Construction of initial data sets for Lorentzian manifolds with lightlike parallel spinors

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Abstract

Lorentzian manifolds with parallel spinors are important objects of study in several branches of geometry, analysis and mathematical physics. Their Cauchy problem has recently been discussed by Baum, Leistner and Lischewski, who proved that the problem locally has a unique solution up to diffeomorphisms, provided that the intial data given on a space-like hypersurface satisfy some constraint equations. In this article we provide a method to solve these constraint equations. In particular, any curve (resp. closed curve) in the moduli space of Riemannian metrics on M with a parallel spinor gives rise to a solution of the constraint equations on $M \times (a,b)$ (resp. $M \times S^1$).

Keywords: Parallel spinor, Lorentzian manifolds, Riemannian manifolds, special holonomy, moduli spaces

1 Introduction

1.1 Parallel spinors

Let $(\overline{N}, \bar{h})$ be a connected $(n+1)$-dimensional oriented and time-oriented Lorentzian manifold with a fixed spin structure. The bundle of complex spinors will be denoted by $\Sigma\overline{N}$. The spinor bundle carries a natural Hermitian form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ of split signature, a compatible connection, and Clifford multiplication. In the present article we search for manifolds with a parallel spinor, i.e. a (non-trivial) parallel section of $\Sigma\overline{N}$. Understanding Lorentzian manifolds with parallel spinors is interesting for several reasons.

The first reason is that Riemannian manifolds with parallel spinors provide interesting structures. We want to briefly sketch some of them: Parallel spinors provide a powerful technique to obtain Ricci-flat metrics on compact manifolds: All known closed Ricci-flat manifolds carry a non-vanishing parallel spinor on a finite covering. Parallel spinors are also linked to "stability", defined in the sense that the given compact Ricci-flat metric cannot be deformed to a positive scalar curvature metric. This condition in turn is linked to dynamical stability of a Ricci-flat metric under Ricci flow: A compact Ricci-flat metric is dynamically

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stable under the Ricci flow if and only if it cannot be deformed to a positive scalar curvature metric ([20] and [23, Theorem 1.1]).

Infinitesimal stability was proven for metrics with a parallel spinor in [39] and local stability (for manifolds with irreducible holonomy) in [15]. Manifolds with parallel spinors provide interesting moduli spaces, see [3]. Furthermore parallel spinors help to understand the space of metrics with non-negative scalar curvature. The stability property explained above implies that every homotopy class in the space of positive scalar curvature metrics which is known to be non-trivial also remains non-trivial in the space of metrics with non-negative scalar curvature, see [35]. It is interesting and challenging to see to which extent it is possible to find Lorentzian analogues to these results.

A second reason to be interested in parallel spinors on arbitrary semi-Riemannian manifolds is that their existence implies that the holonomy is special [27, 28], [11], [22], [13], [8]. Thus the construction of Lorentzian manifolds with parallel spinors provides examples of manifolds with special holonomy.

A third reason is that parallel spinors are relevant in many fields of theoretical physics. For example parallel spinors on Lorentzian manifolds are often viewed as generators for the odd symmetries in a supersymmetric theory (see e.g. [16]). Parallel spinors are also important in several physical theories of Kaluza-Klein type, i.e. involving additional compactified dimensions. Mathematically we model them by a semi-Riemannian submersion $\pi : T \to B$ from a high-dimensional Lorentzian manifold T to a macroscopically visible 3+1-dimensional space-time B. In this context one has to make sure that spectral and other analytic properties of Dirac operators on T are comparable to the corresponding properties on B. This requires the existence of harmonic spinors on the fibers $M := \pi^{-1}(b)$, $b \in B$. If the scalar curvature assumed to be non-negative, these harmonic spinors are parallel which in turn implies that the fibers are Ricci-flat. Varying b yields a family of metrics on M with parallel spinors. One thus obtains a map $B \to \mathcal{M}od_{\|}(M)$, where $\mathcal{M}od_{\|}(M)$ is the (pre-)moduli space of metrics with a parallel spinor on some covering of M, which is also a central ingredient in this article.

1.2 The Cauchy problem for parallel light-like spinors

Important progress about Lorentzian manifolds with parallel spinors was recently achieved by H. Baum, T. Leistner and A. Lischewski [10, 30, 29], see also [9] for associated lecture notes. In particular, these authors showed the well-posedness of an associated Cauchy problem which we will now describe in more detail and which will be the main topic of the present article.

Let again $(\overline{N}, \bar{h})$ be a time- and space-oriented Lorentzian spin manifold, and let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ be the Hermitian product on $\Sigma\overline{N}$ with split signature. The Clifford multiplication on $(\overline{N}, \bar{h})$ will be denoted by $\bullet$.

