THE STRONG LEGENDRE CONDITION AND THE WELL-POSEDNESS OF MIXED ROBIN PROBLEMS ON MANIFOLDS WITH BOUNDED GEOMETRY

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Abstract. Let M be a smooth manifold with boundary and bounded geometry, $\partial_D M \subset \partial M$ be an *open and closed* subset of the boundary of M, P be a second order differential operator on M, and b be a first order differential operator on ∂M. Our operators act on sections of a vector bundle $E \to M$ with bounded geometry. We prove the regularity and well-posedness in the Sobolev spaces $H^s(M; E)$, $s \geq 0$, of the *mixed Dirichlet-Robin boundary value problem*

$$Pu = f \text{ in } M,\ u = 0 \text{ on } \partial_D M,\ \partial_\nu^P u + bu = 0 \text{ on } \partial M \setminus \partial_D M$$

under the following four natural assumptions. First, we assume that P satisfies the strong Legendre condition and the first order terms are small. (In the scalar case, the strong Legendre condition reduces to the uniformly strong ellipticity condition.) Second, we assume that all the coefficients of P and all their covariant derivatives are bounded. Third, we assume that $\Re b \geq 0$ and that there is $\epsilon > 0$ and an open and closed subset $\partial_R M \subset \partial M \setminus \partial_D M$ such that $\Re b \geq \epsilon I$ on $\partial_R M$. Finally, we assume that the distance to $\partial_R M \cup \partial_D M$ is uniformly bounded on M and that $\partial_R M \cup \partial_D M$ intersects all components of ∂M (*i. e.* $(M, \partial_R M \cup \partial_D M)$ has *finite width*).

We include also some extensions of our main result in different directions. First, the finite width assumption is required for the Poincaré inequality on manifolds with bounded geometry, a result for which we give a new, more general proof. Second, we consider also the case when we have a decomposition of the vector bundle E (instead of a decomposition of the boundary). Third, we also consider operators with non-smooth coefficients, but, in this case, we need to limit the range of s. Finally, we also consider the case of uniformly strongly elliptic operators and discuss the equivalence between the *uniform Agmon condition* and the Gårding inequality. The main novelty of our results is that they are formulated on a *non-compact manifold.*

Contents

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1. Introduction

This is the third paper in a series of papers devoted to the spectral and regularity theory of differential operators on a suitable non-compact manifold with boundary M using analytic and geometric methods. In this paper, we extend the well-posedness result of [7], the first paper of the series, from the case of the Laplace operator to that of operators (or systems) with non-smooth coefficients satisfying the strong Legendre condition. Considering systems is important in practical applications. We also take advantage of the general regularity results in [33], the second paper in this series, to obtain results on Robin boundary conditions. The Robin boundary conditions "interpolate" between Dirichlet and Neumann boundary conditions, so are natural to consider.

We have made an extra effort to make this paper readable independently of the previous two papers by recalling the main definitions and results from those papers.

1.1. **Geometric and analytic settings.**

We make several assumptions on the geometry and on the operators. Let us begin by describing our geometric setting. We fix in what follows a smooth m-dimensional Riemannian manifold with *boundary and bounded geometry* (M, g), see Definition 2.6. In particular, ∂M is smooth and a manifold with bounded geometry in its own right. Also, we fix a vector bundle $E \to M$ with a metric and a compatible connection ∇^E. We let R^E denote the curvature of the connection ∇^E. We assume that all the covariant derivatives $\nabla^k R^E$ are bounded. Moreover, we assume our boundary to be *partitioned*, that is, that we are given a disjoint union decomposition

$$\partial M = \partial_D M \sqcup \partial_N M \sqcup \partial_R M \tag{1}$$

where $\partial_D M$, $\partial_N M$ and $\partial_R M$ are (possibly empty) open and closed subsets of ∂M and $\sqcup$ is the *disjoint union.* The indices of the notation reflect that these will become the parts of the boundary where we will impose Dirichlet, Neumann, and Robin boundary conditions, respectively. Let $A \subset \partial M$. Recall from [7] that (M, A) is said to have *finite width* if, in addition to the bounded geometry assumption on (M, g), $\mathrm{dist}(p, A)$ is uniformly bounded in $p \in M$ and A intersects all boundary components of ∂M.

Let us now describe our analytic setting, which, in particular, will describe our operators. All the differential operators considered in this paper will be assumed to have bounded, measurable (*i. e.* L^∞) coefficients. The most important ingredients are a bounded sesquilinear form a on $T^*M \otimes E$ and a first order differential operator b on $E|_{\partial M}$. They give rise to an operator $\tilde{P}_{(a,b)} \colon H^1(M;E) \to H^1(M;E)^*$ by

$$\langle \tilde{P}_{(a,b)}(u), v\rangle := \int_M a(\nabla u, \nabla v)\, \mathrm{dvol}_g + \int_{\partial_R M} (bu, v)_E \, \mathrm{dvol}_{\partial g}, \tag{2}$$

where dvol denotes the volume form with respect to the underlying metric and $\langle\ ,\ \rangle$ denotes the pairing between a space V and its conjugate dual V^*. (The spaces H^1 are recalled in the main body of the paper). See Section 2.2.2 for more details. We note that Gesztesy and Mitrea have considered also non-local operators b, see [31] and the references therein. Let also Q and Q_1 be first order differential operators acting on sections of E. They define linear maps $\tilde{Q}, \tilde{Q}_1^* \colon H^1(M;E) \to H^1(M;E)^*$. First, we let

$$H^1_D(M;E) := \{\, u \in H^1(M;E) \mid u = 0 \text{ on } \partial_D M \,\}. \tag{3}$$

Our regularity and well-posedness results will be for the second order differential operator

$$\tilde{P} := \tilde{P}_{(a,b)} + \tilde{Q} + \tilde{Q}_1^* \colon H^1_D(M;E) \to H^1_D(M;E)^*, \tag{4}$$

which encodes also the Robin boundary conditions. Ignoring these boundary conditions via the restriction $H^1_D(M;E)^* \to H^1_0(M;E)^*$, we obtain the "typical" second order differential operator (in divergence form)

$$P := P_{(a,b)} + Q + Q_1^* \colon H^1_D(M;E) \to H^{-1}(M;E) \simeq H^1_0(M;E)^*. \tag{5}$$

This operator is hence independent of b, unlike $\tilde{P}$.

We will use the operator $\tilde{P}$ to study mixed Dirichlet-Robin boundary conditions, as follows. Let ν be the outward unit vector field at the boundary and ∂_ν^P the conormal derivative associated to P. We consider the *mixed Dirichlet-Robin boundary value problem*:

$$\begin{cases} Pu &= f & \text{in } M, \\ u &= 0 & \text{on } \partial_D M, \\ \partial_\nu^P u + bu &= 0 & \text{on } \partial_N M \sqcup \partial_R M. \end{cases} \tag{6}$$

The relation between this boundary value problem and $\tilde{P}$ is that, in a certain sense that will be made precise below using the maps j_k of Equation (19), we have that $\tilde{P}(u) = (Pu, \partial_\nu^P u + bu)$. (See [33] for a more detailed discussion of the difference between P and $\tilde{P}$ and the role of boundary conditions and [22, 30] for some related results.) We note that the operator ∂_ν^P of the last equation of (6) depends only on a and Q_1. If $P = \Delta$, the Laplacian, then $\partial_\nu^P = \partial_\nu$ is the usual normal derivative.

As in [11, 31], we shall typically assume for our main results that P satisfies the strong Legendre condition, which is the condition that the bilinear form a defining the principal part $P_{(a,b)}$ of the operator P be strongly coercive (Definition 4.1). For scalar operators, the strong Legendre condition is equivalent to the uniform strong ellipticity condition, but, for systems, the strong Legendre condition is more restrictive.

If T is a (possibly unbounded) operator on a Hilbert space $\mathcal{H}$, we shall write $T \geq \epsilon$ if $(T\xi, \xi) \geq \epsilon(\xi, \xi)$ for all ξ in the domain of T and denote $\Re T := \frac{1}{2}(T + T^*)$.

Recall that $\sqcup$ denotes the disjoint union. Our main result is the following well-posedness result.

Theorem 1.1. *Let $\ell \in \mathbb{N}$, $\ell \geq 1$, and assume that:*

(i) $(M, \partial_D M \sqcup \partial_R M)$ *has finite width;*
(ii) $P := P_{(a,b)} + Q + Q_1^*$ *has coefficients in* $W^{\ell,\infty}(M; \mathrm{End}(E))$ *and satisfies the strong Legendre condition;*
(iii) $\Re\, b \geq 0$ *and there is* $\epsilon > 0$ *such that* $\Re\, b \geq \epsilon$ *on* $\partial_R M$.
(iv) There is $\delta > 0$ *depending on* ϵ, a, b, *and* (M, g) *such that* $\Re\,(Q + Q_1^*) \geq -\delta$.

