THE SPINORIAL ENERGY FUNCTIONAL ON SURFACES

BERND AMMANN, HARTMUT WEISS, AND FREDERIK WITT

Abstract. This is a companion paper to [2] where we introduced the spinorial energy functional and studied its main properties in dimensions equal or greater than three. In this article we focus on the surface case. A salient feature here is the scale invariance of the functional which leads to a plenitude of critical points. Moreover, via the spinorial Weierstraß representation it relates to the Willmore energy of periodic immersions of surfaces into $\mathbb{R}^3$.

1. Introduction

Let M^n be a closed spin manifold of dimension n with a fixed spin structure σ. If g is a Riemannian metric on M, we denote by $\Sigma_g M \to M$ the associated spinor bundle. The spinor bundles for all possible choices of g may be assembled into a single fiber bundle $\Sigma M \to M$, the so-called *universal spinor bundle*. A section $\Phi \in \Gamma(\Sigma M)$ determines a Riemannian metric $g = g_\Phi$ and a g-spinor $\varphi = \varphi_\Phi \in \Gamma(\Sigma_g M)$ and vice versa. In particular, one can split the tangent space of ΣM at (g_x, φ_x) into a "horizontal part" $\odot^2 T_x^* M$ and a "vertical" part $(\Sigma_g M)_x$ (see [2] for further explanation). Furthermore, let $S(\Sigma M)$ denote the universal bundle of unit spinors, i.e. $S(\Sigma M) = \{\Phi \in \Sigma M \mid |\Phi| = 1\}$, and $\mathcal{N} = \Gamma(S(\Sigma M))$ its space of *smooth* sections. If we identify Φ with the pair (g, φ) we can consider the *spinorial energy functional*

$$\mathcal{E} : \mathcal{N} \to \mathbb{R}_{\geq 0}, \quad (g, \varphi) \mapsto \frac{1}{2} \int_M |\nabla^g \varphi|_g^2 \, dv^g$$

introduced in [2]. Here, ∇^g denotes the Levi-Civita connection, $|\cdot|_g$ the pointwise norm on spinors in $\Sigma_g M$, and integration is performed with respect to the associated Riemannian volume form dv^g. The functional is invariant under the $\mathbb{Z}_2$-extension of the spin diffeomorphism group and rescales as

$$\mathcal{E}(c^2 g, \varphi) = c^{n-2} \mathcal{E}(g, \varphi) \tag{1}$$

under homothetic change of the metric by $c > 0$. The negative gradient of $\mathcal{E}$ can be viewed as a map

$$Q : \mathcal{N} \to T\mathcal{N}, \quad \Phi \in \mathcal{N} \mapsto \big(Q_1(\Phi), Q_2(\Phi)\big) \in \Gamma(\odot^2 T^* M) \times \Gamma(\varphi^{\perp_g}) \tag{2}$$

(for a curve φ_t with $|\varphi_t| = 1$, $\dot\varphi$ must be pointwise perpendicular to φ). In [2] we showed that for $\Phi = (g, \varphi) \in \mathcal{N}$ we have

$$\begin{aligned} Q_1(\Phi) &= -\tfrac{1}{4} |\nabla^g \varphi|_g^2 g - \tfrac{1}{4} \operatorname{div}_g T_{g,\varphi} + \tfrac{1}{2} \langle \nabla^g \varphi \otimes \nabla^g \varphi \rangle, \\ Q_2(\Phi) &= -\nabla^{g*} \nabla^g \varphi + |\nabla^g \varphi|_g^2 \varphi, \end{aligned} \tag{3}$$

where $T_{g,\varphi} \in \Gamma(T^* M \otimes \odot^2 T^* M)$ is the symmetrisation in the second and third component of the $(3,0)$-tensor defined by $\langle (X \wedge Y) \cdot \varphi, \nabla^g_Z \varphi \rangle$ for X, Y and Z in $\Gamma(TM)$. Further, $\langle \nabla^g \varphi \otimes \nabla^g \varphi \rangle$ is the symmetric 2-tensor defined by $\langle \nabla^g \varphi \otimes \nabla^g \varphi \rangle (X, Y) = \langle \nabla^g_X \varphi, \nabla^g_Y \varphi \rangle$.

As a corollary, the critical points for $n \geq 3$ are precisely the pairs (g, φ) satisfying $\nabla^g \varphi = 0$, i.e. the parallel (unit) spinors. In particular, g must be Ricci-flat and (g, φ) is an absolute minimiser.

The present work investigates the spinorial energy functional on spin surfaces (M_γ, σ) where M_γ is a connected, closed surface of genus γ endowed with a fixed spin structure σ. This differs from the generic case of dimension $n \geq 3$ in several aspects. First, the functional is invariant under rescaling by Eq. (1), which leads to a potentially richer critical point structure in two dimensions. Indeed, we will construct in Section 5.2 certain flat 2-tori with non-minimising critical points which are saddle points in the sense that the Hessian of the functional is indefinite. In particular, these exist for spin structures which do not admit any non-trivial harmonic spinor. Despite the fact that $\mathcal{E}$ does not enjoy any natural convexity property, we note that the existence of the negative gradient flow as shown in [2] still holds in two dimensions. Second, if K_g denotes the Gauß curvature of g, the Schrödinger-Lichnerowicz formula implies

$$\mathcal{E}(g, \varphi) = \tfrac{1}{2} \int_M |D_g \varphi|^2 \, dv^g - \tfrac{1}{4} \int_M K_g \, dv^g, \tag{4}$$

where D_g is the Dirac operator associated with the spinor bundle $\Sigma_g M$. Since the second term in Eq. (4) is topological by Gauß-Bonnet, we obtain immediately the topological lower bound

$$\inf \mathcal{E} \geq \pi |\gamma - 1|.$$

We will show in Theorem 3.9 that we actually have equality. For the infimum we find a trichotomy of well-known spinor field equations. Namely, if P_g is the twistor operator associated with $\Sigma_g M$ (see Section 4.1 for its definition), then (g, φ) attains the infimum if and only if

$$\begin{aligned} P_g \varphi &= 0, & \gamma &= 0 \\ \nabla^g \varphi &= 0, & \gamma &= 1 \\ D_g \varphi &= 0, & \gamma &\geq 2, \end{aligned}$$

which matches the usual trichotomy for Riemann surfaces of positive, vanishing and negative Euler characteristic (Corollary 3.25, Theorem 4.7). Of course, any parallel spinor φ is also harmonic, i.e. $D_g \varphi = 0$. On the other hand, harmonic spinors on M_γ are related to minimal immersions of the universal cover $\widetilde{M}_\gamma$ into $\mathbb{R}^3$ via the spinorial Weierstraß representation (see [18, 22] or alternatively [11] where a nice presentation closer to our article is given). As a result we will be able to construct a plenitude of examples for various spin structures (Theorem 3.19). In particular, with the notable exception of $\gamma = 2$, *for any genus* there exist critical points which are in fact absolute minimisers. Finally, we completely classify the critical points on the sphere (Theorem 4.7) and the flat critical points on the torus (Theorem 5.2).

General conventions. In this article, M_γ will denote the up to diffeomorphism unique closed oriented surface of genus γ. Further, g will always be a Riemannian metric. Rotation on each tangent space by $\pi/2$ in the counterclockwise direction induces a complex structure J which in particular is a g-isometry. More concretely, a local positively oriented g-orthonormal basis (e_1, e_2) satisfies $Je_1 = e_2$ and $Je_2 = -e_1$. Conversely, any complex structure determines a conformal class $[g]$ of Riemannian metrics. We will often tacitly identify (e_1, e_2) with the dual basis (e^1, e^2) via the musical isomorphisms $\sharp$ and $\flat$. The Riemannian volume form ω_g is then locally given by $e_1 \wedge e_2$. Further, the dual complex structure

J^* acting on 1-forms is simply $-\star$, where $\star$ is the usual Hodge operator sending e_1 to e_2 and e_2 to $-e_1$. The Levi-Civita connection associated with g will be written as ∇^g. The Gauß curvature K_g is just half the scalar curvature s_g, i.e. $2K_g = s_g = -2R_g(e_1,e_2,e_1,e_2)$, where R_g denotes the Riemannian $(4,0)$-curvature tensor defined by $R(e_1,e_2,e_1,e_2) = g([\nabla^g_{e_1},\nabla^g_{e_2}]e_1 - \nabla^g_{[e_1,e_2]}e_1, e_2)$. In the sequel we shall often drop any reference to g if the underlying metric is clear from the context. The *divergence* of a tensor T is given by

$$\mathrm{div}_g T = -\sum_{k=1}^{n}(\nabla^g_{e_k}T)(e_k,\cdot\,). \tag{5}$$

Finally, we use the convention $v\odot w := (v\otimes w + w\otimes v)/2$ for the symmetrisation of a $(2,0)$-tensor.

2. Spin geometry

2.1. **Spinors on surfaces.** We recall some spin geometric features of surfaces. Suitable general references are [12, 19].

Every oriented surface admits a spin structure, i.e. a twofold covering of $P_{\mathrm{GL}_+(2)}$, the bundle of positively oriented frames which restricted to a fibre induces the connected 2-fold covering of $\mathrm{GL}_+(2)$. In particular, spin structures on M_γ are classified by elements of $H^1(P_{\mathrm{GL}_+(2)},\mathbb{Z}_2)$ whose restriction to the fibre gives the non-trivial class. From the exact sequence associated with the fibration $\mathrm{GL}_+(2)\to P_{\mathrm{GL}_+(2)}\to M_\gamma$ it follows that spin structures are in $1-1$ correspondence with elements in $H^1(M_\gamma,\mathbb{Z}_2)$. Hence there exist $2^{2\gamma} = \#H^1(M_\gamma,\mathbb{Z}_2)$ isomorphism classes of spin structures on M_γ.

A pair (M_γ,σ) consisting of a genus γ surface and a fixed spin structure σ will be called a *spin surface*. If, in addition, we also fix a metric, we can consider $\Sigma_g M\to M$, the complex bundle of *Dirac spinors* associated with the complex unitary representation (Δ,h) of $\mathrm{Spin}(2)$. Note that the action of ω_g splits Δ into the irreducible $\mp i$ eigenspaces $\Delta_\pm\cong\mathbb{C}$. This gives rise to a global decomposition

$$\Sigma_g = \Sigma_{g+}\oplus\Sigma_{g-}$$

into *positive* and *negative (Weyl) spinors*. Further, since $\Delta_-\cong\bar\Delta_+$, $\Delta\cong\mathbb{C}\oplus\bar{\mathbb{C}}$ carries a *quaternionic structure*. Equivalently, there exists a $\mathrm{Spin}(2)$-equivariant map $\alpha:\Delta\to\Delta$ which interchanges Δ_+ and Δ_- and squares to minus the identity. Hence we can think of Δ as the quaternions $\mathbb{H}$ with real inner product $\langle\cdot\,,\cdot\rangle := \mathrm{Re}\,h$. Locally, we can represent spinors in terms of a local orthonormal basis of the form $(\varphi, e_1\cdot\varphi, e_2\cdot\varphi, \omega\cdot\varphi)$, where φ is a unit spinor and (e_1,e_2) a local positively oriented orthonormal basis. In particular,

$$\nabla_X\varphi = A(X)\cdot\varphi + \beta(X)\omega\cdot\varphi \tag{6}$$

for a uniquely determined endomorphism field $A\in\Gamma(\mathrm{End}(TM))$ and a 1-form $\beta\in\Omega^1(M)$. We also say that the pair (A,β) is *associated* with (g,φ). Note that A and β determine the spinor field φ up to a global constant in the following sense. If φ_1 and φ_2 are unit spinor fields, and if they both solve Eq. (6) for $\varphi=\varphi_i$, then there is a unit quaternion c such that $\varphi_1=\varphi_2 c$. Hence an orbit of the action of the unit quaternions $\mathrm{Sp}(1)$ on unit spinor fields is determined by a pair (A,β) for which a solution to Eq. (6) exists. The question of determining the pairs which can actually arise will be addressed in Section 3.3.

As pointed out above, the choice of a Riemannian metric induces a complex and in fact a Kähler structure on M_γ. In particular, we can make use of the holomorphic picture of spinors on Riemann surfaces [3, 16]. Here, spin structures on $(M_\gamma,[g])$ are in 1–1 correspondence with holomorphic square roots λ of the canonical line bundle $\kappa_\gamma = T^*M^{1,0}$, i.e. $\lambda\otimes\lambda \cong \kappa_\gamma$ as *holomorphic* line bundles. The corresponding spinor bundle is given by

$$\Sigma_g = \Lambda^* TM^{1,0} \otimes \lambda \cong \lambda \oplus \lambda^*$$

where we used the identification $TM^{1,0} \cong T^*M^{0,1}$ as *complex* line bundles. Clifford multiplication is then given by $v\cdot\varphi = \sqrt{2}(v\wedge\varphi - \iota(v^*)\varphi)$ where $v \in T^*M^{0,1}$ and $\iota(v^*)$ denotes contraction with the hermitian adjoint of v. The resulting even/odd-decomposition $\Sigma_g = \lambda\oplus\lambda^*$ is just the decomposition into positive and negative spinors.