Note that for any spinor φ on a Lorentzian manifold one defines the *Lorentzian Dirac current* $V_\varphi \in \Gamma(T\overline{N})$ of φ by requiring that

$$\bar{h}(X, V_\varphi) = -\langle\!\langle X \bullet \varphi, \varphi \rangle\!\rangle$$

holds for all $X \in T\overline{N}$. Recall that on Lorentzian manifolds the Clifford action of vector fields on spinors is symmetric, in contrast to the Riemannian case, where it is anti-symmetric.

Manifold	Metric	Dimension	Type	Clifford multiplication	Scalar product on spinors
$\overline{N}$	$\bar{h}$	$n+1$	Lorentzian	$\bullet$	$\langle\!\langle \cdot, \cdot \rangle\!\rangle$
N	h	n	Riemannian	$\star$	$\langle \cdot, \cdot \rangle$
M	g	$m = n-1$	Riemannian	$\cdot$	$\langle \cdot, \cdot \rangle$

Submanifold	Weingarten map
$N \subset \overline{N}$	$\overline{W}$
$M \subset N$	W

Figure 1: Notation for manifolds, Clifford multiplications and Weingarten maps

If φ is parallel, then V_φ is parallel as well. One can show that V_φ is a future oriented causal vector [9, Sec. 1.4.2, Prop. 2]. Thus V_φ is either time-like or light-like everywhere. In the time-like case, the Lorentzian manifold locally splits as a product $(N,h) \times (\mathbb{R}, -dt^2)$, where (N,h) is a Riemannian manifold with a parallel spinor. So with respect to a suitable Cauchy hypersurface the understanding of such metrics directly relies on the corresponding results on Riemannian manifolds.

In this article we are concerned with the case, that φ is a parallel spinor with a light-like Dirac-current V_φ. This problem was studied in [10], [30], and [29].

Let N be a spacelike hypersurface in this Lorentzian manifold with induced metric h. For a future-oriented (time-like) normal vector field T with $\bar{h}(T,T) = -1$ we define the Weingarten map $\overline{W} := -\bar{\nabla}T$. We use the symbol $\overline{W}$ instead of W, as the latter one will be used for the Weingarten map of hypersurfaces M in N, which will play a central role in our article. The Hermitian product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the Clifford multiplication $\bullet$ on $\Sigma\overline{N}$ induce a positive definite Hermitian product and a Clifford multiplication $\star$ on $\Sigma\overline{N}|_N$ which can be characterized by the formulas

$$\langle \varphi, \psi \rangle = \langle\!\langle T \bullet \varphi, \psi \rangle\!\rangle, \qquad X \star \varphi = iT \bullet X \bullet \varphi$$

for all $x \in N$, $\varphi, \psi \in \Sigma_x\overline{N}$, $X \in T_xN$. The spin structure on $\overline{N}$ induces a spin structure on the spacelike hypersurface N. Let ΣN be the associated spinor bundle of N. With standard tools about Clifford modules, one sees that there is a bundle monomorphism $\iota : \Sigma N \to \Sigma\overline{N}|_N$ over the identity of N such that the Clifford multiplication and the Hermitian product on ΣN are mapped to $\star$ and $\langle \cdot, \cdot \rangle$ on $\Sigma\overline{N}|_N$. The bundle monomorphism ι is a bundle isomorphism if and only if n is even. However note that ι is not parallel, i.e. the connection is not preserved under ι. More precisely $\nabla\iota$ is a linear pointwise expression in $\overline{W}$. From now on we identify ΣN with its image in $\Sigma\overline{N}|_N$ under ι, taking the connection from ΣN. As already mentioned above we will also consider hypersurfaces M of N and the spinor bundle ΣM of M. However the relation between spinors on M and on N is a bit easier than between spinors on $\overline{N}$ and on N. One can work with an embedding ΣM into $\Sigma N|_M$ which preserves the scalar product $\langle \cdot, \cdot \rangle$. We can even choose the embedding such that the Clifford multiplications coincide, however in the literature another embedding is often used which does no longer preserve the Clifford multiplication. In our application in Section 5 it

depends on the parity of n which embedding is more convenient for us, see also Subsection 2.3 and Appendix B for further information about hypersurfaces and spinors. Thus we want to use $\cdot$ for the Clifford multiplication on M in contrast to $\star$ which is used for the Clifford multiplication on N.