Then, for all $k \in \mathbb{N}$, $1 \leq k \leq \ell$, $\tilde{P}u := (Pu, (\partial_\nu^P u + bu)|_{\partial M \setminus \partial_D M})$ *defines an isomorphism*

$$\widetilde{P} \colon H^{k+1}(M; E) \cap \{\, u|_{\partial_D M} = 0 \,\} \to H^{k-1}(M; E) \oplus H^{k-1/2}(\partial M \setminus \partial_D M; E).$$

This theorem follows directly from Theorem 4.7 and 4.8. In fact, it does not matter what b is on $\partial_D M$. In particular, the condition $\Re\, b \geq 0$ is necessary only on $\partial M \smallsetminus \partial_D M$.

1.2. **Comments.** The proof of our main result, Theorem 1.1 combines the Poincaré inequality with regularity. More precisely, by replacing $H^{k-1}(M; E) \oplus H^{k-1/2}(M \setminus \partial_D M; E)$ with $H_D^1(M; E)^*$ as the range for $\widetilde{P}$, our theorem makes sense also for $k = 0$. This pattern of proof follows the classical case [4, 7, 18, 23, 38, 40, 43, 51, 53]. Using the trace theorem [34], we can also consider non-homogeneous Dirichlet boundary conditions in $H^{k+1/2}(\partial_D M; E)$. What is essentially different in the non-compact case is how these two steps (Poincaré inequality and regularity) are dealt with.

For instance, the Poincaré inequality follows from the finite width assumption, using the results from [7]. We moreover know, from that paper, that the assumption that $(M, \partial_D M \sqcup \partial_R M)$ has finite width is necessary in general. Indeed, if M is a subset of $\mathbb{R}^n$ with the induced metric such that $(M, \partial M)$ is not of finite width, then the theorem is not true anymore. A counterexample is provided by a domain that coincides with a cone in a neighborhood of infinity. The finite width assumption on $(M, \partial_D M \sqcup \partial_R M)$ is needed in order to obtain the Poincaré inequality, which implies the special case $k = 0$ of our theorem, see Theorem 4.7 (and is essentially equivalent to it).

For regularity, we can use either positivity (or coercivity) or a uniform version of the Shapiro-Lopatinski conditions. We refer the reader to [33], where this issue is dealt with in detail.

The reader may have wondered what happens in the strongly elliptic case (for systems). In that case, the coercivity (*i. e.* the Gårding inequality) is equivalent to the *uniform Agmon* condition for the boundary conditions, see Subsection 4.3.5. If the uniform Agmon condition is satisfied, then one obtains the analog of Theorem 1.1 for P replaced with $P + R$, for some real, large enough $R > 0$.

1.3. **Earlier results and novelty of our results.** Recently, many results on Robin boundary conditions were obtained, almost all devoted to *bounded domains with non-smooth boundaries.* This is the case with the nice papers by Dancer and Daners [21], Daners [22], and Gesztesy and Mitrea [30, 31], to which we refer for more references. As seen from our result, our focus is rather on *unbounded* domains, but we assume a smooth boundary. This allows us also to obtain certain regularity results for our problem that do not make sense in the Lipschitz case. In fact, our main theorem, Theorem 1.1, is new even in the case of pure Dirichlet or pure Neumann boundary conditions.

One of the new issues that one has to deal with in the case of unbounded domains is the Poincaré inequality. The L^1–Poincaré inequality for scalar functions and for $\partial_D M = \partial M$ was proved in [47]. The form that we need is in [7]. In view of its importance and for further applications, we extended the Poincaré inequality from [7] to functions vanishing on suitable subsets $A \subset \partial M$ by using a new proof based on uniform coverings. The extension is that we no longer assume that A be an open and closed subset of ∂M, but we need a slightly stronger condition than that of (M, A) being of finite width.

Theorem 1.1 was proved in [7] for $P = P_{(g,0)} = \Delta_g \geq 0$, where g is the metric and $P_{(a,b)}$ is as defined in Equation (2) above. If $P = P_{(a,0)}$ with $E = \underline{\mathbb{C}}$ (the one dimensional trivial bundle) and a is real and smooth, Theorem 1.1 then follows also from the results of [7] by replacing the metric g with the equivalent metric a, since in the scalar case the strong Legendre condition is equivalent to the condition that P be uniformly strongly elliptic. The general case, namely $P = P_{(a,b)} + Q + Q_1^*$, where Q and Q_1 are first order differential operators, presents the following additional difficulties:

(i) If $Q + Q_1^* \neq 0$, we cannot reduce P to a Laplacian, even if a is smooth;
(ii) a may be not be real;
(iii) a may not be smooth;
(iv) the bundle E may be topologically non-trivial or of higher dimension.

The first three extensions are relatively easy to deal with. We deal with $Q \neq 0$ or $Q_1 \neq 0$ by assuming that the negative part of $Q + Q^* + Q_1 + Q_1^*$ is small enough (Condition (iv) of Theorem 1.1). The case when a is not real simply requires to use a complex version of the Lax-Milgram Lemma. In the case a is not smooth, we simply restrict the regularity of the resulting solution. These three extensions of the results in [7] *do not* follow from the results of that paper, but can be obtained using the methods of that paper and those in [33], once the additional background material in Section 2 is taken into account.

The last extension, (iv), (to E non-trivial, that is, to systems) causes the most headaches and, so far, to the best of our knowledge, is not dealt with in a completely satisfactory way anywhere. To extend our results to systems, we chose to consider the condition that P satisfies the strong Legendre condition. This condition is satisfied by the Hodge Laplacian $dd^* + d^*d$, but is often too restrictive for applications. The weaker condition (that P be uniformly strongly elliptic) is satisfied in many applications, but it seems that for systems is does not provide results as strong as the ones that one obtains for scalar equations. Nevertheless, one can obtain coercivity under some additional assumptions, see 4.3.5.

We have also included Robin boundary conditions. Except for a few results and definitions that we recall from [7, 33], the first two papers of this series, our paper can be read independently of those papers, as we recall in Section 2 the most important definitions and results from those papers.

1.4. **Contents of the article.** The article is organized as follows. Section 2 is devoted to preliminaries, including a discussion of Sobolev spaces, of differential operators on Riemannian manifolds from a global point of view, and to some background material on manifolds with bounded geometry from [7]. The proof of the

Poincaré inequality is in Section 3. The last section contains the proofs of our main results, which, in turn, yield Theorem 1.1. We also discuss there some extensions of our results in Subsection 4.3, including the uniform Agmon condition.

2. Background, notation, and preliminary results

We recall here some basic material, for the benefit of the reader. We also use this opportunity to fix the notation. For instance, *M will always denote a smooth m-dimensional Riemannian manifold, possibly with boundary.* The metric of M will be denoted by g and the associated volume form will be denoted by dvol_g. The boundary is denoted by ∂M, and is assumed to be smooth, for simplicity, although some intermediate results may hold in greater generality. We assume that the boundary is partitioned as in Equation 1. See [7] or [33] for concepts and notation not fully explained here.

2.1. **General notations and definitions.**

We begin with the most standard concepts and some notation.

2.1.1. *Vector bundles.* Let $E \to M$ be a smooth real or complex vector bundle endowed with metric $(.,.)_E$ and a connection

$$\nabla^E \colon \Gamma(M; E) \to \Gamma(M; E \otimes T^*M).$$

We assume that ∇^E is metric preserving, which means that

$$X(\xi, \eta)_E = (\nabla_X \xi, \eta)_E + (\xi, \nabla_X \eta)_E.$$

We endow the tangent bundle $TM \to M$ with the Levi-Civita connection.

Definition 2.1. A vector bundle $E \to M$ with given connection *has totally bounded curvature* if its curvature and all its covariant derivatives are bounded (that is, $\|\nabla^k R^E\|_\infty < \infty$ for all k). If TM has totally bounded curvature, we shall then say that M has *totally bounded curvature*.