2.2. **Dirac operators.** Associated with any spin structure is the Dirac operator

$$D_g : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M)$$

which is locally given by $D_g\varphi = e_1\cdot\nabla^g_{e_1}\varphi + e_2\cdot\nabla^g_{e_2}\varphi$. We have the useful formulæ

$$\omega\cdot D\varphi = -D(\omega\cdot\varphi) \quad \text{and} \quad \langle\omega\cdot D\varphi, D\varphi\rangle + \langle D\varphi, \omega\cdot D\varphi\rangle = 0.$$

In particular, for $a, b\in\mathbb{R}$ with $a^2+b^2=1$ we obtain

$$|D(a\varphi + b\omega\cdot\varphi)|^2 = a^2|D\varphi|^2 + b^2|\omega D\varphi|^2 = |D\varphi|^2. \tag{7}$$

In terms of the pair (A,β) determined by φ we have

$$D\varphi = \sum_{k=1}^{2} e_k\cdot A(e_k)\cdot\varphi + \beta(e_k)e_k\cdot\omega\cdot\varphi = \mathrm{Tr}A\,\varphi + \mathrm{Tr}(A\circ J)\omega\cdot\varphi - (\beta\circ J)^\sharp\cdot\varphi. \tag{8}$$

Moreover, restriction of D_g to $\Sigma_{g\pm}$ gives rise to the operators $D_g^\pm : \Gamma(\Sigma_{g\pm}) \to \Gamma(\Sigma_{g\mp})$.

A remarkable fact we shall use repeatedly is the conformal equivariance of D in the following sense [16]. If for $u\in C^\infty(M)$ we consider the metric $\tilde g = e^{2u}g$ conformally equivalent to g, we have a natural bundle isometry $\Sigma_g \to \Sigma_{\tilde g}$ sending φ to $\tilde\varphi$. Furthermore,

$$\tilde D\tilde\varphi = e^{-3u/2}\widetilde{De^{u/2}\varphi} \tag{9}$$

where we let $\tilde D = D_{\tilde g}$. Note that for a vector field X we have $\widetilde{X\cdot\varphi} = \tilde X\cdot\tilde\varphi$ if $\tilde X = e^{-u}X$ [7, (1.15)]. In particular, the dimension of the space of *harmonic spinors*, $\ker D$, as well as the spaces of (complex) *positive* and *negative harmonic spinors*, $\ker D^+$ and $\ker D^+$, are conformal invariants. This is also manifest in terms of the holomorphic description above. Namely, after choosing a complex structure, i.e. a conformal class on M_γ, and a holomorphic square root λ of κ_γ, we have

$$D\varphi = \sqrt{2}(\bar\partial_\lambda + \bar\partial_\lambda^*)\varphi$$

where $\bar\partial_\lambda : \Gamma(\lambda)\to\Gamma(T^*M^{0,1}\otimes\lambda)$ is the induced Cauchy-Riemann operator on λ whose formal adjoint is $\bar\partial^*_\lambda$. In particular, a positive Weyl spinor φ is harmonic if and only if the corresponding section of λ is holomorphic. Note that $\operatorname{coker} D^+ \cong \ker D^-$ so that $\dim\ker D^+ = \dim\ker D^-$ by the Atiyah-Singer index theorem. An explicit isomorphism is provided by the quaternionic structure from Section 2.1 which maps positive harmonic spinors to negative ones and vice versa.

2.3. **Bounding and non-bounding spin structures.** The orientation-preserving diffeomorphism group $Diff_+(M_\gamma)$ acts on the bundle of oriented frames and therefore permutes the possible spin structures on M_γ by its action on $H^1(P_{\mathrm{GL}_+(2)}, \mathbb{Z}_2)$ resp. $H^1(M_\gamma, \mathbb{Z}_2)$. There are precisely two orbits, namely the orbits of *bounding* and *non-bounding* spin structures. They contain $2^{\gamma-1}(2^\gamma+1)$ respectively $2^{\gamma-1}(2^\gamma-1)$ elements [3]. In particular, on the 2-torus where $\gamma=1$, there is a unique non-bounding spin structure and three bounding ones. These two orbits correspond to the two spin cobordisms classes of M_γ [21]. Recall that in general, a spin manifold (M,σ) is *spin cobordant to zero* if there exists an orientation preserving diffeomorphism to the boundary of some compact manifold so that the naturally induced spin structure on the boundary (see for instance [19, Proposition II.2.15]) is identified with σ under this diffeomorphism. Numerically, we can distinguish these two orbits as follows. Fix a complex structure on M_γ and identify the set of spin structures with the holomorphic square roots $\mathcal{S}(M_\gamma)$ of the resulting canonical line bundle κ_γ. Let $d^+(g) := \dim_{\mathbb{C}} \ker D^+ = \dim_{\mathbb{C}} H^0(M_\gamma, \lambda)$. Then

$$\varrho : \mathcal{S}(M_\gamma) \to \mathbb{Z}_2, \quad \varrho(\lambda) \equiv d^+(g) \mod 2$$

is a quadratic function whose associated bilinear form corresponds to the cup product on $H^1(M_\gamma, \mathbb{Z}_2)$. Moreover, $\varrho(\lambda)=0$ if and only if λ corresponds to a bounding spin structure [3]. For instance, it is well-known that on a torus, $d^+(g)$ is either 0 or 1 [16]. Therefore, the three bounding spin structures do not admit positive harmonic spinors (regardless of the conformal structure), while the non-bounding one (the generator of the spin cobordism class) admits a harmonic spinor. As a further application, we note that $d(g) = \dim_{\mathbb{C}} \ker D = 2d^+(g)$ is divisible by 4 if and only if Σ is a bounding spin structure.

2.4. **The spinorial Weierstraß representation.** Let (M_γ, σ, g, H) be a spin surface with fixed Riemannian metric g and $H \in C^\infty(M_\gamma)$. The universal covering of M_γ will be denoted by $\widetilde{M}_\gamma$. Let $S_{\gamma,\sigma,g,H}$ be the set of solutions of

$$D\varphi = H\varphi \qquad |\varphi| \equiv 1 \tag{10}$$

on M_γ. Finally, let $\mathrm{Imm}_{\gamma,g,H}$ be the set of all periodic isometric immersions $F : \widetilde{M}_\gamma \to \mathbb{R}^3$ with mean curvature $H : M_\gamma \to \mathbb{R}$. Here, periodic refers to the existence of a 'period homomorphism' $P : \pi_1(M_\gamma) \to \mathbb{R}^3$ with $F([\alpha]\cdot x) = F(x) + P([\alpha])$ for all $x \in \widetilde{M}_\gamma$ and $[\alpha] \in \pi_1(M_\gamma)$ which we view as a Deck transformation of $\widetilde{M}_\gamma \to M_\gamma$. Note that $\mathbb{R}^3$ acts on $\mathrm{Imm}_{\gamma,g,H}/\mathbb{R}^3$ by translations. The spinorial Weierstraß representation is the statement that there is a natural covering map

$$W : S_{\gamma,\sigma,g,H} \to \mathrm{Imm}_{\gamma,g,H}/\mathbb{R}^3$$

with $W(\varphi) = W(\psi)$ if and only if $\varphi = \pm\psi$. It follows from Smale-Hirsch-theory (which is a special case of the h-principle for immersions of surfaces in 3-dimensional manifolds) that the image of W is exactly one connected component of $\mathrm{Imm}_{\gamma,g,H}/\mathbb{R}^3$. This establishes a bijection from the set of isomorphism classes of spin structures on M_γ to the set of connected components of $\mathrm{Imm}_{\gamma,g,H}/\mathbb{R}^3$.
We briefly sketch the constructions of W and of its 'inverse', for details we refer to [4] and [11] which express these constructions in modern language.
The 'inverse' of the map W can be described as follows: We fix a parallel spinor Ψ on $\mathbb{R}^3$ of constant length 1. If $\widetilde{M}_\gamma \to \mathbb{R}^3$ is a periodic isometric immersion then the 'restriction' $\psi := \Psi|_{\widetilde{M}_\gamma}$ is a periodic unit spinor on $\widetilde{M}_\gamma$, well-defined up to a sign.

One can show that ψ is the pullback of a unit spinor on M_γ under the universal covering map, provided that M_γ is equipped with the right choice of spin structure. To describe the map W itself, one shows that a unit spinor φ on M_γ yields a linear isometric embedding $A_x : T_xM_\gamma \to \mathbb{R}^3$ for every $x \in M_\gamma$. The equation (10) is then equivalent to the fact that these maps A_x integrate, i.e. that there is a map $F : \widetilde{M}_\gamma \to \mathbb{R}^3$ with $d_xF = A_x$.
The Weierstraß representation has a long history. For minimal surfaces it can be traced back to Weierstraß' work on conformal parametrisations. For arbitrary surfaces the history is less clear, as surface representations in terms of different data were proven. The earliest reference known to us which shows that surfaces can be represented by solutions of $D\varphi = H\varphi$ is the preprint [18] based on N. Schmitt's thesis [22]. Necessity of (10) was certainly known before, see for instance [23, 24], and related results were already obtained in [17] and reportedly by Eisenhart and Abresch.

3. Critical points

3.1. **The Euler-Lagrange equation.** First we express the negative gradient of $\mathcal{E}$ in Eq. (3) in terms of A and β as defined by Eq. (6). We write $|A|$ for the induced g-norm of A, i.e. $|A|^2 = \operatorname{Tr} A^tA$. Further, for a symmetric 2-tensor h we denote by $h_0 = h - \frac{1}{2}\operatorname{Tr} h \cdot g$ its traceless part.

Proposition 3.1. *The negative gradient of $\mathcal{E}$ is given by*

$$Q_1(g,\varphi) = -\tfrac{1}{4}(\nabla_{J(\,\cdot\,)}\beta)^{sym} + \tfrac{1}{2}(A^tA + \beta\otimes\beta)_0$$
$$Q_2(g,\varphi) = -(\operatorname{div} A)\cdot\varphi - (\operatorname{div}\beta)\,\omega\cdot\varphi.$$

Proof. First, with $A(e_i) = \sum_k A_{ki}e_k$ for a g-orthonormal basis (e_1, e_2),

$$\begin{aligned}\langle\nabla\varphi\otimes\nabla\varphi\rangle &= \sum_{i,j}\langle\nabla_{e_i}\varphi, \nabla_{e_j}\varphi\rangle e_i\otimes e_j\\
&= \sum_{i,j}\langle A(e_i)\cdot\varphi + \beta(e_i)\omega\cdot\varphi, A(e_j)\cdot\varphi + \beta(e_j)\omega\cdot\varphi\rangle e_i\otimes e_j\\
&= \sum_{i,j}\big(\langle A(e_i)\cdot\varphi, A(e_j)\cdot\varphi\rangle + \beta(e_i)\beta(e_j)\big)e_i\otimes e_j\\
&= \sum_{i,j}\big(\sum_k A_{ki}A_{kj} + \beta(e_i)\beta(e_j)\big)e_i\otimes e_j\\
&= A^tA + \beta\otimes\beta\end{aligned}$$

and

$$|\nabla\varphi|^2 = \operatorname{Tr}\langle\nabla\varphi\otimes\nabla\varphi\rangle = \operatorname{Tr}(A^tA) + \operatorname{Tr}(\beta\otimes\beta) = |A|^2 + |\beta|^2.$$

On the other hand, $\langle X\wedge Y\cdot\varphi, A(Z)\cdot\varphi\rangle = 0$ and $\langle X\wedge Y\cdot\varphi, \omega\cdot\varphi\rangle = \omega(X,Y)$, using the convention $e_1\wedge e_2 = e_1\otimes e_2 - e_2\otimes e_1$. This implies

$$T_{g,\varphi}(X,Y,Z) = \tfrac{1}{2}\omega(X,Y)\beta(Z) + \tfrac{1}{2}\omega(X,Z)\beta(Y)$$

and therefore

$$\begin{aligned}\operatorname{div} T_{g,\varphi} &= -\tfrac{1}{2}\sum_{i,k,l}\big(\omega(e_i,e_k)(\nabla_{e_i}\beta)(e_l) + \omega(e_i,e_l)(\nabla_{e_i}\beta)(e_k)\big)e_k\otimes e_l\\
&=(\nabla_{e_2}\beta)(e_1)e_1\otimes e_1 - (\nabla_{e_1}\beta)(e_2)e_2\otimes e_2 + \big((\nabla_{e_2}\beta)(e_2) - (\nabla_{e_1}\beta)(e_1)\big)e_1\odot e_2\\
&=(\nabla_{J(\,\cdot\,)}\beta)^{sym}.\end{aligned}\tag{11}$$

Next we work pointwise with a synchronous frame. Since vector fields anticommute with ω,

$$\begin{aligned}\nabla^*\nabla\varphi =& -\sum_{i=1}^{2}(\nabla_{e_i}\nabla_{e_i}\varphi - \nabla_{\nabla_{e_i}e_i}\varphi)\\ =& -\sum_{i=1}^{2}\Big(A(e_i)\cdot A(e_i)\cdot\varphi + \beta(e_i)\big(A(e_i)\cdot\omega + \omega\cdot A(e_i)\big)\cdot\varphi + \beta(e_i)^2\omega\cdot\omega\cdot\varphi\\ &+ \nabla_{e_i}\big(A(e_i)\big)\cdot\varphi + \nabla_{e_i}\big(\beta(e_i)\big)\omega\cdot\varphi\Big)\\ =&(|A|^2+|\beta|^2)\varphi + (\operatorname{div}A)\cdot\varphi + (\operatorname{div}\beta)\,\omega\cdot\varphi.\end{aligned}$$

Since $Q_2(g,\varphi)$ is orthogonal to φ we must have

$$Q_2(g,\varphi) = -(\operatorname{div}A)\cdot\varphi - (\operatorname{div}\beta)\,\omega\cdot\varphi,$$

whence the assertion. □

In terms of the pair (A,β) we can now characterise a critical point as follows.

