The spectrum might be entirely positive (for example the conformal Laplacian Y_g on the sphere) in which case $\lambda_1^-(P_L)$ is not defined. Similarly, $\lambda_1^+(P_L)$ is not defined if the spectrum of (P_L) is negative.

3.4. **Analytical facts about** (M_∞, g_∞)**.** The analysis of non-compact manifolds as (M_∞, g_∞) is more complicated than in the compact case. Nevertheless (M_∞, g_∞) is an asymptotically cylindrical manifold, and for such manifolds an extensive literature is available. One possible approach would be Melrose's b-calculus [31]: our cylindrical manifold is such a b-manifold, but for simplicity and self-containedness we avoid this theory. We will need some few properties that we will summarize in the following proposition.

We assume in the whole section that P is a conformally covariant elliptic operator that is invertible on $\mathbb{S}^{n-1} \times \mathbb{R}$, and we write $P_\infty := P_{g_\infty}$ for the operator acting on sections of the bundle V over (M_∞, g_∞).

Proposition 3.1. *P_∞ extends to a bounded operator from $\Gamma_{H^k(M_\infty, g_\infty)}(V) \to \Gamma_{L^2(M_\infty, g_\infty)}(V)$ and it satisfies the following regularity estimate*

$$\|(\nabla^\infty)^s u\|_{L^2(M_\infty, g_\infty)} \leq C(\|u\|_{L^2(M_\infty, g_\infty)} + \|P_\infty u\|_{L^2(M_\infty, g_\infty)}) \tag{9}$$

for all $u \in \Gamma_{H^k(M_\infty, g_\infty)}(V)$ and all $s \in \{0, 1, \ldots, k\}$. The operator $P_\infty : \Gamma_{H^k(M_\infty, g_\infty)}(V) \to \Gamma_{L^2(M_\infty, g_\infty)}(V)$ is self-adjoint in the sense of an operator in $\Gamma_{L^2(M_\infty, g_\infty)}(V)$.

The proof of the proposition will be sketched in the appendix.

Proposition 3.2. *The essential spectrum of P_∞ coincides with the essential spectrum of the P-operator on the standard cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. Thus the essential spectrum of P_∞ is contained in $(-\infty, -\sigma_P] \cup [\sigma_P, \infty)$.*

This proposition follows from the characterization of the essential spectrum in terms of Weyl sequences, a well-known technique which is for example carried out and well explained in [13].

The second proposition states that the spectrum of P_∞ in the interval $(-\sigma_P, \sigma_P)$ is discrete as well. Eigenvalues of P_∞ in this interval will be called *small eigenvalues of* P_∞. Similarly to above we use the notation $\lambda_j^\pm(P_\infty)$ for the small eigenvalues of P_∞.

3.5. **The kernel.** Having recalled these well-known facts we will now study the kernel of conformally covariant operators.

If g and $\tilde{g} = f^2$ are conformal metrics on a compact manifold M, then

$$\varphi \mapsto f^{-\frac{n-k}{2}} \varphi$$

obviously defines an isomorphism from $\ker P_g$ to $\ker P_{\tilde{g}}$. It is less obvious that a similar statement holds if we compare g_0 and g_∞ defined before:

Proposition 3.3. *The map*

$$\begin{array}{rcl} \ker P_0 & \to & \ker P_\infty \\ \varphi_0 & \mapsto & \varphi_\infty = F_\infty^{-\frac{n-k}{2}} \varphi_0 \end{array}$$

is an isomorphism of vector spaces.