2.1.2. *Sobolev spaces.* Let us recall the basic definitions related to Sobolev spaces. See [6, 10, 24, 35, 37, 52] for related results. The L^p-norm $\|u\|_{L^p(M;E)}$ of a measurable section of u of E is then

$$\|u\|^p_{L^p(M;E)} := \int_M |u(x)|^p_E \, \mathrm{dvol}_g(x)\,, \quad \text{if} \quad 1 \le p < \infty\,, \text{ and}$$
$$\|u\|_{L^\infty(M;E)} := \operatorname*{ess\text{-}sup}_{x \in M} |u(x)|_E\,,$$

as usual. Let $\ell \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We define $L^p(M; E) := \{u \mid \|u\|_{L^p(M;E)} < \infty\}$ and

$$W^{\ell,p}(M; E) := \{\, u \mid \nabla^j u \in L^p(M; E \otimes T^{*\otimes j} M)\,,\ \forall\, j \le \ell\,\}.$$

We let $W^{\infty,p} := \bigcap_\ell W^{\ell,p}$.

If M has a smooth boundary ∂M and $\partial_D M \subset \partial M$ is an open and closed subset of ∂M, we define

$$W^{\ell,p}_D(M; E) := \mathrm{closure}_{W^{\ell,p}(M;E)} C^\infty_c(M \setminus \partial_D M; E)\,, \tag{7}$$

the closure in $W^{\ell,p}(M; E)$ of the space of smooth sections of $E \to M$ that have compact support not intersecting $\partial_D M$. As usual, we shall use the notation

$$H^\ell(M; E) := W^{\ell,2}(M; E) \quad \text{and} \quad H^\ell_D(M; E) := W^{\ell,2}_D(M; E) \tag{8}$$

in the Hilbert space case ($p = 2$). If $\partial_D M = \partial M$, we simply write $W_0^{\ell,p}(M;E) := W_D^{\ell,p}(M;E)$ and $H_0^{\ell}(M;E) := W_0^{\ell,2}(M;E)$. For manifolds with bounded geometry, these spaces can be characterized using the trace theorem, see [34].

As in [30, 33], we denote by V^* the *complex conjugate* dual of the Banach space V. If $-s \in \mathbb{N}$, we define $H^s(M;E) \simeq H_0^{-s}(M;E)^*$. If M has no boundary and $s \in \mathbb{R}$, then the spaces $H^s(M;E)$ are defined by interpolating the spaces $H^{\ell}(M;E)$, $\ell \in \mathbb{Z}$. See [18, 36, 40, 51] for the case of manifolds with boundary.

2.2. **Differential operators.** We recall now differential operators on manifolds from a global point of view.

2.2.1. *General definitions.* A *differential operator* of order k is an expression of the form $P := \sum_{j=0}^{k} a_j \nabla^j$, with a_j a section of $\mathrm{End}(E) \otimes TM^{\otimes j}$. A differential operator $P = \sum_{j=0}^{k} a_j \nabla^j$ will be said to *have coefficients in* $W^{\ell,\infty}$ for $\ell \in \mathbb{Z}_+ \cup \{\infty\}$ if $a_j \in W^{\ell,\infty}(M; \mathrm{End}(E) \otimes TM^{\otimes j})$ for all $0 \leq j \leq k$. If $\ell = 0$, we shall say that P has *bounded* coefficients. If $\ell = \infty$, we shall say that P has *totally bounded* coefficients. We then obtain a bounded operator

$$P = \sum_{j=0}^{k} a_j \nabla^j \colon W^{\ell+k,p}(M;E) \to W^{\ell,p}(M;E), \quad \ell \geq 0.$$

2.2.2. *Bilinear forms and operators in divergence form.* We now consider differential operators in divergence form, which will allow us to treat the Robin boundary conditions on same footing as the Dirichlet boundary conditions. See [33] for more details. See also [22, 31]. Assume that, for each $x \in M$, we have a sesquilinear map $a_x \colon T_x^*M \otimes E_x \times T_x^*M \otimes E_x \to \mathbb{C}$. The family (a_x) defines a section a of the bundle $((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))'$. We say that the section $a = (a_x)_{x \in M}$ is a *bounded, measurable sesquilinear form on* $T^*M \otimes E$ if it is an L^∞-section of $((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))'$. Let us also consider a first order differential operator b on $E|_{\partial M}$. The *Dirichlet form* $B_{(a,b)}$ on $H_D^1(M;E)$ associated to such a bounded family of sesquilinear forms a and endomorphism section b is then

$$B_{(a,b)}(u,v) := B_{(a,b)}^g(u,v) := \int_M a(\nabla u, \nabla v)\, \mathrm{dvol}_g + \int_{\partial M \smallsetminus \partial_D M} (bu, v)_E\, \mathrm{dvol}_{\partial g}\,. \tag{9}$$

Note that $\langle \tilde{P}_{(a,b)}(u), v \rangle = B_{(a,b)}(u,v)$ by (2). If Q is a first order differential operator with bounded coefficients, then it also defines a continuous map $\tilde{Q} \colon H_D^1(M;E) \to L^2(M;E) \subset H_D^1(M;E)^*$. The adjoint $\tilde{Q}^*$ of $\tilde{Q}$ will then map $\tilde{Q}^* \colon H_D^1(M;E) \to H_D^1(M;E)^*$ as well. The sesquilinear form $B_{(a,b)}$ and the differential operators Q and Q_1 then define the differential operators $\tilde{P}_{(a,b)}$, $\tilde{P}$, and P of Equations (2-5).

Definition 2.2. We shall say that $\tilde{P} = \tilde{P}_{(a,b)} + \tilde{Q} + \tilde{Q}_1^* \colon H_D^1(M;E) \longrightarrow H_D^1(M;E)^*$ and $P := P_{(a,b)} + Q + Q_1^*$ are second order *differential operators in divergence form* if a is a bounded, measurable sesquilinear form on $T^*M \otimes E$, $b = b_1 + b_2$ is a first order differential operator on $E|_{\partial M}$, with b_1 with $W^{1,\infty}$ coefficients and b_2 a bounded, measurable endomorphism of $E|_{\partial M}$, and Q and Q_1 are first order differential operators with bounded coefficients. In particular, P will have bounded coefficients.

Remark 2.3. We have by definition

$$\langle \tilde{P}u, v \rangle := B(u,v) := (\nabla u, \nabla v) + (bu, v)_{\partial M \smallsetminus \partial_D M} + (Qu, v) + (u, Q_1 v) \tag{10}$$

where, we recall, $\langle .,. \rangle$ denotes the dual pairing and $(.,.)_N$ denotes the L^2-product on the manifold N (in case $N = M$ we omit the index).

2.2.3. *Boundary value problems.* We are interested in differential operators in divergence form $\tilde{P}\colon H^1_D(M;E) \to H^1_D(M;E)^*$ because we have the equivalence of the following two problems

(i) The operator $\tilde{P}\colon H^1_D(M;E) \to H^1_D(M;E)^*$ is an isomorphism.
(ii) For each $F \in H^1_D(M;E)^*$ and $h \in H^{1/2}(\partial_D M;E)$, the "weak" problem

$$\begin{cases} \tilde{P}(u)(v) = F(v) & \text{for all } v \in H^1_D(M;E) \\ \qquad u = h & \text{on } \partial_D M \end{cases}$$

has a unique solution $u \in H^1(M;E)$, depending continuously on F and h.

The well-posedness of these problems implies then the well-posedness of

(iii) Let $f \in L^2(M;E)$, $h \in H^{3/2}(\partial_D M;E)$, and $h_1 \in H^{1/2}(\partial M \setminus \partial_D M;E)$. The boundary value problem

$$\begin{cases} Pu &= f \quad \text{in} \quad M \\ u &= h \quad \text{on} \quad \partial_D M \\ \partial^P_\nu u + bu &= h_1 \quad \text{on} \quad \partial M \setminus \partial_D M \end{cases} \tag{11}$$

has a unique solution $u \in H^2(M;E)$, depending continuously on f, h, and h_1. This is obtained by taking $F(v) := \int_M (f,v)_E \,\mathrm{dvol}_g + \int_{\partial M \setminus \partial_D M} (g,v)_E \,\mathrm{dvol}_{\partial g}$ and using higher regularity. See Corollary 4.10 below.

For higher regularity of the data, we obtain the usual formulation of mixed boundary value problems. See Subsection 4.2. In particular, see [22, 31, 33] for the need of Q_1^* in the statement of the main theorem (Theorem 1.1) and for how Q_1^* affects the boundary conditions (*i. e.* the boundary operator ∂^P_ν). Note that the well-posedness of Problem (11) implies right away that of Problem (6). The converse is also true in view of the trace theorem of [34], since ∂M was assumed to be smooth.

The best way to understand the operator ∂^P_ν is using boundary triples [15, 46]. See [22, 33] for explicit definitions of ∂^P_ν in local coordinates.