Corollary 3.2. *A pair (g,φ) is a critical point of $\mathcal{E}$ if and only if*

$$\operatorname{div}\beta = 0,\quad \operatorname{div}A = 0,\quad (\nabla_{J(\,\cdot\,)}\beta)^{sym} = 2(A^tA + \beta\otimes\beta)_0. \tag{12}$$

In particular, if (g,φ) is critical, then

(i) $\operatorname{Tr}Q_1(g,\varphi) = \star d\beta/4 = 0$*, hence β is a harmonic 1-form.*
(ii) $\nabla_{J(\,\cdot\,)}\beta$ *is traceless symmetric, i.e.* $(\nabla_{J(\,\cdot\,)}\beta)_0 = 0$ *and* $(\nabla_{J(\,\cdot\,)}\beta)^{sym} = \nabla_{J(\,\cdot\,)}\beta$.
(iii) $\nabla_{J(X)}\beta(Y) = \nabla_X\beta(J(Y))$.
(iv) $\operatorname{div}(\beta\otimes\beta)_0 = 0$

Proof. Eq. (12) follows directly from Proposition 3.1. For (i), we note that

$$\operatorname{Tr}\operatorname{div}T_{g,\varphi} = (\nabla_{e_2}\beta)(e_1) - (\nabla_{e_1}\beta)(e_2) = -\star d\beta, \tag{13}$$

whence $4\operatorname{Tr}Q_1 = \star d\beta$ from Eq. (3). For (ii) and (iii) we note that in an orthonormal frame the anti-symmetric part of $\nabla_{J(\,\cdot\,)}\beta$ is given by

$$(\nabla_{J(e_2)}\beta)(e_1) - (\nabla_{J(e_1)}\beta)(e_2) = -(\nabla_{e_1}\beta)(e_1) - (\nabla_{e_2}\beta)(e_2) = \operatorname{div}\beta.$$

Hence $\nabla_{J(\,\cdot\,)}\beta$ is symmetric if and only if $\operatorname{div}\beta = 0$. Since $\nabla\beta$ is symmetric if and only if $d\beta = 0$,

$$\nabla_{J(X)}\beta(Y) = \nabla_{J(Y)}\beta(X) = \nabla_X\beta(J(Y))$$

if (g,φ) is critical. To prove (iv) we observe $\operatorname{Tr}\beta\otimes\beta = |\beta|^2$ so that $(\beta\otimes\beta)_0 = \beta\otimes\beta - \frac{1}{2}|\beta|^2 g$. Now in a synchronous frame

$$\begin{aligned}\operatorname{div}\beta\otimes\beta &= -(\nabla_{e_1}\beta)(e_1)\beta - \beta(e_1)\nabla_{e_1}\beta - (\nabla_{e_2}\beta)(e_2)\beta - \beta(e_2)\nabla_{e_2}\beta\\ &= (\operatorname{div}\beta)\beta - \nabla_{\beta^\sharp}\beta,\end{aligned}$$

whence $\operatorname{div}\beta\otimes\beta = -\nabla_{\beta^\sharp}\beta$ if $\operatorname{div}\beta = 0$. Moreover,

$$\begin{aligned}\operatorname{div}|\beta|^2 g &= -d|\beta|^2 = -2g(\nabla\beta,\beta)\\ &= -2\sum_{i,j}(\nabla_{e_i}\beta)(e_j)\beta(e_j)e_i\\ &= -2\sum_{i,j}((\nabla_{e_j}\beta)(e_i) + d\beta(e_i,e_j))\beta(e_j)e_i\\ &= -2\nabla_{\beta^\sharp}\beta + 2\iota_{\beta^\sharp}d\beta.\end{aligned}$$

Consequently, $\operatorname{div}|\beta|^2 g = -2\nabla_{\beta^\sharp}\beta$ if $d\beta = 0$, whence the assertion. □

Remark 3.3.
(i) The proof of properties (ii) to (iv) solely uses the harmonicity of β.
(ii) The identity (7) induces a circle action which preserves the functional $\mathcal{E}$. Together with the quaternionic action on Δ we see that there is a $\mathrm{U}(2) = S^1 \times_{\mathbb{Z}_2} \mathrm{SU}(2)$-action which preserves the functional and therefore acts on the critical points (cf. also [2, Section 4.1.3, Table 2]).

The condition that $Q_1(g,\varphi)$ is trace-free or equivalently, that the associated 1-form β is closed, can be interpreted as follows. As pointed out in Section 2.1, there is a natural bundle isometry $\mathcal{C} : \Sigma_g \to \Sigma_{\tilde g}$ between conformally equivalent metrics $\tilde g = e^{2u} g$, $u \in C^\infty(M)$. Hence, for $(g,\varphi) \in \mathcal{N}$ we can consider the associated *spinor conformal class* $[g,\varphi] := \{(\tilde g,\tilde\varphi) \,|\, \tilde g = e^{2u} g,\ \tilde\varphi = \mathcal{C}\varphi\}$.

Proposition 3.4. *The following statements are equivalent:*
(i) $(g,\varphi) \in \mathcal{N}$ *is an absolute minimiser in its spinor conformal class.*
(ii) $d\beta = 0$.
(iii) $\operatorname{Tr} Q_1(g,\varphi) = 0$.

Furthermore, in any spinor conformal class there exists an absolute minimiser which is unique up to homothety. In particular, any spinor conformal class contains a unique absolute minimiser of total volume one.

Proof. The equivalence between (ii) and (iii) is just Proposition 3.1. For (ii) $\Rightarrow$ (i) assume that β associated with (g,φ) satisfies $d\beta = 0$. For any $(\tilde g,\tilde\varphi) \in [g,\varphi]$ we find

$$|\tilde D\tilde\varphi|^2 = e^{-3u}|De^{u/2}\varphi|^2 = e^{-2u}|D\varphi + \tfrac12 \operatorname{grad} u\cdot\varphi|^2$$

by Eq. (9). For all $u \in C^\infty(M)$ this and Eq. (8) gives

$$\begin{aligned}\int_M |\tilde D\varphi|^2 d\tilde v &= \int_M |D\varphi|^2 + \tfrac14|du|^2 + \langle D\varphi, \operatorname{grad} u\cdot\varphi\rangle dv\\ &= \int_M |D\varphi|^2 + \tfrac14|du|^2 - \langle(\beta\circ J)^\sharp\cdot\varphi, \operatorname{grad} u\cdot\varphi\rangle dv\\ &= \int_M |D\varphi|^2 + \tfrac14|du|^2 + (\star\beta, du) dv\\ &= \int_M |D\varphi|^2 + \tfrac14|du|^2 + (\star d\beta, u) dv\\ &= \int_M |D\varphi|^2 + \tfrac14|du|^2 dv\\ &\geq \int_M |D\varphi|^2 dv. \end{aligned} \tag{14}$$

Further, this yields that $\int_M |du|^2/4 + (\star d\beta, u) dv \geq 0$ for an absolute minimiser. Taking $u = -\star d\beta$ shows that β associated with an absolute minimiser must be closed, hence (i) $\Rightarrow$ (ii). Finally, equality holds in (14) if and only if u is constant. To prove existence of an absolute minimiser we first note that for the 1-form $\tilde\beta$ associated with $(\tilde g,\tilde\varphi) \in [g,\varphi]$ we have $\tilde\beta(\tilde X) = e^{-u}\tilde\beta(X) = \langle\tilde\nabla_{\tilde X}\tilde\varphi, \tilde\omega\cdot\tilde\varphi\rangle$. On the other hand,

$$\langle\tilde\nabla_{\tilde X}\tilde\varphi, \tilde\omega\cdot\tilde\varphi\rangle = e^{-u}\beta(X) + \tfrac12\langle X\cdot\operatorname{grad} e^{-u}\cdot\varphi, \omega\cdot\varphi\rangle$$

by [7, (1.15)]. The latter term equals $J(X)(e^{-u})/2 = de^{-u}(J(X))/2$ which implies

$$\tilde\beta = \beta - \tfrac12 \star du.$$

If $\beta = H(\beta) \oplus d[\beta] \oplus \delta\{\beta\}$ is the Hodge decomposition of β for a function $[\beta]$ and a 2-form $\{\beta\}$, then $d\tilde{\beta} = d(\delta\{\beta\} - \frac{1}{2} \star du)$. Taking $u = -2 \star \{\beta\}$ yields that $d\tilde{\beta} = 0$. □

3.2. **Curvature.** Next we investigate the relationship between A, β and the Gauß curvature K of g. The basic link between curvature, spinors and 1-forms are the formulæ of Weitzenböck type

$$D^2\varphi = \nabla^*\nabla\varphi + \frac{1}{2}K \cdot \varphi \quad \text{and} \quad \Delta\beta = \nabla^*\nabla\beta + K \cdot \beta. \tag{15}$$

In particular, if (g, φ) is a critical and g is flat, β is necessarily parallel. We shall need a technical lemma first.

Lemma 3.5. *Let $\Phi = (g, \varphi) \in \mathcal{N}$. Then $\langle D^2\varphi, \varphi\rangle = |D\varphi|^2 - \star d\beta$.*

Proof. A pointwise computation with a synchronous frame implies

$$\begin{aligned}
\operatorname{Tr}\operatorname{div} T_{g,\varphi} &= -\sum_{j,k=1}^{n} (\nabla_{e_j} T_\varphi)(e_j, e_k, e_k) \\
&= -\sum_{k,j=1}^{n} e_j \langle e_j \cdot e_k \cdot \varphi, \nabla_{e_k}\varphi\rangle - \sum_{k=1}^{n} e_k . \langle \varphi, \nabla_{e_k}\varphi\rangle \\
&= -\sum_{k,j=1}^{n} \langle e_j \cdot e_k \cdot \nabla_{e_j}\varphi, \nabla_{e_k}\varphi\rangle - \sum_{k=1}^{n} \langle e_j \cdot e_k \cdot \varphi, \nabla_{e_j}\nabla_{e_k}\varphi\rangle \\
&\quad - |\nabla\varphi|^2 + \langle \varphi, \nabla^*\nabla\varphi\rangle \\
&= \langle D^2\varphi, \varphi\rangle - |D\varphi|^2.
\end{aligned}$$

On the other hand, as already observed in Eq. (13), $\operatorname{Tr}\operatorname{div} T_{g,\varphi} = - \star d\beta$, whence the result in view of Proposition 3.1. □

In terms of the associated pair (A, β), the equations in (15) read as follows.

Proposition 3.6. *Let $(g, \varphi) \in \mathcal{N}$. Then*
(i) $K = 4\det A - 2 \star d\beta$
(ii) $K \star \beta = \operatorname{div} \nabla_{J(\,\cdot\,)}\beta$.

Proof. (i) Since we always have $\langle \nabla^*\nabla\varphi, \varphi\rangle = |\nabla\varphi|^2$ for a unit spinor we get

$$\tfrac{K}{2} = |D\varphi|^2 - |\nabla\varphi|^2 - \star d\beta$$

from Lemma 3.5 and the Schrödinger-Lichnerowicz formula. Locally,

$$\begin{aligned}
|D\varphi|^2 = |\sum_i e_i \cdot \nabla_{e_i}\varphi|^2 &= \sum_{i,j} \langle e_i \cdot \nabla_{e_i}\varphi, e_j \cdot \nabla_{e_j}\varphi\rangle \\
&= |\nabla\varphi|^2 + \sum_{i \neq j} \langle e_i \cdot \nabla_{e_i}\varphi, e_j \cdot \nabla_{e_j}\varphi\rangle
\end{aligned}$$

and therefore

$$\begin{aligned}
K + 2 \star d\beta &= 4\langle e_1 \cdot \nabla_{e_1}\varphi, e_2 \cdot \nabla_{e_2}\varphi\rangle \\
&= 4\langle e_1 \cdot A(e_1) \cdot \varphi + e_1 \cdot \beta(e_1)\omega \cdot \varphi, e_2 \cdot A(e_2) \cdot \varphi + e_2 \cdot \beta(e_2)\omega \cdot \varphi\rangle \\
&= 4\langle e_1 \cdot A(e_1) \cdot \varphi - \beta(e_1)e_2 \cdot \varphi, e_2 \cdot A(e_2) \cdot \varphi + \beta(e_2)e_1 \cdot \varphi\rangle \\
&= 4\langle e_1 \cdot A(e_1) \cdot \varphi, e_2 \cdot A(e_2) \cdot \varphi\rangle \\
&= 4\langle -A_{11}\varphi + A_{21}e_1 \cdot e_2 \cdot \varphi, -A_{12}e_1 \cdot e_2 \cdot \varphi - A_{22}\varphi\rangle \\
&= 4(A_{11}A_{22} - A_{21}A_{12}) = 4\det A,
\end{aligned}$$

where (A_{ij}) is the matrix of A with respect to the basis $\{e_1, e_2\}$.