For any spinor $\varphi \in \Gamma(\Sigma N)$ we associate — see e.g. [29, (1.7)] — its Riemannian Dirac current $U_\varphi \in \Gamma(TN)$ by requiring

$$h(U_\varphi, X) = -i\langle X \star \varphi, \varphi\rangle, \qquad \forall X \in TN. \tag{1}$$

By Lemma 20 in Appendix A we see, that the spinor φ satisfies $h(U_\varphi, U_\varphi) = \|\varphi\|^4$ if and only if we have for any $p \in N$

$$i\varphi(p) \in \{V \star \varphi \mid V \in T_pN\}. \tag{2}$$

Now, if we assume that $(\overline{N}, \bar{h})$ carries a non-vanishing parallel spinor, then this spinor induces a spinor φ on (N, h) such that

$$\nabla^N_X \varphi = \frac{i}{2}\overline{W}(X) \star \varphi, \qquad \forall X \in TN, \tag{3}$$

$$U_\varphi \star \varphi = i u_\varphi \varphi, \tag{4}$$

where $U_\varphi \in \Gamma(TN)$ is defined as above and where

$$u_\varphi := \sqrt{h(U_\varphi, U_\varphi)}, \tag{5}$$

see [10, (4) and following] or [29, (1.6) and (1.8)].

Note that the constraint equations (1)–(5) are not independent from each others. We comment on this in Appendix A.

Conversely, if (N, h) is given as an abstract Riemannian spin manifold, and if $\overline{W} \in \Gamma(\mathrm{End}(TN))$, and $\varphi \in \Gamma(\Sigma N)$ satisfy (2), (3) and (4) with U_φ and u_φ defined by (1) and (5), then there is a Lorentzian manifold $(\overline{N}, \bar{h})$ with N as a Cauchy hypersurface and a parallel spinor, such that h is the induced Riemannian metric, $\overline{W}$ the Weingarten map and φ is induced by a parallel spinor on $(\overline{N}, \bar{h})$, see [29, Consequence of Theorems 2 and 3]. This was proven by solving the associated wave equations by using the technique of symmetric hyperbolic systems. A simpler approach, going back to a remark by P. Chrusciel was later given in [36, Chap. 4].

The question arises on how to solve these constraint equations. In the present article we will describe a new method to obtain solutions of these constraint equations. We will see how any smooth curve in the (pre-)moduli space of closed $m = (n-1)$-dimensional Riemannian manifolds with a parallel spinor together with a scaling function yields a solution to the constraint equations, see our Main Construction 15 in Section 6. And thus the well-definedness of the Cauchy problem implies the existence of an associated $(n+1)$-dimensional Lorentzian with a parallel spinor. Such a relation between families of metrics with special holonomy and solutions of the constraint equations was already conjectured by Leistner and Lischewski, see [29]. We essentially show that the conditions in [29, Table 1] is satisfied if and only if the divergence condition (31) in our Appendix D is satisfied.

We also derive versions of this construction to obtain initial data on a compact Cauchy hypersurface (without boundary). A first idea is to use a smooth

closed curve in the (pre-)moduli space of closed m-dimensional Riemannian manifolds with a parallel spinor together with a scaling function. However this will in general not lead to a solution of the original constraint equation, but to a twisted version thereof, see Main Construction 17. This will lead to a Lorentzian manifold with a parallel twisted spinor. Here the twist bundle is always a complex line bundle with a flat connection.

In special cases – more precisely assuming the "fitting condition" introduced in Section 6 we however obtain solutions of the constraint equations in the original (i.e. untwisted) sense, see Main Construction 16.

We consider it as remarkable that compared to other classical diffeomorphism invariant Cauchy problems in Lorentzian geometry, e.g. the Cauchy problem for Ricci-flat metrics, we get a large quantity of solutions to the constraint equations. Furthermore it is amazing that the solutions in our situation correspond to curves in the moduli space, while the set of solutions of the constraints in classical problems have no similar description. An important input for our article was the smooth manifold structure of the premoduli space $\mathcal{M}od_{\parallel}(M)$ and the fact that the BBGM connection preserves parallel spinors along divergence free Ricci-flat families of metrics. Ricci-flat deformations of a metric thus preserve the dimension of the space of parallel spinors. The infinitesimal version of this should be seen as some kind of Hodge theory: infinitesimal Ricci-flat deformations can be viewed as elements of $\ker(D^{T^*M})^2$ while infinitesimal deformations with parallel spinors can be viewed as elements of $\ker D^{T^*M}$ — and on compact manifolds Hodge theory tells us that $\ker(D^{T^*M})^2 = \ker D^{T^*M}$. Let us also compare the results of this paper to the tightly related recent preprint [2]. While the current paper constructs initial data for Lorentzian manifolds with a parallel spinor, one of the major themes in [2] is to get topological obstructions for a closed spin manifold to be a spacelike hypersurface in a Lorentzian manifold with a parallel spinor. We also mentioned that the special case of $(3+1)$-dimensional Lorentzian manifolds (i.e. initial data on 3-dimensional manifolds) was studied in the recent preprint [32].