Proof. Suppose $\varphi_0 \in \ker P_0$. Using standard regularity results it is clear that $\sup |\varphi_0| < \infty$. Then

$$\begin{aligned} \int_{M_\infty} |\varphi_\infty|^2 \, dv^{g_\infty} &\leq \int_{M \setminus B_y(1/2)} |\varphi_\infty|^2 \, dv^{g_\infty} + \sup |\varphi_0|^2 \int_{B_y(1/2)} F_\infty^{-(n-k)} \, dv^{g_\infty} \\ &\leq 2^k \int_{M \setminus B_y(1/2)} |\varphi_0|^2 \, dv^{g_0} + \sup |\varphi_0|^2 \omega_{n-1} \int_0^{1/2} \frac{r^{n-1}}{r^k} \, dr < \infty. \end{aligned} \tag{10}$$

Here we used that up to lower order terms dv^{g_∞} coincides with the product measure of the standard measure on the sphere with the measure $d(\log r) = \frac{1}{r} \, dr$. Furthermore, formula (6) implies $P_\infty \varphi_\infty = 0$. Hence the map is well-defined. In order to show that it is an isomorphism we show that the obvious inverse $\varphi_\infty \mapsto \varphi_0 := F_\infty^{\frac{n-k}{2}} \varphi_\infty$ is well defined. To see this we start with an L^2-section in the kernel of P_∞.

We calculate

$$\int_M F_\infty^k |\varphi_0|^2 \, dv^{g_0} = \int_{M_\infty} |\varphi_\infty|^2 \, dv^{g_\infty}.$$

Using again (6) we see that this section satisfies $P_0 \varphi_0 = 0$ on $M \setminus \{y\}$. Hence condition (5) is satisfied, and together with the removal of singularity lemma (Lemma 2.1) one obtains that the inverse map is well-defined. The proposition follows. □

4. Proof of the main theorem

4.1. **Stronger version of the main theorem.** We will now show the following theorem.

Theorem 4.1. *Let P be a conformally covariant elliptic operator of order k, of bidegree $((n-k)/2, (n+k)/2)$, on manifolds of dimension $n > k$. We assume that P is invertible on $\mathbb{S}^{n-1} \times \mathbb{R}$.*

If $\liminf_{L \to \infty} |\lambda_j^\pm(P_L)| < \sigma_P$, then

$$\lambda_j^\pm(P_L) \to \lambda_j^\pm(P_\infty) \in (-\sigma_P, \sigma_P) \qquad \text{for } L \to \infty.$$

In the case $\mathrm{Spec}(P_{g_0}) \subset (0, \infty)$ the theorem only makes a statement about λ_j^+, and conversely in the case that $\mathrm{Spec}(P_{g_0}) \subset (-\infty, 0)$ it only makes a statement about λ_j^-.

Obviously this theorem implies Theorem 1.3.

4.2. **The supremum part of the proof of Theorem 4.1.** At first we prove that

$$\limsup_{L\to\infty}(\lambda_j^+(P_L)) \le \lambda_j^+(P_\infty). \tag{11}$$

Let $\varphi_1,\ldots,\varphi_j$ be sequence of L^2-orthonormal eigenvectors of P_∞ to eigenvalues $\lambda_1^+(P_\infty),\ldots,\lambda_j^+(P_\infty)\in[-\bar\lambda,\bar\lambda]$, $\bar\lambda<\sigma_P$. We choose a cut-off function $\chi: M\to[0,1]$ with $\chi(x)=1$ for $-\log(d(x,y))\le T$, $\chi(y)=0$ for $-\log(d(x,y))\ge 2T$, and $|(\nabla^\infty)^s\chi|_{g_\infty}\le C_s/T^s$ for all $s\in\{0,\ldots,k\}$.

Let φ be a linear combination of the eigenvectors $\varphi_1,\ldots,\varphi_j$. From Proposition 3.1 we see that

$$\|(\nabla^\infty)^s\varphi\|_{L^2(M_\infty,g_\infty)}\le C\|\varphi\|_{L^2(M_\infty,g_\infty)}$$

where C only depends on (M_∞,g_∞). Hence for sufficiently large T

$$\|P_\infty(\chi\varphi)-\chi P_\infty\varphi\|_{L^2(M_\infty,g_\infty)}\le kC/T\|\varphi\|_{L^2(M_\infty,g_\infty)}\le 2kC/T\|\chi\varphi\|_{L^2(M_\infty,g_\infty)}$$