(ii) Computing in a synchronous frame yields

$$\begin{aligned}\operatorname{div} \nabla_{J(\,\cdot\,)}\beta &= -\nabla_{e_1}\nabla_{J(e_1)}\beta - \nabla_{e_2}\nabla_{J(e_2)}\beta\\ &= -\nabla_{e_1}\nabla_{e_2}\beta + \nabla_{e_2}\nabla_{e_1}\beta = -R(e_1,e_2)\beta.\end{aligned}$$

Since $R(e_1, e_2)\beta = -K \star \beta$, (ii) follows. □

Corollary 3.7. *If $(g, \varphi) \in \mathcal{N}$ is a critical point of $\mathcal{E}$, then*

(i) $K = 4 \det A$.
(ii) $K \star \beta = 2 \operatorname{div}(A^t A)_0$.
(iii) $2|\nabla\varphi|^2 \geq |K|$.

Proof. The first two statements are immediate consequences of Corollary 3.2 and the previous proposition. Further, by Lemma 3.5, the second line of Eq. (3), and the assumption $0 \leq |D\varphi|^2 = |\nabla\varphi|^2 + K/2$, while

$$|D\varphi|^2 = |\sum_{i=1}^{2} e_i \cdot \nabla_{e_i}\varphi|^2 \leq \Big(\sum_{i=1}^{2} 1 \cdot |\nabla_{e_i}\varphi|\Big)^2 \leq 2|\nabla\varphi|^2, \tag{16}$$

whence (iii). □

3.3. **Integrability of (A, β).** Next we address the question for which pairs (A, β) a solution to Eq. (6) exists. Towards that end we introduce the Clifford algebra valued 1-form $\Gamma(X) \coloneqq A(X) + \beta(X)\omega$ and define the connection

$$\widetilde{\nabla}_X\varphi \coloneqq \nabla_X\varphi - A(X)\cdot\varphi - \beta(X)\omega\cdot\varphi = \nabla_X\varphi - \Gamma(X)\cdot\varphi.$$

A solution to Eq. (6) exists if and only if if we have a non-trivial $\widetilde{\nabla}$-parallel spinor field. In fact this is equivalent to the triviality of the spinor bundle in the sense of flat bundles for we may regard ΣM as a "quaternionic" line bundle. This in turn is equivalent to the vanishing of the curvature $R^{\widetilde{\nabla}}$ and the triviality of the associated holonomy map $\pi_1(M, p) \to \operatorname{Aut}(\Sigma_p M)$. We have

$$\begin{aligned}
&R^{\widetilde{\nabla}}(X,Y)\varphi = (\widetilde{\nabla}_X\widetilde{\nabla}_Y - \widetilde{\nabla}_Y\widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]})\varphi\\
=&\widetilde{\nabla}_X(\nabla_Y\varphi - \Gamma(Y)\cdot\varphi) - \widetilde{\nabla}_Y(\nabla_X\varphi - \Gamma(X)\cdot\varphi) - \nabla_{[X,Y]}\varphi + \Gamma([X,Y])\cdot\varphi\\
=&R^{\nabla}(X,Y)\varphi - \nabla_X(\Gamma(Y)\cdot\varphi) + \nabla_Y(\Gamma(X)\cdot\varphi) - \Gamma(X)(\nabla_Y\varphi - \Gamma(Y)\cdot\varphi)\\
&+\Gamma(Y)(\nabla_X\varphi - \Gamma(X)\cdot\varphi) + \Gamma(\nabla_X Y)\cdot\varphi - \Gamma(\nabla_Y X)\cdot\varphi\\
=&R^{\nabla}(X,Y)\varphi - (\nabla_X\Gamma)(Y)\cdot\varphi + (\nabla_Y\Gamma)(X)\cdot\varphi + \Gamma(X)\Gamma(Y)\cdot\varphi - \Gamma(Y)\Gamma(X)\cdot\varphi\\
=&R^{\nabla}(X,Y)\varphi - d\Gamma(X,Y)\cdot\varphi + [\Gamma(X),\Gamma(Y)]\varphi,
\end{aligned}$$

where $d\Gamma$ denotes the skew-symmetric part of the covariant derivative $\nabla\Gamma$, i.e.

$$d\Gamma(X,Y) \coloneqq (\nabla_X\Gamma)(Y) - (\nabla_Y\Gamma)(X) = (\nabla_X A)(Y) - (\nabla_Y A)(X) + d\beta(X,Y)\omega. \tag{17}$$

Similarly, we define $dA(X,Y) \coloneqq (\nabla_X A)(Y) - (\nabla_Y A)(X)$. Now for an oriented orthonormal basis (e_1, e_2) we find

$$\begin{aligned}[\Gamma(e_1),\Gamma(e_2)] =&[A(e_1),A(e_2)] + 2\beta(e_2)A(e_1)\omega - 2\beta(e_1)A(e_2)\omega\\ =&2(\det A)\omega - 2\beta(e_2)J(A(e_1)) + 2\beta(e_1)J(A(e_2)).\end{aligned}$$

Since $2R^{\nabla}(e_1,e_2)\varphi = K\omega\cdot\varphi$ we finally get

$$R^{\widetilde{\nabla}}(e_1,e_2)\varphi = -\frac{1}{2}K\omega\cdot\varphi - dA(e_1,e_2)\varphi - d\beta(e_1,e_2)\omega\cdot\varphi$$
$$+2(\det A)\omega\cdot\varphi - 2\beta(e_2)J(A(e_1))\varphi + 2\beta(e_1)J(A(e_2))\varphi.$$

Since $K = 4\det A - 2\star d\beta$ by Proposition 3.6, this vanishes for all φ if and only if $dA(e_1,e_2) = -2\beta(e_2)J(A(e_1)) + 2\beta(e_1)J(A(e_2))$. Since M is Kähler, $\nabla J = 0$, hence $\nabla_X(A\circ J)(Y) = (\nabla_X A)(JY)$. Writing the previous expression invariantly yields the following

Proposition 3.8. *If the pair (A,β) arises from a spinor field as in* (6)*, then*

$$\operatorname{div}(A\circ J) = -2(J\circ A\circ J)(\beta^\sharp).$$

Conversely, if the integrability condition of Proposition 3.8 is satisfied, then there exists a *local* solution φ to Eq. (6). Moreover, φ is uniquely determined up to multiplication by a unit quaternion from the right.

3.4. **Absolute minimisers.** In dimension $n \geq 3$ the only critical points of the spinorial energy functional $\mathcal{E}$ are absolute minimisers with $\mathcal{E}(g,\varphi) = 0$ [2]. This stands in sharp contrast to the surface case.

Theorem 3.9. *On a spin surface (M_γ,σ) we have*

$$\inf\mathcal{E} = \pi|\gamma - 1|.$$

Proof. The lower bound $\inf\mathcal{E} \geq \pi|\gamma-1|$ follows directly from the Schrödinger-Lichnerowicz and Gauß-Bonnet formulæ, for

$$\tfrac{1}{2}\int_{M_\gamma}|\nabla\varphi|^2 \geq -\tfrac{1}{4}\int_{M_\gamma}K = \pi(\gamma-1) \tag{18}$$

which gives the estimate for $\gamma \geq 1$. For the sphere, we use (iii) of Corollary 3.7 to obtain

$$2\pi = \frac{1}{2}\int_{S^2}K \leq \int_{S^2}|\nabla\varphi|^2. \tag{19}$$

Further, the results of Section 4 show that this lower bound is actually attained on the sphere. For genus $\gamma \geq 1$ we show the existence of "almost-minimisers", i.e. for every $\varepsilon > 0$ there is a unit spinor (g,φ) such that $\mathcal{E}(g,\varphi) \leq \pi|\gamma-1| + \varepsilon$. There is a standard strategy for their construction by gluing together 2-tori with small Willmore energy in a flat 3-torus (T^3,g_0) and restricting the parallel spinors of T^3 to the resulting surface, see also [13] and [25] (which we discuss further in Example 3.15) for related constructions.

To start with we define the Willmore energy of a piecewise smoothly embedded surface $F: M \to T^3$ by

$$\mathcal{W}(F) := \frac{1}{2}\int_{F(M)}H^2 dv^g.$$

Here, H is the mean curvature of $F(M)$ in (T^3,g_0) and integration is performed with respect to the volume element dv^g associated to the restriction of the Euclidean metric to $F(M)$. For sake of concreteness, consider a square fundamental domain of the torus in $\mathbb{R}^3$, fix $\rho > 0$ and consider two flat disks of radius ρ inside that domain which are parallel to the (x_1,x_3)-plane and are at small distance from each other. We want two replace the disjoint union of the disks of radius $\rho/2$ by a catenoidal neck and retain the vertical annular pieces. The result of this process will be called a *handle of radius ρ*.

Lemma 3.10. *For all $\varepsilon > 0$ there exists a handle of radius ρ which has Willmore energy less than ε.*

Proof. Since the Willmore energy is scaling invariant it suffices to construct a model handle with Willmore energy less than ε for some radius $\rho(\varepsilon) > 0$. The solution for the given radius ρ is then simply obtained by rescaling. We construct a model handle as a surface of revolution. It will be composed of a catenoidal part, a spherical part and a flat annular part. More precisely, let $L > 0$ and consider the curve $\gamma = (\gamma_1, \gamma_2) : [0, \infty) \to \mathbb{R} \times (0, \infty)$ defined by

$$\gamma(u) = \begin{cases} (\operatorname{arsinh}(u), \sqrt{1+u^2}) & , 0 \le u \le L \\ (a,b) + R\big(\cos(\frac{u-L}{R} - \alpha), \sin(\frac{u-L}{R} - \alpha)\big) & , L \le u \le L + \alpha R \\ \big(a + R, b + u - (L + \alpha R)\big) & , L + \alpha R \le u < \infty \end{cases}$$

where we have set $(a,b) = (\operatorname{arsinh}(L) - L\sqrt{1+L^2}, 2\sqrt{1+L^2})$, $R = 1 + L^2$ and $\alpha = \arcsin(1/\sqrt{1+L^2})$. Consider the surface of revolution around the x_1-axis defined by

$$F(u,v) = \big(\gamma_1(u), \cos(v)\gamma_2(u), \sin(v)\gamma_2(u)\big)$$

where $u \in [0,\infty)$, $v \in [0, 2\pi)$. This surface is a piecewise smooth C^1-surface with Willmore energy

$$\mathcal{W}(F) = \frac{\pi}{\sqrt{1+L^2}}$$

which is precisely the Willmore energy of the spherical piece, the catenoid and the flat piece being minimal. We double this surface along the boundary $\{x_1 = 0\}$ and intersect with the region $\{x_2^2 + x_3^2 \le 4b^2\}$ to get a handle of radius $\rho(L) = 2b$ with Willmore energy $2\pi/\sqrt{1+L^2} < \varepsilon$ for L big enough. This piecewise smooth handle may be approximated by smooth handles with respect to the $W^{2,2}$-topology to yield the desired smooth handle.

Remark 3.11. Fix $\rho > 0$ and consider the handle of radius ρ with Willmore energy $\varepsilon = 4\pi/\sqrt{1+L^2}$ which we obtain by rescaling the handle constructed above by $2b$. Then the distance between the flat annular pieces is given by

$$2\frac{a+R}{2b} = \frac{1}{2}\left(\frac{\operatorname{arsinh}(L)}{\sqrt{1+L^2}} + \sqrt{1+L^2} - L\right)$$

which goes to zero as $\varepsilon \to 0$ (i.e. $L \to \infty$).

Lemma 3.12. *For a compact connected surface M_γ of genus $\gamma \ge 1$ with a fixed spin structure σ, there is a flat torus (T^3, g_0) and an embedding $F : M_\gamma \to T^3$ such that $\mathcal{W}(F) \le \varepsilon$ and such that the spin structure on M_γ induced by this embedding is the given spin structure σ.*

Proof. Since orientation preserving diffeomorphisms act transitively on both bounding and non-bounding spin structures, it is enough to show the lemma for only one bounding or non-bounding spin structure.
We deal with the case $\gamma = 1$ first. For the non-bounding spin structure we may simply take T_n to be any totally geodesic 2-torus in a flat torus (T^3, g_0). This embedding has zero Willmore energy and the induced spin structure on T_n is the non-bounding one. For a bounding spin structure we choose an embedding $D^2 \subset T^2$ and let $S^1 = \partial D^2$. Let S^1_δ denote the circle of length $\delta > 0$ and set $T_b := S^1 \times S^1_\delta \subset$

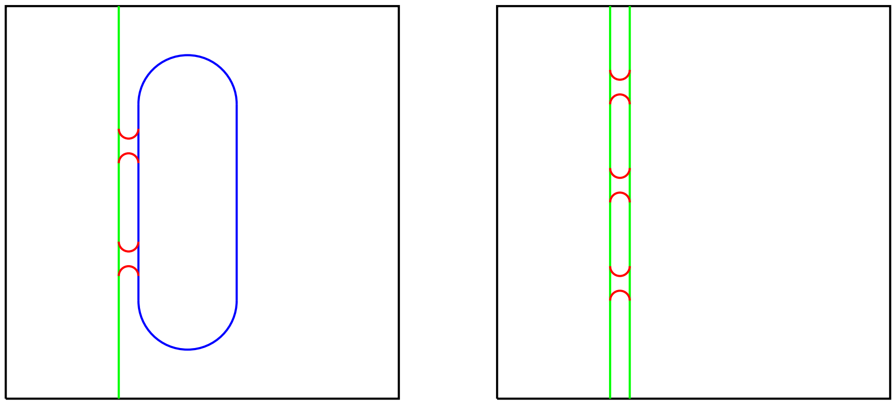

FIGURE 1. Surfaces with almost minimisers. The left-hand picture shows a torus with a non-bounding spin structure, drawn in green, and a torus with a bounding spin structure, drawn in blue. These surfaces are connected by necks drawn in red. The right-hand picture shows two tori with a non-bounding spin structure, drawn in green, connected by necks drawn in red.