The structure of the article is as follows. In Section 2 we fix some conventions, recall and extend known facts about Clifford modules and about spinors on hypersurfaces. We also explain how to differentiate a spinorial expression such as the Levi-Civita derivative $\nabla^g\varphi$ of a spinor φ with respect to the Riemannian metric g. The known results for defining this differential along a path of metrics is recalled in Subsection 2.4, this includes the BBGM parallel transport and the associated connection which arises from the universal spinor bundle construction; the concrete calculations are a central tool of the article and are carried out in Section 3. In Section 4 we show that for divergence free Ricci-flat deformations, the BBGMconnection preserves parallel spinors, a result which we consider an interesting outcome of the article independently of the application to the constraint equations, mainly discussed in the article. In Section 5 we use this construction to obtain solutions of the constraint equations (1) – (5). In Section 6 we establish the connection to curves in (pre-)moduli spaces and we also discuss on how to obtain solutions on compact Cauchy hypersurfaces. The article ends with several appendices where we provide some details about facts which are already well-known, but where adequate literature was not available. We hope that these appendices help to make the article sufficiently self-contained.

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2 Preliminaries

2.1 Conventions

All Hermitian scalar products in this article are complex linear in the first entry and complex anti-linear in the second one. Let E be a Riemannian vector bundle over a compact Riemannian manifold (M, g) equipped with a metric connection ∇ and $k \in \mathbb{N}_0$. Then the Sobolev norm H^k of a section $s \in \Gamma(E)$ is

$$\|s\|_{H^k} = \left(\sum_{l=0}^{k} \int_M |\nabla^l s|^2 dV \right)^{1/2}.$$

where dV is the volume element of g. As usual, we denote $L^2 = H^0$. We will write $H^k(g)$ if we wish to emphasise the dependence of the norm on g.

We use the symbol $\odot^2$ for the symmetrized tensor product, i.e. for a finite-dimensional real vector space V the space of symmetric bilinear forms $V \times V \to \mathbb{R}$ is denoted by $\odot^2 V^*$. Let $SV^* \subset \odot^2 V^*$ be the subset of positive definite symmetric forms on V. Applying this fiberwise to the tangent bundle TM we obtain the vector bundle $\odot^2 T^*M$ and the bundle $ST^*M =: SM$.

Definition 1. *On a Riemannian manifold M the Einstein operator is the elliptic differential operator $\Delta_E : \Gamma(\odot^2 T^*M) \to \Gamma(\odot^2 T^*M)$, given by*

$$\Delta_E h = \nabla^* \nabla h - 2\mathring{R}h, \qquad \mathring{R}h(X,Y) := h(R_{e_i,X}Y, e_i)$$

where $\{e_1, \dots e_n\}$ is a local orthonormal frame. Here, the curvature tensor is defined with the sign convention such that $R_{X,Y}Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z$.

The Einstein operator is linked to the deformation theory of Ricci-flat metrics as follows: Let g be a Ricci-flat metric and g_s a smooth family of Ricci-flat metrics through $g_0 = g$, then

$$\frac{d}{ds}\Big|_{s=0} g_s = h + \mathcal{L}_X g$$

where X is a vector field and $h \in \ker(\mathrm{div}_g) \cap \ker(\Delta_E)$, i.e. the essential part of a Ricci-flat deformation is an element in $\ker(\Delta_E)$. In particular, if g_s is a family of metrics with a parallel spinor, the essential part of its s-derivative is contained in $\ker(\Delta_E)$. For more details on the deformation theory of Ricci-flat metrics, see [12, Chapter 12 D].

2.2 Some facts about Clifford modules

In this subsection we briefly summarize some facts about representations of Clifford algebras. The interested reader might consult the first Chapter of the book by Lawson and Michelsohn [26] for further details.

Let $(E_1, \ldots, E_n)$ be the canonical basis of $\mathbb{R}^n$. The complexified Clifford algebra for $\mathbb{R}^n$ with the Euclidean scalar product will be denoted by $\mathbb{C}l_n$. In this section all Clifford multiplications are written as $\cdot$, independently on n; we use the convention that $\cdot$ does not depend on n. In the following section we will then explain that the associated bundle construction then turns this Clifford multiplication both into the Clifford multiplication $\cdot$ on M and the Clifford multiplication $\star$ on N.