for sufficiently large T as $\|\chi\varphi\|_{L^2(M_\infty,g_\infty)}\to\|\varphi\|_{L^2(M_\infty,g_\infty)}$ for $T\to\infty$. The section $\chi\varphi$ can be interpreted as a section on (M,g_L) if $L>2T$, and on the support of $\chi\varphi$ we have $g_L=g_\infty$ and $P_\infty(\chi\varphi)=P_L(\chi\varphi)$. Hence standard Rayleigh quotient arguments imply that if P_∞ has m eigenvalues (counted with multiplicity) in the interval $[a,b]$ then P_L has m eigenvalues in the interval $[a-2kC/T,b+2kC/T]$. Taking the limit $T\to\infty$ we obtain (11).

By exchanging some obvious signs we obtain similarly

$$\limsup_{L\to\infty}(-\lambda_j^-(P_L))\le -\lambda_j^-(P_\infty). \tag{12}$$

4.3. **The infimum part of the proof of Theorem 4.1.** We now prove

$$\liminf_{L\to\infty}(\pm\lambda_j^\pm(P_L))\ge\pm\lambda_j^\pm(P_\infty). \tag{13}$$

We assume that we have a sequence $L_i\to\infty$, and that for each i we have a system of orthogonal eigenvectors $\varphi_{i,1},\ \ldots,\ \varphi_{i,m}$ of P_{L_i}, i.e. $P_{L_i}\varphi_{i,\ell}=\lambda_{i,\ell}\varphi_{i,\ell}$ for $\ell\in\{1,\ldots,m\}$. Furthermore we suppose that $\lambda_{i,\ell}\to\bar\lambda_\ell\in(-\sigma_P,\sigma_P)$ for $\ell\in\{1,\ldots,m\}$.

Then

$$\psi_{i,\ell}:=\left(\frac{F_{L_i}}{F_\infty}\right)^{\frac{n-k}{2}}\varphi_{i,\ell}$$

satisfies

$$P_\infty\psi_{i,\ell}=h_{i,\ell}\psi_{i,\ell}\qquad\text{with}\qquad h_{i,\ell}:=\left(\frac{F_{L_i}}{F_\infty}\right)^k\lambda_{i,\ell}.$$

Furthermore

$$\|\psi_{i,\ell}\|^2_{L^2(M_\infty,g_\infty)}=\int_M\left(\frac{F_{L_i}}{F_\infty}\right)^{-k}|\varphi_{i,\ell}|^2\,dv^{g_{L_i}}\le\sup_M|\varphi_{i,\ell}|^2\int_M\left(\frac{F_{L_i}}{F_\infty}\right)^{-k}dv^{g_{L_i}}$$

Because of $\int_M\left(\frac{F_{L_i}}{F_\infty}\right)^{-k}dv^{g_L}\le C\int r^{n-1-k}\,dr<\infty$ (for $n>k$) the norm $\|\psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)}$ is finite as well, and we can renormalize such that

$$\|\psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)}=1.$$

Lemma 4.2. *For any $\delta>0$ and any $\ell\in\{0,\ldots,m\}$ the sequence*

$$\Big(\|\psi_{i,\ell}\|_{C^{k+1}(M\setminus B_y(\delta),g_\infty)}\Big)_i$$

is bounded.