$T^2 \times S^1_\delta$. Then T_b has arbitrarily small Willmore energy for δ small enough, and the induced spin structure on T_b is bounding. Note that we may slightly flatten the circle $S^1 \subset T^2$ in order to make it contain a line segment. Then T_b contains a flat disk which will be useful later for gluing in a handle.
In the higher genus case we use the tori T_b and T_n constructed above as building blocks which we connect by handles with small Willmore energy. The construction is illustrated in Fig. 1. If σ is a non-bounding spin structure, we align a copy of T_n and a copy of T_b in such a way that T_n is parallel and at small distance to a flat disk inside T_b. Then we connect T_n and T_b by $\gamma - 1$ handles. If σ is a bounding spin structure, we take two parallel copies of T_n at small distance, and call them T_n' and T_n''. Then we connect T_n' and T_n'' by $\gamma - 1$ handles. According to Lemma 3.10 this can be done without introducing more than an arbitrarily small amount of Willmore energy. The resulting surface has genus γ and carries a non-bounding spin-structure in the first, and a bounding spin structure in the second case.

We return to the proof of Theorem 3.9. With the notations of the lemma and the proposition we set $g \coloneqq F^*(g_0)$. Further, we restrict a parallel spinor of unit length on T^3 to $F(M_\gamma)$ and pull it back to a spinor φ on (M_γ, σ, g). As in [11] it follows that $D\varphi = H\varphi$, whence

$$\frac{1}{2}\int_{M_\gamma} |D\varphi|^2 dv^g = \frac{1}{2}\int_{M_\gamma} H^2 dv^g = \mathcal{W}(F) \le \varepsilon$$

and thus

$$\mathcal{E}(g,\varphi) = \frac{1}{2}\int_{M_\gamma} |D\varphi|^2 - \frac{1}{4}\int_{M_\gamma} K \le \varepsilon - \frac{\pi}{2}\chi(M) = \varepsilon + \pi|\gamma - 1|$$

as claimed. □

From (18), the Schrödinger-Lichnerowicz formula and the results from Section 2.4 we immediately deduce the

Corollary 3.13. *If $\gamma \geq 1$, then $\mathcal{E}(g,\varphi) = \pi|\gamma-1|$ if and only if $D_g\varphi = 0$, that is, φ is a harmonic spinor of unit length. In particular, absolute minimisers of $\mathcal{E}$ over M_γ correspond to minimal isometric immersions of the universal covering of M_γ.*

Remark 3.14. In the case of the sphere ($\gamma = 0$) equality holds if and only if φ is a so-called *twistor spinor*, see Section 4. Furthermore, as a consequence of the Schrödinger-Lichnerowicz and Gauß-Bonnet formula, a unit spinor on the torus ($\gamma = 1$) is harmonic if and only if it is parallel.

Example 3.15. For any $\gamma \geq 3$ there exists a triply periodic orientable minimal surface M in $\mathbb{R}^3$ such that if Γ denotes the lattice generated by its three periods, the projection of M to the flat torus $T^3 = \mathbb{R}^3/\Gamma$ is M_γ [25, Theorem 1]. Since the normal bundle of M_γ in T^3 is trivial there exists a natural induced spin structure which we claim to be a bounding one. To see this we need to analyse the construction in [25] which is a refinement of the construction used in Lemma 3.12. In a first step one starts with two flat minimal 2-dimensional tori T_1 and T_2 inside the flat 3-dimensional torus T^3. One can assume that T_1 and T_2 are parallel. The trivial spin structure on T^3 admits parallel spinors which we can restrict to parallel spinors on T_1 and T_2. In particular, both T_1 and T_2 carry the non-bounding spin structure so that the disjoint union $T_1 \amalg T_2$ carries a bounding spin structure. Namely, $T_1 \amalg T_2$ is the boundary of any connected component of $T^3 \setminus (T_1 \amalg T_2)$, and this even holds in the sense of spin manifolds, cf. also the discussion in [19, Remark II.2.17]. In a second step, small catenoidal necks are glued in between T_1 and T_2 but this does not affect the nature of the spin structure which thus remains a bounding one.

Using the conformal equivariance (9) of the Dirac operator gives a further corollary. Namely, $\mathcal{E}(g,\varphi) = \pi(\gamma-1)$ for $(g,\varphi) \in \mathcal{N}$ if and only if there is metric $\tilde{g}$ with nowhere vanishing spinor $\tilde{\varphi}$ with $D_{\tilde{g}}\tilde{\varphi} = 0$. Indeed, for $g = |\tilde{\varphi}|^4_{\tilde{g}}\tilde{g}$ the rescaled spinor $\varphi = \tilde{\varphi}/|\tilde{\varphi}|_g$ is in the kernel of D_g and of unit norm.

Corollary 3.16. *For $\gamma \geq 1$ absolute minimisers on a spin surface correspond to nowhere vanishing harmonic spinors on Riemann surfaces.*

For a generic conformal class on M_γ nowhere vanishing harmonic spinors do *not* exist. More exactly, it was proven in [1] that the set $\mathcal{M}$ of all metrics g with $\dim_{\mathbb{C}} \ker D_g \leq 2$ is open in the C^1-topology and dense in the C^∞-topology in the set of all metrics. Thus, with similar arguments as in Lemma 3.18 below it follows that nowhere vanishing harmonic spinors cannot exist in conformal classes $[g]$ with $g \in \mathcal{M}$. However, examples do exist for special cases such as hyperelliptic Riemann surfaces, where complex techniques can be used when regarding harmonic spinors as holomorphis sections as explained in Section 2.2. For instance, Bär and Schmutz [5] could compute the dimension of the space of harmonic spinors for any spin structure based on earlier work by Martens [20] and Hitchin [16]. Complex geometry also gives us control on the zero set of the spinors, as highlighted by the following example.

Example 3.17. Recall that hyperelliptic surfaces are precisely the Riemann surfaces of genus $\gamma \geq 2$ which arise as two-sheeted branched coverings of the complex projective line (see for instance [14, Paragraph §7 and 10]). There are exactly $2(\gamma-1)$ branch points $w_1,\ldots,w_{2(\gamma+1)}$, the so-called *Weierstraß points*. For any such Weierstraß point w, the divisor $2(\gamma-1)w$ defines the canonical line bundle κ of M_γ, and λ defined by $(\gamma-1)w$ is a holomorphic square root. In particular, there exists a holomorphic section $\varphi_0 \in H^0(M_\gamma, \mathcal{O}(\lambda))$ – a positive harmonic spinor – whose divisor of zeroes is precisely $(\gamma-1)w$, that is, φ_0 has a unique zero of order $\gamma-1$ at w. Furthermore, on a hyperelliptic Riemann surface there exists a meromorphic function f on M with a pole of order 2 at w and a double zero elsewhere, say at $p \in M$. Hence, if the *genus of M is odd*, then $\varphi_1 = f^{(\gamma-1)/2}\varphi_0$ is a holomorphic section which has a unique zero at p. Regarding φ_1 as a negative harmonic spinor via the quaternionic structure therefore gives a non-vanishing harmonic spinor $\varphi_0 \oplus \varphi_1 \in \Gamma(\Sigma_g)$. Rescaling by its norm gives finally the desired absolute minimiser. Note that $\dim_{\mathbb{C}} H^0(M_\gamma, \mathcal{O}(\lambda)) = (\gamma+1)/2$ (see for instance [14, Theorem 14]) so that λ corresponds to a non-bounding spin structure if $\gamma \equiv 1 \mod 4$, and to a bounding spin structure if $\gamma \equiv 3 \mod 4$.

As already mentioned above, there are obstructions against absolut minimisers.

Lemma 3.18. *If (g,φ) is an absolute minimiser over M_γ with $\gamma \geq 2$, then $d(g) = \dim_{\mathbb{C}} \ker D_g \geq 4$.*

Proof. As noted in Section 2.3, $d(g)$ is even, so it remains to rule out the case $d(g) = 2$. Viewing $\Sigma M \to M$ as a quaternionic line bundle with scalar multiplication from the right, $\ker D_g$ inherits a natural quaternionic vector space structure. In particular, it is a 1-dimensional quaternionic subspace if $d(g) = 2$. Since $D(1+i\omega)\varphi = D\varphi - i\omega D\varphi = 0$ there is a quaternion q with $(1+i\omega)\varphi = \varphi q$. If $q \neq 0$, then $(1+i\omega)\varphi$ is a nowhere vanishing section of the complex line bundle Σ_+ and thus yields a holomorphic trivialisation of the holomorphic tangent bundle via the holomorphic description of harmonic spinors in Section 2.1. In particular, $\gamma = 1$. If $q = 0$, then φ is a nowhere vanishing section of $\Sigma_- \cong \overline{\Sigma}_+$ and a similar argument applies. □

Summarising, we obtain the following theorem concerning existence respectively non-existence of absolute minimisers.

Theorem 3.19. *On (M_γ, σ) the infimum of $\mathcal{E}$*

- (i) *is attained in the cases*
  - (a) $\gamma = 1$ *and σ is the non-bounding spin structure.*
  - (b) $\gamma \geq 3$ *and σ is a bounding spin structure.*
  - (c) $\gamma \geq 5$ *with* $\gamma \equiv 1 \mod 4$ *and σ is a non-bounding spin structure.*
- (ii) *is not attained in the cases*
  - (a) $\gamma = 1$ *and σ is a bounding spin structure.*
  - (b) $\gamma = 2$
  - (c) $\gamma = 3, 4$ *and σ is a non-bounding spin structure.*

Remark 3.20.

- (i) It remains unclear whether the infimum is attained for a non-bounding spin structure on surfaces of genus $\gamma \geq 6$ and $\gamma \not\equiv 1 \mod 4$.
- (ii) In the case of the sphere ($\gamma = 0$) the infimum of $\mathcal{E}$ is always attained. This will be discussed in Section 4.

Proof of Theorem 3.19. (i) The non-bounding spin structure on T^2 is the one which admits parallel spinors, while (b) and (c) follow from Example 3.15 and Example 3.17 respectively.

(ii) From Section 2.3 we know that $d(g)$ must be divisible by 4 if σ is bounding while from Hitchin's bound $d(g) \leq \gamma + 1$ [16]. Therefore, under the conditions stated in

(a) or (b), $d(g) \leq 3$ for any metric g on M_γ so that for a bounding σ we necessarily have $d(g) = 0$. If $\gamma \geq 2$ we have $d(g) \geq 4$ by Section 2.3 and moreover, $d(g) \equiv 2 \bmod 4$ if σ is non-bounding. Hence $d(g) \geq 6$ which is impossible if $\gamma \leq 4$. □

Finally, we characterise the absolute minimisers in terms of A and β. First we note that J induces a natural complex structure on $T^*M \otimes TM$ defined by

$$i(\alpha \otimes v) = i\alpha \otimes v = \alpha \otimes iv := \alpha \otimes Jv.$$

Equipped with this complex structure, $T^*M \otimes TM$ becomes a complex rank 2 bundle, and we have the complex linear bundle isomorphism

$$T^*M \otimes TM \cong TM^{1,0} \otimes_{\mathbb{C}} (T^*M \otimes \mathbb{C}), \quad \alpha \otimes v \mapsto \alpha \otimes \tfrac{1}{2}(v - iJv). \tag{20}$$

In this way, considering A as a TM-valued 1-form, the decomposition $\Omega^1(TM) \cong \Omega^{1,0}(TM^{1,0}) \oplus \Omega^{0,1}(TM^{1,0})$ gives a decomposition

$$A = A^{1,0} + A^{0,1}.$$

Since $T^*M^{1,0} \otimes_{\mathbb{C}} TM^{1,0}$ is trivial we may identify $A^{1,0}$ with a smooth function $f : M \to \mathbb{C}$. Further, on any Kähler manifold $TM^{0,1} \cong T^*M^{1,0}$ so we may identify $A^{0,1}$ with a quadratic differential $q \in \Gamma(\kappa_\gamma^2)$. Finally, $\bar\partial f \in \Omega^{0,1}(M_\gamma) \cong \Gamma(TM_\gamma^{1,0})$ and $\bar\partial q \in \Omega^{0,1}(\kappa_\gamma^2) \cong \Gamma(TM_\gamma^{0,1})$.