2.3. **Manifolds with boundary and bounded geometry.** We first recall some basic material on manifolds with boundary and bounded geometry from [7], to which we refer for more details (see also [26, 48]). As in [7], we will only assume that our manifolds are paracompact (thus we will *not require* our manifolds to be second countable).

If $x, y \in M$, then $\mathrm{dist}(x,y)$ denotes the distance between x and y with respect to the metric g. If $N \subset M$, then

$$U_r(N) := \{x \in M \mid \exists y \in N,\ \mathrm{dist}(x,y) < r\} \tag{12}$$

will denote the r-neighborhood of N, that is, the set of points of M at distance $< r$ to N. Thus, if E is a Euclidean space, then $B^E_r(0) := U_r(\{0\}) \subset E$ is simply the ball of radius r centered at 0.

Let N be a hypersurface in M, *i. e.* a submanifold with $\dim N = \dim M - 1$. We assume that N carries a globally defined normal vector field ν of unit length, simply called a *unit normal field*, which will be fixed from now on. The Levi–Civita connection for the induced metric on N will be denoted by ∇^N. The symbol II^N will denote the *second fundamental form* of N (in M: $\mathrm{II}^N(X,Y)\nu := \nabla_X Y - \nabla^N_X Y$).

Let $\exp_p^M : T_pM \to M$ be the exponential map at p associated to the metric and

$$r_{\mathrm{inj}}(p) := \sup\{r \mid \exp_p^M : B_r^{T_pM}(0) \to M \text{ is a diffeomorphism onto its image}\}$$
$$r_{\mathrm{inj}}(M) := \inf_{p\in M} r_{\mathrm{inj}}(p).$$

Definition 2.4. A Riemannian manifold without boundary (M, g) is said to be of *bounded geometry* if $r_{\mathrm{inj}}(M) > 0$ and if M has *totally bounded curvature*.

If M has boundary, then $r_{\mathrm{inj}}(M) = 0$. Let $\exp^\perp(x, t) := \exp_x^M(t\nu_x)$.

Definition 2.5. Let (M^m, g) be a Riemannian manifold of bounded geometry with a hypersurface $H = H^{m-1} \subset M$ and a unit normal field ν on H. We say that H is a *bounded geometry hypersurface* in M if the following conditions are fulfilled:

(i) H is a closed subset of M;
(ii) $\|(\nabla^H)^k \mathrm{I\!I}^H\|_{L^\infty} < \infty$ for all $k \geq 0$;
(iii) $\exp^\perp : H \times (-\delta, \delta) \to M$ is a diffeomorphism onto its image for some $\delta > 0$.

As we have shown in [7], we have that the Riemannian manifold $(H, g|_H)$ in the above definition is then a manifold of bounded geometry. See also [25, 26] for a larger class of submanifolds of manifolds with bounded geometry. We shall denote by r_∂ the largest value of δ satisfying the last condition of the last definition. Recall from [48] the following definition (the precise form below is from [7]):

Definition 2.6. A Riemannian manifold M with (smooth) boundary has *bounded geometry* if there is a Riemannian manifold $\widehat{M}$ with bounded geometry satisfying

(i) M is contained in $\widehat{M}$;
(ii) ∂M is a bounded geometry hypersurface in $\widehat{M}$.

Example 2.7. Lie manifolds have bounded geometry [8, 9]. It follows that Lie manifolds with boundary are manifolds with boundary and bounded geometry.

For our well-posedness results, we shall also need to assume that $M \subset U_R(\partial_D M \cup \partial_R M)$, for some $0 < R < \infty$, and hence, in particular, that $\partial_D M \cup \partial_R M \neq \emptyset$.

Definition 2.8. If M is a manifold with boundary and bounded geometry, if $A \subset \partial M$ and $M \subset U_R(A)$, for some $0 < R < \infty$, we shall say that (M, A) *has finite width*.

Since we let $\mathrm{dist}(x, y) = \infty$ if x and y belong to different components of M, the condition that (M, A) have finite width then implies, in particular, that A intersects every component of M. See [5, 13, 14, 26, 28, 29, 39] for applications of manifolds of bounded geometry.

Vector bundles with totally bounded curvature defined on manifolds with bounded geometry are called *vector bundles with bounded geometry.*

3. The Poincaré inequality

We now give a new proof of the Poincaré inequality in [7, 47] and generalize it by allowing more general subsets of the boundary where the function vanishes. We assume from now on that M is a Riemannian manifold with boundary and bounded geometry.

3.1. **A uniform Poincaré inequality for bounded domains.** We shall need the following extension of the Poincaré inequality (see [17, 20] or [27, §5.8.1]), which is proved (essentially) in the same way as the classical result.

Proposition 3.1. *Assume that Ω is a connected domain of finite volume in a Riemannian manifold (M,g) such that $H^1(\Omega) \to L^2(\Omega)$ is a compact operator. Let $K \subset L^2(\Omega)^*$ be a bounded, weakly closed set of continuous linear functionals such that $L(1) \neq 0$ for all $L \in K$. Then there is $C > 0$ such that, for any $f \in H^1(\Omega)$ and any $L \in K$, we have*

$$\int_\Omega |f|^2 \,\mathrm{dvol}_g \,\leq\, C\Big(\int_\Omega |\nabla f|^2 d\,\mathrm{vol}_g + |L(f)|^2 \Big).$$

Proof. Let us assume, by contradiction, that the contrary is true. Then we can find a sequence $f_n \in H^1(\Omega)$ and a sequence $L_n \in K$ such that

$$\int_\Omega |f_n|^2 d\,\mathrm{vol}_g \,>\, n\Big(\int_\Omega |\nabla f_n|^2 d\,\mathrm{vol}_g + |L_n(f_n)|^2 \Big). \tag{13}$$

By replacing f_n with $\|f_n\|^{-1}_{H^1(\Omega)} f_n$, we may assume that $\|f_n\|_{H^1(\Omega)} = 1$. Then Equation (13) gives that $\nabla f_n \to 0$ in $L^2(\Omega)$ in norm and that $L_n(f_n) \to 0$.

Since the unit ball in a Hilbert space is a weakly compact set (by the Alaoglu-Bourbaki theorem) and we are dealing with a separable Hilbert space (so the weak topology on the unit ball of $H^1(\Omega)$ is metrisable), the sequence f_n has a subsequence weakly converging in $H^1(\Omega)$ to some $v \in H^1(\Omega)$. We replace the original sequence with that sequence. Then ∇f_n converges weakly to ∇v in $L^2(\Omega)$, since $\nabla\colon H^1(\Omega) \to L^2(\Omega)$ is continuous. Therefore $\nabla v = 0$ since we have seen that $\nabla f_n \to 0$ in $L^2(\Omega)$ in norm. Since Ω is connected, it follows that v is a constant.

Since $H^1(\Omega) \to L^2(\Omega)$ is a compact operator (by assumption), we obtain that f_n converges to v in norm in $L^2(\Omega)$. Since K was assumed to be bounded and weakly closed, it is weakly compact. We thus have that, by passing to a subsequence, we may also assume that L_n converges weakly to some $L \in K$. We thus obtain that $L_n(f_n) \to L(v)$, and hence $L(v) = 0$. Since v is a constant and $L(1) \neq 0$ (since $L \in K$), we obtain $v = 0$. This gives

$$1 = \|f_n\|^2_{H^1(\Omega)} = \|f_n\|^2_{L^2(\Omega)} + \|\nabla f_n\|^2_{L^2(\Omega)} \to \|v\|^2_{L^2(\Omega)} + 0 = 0,$$

which is a contradiction. □

We can replace the assumption that $K \subset L^2(\Omega)^*$ be a bounded, weakly closed set of continuous linear functionals with the assumption that $K \subset H^1(\Omega)^*$ be a (norm) compact subset. We shall need the following two corollaries (which hold in greater generality, but, for simplicity, we state the case that we need).

Corollary 3.2. *Let Ω be an open ball in $\mathbb{R}^n$ and $\epsilon > 0$. Then there exists $C > 0$ such that, for any $B \subset \Omega$ a subset of measure $\geq \epsilon$, we have*

$$\int_\Omega |f|^2 dx \,\leq\, C\Big(\int_\Omega |\nabla f|^2 dx + \int_B |f|^2 dx \Big)$$

for all $f \in H^1(\Omega)$.