Proof of the lemma. After removing finitely many i, we can assume that $\lambda_i \le 2\bar\lambda$ and $e^{-L_i} < \delta/2$. Hence $F_L = F_\infty$ and $h_i = \lambda_i$ on $M \setminus B_y(\delta/2)$. Because of

$$\int_{M\setminus B_y(\delta/2)} |(P_\infty)^s \psi_i|^2 \, dv^{g_\infty} \le (2\bar\lambda)^{2s} \int_{M\setminus B_y(\delta/2)} |\psi_i|^2 \, dv^{g_\infty} \le (2\bar\lambda)^{2s}$$

we obtain boundedness of ψ_i in the Sobolev space $H^{sk}(M \setminus B_y(3\delta/4), g_\infty)$, and hence, for sufficiently large s boundedness in $C^{k+1}(M \setminus B_y(\delta), g_\infty)$. The lemma is proved. $\square$

Hence after passing to a subsequence $\psi_{i,\ell}$ converges in $C^{k,\alpha}(M \setminus B_y(\delta), g_\infty)$ to a solution $\bar\psi_\ell$ of

$$P_\infty \bar\psi_\ell = \bar\lambda_\ell \bar\psi_\ell.$$

By taking a diagonal sequence, one can obtain convergence in $C^{k,\alpha}_{\mathrm{loc}}(M_\infty)$ of $\psi_{i,\ell}$ to $\bar\psi_\ell$. It remains to prove that $\bar\psi_1,\ldots,\bar\psi_m$ are linearly independent, in particular that any $\bar\psi_\ell \ne 0$. For this we use the following lemma.

Lemma 4.3. *For any $\varepsilon > 0$ there is δ_0 and i_0 such that*

$$\Big\|\psi_{i,\ell}\Big\|_{L^2(B_y(\delta_0),g_\infty)} \le \varepsilon \Big\|\psi_{i,\ell}\Big\|_{L^2(M_\infty,g_\infty)}$$

for all $i \ge i_0$ and all $\ell \in \{0,\ldots,m\}$. In particular,

$$\Big\|\psi_{i,\ell}\Big\|_{L^2(M\setminus B_y(\delta_0),g_\infty)} \ge (1-\varepsilon)\Big\|\psi_{i,\ell}\Big\|_{L^2(M_\infty,g_\infty)}.$$

Proof of the lemma. Because of Proposition 3.1 and

$$\|P_\infty \psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)} \le |\bar\lambda_\ell| \, \|\psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)} = |\bar\lambda_\ell|$$

we get

$$\|(\nabla^\infty)^s \psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)} \le C$$

for all $s \in \{0,\ldots,k\}$. Let χ be a cut-off function as in Subsection 4.2 with $T = -\log\delta$. Hence

$$\|P_\infty\Big((1-\chi)\psi_{i,\ell}\Big) - (1-\chi)P_\infty(\psi_{i,\ell})\|_{L^2(M_\infty,g_\infty)} \le \frac{C}{T} = \frac{C}{-\log\delta}. \tag{14}$$

On the other hand $(B_y(\delta)\setminus\{y\}, g_\infty)$ converges for suitable choices of basepoints for $\delta \to 0$ to $\mathbb{S}^{n-1}\times(0,\infty)$ in the C^∞-topology of Riemannian manifolds with basepoints. Hence there is a function $\tau(\delta)$ converging to 0 such that

$$\|P_\infty\Big((1-\chi)\psi_{i,\ell}\Big)\|_{L^2(M_\infty,g_\infty)} \ge (\sigma_p - \tau(\delta))\|(1-\chi)\psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)}. \tag{15}$$

Using the obvious relation

$$\|(1-\chi)P_\infty(\psi_{i,\ell})\|_{L^2(M_\infty,g_\infty)} \le |\lambda_{i,\ell}| \, \|(1-\chi)\psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)}$$

we obtain with (14) and (15)

$$\|\psi_{i,\ell}\|_{L^2(B_y(\delta^2),g_\infty)} \le \|(1-\chi)\psi_{i,\ell}\|_{L^2(M_\infty,g_\infty)} \le \frac{C}{|\log\delta|(\sigma_P - \tau(\delta) - |\lambda_{i,\ell}|)}.$$

The right hand side is smaller than ε for i sufficiently large and δ sufficiently small. The main statement of the lemma then follows for $\delta_0 := \delta^2$. The Minkowski inequality yields.