We define the complex volume element $\omega_n^{\mathbb{C}}$ as

$$\omega_n^{\mathbb{C}} := \epsilon_n E_1 \cdot E_2 \cdot \ldots \cdot E_n \in \mathbb{C}l_n,$$

where $\epsilon_{4k} = \epsilon_{4k+3} = 1$, $\epsilon_{4k+1} = -i$ and $\epsilon_{4k+2} = i$. The choices imply $(\omega_n^{\mathbb{C}})^2 = 1$. The choice of sign for ϵ_n in the literature varies between different sources, our particular and unconventional choice yields an easy formulation of Lemma 2.

For n even, there is a unique irreducible complex represention of $\mathbb{C}l_n$. Here and in the following "unique" will always "unique up to isomorphism of representation". This unique representation will be denoted by Σ_n. It comes with a grading $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ given by the eigenvalues ± 1 of $\omega_n^{\mathbb{C}}$.

For n odd there are two irreducible representations, and as $\omega_n^{\mathbb{C}}$ is central, Schur's Lemma implies that $\omega_n^{\mathbb{C}}$ acts either as the identity or minus the identity which allows us to distinguish the two representations. For n odd, we will assume that Σ_n is the irreducible representation on which $\omega_n^{\mathbb{C}}$ acts as the identity. The other representation is called $\Sigma_n^{\#}$. In the following $\Sigma_n^{(\#)}$ denotes either Σ_n or $\Sigma_n^{\#}$ for odd n and Σ_n for even n.

The standard inclusion $\mathbb{R}^{n-1} \to \mathbb{R}^n$, $x \mapsto (x, 0)$ induces an inclusion $\mathbb{C}l_{n-1} \to \mathbb{C}l_n$. This turns Σ_n into a $\mathbb{C}l_{n-1}$-module. In particular, the Clifford action of $X \in \mathbb{R}^{n-1}$ is the same if X is viewed as an element of $\mathbb{C}l_{n-1}$ or $\mathbb{C}l_n$. [1]

Thus, if n is odd, we can choose (and fix from now on) isometric isomorphisms of $\mathbb{C}l_{n-1}$-modules $J_n : \Sigma_{n-1} \to \Sigma_n$ and $J_n^{\#} : \Sigma_{n-1} \to \Sigma_n^{\#}$. If n is even, then we can choose (and fix from now on) an isometric isomorphism of $\mathbb{C}l_{n-1}$-modules $I_n : \Sigma_{n-1} \oplus \Sigma_{n-1}^{\#} \to \Sigma_n$.

Lemma 2. *For n odd, $J_n : \Sigma_{n-1} \to \Sigma_n$ and $J_n^{\#} : \Sigma_{n-1} \to \Sigma_n^{\#}$ are $\mathbb{C}l_{n-1}$-linear isomorphisms of complex $\mathrm{Spin}(n-1)$-representations and they satisfy*

$$e_n \cdot J_n(\varphi) = iJ_n(\omega_{n-1}^{\mathbb{C}}\varphi), \tag{6}$$
$$e_n \cdot J_n^{\#}(\varphi) = -iJ_n^{\#}(\omega_{n-1}^{\mathbb{C}}\varphi). \tag{7}$$

for all $\varphi \in \Sigma_{n-1}$. For n even there are isometric monomorphisms of complex $\mathrm{Spin}(n-1)$-representations $J_n^{\pm} : \Sigma_{n-1} \to \Sigma_n$ and $J_n^{\#,\pm} : \Sigma_{n-1}^{\#} \to \Sigma_n$, satisying

$$e_n \cdot J_n^{\pm}(\varphi) = \pm iJ_n^{\pm}(\varphi), \tag{8}$$
$$e_n \cdot J_n^{\#,\pm}(\varphi) = \pm iJ_n^{\#,\pm}(\varphi). \tag{9}$$

for all $\varphi \in \Sigma_{n-1}^{(\#)}$ and for any $\pm \in \{+,-\}$.

Proof. If n is odd, then we see that $\epsilon_n = -i\epsilon_{n-1}$, thus $\omega_n^{\mathbb{C}} = -i\omega_{n-1}^{\mathbb{C}} \cdot e_n = -ie_n \cdot \omega_{n-1}^{\mathbb{C}}$. This implies $e_n \cdot \omega_n^{\mathbb{C}} = i\omega_{n-1}^{\mathbb{C}}$. Hence, we obtain (6):

$$e_n \cdot J_n(\varphi) = e_n \cdot \omega_n^{\mathbb{C}} \cdot J_n(\varphi) = i\omega_{n-1}^{\mathbb{C}} J_n(\varphi) = J_n(i\omega_{n-1}^{\mathbb{C}}\varphi).$$

[1] Here we do not follow the convention chosen by Bär, Gauduchon and Moroianu [6], where vectors in $\mathbb{R}^{n-1}$ have two different Clifford actions on Σ_n, depending on whether it is seen as an element in $\mathbb{C}l_{n-1}$ or $\mathbb{C}l_n$-action.