Proof. We consider $K := \{L \in L^2(\Omega)^* \mid \|L\| \leq \mathrm{vol}(\Omega)^{\frac{1}{2}}, L(1) \geq \epsilon\}$ which is norm closed, convex, and bounded. Hence it is weakly compact. Then $L(u) := \int_B u \,\mathrm{dvol}_g$

is in K, whenever $B \subset \Omega$ is a subset of measure $\geq \epsilon$. Proposition 3.1 then gives

$$\int_\Omega |f|^2 dx \,\leq\, C\Big(\int_\Omega |\nabla f|^2 dx + \Big|\int_B f dx\Big|^2 \Big),$$

for some C independent of $f \in H^1(\Omega)$ and of B (of measure $\geq \epsilon$). The result then follows from the Cauchy-Schwarz inequality applied to f and the characteristic function of B: $|\int_B f dx|^2 \leq \Big(\int_B dx\Big) \int_B |f|^2 dx \leq \mathrm{vol}(\Omega) \int_B |f|^2 dx$. □

Similarly, we obtain the following corollary.

Corollary 3.3. *Let Ω be an open ball in $\mathbb{R}^n$, $\Omega = \Omega' \times [0, r]$, and $\epsilon > 0$. Then there exists $C > 0$ such that, for any $B \subset \Omega \times \{0\}$ a subset of measure $\geq \epsilon$, we have*

$$\int_\Omega |f|^2 dx \,\leq\, C\Big(\int_\Omega |\nabla f|^2 dx + \int_B |f|^2 dx' \Big).$$

3.2. **Proof of the Poincaré inequality.** Next we globalize the above inequalities to manifolds M with boundary and bounded geometry. We assume — following Definition 2.6 — that M is embedded in a boundaryless manifold $\widehat{M}$ of the same dimension, of bounded geometry and without boundary, such that ∂M is a bounded geometry hypersurface in $\widehat{M}$. Recall that $U_r(A)$ denotes the set of points of M at distance $< r$ to A. We use the notation in [7], which we recall now: Let $\{p_\gamma\}_{\gamma\in I}$ be a subset of M and $0 < 3r < \min\{r_{\mathrm{inj}}(M), r_{\mathrm{inj}}(\partial M), r_\partial\}$, where r_∂ is the largest value of δ satisfying the last condition of Definition 2.5 for $H = \partial M$ and for M replaced with $\widehat{M}$. We let $W_\gamma := W_\gamma(r) := U_r(p_\gamma)$, if p_γ is an interior point of M; otherwise we let $W_\gamma := W_\gamma(r) := \exp^\perp(B_r^{T\partial M}(0) \times [0, r))$.

Definition 3.4. Let (M^m, g) be a manifold with boundary and bounded geometry and assume $0 < 3r < \min\{r_{\mathrm{inj}}(M), r_{\mathrm{inj}}(\partial M), r_\partial\}$ as above. A subset $\{p_\gamma\}_{\gamma\in I}$ is called an *r-covering subset of M* if the following conditions are satisfied:

(i) For each $R > 0$, there exists $N_R \in \mathbb{N}$ such that, for each $p \in M$, the set $\{\gamma \in I |\ \mathrm{dist}(p_\gamma, p) < R\}$ has at most N_R elements.
(ii) For each $\gamma \in I$, we have either $p_\gamma \in \partial M$ or $d(p_\gamma, \partial M) \geq r$.
(iii) $M \subset \bigcup_{\gamma=1}^\infty W_\gamma(r)$.

We have the following Poincaré-type inequality, which allows us to consider more general Dirichlet boundary conditions.

Theorem 3.5. *Let (M, g) be a Riemannian manifold with boundary of bounded geometry, $E \to M$ be a hermitian vector bundle with a metric preserving connection, and $A \subset \partial M$ be a measurable subset. We assume that there exists an r-covering subset $\{p_\gamma\}_{\gamma\in I}$ and $S \subset \{\gamma \in I \mid p_\gamma \in \partial M\}$ satisfying the following properties:*

(i) $\mathrm{dist}(x, S)$ *is bounded on M;*
(ii) there exists $\epsilon > 0$ such that, for any $\gamma \in S$, $\mathrm{vol}_{\partial M}(A \cap W_\gamma) \geq \epsilon\, \mathrm{vol}_{\partial M}(W_\gamma)$.

Then there exists $C_{M,A} > 0$ such that

$$\int_M |f|^2\, \mathrm{dvol}_g \leq C_{M,A}\Big(\int_M |\nabla f|^2\, \mathrm{dvol}_g + \int_A |f|^2\, \mathrm{dvol}_{\partial g} \Big),$$

for any smooth, compactly supported section f of E.

Proof. The vector bundle case follows from the scalar case by Kato's inequality, see the end of the proof. So let us assume in the beginning that E is the trivial real line bundle and hence that f is a smooth, real-valued, compactly supported function on M.

Let us assume, for the simplicity of notation, that we have a countable set of indices γ for our r-covering set, which is equivalent to having a countable basis of topology. We first enlarge the given set $\{p_\gamma\}$ to be an $r/3$-covering set (but still use r to define the sets W_γ; we need that in order to ensure that two neighboring W_γ's will meet in a large enough set). Let $S_0 := S \subset \mathbb{N}$. Define, by induction, $S_{\ell+1}$ to be the set of $\gamma \in \mathbb{N} \setminus \bigcup_{j=0}^{\ell} S_j$ such that p_γ is at distance at most $2r/3$ to S_ℓ. Then, for N large enough, we have $\mathbb{N} = S_0 \cup S_1 \cup \ldots \cup S_N$, since there exists (by assumption) $R > 0$ such that $\mathrm{dist}(x, S) \le R$, for all $x \in M$. For each $\gamma \in S_{\ell+1}$, $\ell \ge 0$, we choose a *predecessor* $\pi(\gamma) \in S_\ell$ such that $\mathrm{dist}(p_\gamma, p_{\pi(\gamma)}) \le 2r/3$.

Below, $C > 0$ is a constant (close to 1) that yields a comparison of the volume elements on M and on the coordinate charts $\kappa_\gamma := \kappa_{p_\gamma}$ corresponding to the r-covering defined by the r-covering set $\{p_\gamma\}_{\gamma\in\mathbb{N}}$:

$$\begin{cases} \kappa_p\colon B_r^{m-1}(0) \times [0, r) \to M, & \kappa_p(x,t) := \exp^\perp(\exp_p^{\partial M}(x), t), \quad \text{if } p \in \partial M \\ \kappa_p\colon B_r^m(0) \to M, & \kappa_p(v) := \exp_p^M(v), \quad \text{otherwise.} \end{cases} \tag{14}$$

(So W_γ is the image of κ_{p_γ}.) The constant C depends only on r and M, but not on $\gamma \in \mathbb{N}$, since M has bounded geometry and we have chosen r less than the injectivity radius of M. If $\gamma \in S_0 := S$, then Corollary 3.3 gives for $\Omega := B_r^{m-1}(0) \times [0, r)$ and $B := \kappa_\gamma^{-1}(A \cap W_\gamma) \subset B_r^{m-1}(0) \times \{0\}$

$$\begin{aligned} \int_{W_\gamma} |f|^2 \,\mathrm{dvol}_g \le C \int_\Omega |f \circ \kappa_\gamma|^2 dx &\le C C_\Omega \Big(\int_\Omega |\nabla^E (f\circ\kappa_\gamma)|^2 dx + \int_B |f\circ\kappa_\gamma|^2 \,\mathrm{dvol}_{\partial g} \Big) \\ &\le C C' C_\Omega \Big(\int_{W_\gamma} |\nabla f|^2 dx + \int_{W_\gamma \cap A} |f|^2 \,\mathrm{dvol}_{\partial g} \Big). \end{aligned} \tag{15}$$

Here ∇^E is the covariant derivative with respect to the euclidean metric and C' is the constant in the equivalence of the local H^1-norms with respect to the euclidean metric and g. Again, since (M, g) is of bounded geometry, this constant does not depend on γ. On the other hand, by the bounded geometry assumption and the choice of the W_γ, if $\gamma \notin S_0$, the sets W_γ and W_β will intersect in a set of volume (or measure) $\ge \epsilon$ for some $\epsilon > 0$ independent of γ and β if $\mathrm{dist}(p_\gamma, p_\beta) \le 2r/3$. Then using Corollary 3.2 (for $\Omega := B_r^m(0)$ and $B := \kappa_\gamma^{-1}(W_\gamma \cap W_\beta) \subset B_r^m(0)$, and $\beta = \pi(\gamma)$) and similar calculations to (15) we obtain

$$\int_{W_\gamma} |f|^2 \,\mathrm{dvol}_g \le C \Big(C_\Omega \int_{W_\gamma} |\nabla f|^2 \,\mathrm{dvol}_g + \int_{W_{\pi(\gamma)}} |f|^2 \,\mathrm{dvol}_g \Big). \tag{16}$$