Lemma 3.21. *Modulo these isomorphisms we have*

$$-\frac{1}{2}\operatorname{div} A^{1,0} = \bar\partial f \quad \textit{and} \quad -\frac{1}{2}\operatorname{div} A^{0,1} = \overline{\bar\partial q}.$$

In particular, $\operatorname{div} A^{1,0} = 0$ *if and only if* $\bar\partial f = 0$ *and* $\operatorname{div} A^{0,1} = 0$ *if and only if* $\bar\partial q = 0$.

Proof. If we write

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

in terms of a positively oriented local orthonormal frame $\{e_i\}$, then

$$A^{1,0} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} a+d & -b+c \\ b-c & a+d \end{pmatrix}, \quad A^{0,1} = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} a-d & b+c \\ b+c & -a+d \end{pmatrix}. \tag{21}$$

Hence $A^{1,0}$ is the sum of the trace and skew-symmetric part of A, while $A^{0,1}$ is the traceless symmetric part of A. Now fix a local holomorphic coordinate $z = x + iy$ and assume that $\{e_i\}$ is synchronous at $z = 0$, i.e. $e_1(0) = \partial_x(0)$ and $e_2(0) = \partial_y(0)$. In particular, $\partial_z = (\partial_x - i\partial_y)/2$ corresponds to e_1 under the identification (20). From (21)

$$A^{1,0} = (\alpha + i\beta)\, dz \otimes \partial_z \quad \text{and} \quad A^{0,1} = (\gamma - i\delta)\, dz \otimes \partial_{\bar z},$$

whence $f = \alpha + i\beta$ and $q = (\gamma - i\delta)\, dz^2$. Then at $z = 0$,

$$\operatorname{div} A^{1,0} = (-e_1(\alpha) + e_2(\beta))e_1 - (e_2(\alpha) + e_1(\beta))e_2$$

and

$$\operatorname{div} A^{0,1} = -(e_1(\gamma) + e_2(\delta))e_1 + (e_2(\gamma) - e_1(\delta))e_2.$$

Computing $\bar\partial f = \partial_{\bar z}(\alpha + i\beta)\, d\bar z$ and $\bar\partial q = \partial_{\bar z}(\gamma - i\delta)\, d\bar z \otimes dz^2$ gives immediately the desired result. □

Remark 3.22. In particular, for a critical point (g,φ) the symmetric $(2,0)$-tensor associated with $A^{0,1}$ is a *tt-tensor*, that is, traceless and *transverse* (divergence-free). For $\gamma \geq 2$, the previous lemma therefore recovers the standard identification of the space of tt-tensors with the tangent space of Teichmüller space given by holomorphic quadratic differentials.

We are now in a position to give an alternative characterisation of absolute minimisers if $\gamma \geq 1$. The case of the sphere will be handled in Theorem 4.7.

Proposition 3.23. *Let $\gamma \geq 1$. The following statements are equivalent:*

(i) *(g,φ) is an absolute minimiser.*
(ii) *$\nabla_X\varphi = A(X)\cdot\varphi$ for a traceless symmetric endomorphism A.*
(iii) *(g,φ) is critical and $\beta = 0$.*

Remark 3.24. In particular, we recover the equivalence (ii)⇔(iii) of [11, Theorem 13] for the case $H = 0$.

Proof. By Theorem 3.9, (g,φ) is an absolute minimiser if and only if $D\varphi = 0$. From (8) this is tantamount to $\mathrm{Tr}A = 0$, $\mathrm{Tr}(A\circ J) = 0$ and $\beta = 0$. The trace conditions are equivalent to A being symmetric and traceless whence the equivalence between (i) and (ii). Furthermore, (ii) immediately forces $\beta = 0$. Conversely, (iii) together with the critical point equation in Proposition 3.1 implies $2A^tA = |A|^2\mathrm{Id}$, whence $2|A|^2 = |K|$ by Corollary 3.7 (i). In particular, $2|A|^2 = -K$ on the open set $U = \{x \in M_\gamma : K(x) < 0\}$. Assume that U is non-empty and not dense in M_γ, i.e. $\bar{U} \subset M_\gamma \smallsetminus \{p\}$ for some $p \in M_\gamma$. Without loss of generality we may also assume U to be connected. On its boundary the curvature vanishes so that in particular, $|A| = 0$ on ∂U. Further, $|D\varphi|^2 = |A|^2 + K/2 = 0$ on U as a simple computation in an orthonormal frame using Eq. (8) reveals. As before, $D\varphi = 0$ implies that A is traceless symmetric and divergence-free over U. In particular, A corresponds to a holomorphic quadratic differential by Lemma 3.21. Since every holomorphic line bundle on the non-compact Riemann surface $M_\gamma \smallsetminus \{p\}$ is holomorphically trivial (see for instance [8, Theorem 30.3]), over U the coefficients of A arise as the real and imaginary part of a holomorphic function and are therefore harmonic. However, they are continuous on $\bar{U}$ and vanish on the boundary, hence $A = 0$ by the maximum principle. In particular, $K = 0$ on U, a contradiction. This leaves us with two possibilities. Either U is dense in M_γ or U is empty. By Gauß-Bonnet the second case can only happen for genus 1 and g must be necessarily flat. In any case, φ is harmonic and therefore defines an absolute minimiser. □

Corollary 3.25. *Let $\gamma \geq 1$. If (g,φ) is an absolute minimiser, then A is a tt-tensor. Furthermore, $K \equiv 0$ if $\gamma = 1$ and $K \leq 0$ with only finitely many zeroes if $\gamma \geq 2$.*

Remark 3.26.

(i) As we will see in Section 5 there exist flat critical points which are not absolute minimisers.
(ii) If $\gamma \geq 2$ in Proposition 3.23 (iii), it suffices to assume that $|\beta| = const$. Indeed, $(\beta\otimes\beta)_0$ induces a holomorphic section of κ_γ^2 and has therefore at least one zero. At such a zero, $(\beta\otimes\beta)_0 = 0$, hence $\beta = 0$ in this point and thus everywhere.
(iii) If $\beta = 0$, then Proposition 3.8 implies $\mathrm{div}(A\circ J) = 0$. However, this does not yield an extra constraint as $\mathrm{div}(A\circ J) = \mathrm{div}\,A$ for A symmetric.

4. Critical points on the sphere

In this section we completely classify the critical points in the genus 0 case where M_γ is diffeomorphic to the sphere. In particular, up to isomorphism there is only one spin structure for S^2 is simply-connected.

4.1. **Twistor spinors.** For a general Riemannian spin manifold (M^n, σ, g) with spinor bundle $\Sigma_g M^n \to M^n$, a *Killing spinor* $\varphi \in \Gamma(\Sigma_g M^n)$ satisfies

$$\nabla_X \psi = \lambda X \cdot \psi$$

for any vector field $X \in \Gamma(TM)$ and some fixed $\lambda \in \mathbb{C}$, the so-called *Killing constant*. In particular, the underlying Riemannian manifold is Einstein with $\mathrm{Ric} = 4\lambda^2 g$ so that λ is either real or purely imaginary. If M is compact and connected, only Killing spinors of *real* type, where $\lambda \in \mathbb{R}$, can occur [7, Theorem 9 in Section 1.5]. More generally we can consider *twistor spinors*. By definition, these are elements of the kernel of the *twistor operator* $T_g = \mathrm{pr}_{\ker\mu} \circ \nabla$, where $\mathrm{pr}_{\ker\mu} : \Gamma(T^*M \otimes \Sigma) \to \Gamma(\ker\mu)$ is projection on the kernel of the Clifford multiplication $\mu : T^*M \otimes \Sigma M \to \Sigma M$. Equivalently, a twistor spinor satisfies

$$\nabla_X \varphi = -\tfrac{1}{n} X \cdot D\varphi$$

for all $X \in \Gamma(TM)$. The subsequent alternative characterisation will be useful for our purposes. The following proposition was stated (in slightly different form) in [10], see e.g. [7, Theorem 2 in Section 1.4] for a proof.

Proposition 4.1. *On a Riemannian spin manifold M^n the following conditions are equivalent:*

(i) *φ is a twistor spinor.*
(ii) *$X \cdot \nabla_X \varphi$ does not depend on the unit vector field X.*

Example 4.2. (cf. [7, Example 2 in Section 1.5]) On the round sphere S^n there are Killing spinors $\psi_\pm \neq 0$ with $\lambda_\pm = \pm\frac{1}{2}$. Furthermore, $\varphi_{ab} = a\psi_+ + b\psi_-$ for constants $a, b \in \mathbb{R}$ are twistor spinors which are not Killing for $ab \neq 0$. Indeed, Killing spinors must have constant length, while φ_{ab} will have zeroes in general. If n is even, then a spinor ψ_+ is a Killing spinor for the Killing constant $\frac{1}{2}$ if and only if $\psi_- \coloneqq \omega \cdot \psi_+$ is a Killing spinor for $-\frac{1}{2}$. Moreover, if $n \equiv 2 \mod 4$, then these $\psi_\pm$ are pointwise orthogonal. In this particular case $\varphi_{ab} = a\psi_+ + b\psi_-$ is a twistor spinor of constant length.

Using [15] the following lemma is straightforward.

Lemma 4.3. *Let (M_γ, σ) be a spin surface and $(g, \varphi) \in \mathcal{N}$. Then the following conditions are equivalent.*

(i) *φ is a twistor spinor.*
(ii) *There exist $a, b \in \mathbb{R}$ such that $\nabla_X \varphi = aX \cdot \varphi + bJ(X) \cdot \varphi$ for all $X \in \Gamma(TM)$.*
(iii) *There exist $\alpha \in \mathbb{R}$ and a unit Killing spinor ψ such that*

$$\varphi = \cos\alpha\, \psi + \sin\alpha\, \omega \cdot \psi.$$

Furthermore, the Killing constant λ of ψ is given by $\lambda = \sqrt{a^2 + b^2}$.

Remark 4.4. Note that for (iii), $\omega \cdot \psi$ is a Killing spinor with Killing constant $-\lambda$.

In terms of the associated pair (A, β) we have $A = a\mathrm{Id} + bJ$ and $\beta = 0$ for a twistor spinor. Hence Lemma 4.3 together with Proposition 3.6 immediately implies:

Corollary 4.5. *Let φ be a g-twistor spinor of unit length. Then g has non-negative constant Gauß curvature $K = 4(a^2 + b^2)$. In particular, $K = 0$ if and only if φ is a parallel spinor.*

Remark 4.6. The previous corollary is a special case of [10, Theorem 1, p. 69] and [10, Theorem 3, p. 71] where it was shown to hold in any dimension.

4.2. **Critical points on the sphere.** Next we completely describe the set of critical points on the sphere.

Theorem 4.7. *On $M_0 = S^2$, the following statements are equivalent:*

(i) *(g,φ) is a critical point of $\mathcal{E}$.*
(ii) *$\mathcal{E}(g,\varphi) = \pi$, i.e. (g,φ) is an absolute minimiser.*
(iii) *φ is a twistor spinor, i.e.*

$$\nabla_X\varphi = aX\cdot\varphi + bJ(X)\cdot\varphi \tag{22}$$

for constants $a, b \in \mathbb{R}$.

(iv) *There is a unit-length Killing spinor ψ on (S^2, g) and $\alpha \in \mathbb{R}$ such that*

$$\varphi = \cos\alpha\,\psi + \sin\alpha\,\omega\cdot\psi \tag{23}$$

Moreover, any of these conditions implies that the Gauß curvature of g is a positive constant.