$$\|\psi_{i,\ell}\|_{L^2(M\setminus B_y(\delta^2),g_\infty)} \ge 1 - \|\psi_{i,\ell}\|_{L^2(B_y(\delta^2),g_\infty)} \ge 1-\varepsilon.$$

$\square$

The convergence in $C^1(M \setminus B_y(\delta_0))$ implies strong convergence in $L^2(M\setminus B_y(\delta_0), g_\infty)$ of $\psi_{i,\ell}$ to $\bar\psi_\ell$. Hence

$$\|\bar\psi_\ell\|_{L^2(M\setminus B_y(\delta_0),g_\infty)} \ge 1-\varepsilon,$$

and thus $\|\bar\psi_\ell\|_{L^2(M_\infty,g_\infty)} = 1$. The orthogonality of these sections is provided by the following lemma, and the inequality (13) then follows immediately.

Lemma 4.4. *The sections $\bar\psi_1,\ldots,\bar\psi_m$ are orthogonal.*

Proof of the lemma. The sections $\varphi_{i,1},\ldots,\varphi_{i,\ell}$ are orthogonal. For any fixed δ_0 (given by the previous lemma), it follows for sufficiently large i that

$$\begin{aligned}\Big|\int_{M\setminus B_y(\delta_0)}\langle\psi_{i,\ell},\psi_{i,\tilde\ell}\rangle\,dv^{g_\infty}\Big| &= \Big|\int_{M\setminus B_y(\delta_0)}\langle\varphi_{i,\ell},\varphi_{i,\tilde\ell}\rangle\,dv^{g_{L_i}}\Big|\\ &= \Big|\int_{B_y(\delta_0)}\langle\varphi_{i,\ell},\varphi_{i,\tilde\ell}\rangle\,dv^{g_{L_i}}\Big|\\ &= \Big|\int_{B_y(\delta_0)}\underbrace{\left(\frac{F_{L_i}}{F_\infty}\right)^k}_{\le 1}\langle\psi_{i,\ell},\psi_{i,\tilde\ell}\rangle\,dv^{g_\infty}\Big|\\ &\le\varepsilon^2\end{aligned}\tag{16}$$

Because of strong L^2 convergence on $M\setminus B_y(\delta_0)$ this implies

$$\Big|\int_{M\setminus B_y(\delta_0)}\langle\bar\psi_\ell,\bar\psi_{\tilde\ell}\rangle\,dv^{g_\infty}\Big|\le\varepsilon^2\tag{17}$$

for $\tilde\ell\neq\ell$, and hence in the limit $\varepsilon\to 0$ (and $\delta_0\to 0$) we get the orthogonality of $\bar\psi_1,\ldots,\bar\psi_m$. □

Appendix A. Analysis on (M_∞, g_∞)

The aim of this appendix is to sketch how to prove Proposition 3.1. All properties in this appendix are well-known to experts, but explicit references are not evident to find. Thus this summary might be helpful to the reader.

The geometry of (M_∞,g_∞) is asymptotically cylindrical. The metric g_∞ is even a b-metric in the sense of Melrose [31], but to keep the presentation simple, we avoid the b-calculus.

If $(r,\gamma)\in\mathbb{R}^+\times\mathbb{S}^{n-1}$ denote polar normal coordinates with respect to the metric g_0, and if we set $t:=-\log r$, then (t,γ) defines a diffeomorphism $\alpha: B_y^{(M,g_0)}(1/2)\setminus\{y\}\to[\log 2,\infty)\times\mathbb{S}^{n-1}$ such that $(\alpha^{-1})^*g_\infty = dt^2+h_t$ for a family of metrics such that $(\alpha^{-1})^*g_\infty$, all of its derivatives, its curvature, and all derivatives of the curvature tend to the standard metric on the cylinder, and the speed of the convergence is majorised by a multiple of e^t. Thus the continuity of the coefficients property implies, that P_∞ extends to a bounded operator from $\Gamma_{H^k(M_\infty,g_\infty)}(V)\to\Gamma_{L^2(M_\infty,g_\infty)}(V)$.