Iterating Equation (16) and using Equation (15) we obtain that there exists $C_k > 0$ such that for $\gamma \in S_k$ we have

$$\int_{W_\gamma} |f|^2 \,\mathrm{dvol}_g \le C_k \Big(\sum_{j=0}^{k} \int_{W_{\pi^j(\gamma)}} |\nabla f|^2 \,\mathrm{dvol}_g + \int_{W_{\pi^k(\gamma)} \cap A} |f|^2 \,\mathrm{dvol}_{\partial g} \Big). \tag{17}$$

(This equation reduces to Equation (15) if $k = 0$.) Since our cover is uniform, there exists $N_0 \in \mathbb{N}$ such that no point in M belongs to more than N_0 sets of the form W_γ. We can also assume that the $C_0 \le C_1 \le \ldots \le C_N$ (recall that we stop at N). Using these observations and summing up (17) over γ, we obtain

$$\begin{aligned}\int_M |f|^2 \,\mathrm{dvol}_g &\le \sum_{\gamma=1}^{\infty} \int_{W_\gamma} |f|^2 \,\mathrm{dvol}_g \\ &\le C_N \sum_{\gamma=1}^{\infty} \Big(\sum_{j=0}^{k} \int_{W_{\pi^j(\gamma)}} |\nabla f|^2 \,\mathrm{dvol}_g + \int_{W_{\pi^k(\gamma)} \cap A} |f|^2 \,\mathrm{dvol}_{\partial g} \Big) \\ &\le (N+1) N_0 C_N \Big(\int_M |\nabla f|^2 \,\mathrm{dvol}_g + \int_A |f|^2 \,\mathrm{dvol}_{\partial g} \Big),\end{aligned}$$

which is the desired inequality in the scalar case where $C_{M,A} = (N+1)N_0 C_N$ (note that k depends on x, but we have $k \le N$, which explains the factor $N+1$ in $C_{M,A}$).

For general vector bundles E with metric connection, we have the Kato inequality $|\nabla |f|_E| \le |\nabla f|_E$. Using then the inequality just proved for f replaced by $|f|$ we have

$$\begin{aligned}\int_M |f|^2 \,\mathrm{dvol}_g &\le C_{M,A} \Big(\int_M |\nabla |f||^2 \,\mathrm{dvol}_g + \int_A |f|^2 \,\mathrm{dvol}_{\partial g} \Big) \\ &\le C_{M,A} \Big(\int_M |\nabla f|^2 \,\mathrm{dvol}_g + \int_A |f|^2 \,\mathrm{dvol}_{\partial g} \Big).\end{aligned}$$

This completes the proof. □

Example 3.6. Let $M = [0,1] \times \mathbb{R}$. Then $A = \bigcup_{k \in \mathbb{Z}} \{0\} \times [2k, 2k+1]$ satisfies the assumptions of our theorem, however, that would not be the case if we replaced A with $\{0\} \times [0, \infty)$.

We obtain the following result (proved for $A = \partial M$ in [47] and, in general, for $A = \partial_D M$ in [7])

Corollary 3.7. *Let (M, g) be a Riemannian manifold with boundary of bounded geometry, let $A \subset \partial M$ be an open and closed subset such that (M, A) has finite width. Moreover, let $E \to M$ be a hermitian vector bundle with a metric preserving connection. Then there exists $C_{M,A} > 0$ such that*

$$\int_M |f|^2 \,\mathrm{dvol}_g \le C_{M,A} \Big(\int_M |\nabla f|^2 \,\mathrm{dvol}_g + \int_A |f|^2 \,\mathrm{dvol}_{\partial g} \Big),$$

for any smooth, compactly supported section f of E.

Proof. This follows right away from Theorem 3.5 by taking any r-covering set $\{p_\gamma\}$ and $S = \{\gamma \mid p_\gamma \in A\}$. □

We have the following extension of the Poincaré inequality

Corollary 3.8. *Let us keep the assumptions of Corollary 3.7. Then*

$$\int_M |f|^2 \,\mathrm{dvol}_g \le C_{M,A}^k \int_M |\nabla^k f|^2 \,\mathrm{dvol}_g,$$

for any $f \in H^k(M; E)$, vanishing of order k at A.

Proof. Both the left hand side and the right hand side are continuous with respect to the H^k-norm. We have that $\mathcal{C}_c^\infty(M \setminus A; E)$ is dense in $\{f \in H^k(M; E) \mid \partial_\nu^j u = 0 \text{ on } A, j \le k-1\}$ (see [7] and the references therein, for instance). The proof is then obtained by iterating Corollary 3.7. □

4. Well-posedness

We now prove our well-posedness results, under the assumption that P satisfies the strong Legendre condition, that $(M, \partial_D M \cup \partial_R M)$ has finite width, and that $E \to M$ has totally bounded curvature (in which case, we recall, E is said to have bounded geometry). See Subsection 4.3 for an extension of our results to the case when we have a decomposition $E|_{\partial M} = E_D \oplus E_R \oplus E_N$ of the vector bundle $E|_{\partial M}$, instead of a decomposition of the boundary ∂M.

Recall that, by the definition of finite width, our assumption that $(M, \partial_D M \cup \partial_R M)$ has finite width implies, in particular, that M is of bounded geometry. Also, recall that we assume that all our differential operators have bounded coefficients.

4.1. Coercivity.

In order to study the invertibility of operators like $\tilde{P}$, one often uses "strong coercivity." An easy way to obtain strongly coercive operators is to combine the "strong Legendre condition" with the Poincaré inequality. See, however, Subsection 4.3 for a discussion of uniformly strongly elliptic operators and of the Gårding's inequality. We now recall the needed concepts, using the terminology of [1, 19]. See also [30, 31, 41, 51].

Definition 4.1. Let a be a bounded, measurable sesquilinear form on $T^*M \otimes E$. We say that a satisfies the *strong Legendre condition* if there exists $\gamma_a > 0$ such that

$$\Re\, a(\zeta, \zeta) \;\geq\; \gamma_a |\zeta|^2, \text{ for all } \zeta \in T^*M \otimes E\,. \tag{18}$$

Note that this is a condition at every $T_x^*M \otimes E_x$ and that it is *uniform* in x. It would be more apropriate then to say that a satisfies the *uniform strong Legendre condition.* For simplicity, we have chosen not to do that. However, in agreement with the standard terminology, we use the terminology *uniformly strongly elliptic* for operators that are strongly elliptic with uniform constants. We can now introduce the operators in which we are interested.

Definition 4.2. Let $\tilde{P} = \tilde{P}_{(a,b)} + \tilde{Q} + \tilde{Q}_1^*$ be a second order (linear) differential operator in divergence form on the vector bundle $E \to M$ (Definition 2.2), with Q and Q_1 first order differential operators (as usual). We shall say that $\tilde{P}$ (or P) satisfies the *strong Legendre condition* if a does. (Recall that it is a standing assumption that $\tilde{P}$ has bounded coefficients.)

Thus P satisfies the strong Legendre condition if, and only if, $P_{(a,b)}$ does. Moreover, if P satisfies the strong Legendre condition, then it is uniformly strongly elliptic. One of our results next amounts to the fact that, if the Poincaré inequality is satisfied, if $P = P_{(a,b)}$ satisfies the strong Legendre condition, if $\Re\, b \geq \epsilon$, $\epsilon > 0$ on $\partial_R M$ and $\Re\, b \geq 0$ on ∂M, and if condition (iii) of Theorem 1.1 is fulfilled, then P will also be "strongly coercive," a concept that we now recall.

Definition 4.3. Let V be a Hilbert space and let $S\colon V \to V^*$ be a bounded operator. We say that S is *strongly coercive* (on V) if there exists $\gamma > 0$ such that

$$\Re\, \langle Su, u\rangle \geq \gamma \|u\|_V^2.$$

In other words, the smooth family $(a_x)_{x\in M}$ of sesquilinear forms $a_x\colon T_x^*M \otimes E \times T_x^*M \otimes E \to \mathbb{C}$ satisfies the strong Legendre condition if, and only if, it is *uniformly strongly coercive.*

Lemma 4.4. *Let us assume that $(M, \partial_D M \sqcup \partial_R M)$ has finite width. Then the semi-norm*

$$|||u|||^2 := \|\nabla u\|^2_{L^2(M;E)} + \int_{\partial_R M} |u|^2_E \, \mathrm{dvol}_{\partial g}$$

is a norm on $H^1_D(M;E)$ that is equivalent to the H^1-norm.