The verification of (7) is analogous. As these morphisms are $\mathbb{C}\mathrm{l}_{n-1}$-linear, and as $\mathrm{Spin}(n-1)$ is contained in $\mathbb{C}\mathrm{l}_{n-1}$, the morphisms are also $\mathrm{Spin}(n-1)$-equivariant.

If n is even, then we define

$$J_n^{(\#),\pm}(\varphi) := \frac{1}{\sqrt{2}}\left(I_n(\varphi) \mp ie_n \cdot I_n(\varphi)\right).$$

Equations (8) and (9) are obvious. The morphisms are no longer $\mathbb{C}\mathrm{l}_{n-1}$-linear. However e_n commutes with $(\mathbb{C}\mathrm{l}_{n-1})_0$ which is defined as the even part of $\mathbb{C}\mathrm{l}_{n-1}$, i.e. the sub-algebra generated by elements of the form $X \cdot Y$ with $X, Y \in \mathbb{R}^{n-1}$. As a consequence the morphisms are $(\mathbb{C}\mathrm{l}_{n-1})_0$-linear. As $\mathrm{Spin}(n-1)$ is contained in $(\mathbb{C}\mathrm{l}_{n-1})_0$, the morphisms are $\mathrm{Spin}(n-1)$-equivariant.

To show that the morphisms $J_n^{(\#),\pm}$ are isometric (and thus also injective) it suffices to show $I_n(\varphi) \perp ie_n \cdot I_n(\varphi)$, where $\varphi \in \Sigma_{n-1}$ or $\varphi \in \Sigma_{n-1}^{\#}$. We argue only for $\varphi \in \Sigma_{n-1}$, as the other case is completely analogous. As $\omega_{n-1}^{\mathbb{C}}$ and e_n anticommute, $ie_n \cdot I_n(\varphi)$ is an eigenvector of $\omega_{n-1}^{\mathbb{C}}$ to the eigenvalue -1, while $I_n(\varphi)$ is an eigenvector to the eigenvalue 1. As $\omega_{n-1}^{\mathbb{C}}$ is self-adjoint, orthogonality follows. □

Remark 3. Let n be even. Clifford multiplication with e_n anticommutes with all odd elements in $\mathbb{C}\mathrm{l}_{n-1}$, in particular with $\omega_{n-1}^{\mathbb{C}}$ and vectors $\mathbb{R}^{n-1} \subset \mathbb{C}\mathrm{l}_{n-1}$. Thus, the map $\varphi \mapsto I_n^{-1}(e_n \cdot I_n(\varphi))$ restricts to a vector space isomorphism $A : \Sigma_{n-1} \to \Sigma_{n-1}^{\#}$ which anticommutes with Clifford multiplication by vectors in $\mathbb{R}^{n-1}$.

2.3 Spinors on hypersurfaces

In this section we want to describe how one can restrict a spinor on an n-dimensional Riemannian spin manifold (N, h) to a spinor on an oriented hypersurface M carrying the induced metric g. As, this restriction is local, we can assume — by restricting to a tubular neighborhood of M and using Fermi coordinates, i.e. normal coordinates in normal directions — that $N = M \times (a, b)$ and $h = g_s + ds^2$ where $s \in (a, b)$.

Conversely, given a family of metrics g_s and spinors $\varphi_s \in \Gamma(\Sigma^{g_s} M)$, $s \in (a, b)$, we want to obtain a spinor on $N := M \times (a, b)$, $h = g_s + ds^2$.

Note that this description does not include the passage from a Lorentzian manifold to a spacelike hypersurface and vice versa, described in the introduction, which plays an important role in the work of Baum, Leistner and Lischewski. For this Lorentzian version analogous techniques are well presented in the lecture notes [9, Sec. 1.5].

For the Riemannian case, which is the goal of this section, a standard reference is [6]. However for the purpose of our article it seems to be more efficient to choose some other convention, following e.g. [1, Sec. 5.3]. To have a fluently readable summary, we do not include detailed proofs in this section. For self-containedness we include them in Appendix B.

In the following let $T_M N$ be the vertical bundle of the projection $N \to (a, b)$, i.e. $T_M N = \coprod_{s \in (a,b)} T(M \times \{s\})$, viewed as a bundle over N.