The formal self-adjointness of P_∞ implies that

$$\int_{M_\infty}\langle\psi,P_\infty\varphi\rangle=\int_{M_\infty}\langle P_\infty\psi,\varphi\rangle\tag{18}$$

holds for $\varphi,\psi\in\Gamma_c(V)$ and as $\Gamma_c(V)$ is dense in H^k, property (18) follows all H^k-sections φ,ψ.

To show Proposition 3.1 it remains to prove the regularity estimate and then to verify that the adjoint of $P_\infty:\Gamma_{H^k(M_\infty,g_\infty)}(V)\to\Gamma_{L^2(M_\infty,g_\infty)}(V)$ has domain $\Gamma_{H^k(M_\infty,g_\infty)}(V)$.

For proving the regularity estimate we need the following local estimate.

Lemma A.1. *Let K be a compact subset of a Riemannian manifold (U,g). Let P be an elliptic differential operator on U of order $k\ge 1$. Then there is a constant $C=C(U,K,P,g)$ such that*

$$\|u\|_{H^k(K,g)}\le C(\|u\|_{L^2(U,g)}+\|Pu\|_{L^2(U,g)}).\tag{19}$$

Here the $H^k(K,g)$-norm is defined via the Levi-Civita connection for g.

This estimate holds uniformly in an ε-neighborhood of P and g in the following sense. Assume that $\tilde P$ is another differential operator, and that the C^0-norm of the coeffcients of $\tilde P-P$ is at most ε, where ε is small. Also assume that $\tilde g$ is ε-close to g in the C^k-topology. Then the estimate (19) *holds for $\tilde P$ instead of P and for $\tilde g$ instead of g and again for a constant $C=C(U,K,P,g,\varepsilon)$.*

Proof of the lemma. We cover the compact set K by a finite number of coordinate neighborhoods $U_1, \dots, U_m$. We choose open sets $V_i \subset U_i$ such that the closure of V_i is compact in U_i and such that $K \subset V_1 \cup \ldots \cup V_m$. One can choose compact sets $K_i \subset V_i$ such that $K = K_1 \cup \ldots \cup K_m$. To prove (19) it is sufficient to prove $\|u\|_{H^k(K_i,g)} \le C(\|u\|_{L^2(V_i,g)} + \|Pu\|_{L^2(V_i,g)})$ for any i.

We write this inequality in coordinates. As the closure of V_i is a compactum in U_i, the transition to coordinates changes the above inequality only by a constant. The operator P, written in a coordinate chart is again elliptic.

We have thus reduced the prove of (19) to the prove of the special case that U and K are open subsets of flat $\mathbb{R}^n$.

The proof of this special case is explained in detail for example in in [33, Corollary III 1.5]. The idea is to construct a parametrix for P, i.e. a pseudodifferential operator of order $-k$ such that $S_1 := QP - \mathrm{Id}$ and $S_2 := PQ - \mathrm{Id}$ are infinitely smoothing operators. Thus Q is bounded from $L^2(U)$ to the Sobolev space $H^k(U)$, in particular $\|Q(P(u))\|_{H^k} \le C\|P(u)\|_{L^2}$. Smoothing operators map the Sobolev space L^2 continuously to H^k. We obtain

$$\|u\|_{H^k(K)} \le \|u\|_{H^k(U)} \le \|Q(P(u))\|_{H^k(U)} + \|S_1(u)\|_{H^k(U)} \le C'\Big(\|P(u)\|_{L^2(U)} + \|u\|_{L^2(U)}\Big).$$

See also [28, III §3] for a good presentation on how to construct and work with such a parametrix.