Proof. Assume (g,φ) is a critical point. Since $H^1(S^2,\mathbb{R}) = 0$, Proposition 3.1 and Corollary 3.2 imply

$$\beta = 0, \quad A^tA = \tfrac{1}{2}|A|^2\mathrm{Id}, \quad \operatorname{div} A = 0,$$

whence $2|A|^2 = |K|$. Since the set of points where $K < 0$ cannot be dense on S^2 by Gauß-Bonnet, it must be empty (cf. the proof of Proposition 3.23). In particular, $2|A|^2 = K$. Since $|\nabla\varphi|^2 = |A|^2$, Gauß-Bonnet again implies

$$\mathcal{E}(g,\varphi) = \tfrac{1}{2}\int_M |\nabla\varphi|^2 = \tfrac{1}{2}\int_M |A|^2 = \tfrac{1}{4}\int_M K = \tfrac{1}{4}\cdot 4\pi = \pi$$

Conversely, this implies that (g,φ) is critical by Theorem 3.9.
Next assume that (ii) holds. The equality $2\pi = \int_M |\nabla\varphi|^2$ gives the pointwise equality $|D\varphi|^2 = 2|\nabla\varphi|^2$, cf. (16) and (19). On the other hand, equality in (16) arises if and only if $e_1\cdot\nabla_{e_1}\varphi = e_2\cdot\nabla_{e_2}\varphi$. Multiplying with $\omega = e_1\cdot e_2$ from the left yields the equation $e_1\cdot\nabla_{e_2}\varphi = -e_2\cdot\nabla_{e_1}\varphi$. Hence for $X = ae_1 + be_2$ with $a^2 + b^2 = 1$ we obtain

$$\begin{aligned} X\cdot\nabla_X\varphi &= a^2e_1\cdot\nabla_{e_1}\varphi + b^2e_2\cdot\nabla_{e_2}\varphi + ab(e_1\cdot\nabla_{e_2}\varphi + e_2\cdot\nabla_{e_2}e_1) \\ &= e_1\cdot\nabla_{e_1}\varphi = e_2\cdot\nabla_{e_2}\varphi. \end{aligned}$$

According to Proposition 4.1, φ is a twistor spinor .
The equivalence between (iii) and (iv) follows directly from Lemma 4.3.
Finally, Eq. (23) states that φ is in the S^1-orbit of a Killing spinor which is clearly a critical point - its associated pair is $A = \lambda\mathrm{Id}$ and $\beta = 0$. Hence (ii) follows. ☐

Corollary 4.8. *Up to rescaling there is exactly one $\mathrm{U}(2) = S^1 \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ orbit of critical points on S^2.*

5. Critical points on the torus

5.1. **Spin structures on tori.** Finally we investigate the genus 1 case, that is we consider a torus $T^2_\Gamma = \mathbb{R}^2/\Gamma$ for a given lattice $\Gamma \subset \mathbb{R}^2$. Here, we have four inequivalent spin structures, three of which are bounding. In the case of a flat metric these can be described uniformly through homomorphisms $\chi : \Gamma \to \mathbb{Z}_2 = \{-1, 1\} = \ker\theta \subset \mathrm{Spin}(2)$ giving rise to an associated bundle $P_\chi := \mathbb{R}^2 \times_\zeta \mathrm{Spin}(2)$. Here, θ is the connected double covering $\mathrm{Spin}(2) \cong S^1 \to \mathrm{SO}(2) \cong S^1$. The quotient map $\mathbb{R}^2 \to T^2_\Gamma$ and the covering θ induce a map $\eta_\chi : P_\chi \to P_{\mathrm{SO}(2)}(T^2_\Gamma)$ which defines a spin structure. In fact, there is a bijection between $\mathrm{Hom}(\Gamma, \mathbb{Z}_2) \cong H^1(T^2_\Gamma; \mathbb{Z}_2)$ and isomorphism classes of spin structures on T^2_Γ such that the non-bounding spin structure corresponds to the trivial homomorphism $\chi \equiv 1$ (see [9] or [6, Section 2.5.1] for further details). For example, the non-bounding spin structure is the trivial spin structure given by $\mathrm{Id}\times\theta : T^2 \times \mathrm{Spin}(2) \to T^2 \times \mathrm{SO}(2)$. Its associated spinor bundle is trivialised by parallel sections in contrast to the spinor bundles associated with the three bounding spin structures which do not admit non-trivial parallel spinors [16]. (Note that for flat metrics a parallel spinor is the same as a harmonic spinor in virtue of the Schrödinger-Lichnerowicz formula.) For an example of a bounding spin structure, consider the Clifford torus inside S^3. If we equip the resulting solid torus with the spin structure induced from its ambient S^3, then the induced spin structure on its boundary, i.e. the Clifford torus, is a bounding spin structure.

5.2. **Non-minimising critical points on tori.** We are going to show that on certain flat tori, critical points which are not absolute minimisers do exist. Examples, which are in fact saddle points, are provided by the following construction.

We begin with two parallel unit spinors ψ_1 and ψ_2 on the Euclidean space $(\mathbb{R}^2, g_0)$ satisfying $\psi_1 \perp \psi_2$ and $\psi_1 \perp \omega\cdot\psi_2$. Then an orthonormal basis of the spinor module Δ is given by $\{\psi_1, \omega\cdot\psi_1, \psi_2, \omega\cdot\psi_2\}$. Thinking of ω as an imaginary unit, we set

$$e^{t\omega} := \cos(t) + \sin(t)\omega$$

for $t \in \mathbb{R}$. In particular, the usual formulæ such as $e^{(s+t)\omega} = e^{s\omega}e^{t\omega}$ or $\nabla e^{t\omega} = \omega e^{t\omega}$ hold. Furthermore, let $\alpha_1, \alpha_2 \in \mathbb{R}^{2*}$. For $\theta \in \mathbb{R}$ consider the unit spinor

$$\varphi(x) = \cos(\theta)e^{\alpha_1(x)\omega}\psi_1 + \sin(\theta)e^{\alpha_2(x)\omega}\psi_2 \tag{24}$$

for which

$$\nabla_{(\cdot)}\varphi(x) = \cos(\theta)\alpha_1(\cdot)(x) \otimes e^{\alpha_1(x)\omega}\omega\cdot\psi_1 + \sin(\theta)\alpha_2(\cdot)(x) \otimes e^{\alpha_2(x)\omega}\omega\cdot\psi_2. \tag{25}$$

As both $\{e_1\cdot\psi_1, e_2\cdot\psi_1\}$ and $\{\psi_2, \omega\cdot\psi_2\}$ span the space orthogonal to ψ_1 and $\omega\cdot\psi_1$, there is a unit vector field V such that $\psi_2 = V\cdot\psi_1$. Parallelity of ψ_1 and ψ_2 imply parallelity of V. The pair (A, β) corresponding to φ in the decomposition (6) is given by the $(1, 1)$-tensor

$$A_x = \cos(\theta)\sin(\theta)(\alpha_2 - \alpha_1) \otimes e^{(\alpha_1(x)+\alpha_2(x)+\pi/2)\omega}V \tag{26}$$

and the *parallel* 1-form

$$\beta = \cos^2(\theta)\alpha_1 + \sin^2(\theta)\alpha_2. \tag{27}$$

In particular, we find $\det A = 0$ in accordance with Proposition 3.6 (i). Indeed, $\omega\cdot V = -V\cdot\omega$ and $Ve^{t\omega} = e^{-t\omega}V$ for $t \in \mathbb{R}$ so that

$$\varphi(x) = \big(\cos(\theta)e^{\alpha_1(x)\omega} + \sin(\theta)e^{\alpha_2(x)\omega}V\big)\psi_1$$

and

$$\nabla_X\varphi(x) = \big(\cos(\theta)\alpha_1(X)e^{\alpha_1(x)\omega} - \sin(\theta)\alpha_2(X)e^{\alpha_2(x)\omega}V\big)\omega\cdot\psi_1. \tag{28}$$

On the other hand,

$$\big(\cos(\theta)e^{-\alpha_1(x)\omega} - \sin(\theta)e^{\alpha_2(x)\omega}V\big)\big(\cos(\theta)e^{\alpha_1(x)\omega} + \sin(\theta)e^{\alpha_2(x)\omega}V\big) = 1,$$

and therefore

$$\psi_1 = \big(\cos(\theta)e^{-\alpha_1(x)\omega} - \sin(\theta)e^{\alpha_2(x)\omega}V\big)\varphi.$$

After substitution into (28) this gives

$$\begin{aligned}\nabla_X\varphi =&\big(\cos^2(\theta)\alpha_1(X) + \sin^2(\theta)\alpha_2(X)\\ &+\cos(\theta)\sin(\theta)\big(\alpha_1(X)-\alpha_2(X)\big)e^{(\alpha_1(x)+\alpha_2(x))\omega}V\big)\omega\cdot\varphi\\ =&\cos(\theta)\sin(\theta)\big(\alpha_2(X)-\alpha_1(X)\big)e^{(\alpha_1(x)+\alpha_2(x)+\pi/2)\omega}V\cdot\varphi\\ &+\big(\cos^2(\theta)\alpha_1(X)+\sin^2(\theta)\alpha_2(X)\big)\omega\cdot\varphi.\end{aligned}$$

Next we compute the negative gradient of $\mathcal{E}$ in (g,φ). This is most easily done by considering the identities in (3) from which $4Q_1(g,\varphi) = -|\nabla\varphi|^2 g - \operatorname{div} T_{g,\varphi} + 2\langle\nabla\varphi\otimes\nabla\varphi\rangle$. Using (25) we compute

$$\langle\nabla\varphi\otimes\nabla\varphi\rangle = \cos^2(\theta)\alpha_1\otimes\alpha_1 + \sin^2(\theta)\alpha_2\otimes\alpha_2.$$

Since

$$|\nabla\varphi|^2 = \operatorname{Tr}\langle\nabla\varphi\otimes\nabla\varphi\rangle = \cos^2(\theta)|\alpha_1|^2 + \sin^2(\theta)|\alpha_2|^2$$

we obtain

$$\tfrac12\langle\nabla\varphi\otimes\nabla\varphi\rangle - \tfrac14|\nabla\varphi|^2 g = \tfrac12\langle\nabla\varphi,\nabla\varphi\rangle_0 = \tfrac12\cos^2(\theta)(\alpha_1\otimes\alpha_1)_0 + \tfrac12\sin^2(\theta)(\alpha_2\otimes\alpha_2)_0.$$

Finally, if $\{e_1,e_2\}$ is the standard basis of $\mathbb{R}^2$, then as in (11)

$$\operatorname{div} T_{g,\varphi}(e_1,e_1) = e_2(\omega\varphi,\nabla_{e_1}\varphi\rangle = e_2(\beta(e_1)) = 0$$

since β is parallel. Next $Q_2(g,\varphi) = -\nabla^*\nabla\varphi + |\nabla\varphi|^2\varphi$ by (3). Again using Eq. (25) we compute

$$\nabla^*\nabla\varphi = |\alpha_1|^2\cos(\theta)e^{\alpha_1(x)\omega}\psi_1 + |\alpha_2|^2\sin(\theta)e^{\alpha_2(x)\omega}\psi_2.$$

Altogether we get for the spinor φ defined by (24) that

$$\begin{aligned}Q_1(g,\varphi) =&\tfrac12\cos^2(\theta)(\alpha_1\otimes\alpha_1)_0 + \tfrac12\sin^2(\theta)(\alpha_2\otimes\alpha_2)_0,\\ Q_2(g,\varphi) =&-|\alpha_1|^2\cos(\theta)e^{\alpha_1(x)\omega}\psi_1 - |\alpha_2|^2\sin(\theta)e^{\alpha_2(x)\omega}\psi_2\\ &+\big(\cos^2(\theta)|\alpha_1|^2+\sin^2(\theta)|\alpha_2|^2\big)\varphi.\end{aligned}$$

For a critical point we need Q_1 and Q_2 to vanish. Now $Q_1(g,\varphi)$ vanishes if and only if $\cos^2(\theta)\alpha_1\otimes\alpha_1 + \sin^2(\theta)\alpha_2\otimes\alpha_2$ is a constant multiple of the Euclidean metric g. This in turn is the case if and only if $\alpha_1 \perp \alpha_2$ and $|\cos(\theta)||\alpha_1| = |\sin(\theta)||\alpha_2|$. Furthermore, $Q_2(g,\varphi) = 0$ if and only if $\nabla^*\nabla\varphi = f\varphi$ for some function $f:\mathbb{R}^2\to\mathbb{R}$, i.e. if

$$\begin{aligned}&|\alpha_1|^2\cos(\theta)e^{\alpha_1(x)\omega}\psi_1 + |\alpha_2|^2\sin(\theta)e^{\alpha_2(x)\omega}\psi_2\\ &\qquad=f(x)\big(\cos(\theta)e^{\alpha_1(x)\omega}\psi_1 + \sin(\theta)e^{\alpha_2(x)\omega}\psi_2\big).\end{aligned}$$

Again this holds if and only if $|\alpha_1|^2 = f(x) = |\alpha_2|^2$.

Summarising, the spinor φ in (24) is a critical point if and only if

$$\alpha_1\perp\alpha_2,\quad |\alpha_1| = |\alpha_2|,\quad (\theta-\pi/4)\in(\pi/2)\mathbb{Z} \tag{29}$$

are satisfied. When does then φ descend to a well-defined spinor on a torus? For $\ell := \pi/|\alpha_1|$ consider first the square torus $T_\ell := \mathbb{R}^2/\Gamma_\ell$ whose lattice is spanned by

$$\gamma_1 = \ell\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \gamma_2 = \ell\begin{pmatrix}1\\-1\end{pmatrix}. \tag{30}$$

Possibly after an additional rotation we may assume without loss of generality that $\alpha_i = |\alpha_1|e^i$ for the standard basis (e^1, e^2) of $\mathbb{R}^{2*}$. If σ_χ is the (necessarily bounding) spin structure defined by the group morphism $\chi_\ell : \Gamma_\ell \to \mathbb{Z}_2$, $\chi_\ell(\gamma_1) = \chi_\ell(\gamma_2) = -1$, then

$$e^{\alpha_1(\gamma)\omega} = e^{\alpha_2(\gamma)\omega} = \chi_\ell(\gamma) \tag{31}$$

so that φ descends to (T^2_ℓ, σ_ℓ) and gives rise to a critical point there. More generally, φ descends to any covering $T_\Gamma = \mathbb{R}^2/\Gamma$ of T_ℓ, where the spin structure on T_Γ is induced by $\chi = \chi_\ell|_\Gamma$. For instance, the double covering $T_{2\ell} \to T^2_\ell$ yields a square torus for which φ descends to a spinor with respect to the non-bounding spin structure defined by $\chi \equiv 1$. Conversely, any torus T_Γ to which φ descends is necessarily a covering of T^2_ℓ. Indeed, assume (31) holds for the spin structure σ_χ on T^2_Γ instead of σ_ℓ on T^2_ℓ, and let $\Gamma_0 = \ker\chi$. In particular, $\Gamma_0 \subset 2\ell\mathbb{Z}^2$. If σ_χ is the non-bounding structure, then $\chi \equiv 1$ and therefore $\Gamma_0 = \Gamma$. Otherwise, there exists $\gamma_0 \in \Gamma$ with $\chi(\gamma_0) = -1$ so that (31) implies

$$\gamma_0 - \frac{\ell}{2}\begin{pmatrix}1\\1\end{pmatrix} \in \ell\mathbb{Z}^2.$$

In particular, Γ is contained in Γ_ℓ.