Proof. Using the trace theorem [34], there is $c > 0$ such that $|||u||| \leq c\|u\|_{H^1}$. The reverse inequality is obtained as follows: Let c_2 be the best constant in the Poincaré inequality of Corollary 3.7 for $A = \partial_D M \cup \partial_R M$ and sections *vanishing on $\partial_D M$*. Then $\|u\|^2_{H^1} \leq (1 + c_2)|||u|||^2$. □

The strong Legendre condition and Poincaré's inequality combine to yield *strong* coercivity:

Proposition 4.5. *Let $P = P_{(a,b)}$ be a second order (linear) differential operator in divergence form on the vector bundle $E \to M$ (see Definition 2.2). Assume that $(M, \partial_D M \sqcup \partial_R M)$ has finite width, that P satisfies the strong Legendre condition, that $\Re\, b := \frac{1}{2}(b + b^*) \geq 0$ on ∂M (as operators), and that there exists $\epsilon > 0$ such that $\Re\, b \geq \epsilon$ on $\partial_R M$, then P is strongly coercive on $H^1_D(M;E)$. (So $Q = Q_1 = 0$ in this result.)*

Proof. The definition of $\tilde{P}_{(a,b)}$, Equation (2), gives for all $u \in H^1_D(M;E)$ that

$$\begin{aligned}\Re\,(\tilde{P}_{(a,b)}u)(u) &= \int_M \Re\, a(\nabla u, \nabla u)\, \mathrm{dvol}_g + \int_{\partial M \smallsetminus \partial_D M} \Re\,(bu, u)\, \mathrm{dvol}_{\partial g} \\ &\geq \gamma_a \|\nabla u\|^2 + \epsilon \int_{\partial_R M} |u|^2 \, \mathrm{dvol}_{\partial g} \geq \min\{\gamma_a, \epsilon\}|||u|||^2 \geq \frac{\min\{\gamma_a, \epsilon\}}{1 + c_2}\, \gamma_a \|u\|^2_{H^1},\end{aligned}$$

where the last step is by Lemma 4.4. The proof is complete. □

The relation $\Re b := \frac{1}{2}(b + b^*) \geq \epsilon$, as operators, means, as customary, that

$$\Re(b\zeta, \zeta) = (\Re b\zeta, \zeta) \geq \epsilon\|\zeta\|^2_{L^2},$$

for all $\zeta \in H^1(\partial_R M; E)$.

We are interested in strongly coercive operators in view of the Lax-Milgram Lemma (see, for example, [32, Section 5.8]).

Lemma 4.6 (Lax–Milgram lemma)**.** *Let $S \colon V \to V^*$ be a strongly coercive map with $\Re\langle Su, u\rangle \geq \gamma\|u\|^2_V$. Then S is invertible and $\|S^{-1}\| \leq \gamma^{-1}$.*

Combining the above results (Proposition 4.5 and the Lax-Milgram Lemma 4.6), we immediately obtain the following theorem which is the analog result of Theorem 1.1 for $k = 0$.

The theorem uses the definitions of P and $\tilde{P}$ explained in Definition 2.2. Recall that $\tilde{P}_{(a,b)}$ is defined by the sesquilinear form a, by the first order differential operator b acting on $E_{\partial_R M}$, by the first order differential operators Q and Q_1, and, finally, by the relation $\tilde{P} = \tilde{P}_{(a,b)} + \tilde{Q} + \tilde{Q}_1^*$. All operators are assumed to have bounded coefficients. Moreover, $P_{(a,b)}$ is the associated second order operator obtained by partial integration from $\tilde{P}_{(a,b)}$ ignoring boundary terms, that is, $P = P_{(a,b)} + Q + Q_1^*$.

Theorem 4.7. *Let (M, g) be a Riemannian manifold with boundary. Assume that:*

(i) $(M, \partial_D M \sqcup \partial_R M)$ has finite width.

(ii) $P = P_{(a,b)} + Qu + Q_1^*$ *satisfies the strong Legendre condition and has bounded coefficients, as usual;*

(iii) $\Re\, b \geq 0$ *and there is* $\epsilon > 0$ *such that* $\Re b \geq \epsilon$ *on* $\partial_R M$*.*

(iv) *there is* $\delta = \delta(a, b, g) \geq 0$ *small enough such that* $\Re(Q + Q_1) \geq -\delta$*.*

Then $\tilde{P}\colon H^1_D(M; E) \to H^1_D(M; E)^*$ *is an isomorphism.*

Note that $\Re(Q+Q_1^*) = \Re(Q+Q_1)$. In particular, the condition $\Re(Q+Q_1) \geq -\delta$ means that

$$\Re((Q + Q_1)\xi, \xi) = \Re\big((Q\xi, \xi) + (\xi, Q_1\xi)\big) \geq -\delta\|\xi\|^2_{H^1},$$

for all $\xi \in H^1_D(M; E)$.

4.2. **Higher regularity.** We continue to assume that M is a smooth manifold with smooth boundary and bounded geometry. In this section, we record what is one of our main applications of the Poincaré inequality, that is, the well-posedness of the mixed Dirichlet-Robin problem on manifolds with finite width in *higher* Sobolev spaces. Even the particular case of the Poisson problem with Neumann or Dirichlet boundary conditions is new in the setting of manifolds with bounded geometry. These results extend the well-posedness result in energy spaces of the previous subsection to higher regularity Sobolev spaces. They follow by combining the well-posedness in energy spaces with the regularity results in [33].

To this end, we assume that P has coefficients in $W^{k,\infty}$, for some fixed $k \geq 1$. We also continue to assume that $\tilde{P} = \tilde{P}_{(a,b)} + \tilde{Q} + \tilde{Q}_1^*$ (again with P and $\tilde{P}$ defined as in Definition 2.2) satisfies the strong Legendre condition and $\Re\, b$ is strictly positive on ∂_R and nonnegative everywere. We have seen then that $\tilde{P}_{(a,b)}$ is strongly coercive.

Let us define

$$j_k\colon H^{k-1}(M; E) \oplus H^{k-1/2}(M \setminus \partial_D M; E) \to H^1_D(M; E)^* \tag{19}$$

by $j_k(f, g)(w) := \int_M (f, w)\, \mathrm{dvol}_g + \int_{\partial M \smallsetminus \partial_D M} (g, w)\, \mathrm{dvol}_{\partial g}$, if $k \geq 1$, $j_0 = id$, if $k = 0$. Note, however, that, for $k = 0$, we have an exact sequence

$$0 \to H^{-1/2}(M \setminus \partial_D M; E) \to H^1_D(M; E)^* \to H^{-1}(M; E) \to 0,$$

which explains our notation. If $\tilde{P}u = j_k(f, g)$, we shall write $\partial^P_\nu u + bu = g$ and $Pu = f$. This explains the difference between $\tilde{P}$ and P. See [33] for more details.

The following result was proved in [33, Corollary 7.5], using that the Neumann and Robin problems satisfy regularity. See also [2, 3, 41, 44].

Theorem 4.8. [33] *Assume that the operator* $P = P_{(a,b)} + Q + Q_1$ *satisfies the strong Legendre condition, that it has coefficients in* $W^{k,\infty}$*,* $k \geq 1$*, and* $\Re\, b$ *is an order zero operator. Then there exists* $c > 0$ *such that*

$$\begin{aligned}\|u\|_{H^{k+1}(M;E)} \leq c\, \Big(& \|Pu\|_{H^{k-1}(M;E)} + \|u\|_{H^1(M;E)} \\ &+ \|u|_{\partial_D M}\|_{H^{k+1/2}(\partial_D M;E)} + \|\partial^P_\nu u + bu\|_{H^{k-1/2}(M\setminus\partial_D M;E)}\Big)\,,\end{aligned}$$

for any $u \in H^1(M; E)$ *such that* $\tilde{P}u \in j_k(H^{k-1}(M; E) \oplus H^{k-1/2}(\partial M \setminus \partial_D M; E))$*. For* $k = 0$ *the statement is trivial (once suitably reformulated).*

The meaning of this result is also that, if $u \in H^1(M; E)$, $u|_{\partial_D M} \in H^{k+1/2}(\partial_D M; E)$, and $\tilde{P}u \in \mathrm{Im}(j_k) = j_k(H^{k-1}(M; E) \oplus H^{k-1/2}(M \setminus \partial_D M; E))$ with $\partial^P_\nu u + bu \in H^{k-1/2}(M \setminus \partial_D M; E)$, then, in fact, $u \in H^{k+1}(M; E)$.