To see the uniformicity, one verifies that

$$\left|\frac{\|u\|_{H^k(K,\tilde g)}}{\|u\|_{H^k(K,g)}} - 1\right| \le C\,\|\tilde g - g\|_{C^k} \le C\varepsilon$$

and

$$\left|\frac{\|\tilde P(u)\|_{L^2(U)}}{\|P(u)\|_{L^2(U)}} - 1\right| \le C\varepsilon\|u\|_{H^k(U)}.$$

The unformicity statement thus follows. $\square$

Proof of the regularity estimate in Proposition 3.1. We write M_∞ as $M_B \cup ([0,\infty) \times S^{n-1})$, such that the metric g_∞ is asymptotic (in the C^∞-sense) to the standard cylindrical metric. The metric g_∞ restricted to $[R-1, R+2] \times S^{n-1}$ then converges in the C^k-topology to the cylindrical metric $dt^2 + \sigma^{n-1}$ on $[0,3] \times S^{n-1}$ for $R \to \infty$. As the coefficients of P_g depend continously on the metric, the P-operators on $[R-1, R+2] \times S^{n-1}$ is in an ε-neighborhood of P, for $R \ge R_0 = R_0(\varepsilon)$. Applying the preceding lemma for $K = [R, R+1] \times S^{n-1}$ and $U = (R-1, R+2) \times S^{n-1}$ we obtain

$$\|\nabla^s u\|_{L^2([R,R+1]\times S^{n-1},g_\infty)} \le C(\|u\|_{L^2((R-1,R+2)\times S^{n-1},g_\infty)} + \|P_\infty u\|_{L^2((R-1,R+2)\times S^{n-1},g_\infty)}). \tag{20}$$

Similarly, applying the lemma to $K = M_B \cup ([0, R_0] \times S^{n-1})$ and $U = M_B \cup ([0, R_0+1) \times S^{n-1})$ gives

$$\|\nabla^s u\|_{L^2(M_B\cup([0,R_0]\times S^{n-1}),g_\infty)} \le C(\|u\|_{L^2(M_B\cup([0,R_0+1)\times S^{n-1}),g_\infty)} + \|P_\infty u\|_{L^2(M_B\cup([0,R_0+1)\times S^{n-1}),g_\infty)}). \tag{21}$$

Taking the sum of estimate (21), of estimate (20) for $R = R_0$, again estimate (20) but for $R = R_0 + 1$, and so for all $R \in \{R_0 + 2, R_0 + 3, \ldots\}$ we obtain (9), with a larger constant C. $\square$

Now we study the domain $\mathcal{D}$ of the adjoint of $P_\infty : \Gamma_{H^k(M_\infty,g_\infty)}(V) \to \Gamma_{L^2(M_\infty,g_\infty)}(V)$.

By definition a section $\varphi : \Gamma_{L^2(M_\infty,g_\infty)}(V)$ is in $\mathcal{D}$ if and only if

$$\Gamma_{H^k(M_\infty,g_\infty)}(V) \ni u \mapsto \int_{M_\infty} \langle P_\infty u, \varphi\rangle \tag{22}$$

is bounded as a map from L^2 to $\mathbb{R}$. For $\varphi \in \Gamma_{H^k(M_\infty,g_\infty)}(V)$ we know that $P_\infty\varphi$ is L^2 and thus property (18) directly implies this boundedness. Thus $\Gamma_{H^k(M_\infty,g_\infty)}(V) \subset \mathcal{D}$.

Conversely assume the boundedness of (22). Then there is a $v \in \Gamma_{L^2(M_\infty,g_\infty)}(V)$ such that $\int_{M_\infty} \langle u, v\rangle = \int_{M_\infty} \langle P_\infty u, \varphi\rangle$, or in other words $P_\infty\varphi = v$ holds weakly. Standard regularity theory implies $\varphi \in \Gamma_{H^k(M_\infty,g_\infty)}(V)$.

We obtain $\Gamma_{H^k(M_\infty,g_\infty)}(V) = \mathcal{D}$, and thus the self-adjointness of P_∞ follows.
Proposition 3.1 is thus shown.

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