For each $s \in I := (a, b)$ let $P_{\mathrm{SO}}(M, g_s)$ be the $\mathrm{SO}(n-1)$-principal bundle of positively oriented orthornormal frames on (M, g_s), and let $P_{\mathrm{SO}}(N)$ be defined as the corresponding $\mathrm{SO}(n)$-bundle over (N, h). We note $\nu :=$

$\partial/\partial s$. The union of the bundles $P_{\mathrm{SO}}(M, g_s)$ is an $\mathrm{SO}(n-1)$-principal bundle over $N = M \times I$, which we will denoted by $P_{\mathrm{SO},M}(N) \to N$ (whose topology and bundle structure is induced by the $\mathrm{SO}(n-1)$-reduction of the $\mathrm{GL}(n-1)$-principal bundle of all frames over N containing ν). We get a map $P_{\mathrm{SO},M}(N) \to P_{\mathrm{SO}}(N)$, mapping $(e_1, \ldots, e_{n-1})$ to $(e_1, \ldots, e_{n-1}, \nu)$ which yields an isomorphism $P_{\mathrm{SO}}(N) \cong P_{\mathrm{SO},M}(N) \times_{\mathrm{SO}(n-1)} \mathrm{SO}(n)$. Any topological spin structure on M yields a $\mathrm{Spin}(n-1)$-principal bundle $P_{\mathrm{Spin},M}(N)$ with a $\mathrm{Spin}(n-1) \to \mathrm{SO}(n-1)$-equivariant map $P_{\mathrm{Spin},M}(N) \to P_{\mathrm{SO},M}(N)$ in the usual way. This map induces

$$P_{\mathrm{Spin}}(N) \coloneqq P_{\mathrm{Spin},M}(N) \times_{\mathrm{Spin}(n-1)} \mathrm{Spin}(n) \to P_{\mathrm{SO}}(N) \cong P_{\mathrm{SO},M}(N) \times_{\mathrm{SO}(n-1)} \mathrm{SO}(n)$$

which is a spin structure on $N = M \times I$. In particular this induces a bijection from the set of spin structures on M (up to isomorphism) to spin structures on N (up to isomorphism).

The complex volume elements $\omega_{n-1}^{\mathbb{C}} \in \mathbb{C}\mathrm{l}_{n-1}$ and $\omega_n^{\mathbb{C}} \in \mathbb{C}\mathrm{l}_n$ then provide associated complex volume elements

$$\omega_M^{\mathbb{C}} = \epsilon_{n-1} e_1 \cdot e_2 \cdot \ldots \cdot e_{n-1} \in \mathbb{C}\mathrm{l}(TM) \text{ and } \omega_N^{\mathbb{C}} = \epsilon_n e_1 \star e_2 \star \ldots \star e_n \in \mathbb{C}\mathrm{l}(TN),$$

where $(e_1, e_2, \ldots)$ is any positively oriented basis in TM resp. TN.

Using the associated bundle construction, we obtain for n odd:

$$\begin{aligned}
\Sigma_M N &\coloneqq P_{\mathrm{Spin},M}(N) \times_{\mathrm{Spin}(n-1)} \Sigma_{n-1} = \coprod_{s \in (a,b)} \Sigma^{g_s} M, \\
\Sigma N &\coloneqq P_{\mathrm{Spin}}(N) \times_{\mathrm{Spin}(n)} \Sigma_n, \\
\Sigma^{\#} N &\coloneqq P_{\mathrm{Spin}}(N) \times_{\mathrm{Spin}(n)} \Sigma_n^{\#}.
\end{aligned}$$

The grading $\Sigma_{n-1} = \Sigma_{n-1}^+ \oplus \Sigma_{n-1}^-$ yields a grading $\Sigma_M N = (\Sigma_M N)^+ \oplus (\Sigma_M N)^-$ which is the grading by eigenvalues ± 1 of $\omega_M^{\mathbb{C}}$.

For n even we have

$$\begin{aligned}
\Sigma_M N &\coloneqq P_{\mathrm{Spin},M}(N) \times_{\mathrm{Spin}(n-1)} \Sigma_{n-1} = \coprod_{s \in (a,b)} \Sigma^{g_s} M, \\
\Sigma_M^{\#} N &\coloneqq P_{\mathrm{Spin},M}(N) \times_{\mathrm{Spin}(n-1)} \Sigma_{n-1}^{\#} = \coprod_{s \in (a,b)} \Sigma^{\#, g_s} M, \\
\Sigma N &\coloneqq P_{\mathrm{Spin}}(N) \times_{\mathrm{Spin}(n)} \Sigma_n,
\end{aligned}$$

and the grading $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ yields a grading $\Sigma N = (\Sigma N)^+ \oplus (\Sigma N)^-$, given by eigenvalues ± 1 of $\omega_N^{\mathbb{C}}$.