To prove Theorem 1.1, we first notice that the assumption that $\Re\, b \geq 0$ implies that $\Re b := \frac{1}{2}(b+b^*)$ is of order zero, since b is of order (at most) one. Theorem 1.1 is therefore a consequence of Theorems 4.7 and 4.8.

4.3. **Applications and extensions.** We include now some consequences and extensions of our main result, Theorem 1.1. For simplicity, we assume here that our differential operators have totally bounded coefficients.

4.3.1. *Splitting of E.* Let us assume that we are given a splitting

$$E|_{\partial M} = E_D \oplus E_R \oplus E_N \tag{20}$$

as a direct sum of three smooth vector bundles with bounded geometry. We denote by p_D, p_R, p_N the associated orthogonal projections $E \to E_D, E_R, E_N$. We then replace the space $H^1_D(M;E)$ with

$$V := \{u \in H^1(M;E) \mid p_D u = 0\}. \tag{21}$$

Up until this point, we had $E_D := E|_{\partial_D M}$, $E_R := E|_{\partial_R M}$, and $E_N := E|_{\partial_N M}$. The more general framework introduced here is needed in order to treat the Hodge-Laplacian.

4.3.2. *Assumptions under the splitting of E.* In general, here is how the assumptions change in the new setting:

(i) The Poincaré inequality becomes the *assumption* that the modified norm

$$|||u|||^2 := \|\nabla u\|^2_M + \|p_R u\|^2_{\partial M}$$

is equivalent to the H^1-norm on V.

(ii) We continue to assume that P has coefficients in $W^{\ell,\infty}$.

(iii) The differential operator b is then assumed to satisfy $\Re b \geq \epsilon p_R$ for some $\epsilon > 0$.

(iv) Also, we continue to assume that $\Re(Q + Q_1^*) \geq -\delta$, for some δ small enough, with δ depending on a, ϵ, and (M, g).

Then Theorem 1.1 remains valid in this setting. This is equivalent to Corollary 4.10 formulated in detail below. Before discussing this theorem, let us notice that condition (i) replacing the Poincaré inequality is somewhat tricky, as seen in the following example.

Example 4.9. Let $M = [0,1]$ with the standard, euclidean metric. Then both $(M, \{0\})$ and $(M, \{1\})$ are of finite width, so they satisfy the Poincaré inequality (for scalar functions, that is, for $E = \mathbb{C}$). Let now $E = \mathbb{C}^2$. It is enough to take $E_R = \{0\}$, but V as in (21). We thus need to specify E_D above $\partial M = \{0, 1\}$. Two seemingly similar choices will give completely different results.

Indeed, let $E_D = \{0\} \oplus \mathbb{C}$ above $\{0\}$. Then Assumption (i) on the equivalence of norms is satisfied if $E_D = \mathbb{C} \oplus \{0\}$ above $\{1\}$, but is not satisfied if $E_D = \{0\} \oplus \mathbb{C}$ above $\{1\}$. The first case corresponds to putting together $(M, \{0\})$ and $(M, \{1\})$, whereas the second case corresponds to putting together $(M, \{0, 1\})$ and $(M, \emptyset)$. In the second case, the Poincaré inequality is clearly not satisfied (since $u = 1$ is allowed).

4.3.3. *Boundary value problems.* Recall the discussion on boundary value problems in Section 2.2.3. As usual, Theorem 1.1 gives results on boundary value problems. We formulate, nevertheless, the result in the more general framework relying on a decomposition of E as in Equation (20).

Corollary 4.10. *We consider the setting of Section 4.3.2. Then the boundary value problem*

$$\begin{cases} Pu = f \in H^{\ell-1}(M;E) & \text{in } M \\ p_D u = h_0 \in H^{\ell+1/2}(\partial M; E_D) & \text{on } \partial M \\ (1-p_D)(\partial_\nu^P u + bu) = h_1 \in H^{\ell-1/2}(\partial M; E_R \oplus E_N) & \text{on } \partial M \end{cases} \tag{22}$$

*is well-posed (*i. e. *it has a unique solution $u \in H^{\ell+1}(M;E)$ that depends continuously on h_0 and h_1).*

4.3.4. *Self-adjointness.* As in [7], we obtain the following corollary.

Corollary 4.11. *Let us assume that P is as in Section 4.3.2 and, moreover, that it has coefficients in $W^{1,\infty}$ and is formally self-adjoint, that is, that $(Pu,v) = (u,Pv)$ for $u,v \in \mathcal{C}_c^\infty(M \smallsetminus \partial M; E)$. Then P with domain*

$$\mathcal{D}(P) := \{u \in H^2(M;E) \mid p_D u = 0,\ (1-p_D)(\partial_\nu^P u + bu) = 0\}$$

is self-adjoint.

See also [21, 22, 30, 31], where bounded domains, but with Lipschitz or more general boundaries, were considered. As in those papers, one obtains also consequences for the corresponding parabolic equations.

4.3.5. *Coercivity in general and Gårding's inequality.* As is well known, results such as Corollary 4.11 are closely related to Gårding's inequality. This inequality is usually obtained for uniformly strongly elliptic operators. Indeed, following [1], we can extend our results to uniformly strongly elliptic operators as follows.

Recall that an operator P is *coercive* on $V \subset H^1(M;E)$ if it satisfies the Gårding inequality, that is, if there exist $\gamma > 0$ and $R \in \mathbb{R}$ such that for all $u \in V$

$$\Re(Pu,u) \geq \gamma \|u\|^2_{H^1(M;E)} - R\|u\|^2_{L^2(M;E)}. \tag{23}$$

Then $P + \lambda$ is strongly coercive for $\Re(\lambda) > R$, and hence Theorems 1.1 and 4.7 remain true for P replaced with $P + \lambda$. Coercive operators on *bounded* domains were characterized by Agmon in [1] as strongly elliptic operators satisfying suitable conditions at the boundary (which we shall call the "Agmon condition."). We shall need a *uniform* version of this condition, to account for the non-compactness of the boundary.

Let $P_x^{(0)}$ be the principal part of the operator P and $C_x^{(0)}$ be the principal part of the boundary conditions $(p_D, (1-p_D)(\partial_\nu^P + b))$ with coefficients frozen at some $x \in \partial M$, as in [33]. Let $B_x^{(0)}$ be the associated Dirichlet bilinear form to $P_x^{(0)}$ equipped with the above boundary conditions (again with coefficients frozen at x). This is as in Equation (9). In particular, we have the projection $p_{D,x}^{(0)} \colon E_x \to (E_D)_x$ that enters in the boundary conditions defined by $C_x^{(0)}$. This defines a boundary value problem on the half-space $T_x^+ M$ and a bilinear form on $T_x^+ M$ that is continuous in the H^1-seminorm $|u|_{H^1} := \|\nabla u\|_{L^2}$.

Definition 4.12. We say that P (or the form B of Equation (10)) satisfies the *uniform Agmon condition (on ∂M)* if it is uniformly strongly elliptic and if there exists $C > 0$ such

$$B_x^{(0)}(u, u) := (P_x^{(0)} u, u) + \int_{T_x \partial M} (b^{(0)} u, u) dx' \geq C |u|_{H^1}^2,$$

for all $x \in \partial M$ and all $u \in \mathcal{C}_c^\infty(T_x^+ M)$ that satisfies $p_{D,x}^{(0)} u = 0$ (on $T_x \partial M = \partial T_x^+ M$).

We have then the following result that is proved, *mutatis mutandis*, as the regularity result in [33], to which we refer for more details.

Theorem 4.13. *We use the notation in 4.3.1, in particular,*

$$V := \{u \in H^1(M; E) \mid p_D u = 0\}.$$

We have that $\tilde{P}$ (equivalently, the form B of Equation (10)) is coercive on V if, and only if, it satisfies the uniform Agmon condition on ∂M.

The idea of the proof, in one direction, is to consider u with a shrinking supports towards x using dilations and to retain the dominant terms. In the other direction, one uses the standard partitions of unity on manifolds with (boundary and) bounded geometry. See [33, 51] for details of this method.

Remark 4.14. The reader may have noticed that our Robin boundary conditions are of the form $\partial_\nu^P + b$. It makes sense, of course, to consider boundary conditions of the form $a\partial_\nu^P + b$, where a is an endomorphism of $E_R \oplus E_N$. If a is invertible, this changes nothing. However, significant differences arise if a is singular. See, for instance, the recent preprint [42] and the references therein.

See [12, 16, 45, 49, 50] for an approach to boundary value problems on non-compact manifolds using pseudodifferential operators and for related recent results.

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