Remark 5.1. From Eq. (26) and Eq. (27) it follows immediately that for a flat critical point on the torus, $\beta^\sharp \in \ker A$. Conversely, any critical point satisfying this condition is necessarily flat, cf. Proposition 5.4.

In conclusion we established the existence of critical points (g_0, φ_0) on any torus T_Γ covering T_ℓ. Its spin structure is determined by the restriction of χ_ℓ to Γ. Finally, we will show the existence of saddle points. First of all, if φ satisfies (31), then

$$\mathcal{E}(g, \varphi) = \frac{\cos^2(\theta)|\alpha_1|^2 + \sin^2(\theta)|\alpha_2|^2}{2}\operatorname{area}(T^2_\ell).$$

Now $\operatorname{area}(T^2_\ell) = 2\ell^2$, and if (g_0, φ_0) is critical, $\cos^2(\theta_0) = \sin^2(\theta_0) = \frac{1}{2}$ and $|\alpha_1| = |\alpha_2| = \pi/\ell$, whence $\mathcal{E}(g_0, \varphi_0) = \pi^2$. Next we construct special curves (g_t, φ_t) through (g_0, φ_0). The metric g_t is obtained through an area-preserving deformation of T^2_ℓ by taking the lattice Γ_t spanned by

$$\gamma_1(t) = \frac{\ell}{2}\begin{pmatrix}1\\1\end{pmatrix}(1+t) \quad \text{and} \quad \gamma_2(t) = \frac{\ell}{2}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{(1+t)}.$$

The spinor will be modified through $\theta = \theta(t) = ct + \theta_0$. Then $\mathcal{E}(g_t, \varphi_t) = f(t)\pi^2$ for

$$f(t) = \cos^2(\theta(t))\frac{1}{(1+t)^2} + \sin^2(\theta(t))(1+t)^2.$$

Since $f''(0) = 8\theta'(0) + 4$, the second derivative takes any real value by suitably choosing the slope c in $\theta(t)$. The non-minimising critical point on T^2_0 is therefore a saddle point, and so are the critical points obtained by taking covers.

5.3. **Classification of flat critical points on the torus.** We are now in a position to classify the flat critical points on the torus. Recall the decomposition $A = A^{1,0} + A^{0,1}$, where $A^{1,0}$ is the trace and skew-symmetric part of A, while $A^{0,1}$ is the symmetric part of A. If A is associated with a critical point, these components correspond to a holomorphic function f and a quadratic differential q, cf. Lemma 3.21. From the coordinate description of (21) one easily verifies the identities

$$\det A = \det A^{1,0} + \det A^{0,1}$$

and

$$(A^{1,0})^t A^{1,0} = \det A^{1,0} \cdot \mathrm{Id}, \qquad (A^{0,1})^2 = -\det A^{0,1} \cdot \mathrm{Id},$$
$$A^{0,1} A^{1,0} = (A^{1,0})^t A^{0,1}, \qquad A^{1,0} A^{0,1} = A^{0,1} (A^{1,0})^t.$$

In particular, these identities imply

$$(A^t A)_0 = 2(A^{1,0})^t A^{0,1} = 2A^{0,1} A^{1,0},$$

so that $(A^t A)_0$ corresponds to the quadratic differential $2fq$.

Theorem 5.2. *A flat critical point on the torus is either an absolute minimiser, i.e. a parallel spinor, or a non-minimising critical point as in Section 5.2.*

Proof. Let (g, φ) be a critical point on $M = T^2$ with vanishing Gauß curvature and associated pair (A, β). The Euler-Lagrange equation implies

$$d\beta = \operatorname{div} \beta = 0$$
$$\operatorname{div} A = \operatorname{div} A^{1,0} + \operatorname{div} A^{0,1} = 0.$$

Furthermore, together with $K = 0$ and Corollary 3.7,

$$\operatorname{div}(A^t A)_0 = 0.$$

On $M = T^2$ we may trivialize $T^*M^{1,0}$ and write $q = h\,dz^2$ for $h = c - id$ globally. Then $\operatorname{div} A = 0$ yields the equation $\partial_{\bar z} f + \partial_z \bar h = 0$. The traceless symmetric endomorphism $(A^t A)_0$ corresponds to the quadratic differential $fq = fh\,dz^2$ and $\operatorname{div}(A^t A)_0 = 0$ yields the holomorphicity of fh. In particular we get $fh = c$ for some constant $c \in \mathbb{C}$. Moreover, by Corollary 3.7 again, $K = 0$ also yields

$$\det A = \det A^{1,0} + \det A^{0,1} = 0.$$

Consequently, $|f|^2 = |h|^2 = |c|$, for $\det A^{1,0} = |f|^2$ and $\det A^{0,1} = -|h|^2$. Rotating the coordinate system if necessary we may assume that c is a non-negative real number. If $c = 0$, then $A = \beta = 0$ and we have an absolute minimiser, so assume from now on that $c > 0$. We want to show that (g, φ) is of the form of the critical points in Section 5.2. Scaling the metric on the spinor bundle appropriately we may assume that $c = 1/4$. Writing $f(x, y) = e^{i\tau(x,y)}/2$ for $\tau : T^2 \to \mathbb{R}$, we have $h = \bar f$ and $\partial_{\bar z} f + \partial_z \bar h = 0$ if and only if $\partial_{\bar z}\tau + \partial_z \tau = \partial_x \tau = 0$, whence $\tau \equiv \tau(y)$. It follows that

$$A = \begin{pmatrix} \cos\tau(y) & 0 \\ \sin\tau(y) & 0 \end{pmatrix}.$$

Further, β is parallel since $K = 0$ and β is harmonic. Since

$$(A^t A + \beta \otimes \beta)_0 = 0$$

we obtain $\beta = dy$. The integrability condition of Proposition 3.8 now reads

$$\nabla_{\partial_y} A(\partial_x) = 2\beta(\partial_y) J(A(\partial_x))$$

which implies $\theta'(y) = 2\beta(\partial_y) = 2$. Hence A and β are as in Eq. (26) and Eq. (27) for $V = \partial_y$, $\theta = \pi/4$, $\alpha_1 = dx + dy$ and $\alpha_2 = dy - dx$. □

Remark 5.3. As we have remarked in Section 5.1, non-trivial parallel spinors only exist for the non-bounding spin structure. However, by the previous proposition, flat critical points can also exist for the bounding spin-structures.

It remains an open question if a critical point on the torus is necessarily flat, but at least we can give a number of equivalent conditions.

Proposition 5.4. *For a critical point (g,φ) on the torus which is associated with (A,β), the following conditions are equivalent.*
- (i) *g is flat.*
- (ii) *$|\beta| = const.$*
- (iii) *$\beta^\sharp \in \ker A$.*

Moreover, any of these conditions implies
$$|A|^2 = |\beta|^2. \tag{32}$$

Proof. If (g,φ) is a flat critical point, then β is parallel and hence has constant length. Conversely, if (g,φ) is critical with $|\beta| = const$, then the Weitzenböck formula on 1-forms (15) and Gauß-Bonnet immediately imply that $\nabla\beta = 0$. Therefore, $(A^tA + \beta\otimes\beta)_0 = 0$, whence $\operatorname{div}(A^tA)_0 = 0$ by Corollary 3.2 (iv). But either $\beta \equiv 0$ so that $\nabla\varphi = 0$ by Proposition 3.23, or β has no zeroes at all and we can apply Corollary 3.7. In both cases it follows $K = 0$.
On the other hand, for a flat critical point (g,φ), $\beta^\sharp \in \ker A$ follows from Remark 5.1. Conversely, let $\beta^\sharp \in \ker A$. If $\beta(x) = 0$, then (15) implies $\nabla^*\nabla\beta(x) = 0$. Otherwise, $\beta(x)$ is a non-trivial element in the kernel of A so that $\det A(x) = K(x) = 0$ by Corollary 3.7 (i). Again, we find $\nabla^*\nabla\beta(x) = 0$ so that β is actually parallel. Then either $\beta \equiv 0$ and g is flat (for in this case (g,φ) is an absolute minimiser), or β is nowhere vanishing so that $K = \det A \equiv 0$.
If any of these equivalent conditions holds, then $(A^tA + \beta\otimes\beta)_0 = 0$ and $\beta^\sharp \in \ker A$, whence
$$\begin{aligned} 0 = \langle \beta^\sharp, (A^tA + \beta\otimes\beta)_0\beta^\sharp\rangle &= |A(\beta^\sharp)|^2 + |\beta|^4 - \tfrac{1}{2}\operatorname{Tr}(A^tA)|\beta|^2 - \tfrac{1}{2}|\beta|^4 \\ &= \tfrac{1}{2}|\beta|^2\big(|\beta|^2 - |A|^2\big). \end{aligned}$$
This implies $|A| = |\beta|$ or $\beta = 0$. In the latter case Proposition 3.23 implies $A = 0$. □

Acknowledgements. The authors thank T. Friedrich for providing interesting references and the referee for carefully reading the manuscript.

References

[1] Ammann, B., Dahl, M., and Humbert, E., *Surgery and harmonic spinors*, Adv. Math. **220** (2009), 523–539.

[2] B. Ammann, H. Weiss, and F. Witt, *A spinorial energy functional: critical points and gradient flow*, Preprint 2012, arXiv:1207.3529.

[3] M. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 47–62.

[4] C. Bär, *Extrinsic bounds for eigenvalues of the Dirac operator*, Ann. Global Anal. Geom. **16** (1998), 573–596.

[5] C. Bär and P. Schmutz, *Harmonic spinors on Riemann surfaces*, Ann. Global Anal. Geom. **10** (1992), 263–273.

[6] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, Teubner-Texte zur Mathematik, **41**. BSB B. G. Teubner, Leipzig, 1981.
[7] H. Baum, T. Friedrich, R. Grunewald and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik, **124**, B. G. Teubner, Stuttgart, 1991.
[8] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, **81**, Springer-Verlag, Berlin, 1981.
[9] T. Friedrich, *Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur*, Colloq. Math. **48** (1984), no. 1, 57–62.
[10] T. Friedrich, *On the conformal relation between twistors and Killing spinors*, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 22, 59–75 (1990). Avaliable on `https://eudml.org/doc/220945`
[11] T. Friedrich, *On the spinor representation of surfaces in Euclidean* 3*-space*, J. Geom. Phys. **28** (1998), no. 1-2, 143–157.
[12] T. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics **25**, AMS, Providence, 2000.
[13] N. Ginoux, J.-F. Grosjean, *Almost harmonic spinors*, C. R. Math. Acad. Sci. Paris **348** (2010), 811–814.
[14] R. Gunning, *Lectures on Riemann surfaces*, Princeton Univ. Press, New Jersey, 1966.
[15] K. Habermann, *The twistor equation of Riemannian manifolds*, J. Geom. Phys. **7**, 469–488 (1990).
[16] N. Hitchin, *Harmonic spinors*, Adv. in Math. **14** (1974), 1–55.
[17] K. Kenmotsu, *Weierstrass formula for surfaces of prescribed mean curvature*, Math. Ann. **245** (1979), 89–99.
[18] R. Kusner and N. Schmitt, *The spinor representation of minimal surfaces*, preprint, http://www.arxiv.org/abs/dg-ga/9512003, 1995.
[19] H. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Univ. Press, New Jersey, 1989.
[20] H. Martens, *Varieties of special divisors on a curve. II*, J. Reine Angew. Math. **233** (1968), 89–100.
[21] J. Milnor, *Remarks concerning spin manifolds*, 1965 Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) pp. 55–62 Princeton Univ. Press, New Jersey.
[22] N. Schmitt, *Minimal surface with planar embedded ends*, Ph.D. dissertation, University of Amherst, 1993.
[23] A. Trautman, *Spinors and the Dirac operator on hypersurfaces. I. General theory*, J. Math. Phys. **33** (1992), 4011–4019.
[24] A. Trautman, *The Dirac operator on hypersurfaces*, Acta Phys. Polonica B **26** (1995), 1283–1310.
[25] M. Traizet, *On the genus of triply periodic minimal surfaces*, J. Differential. Geom. **79** (2008), no. 2, 243–75.

Fakultät für Mathematik, Universität Regensburg, Universitätsstrasse 40, D–93040 Regensburg, Germany
E-mail address: `bernd.ammann@mathematik.uni-regensburg.de`

Mathematisches Seminar der Universität Kiel, Ludewig-Meyn Strasse 4, D–24098 Kiel, Germany
E-mail address: `weiss@math.uni-kiel.de`

Institut für Geometrie und Topologie der Universität Stuttgart, Pfaffenwaldring 57, D–70569 Stuttgart, Germany
E-mail address: `frederik.witt@mathematik.uni-stuttgart.de`