The Clifford multiplications $\mathbb{R}^{n-1} \otimes \Sigma_{n-1}^{(\#)} \to \Sigma_{n-1}^{(\#)}$, $\mathbb{R}^n \otimes \Sigma_n^{(\#)} \to \Sigma_n^{(\#)}$, induce Clifford multiplications $\cdot : T_M N \otimes \Sigma_M^{(\#)} N \to \Sigma_M^{(\#)} N$, $\star : TN \otimes \Sigma^{(\#)} N \to \Sigma^{(\#)} N$. As indicated before we use the symbols "$\cdot$" and "$\star$" to distinguish properly between the Clifford multiplication of M and the one of N.

Furthermore the maps J_n, $J_n^{\#}$, $J_n^{\pm}$, and $J_n^{\#,\pm}$ from Lemma 2 induce vector bundle maps over $\mathrm{id} : N \to N$ which are isometric injective $\mathbb{C}$-linear maps in each fiber. For n odd, we obtain fiberwise isomorphisms

$$J : \Sigma_M N \to \Sigma N \text{ and } J^{\#} : \Sigma_M N \to \Sigma^{\#} N.$$

These maps commute with Clifford multiplication by vectors tangent to M. They satisfy

$$\nu \star J(\varphi) = iJ_n(\omega_M^{\mathbb{C}} \cdot \varphi), \tag{10}$$
$$\nu \star J^{\#}(\varphi) = -iJ^{\#}(\omega_M^{\mathbb{C}} \cdot \varphi). \tag{11}$$

for all $\varphi \in \Sigma_M N$.

On the other hand we get for n even

$$J^{\pm} : \Sigma_M N \to \Sigma N \text{ and } J^{\#,\pm} : \Sigma_M^{\#} N \to \Sigma N.$$

These maps *do not* commute with Clifford multiplication by vectors tangent to M, and they *are not* surjective in any fiber. However, they satisfy

$$\nu \star J^{\pm}(\varphi) = \pm iJ^{\pm}(\varphi), \tag{12}$$
$$\nu \star J^{\#,\pm}(\varphi) = \pm iJ^{\#,\pm}(\varphi). \tag{13}$$

for all $\varphi \in \Sigma_M N$ and for any $\pm \in \{+,-\}$.

The bundle maps defined above do not preserve the (partially defined) Levi-Civita connections on the bundles. They modify the connection by terms depending on the second fundamental form of $M \times \{s\}$ in N. This is made precise in the following lemma.

Lemma 4. *Let $J^{(\#),(\pm)}$ be one of the bundle maps defined above. For $X \in \Gamma(T_M N)$, and $\varphi \in \Gamma(\Sigma_M N)$ resp. $\Gamma(\Sigma_M^{\#} N)$ we have*

$$\nabla_X^N J^{(\#),(\pm)}(\varphi) = J^{(\#),(\pm)}(\nabla_X^{M,g_s}\varphi) + \frac{1}{2} W(X) \star \nu \star J^{(\#),(\pm)}(\varphi), \tag{14}$$

where the Weingarten map W is defined through $g(W(X),Y)\nu = \nabla_X^N Y - \nabla_X^M Y$.

A proof will be given in Appendix B.

In combination with equations (10), (11), (12), and (13) we will see later in this article that families of (M, g_s)-parallel spinors will lead to solutions of the generalized imaginary Killing spinor equation (3) which is one of the constraint equations.

2.4 BBGM connection

Let M be a compact spin manifold. We denote the space of all Riemannian metrics on M by $\mathcal{M}$. For every metric $g \in \mathcal{M}$ we define

$$\mathcal{F}_g = \Gamma(\Sigma^g M),$$

and the disjoint union

$$\mathcal{F} := \coprod_{g \in \mathcal{M}} \mathcal{F}_g.$$

One can equip $\mathcal{F}$ and $\mathcal{M}$ naturally with the structure of a Fréchet bundle $\pi^{\mathcal{F}} : \mathcal{F} \to \mathcal{M}$. This bundle structure is needed to define a connection on this bundle. The oldest references that we used are by Bourguignon and Gauduchon [14] resp. by Bär, Gauduchon and Moroianu [6], this is why we choose the abbreviation BBGM for the four names. However we were told that there was

also work by Bismut. The concepts were later properly formalized under the name 'universal spinor bundle' in [4] and [31], where a finite-dimensional fiber bundle with a partial connection is constructed whose sections correspond to the elements of $\mathcal{F}$ such that the parallel transport corresponds to the BBGM parallel transport. The connection is given in terms of horizontal spaces $\mathcal{H}_{(g,\varphi)}$, i.e. vector spaces satisfying

$$T_{(g,\varphi)}\mathcal{F} = \mathcal{H}_{(g,\varphi)} \oplus \Gamma(\Sigma^g M)$$

in the sense of topological vector spaces such that

$$d\pi^{\mathcal{F}}|_{(g,\varphi)} : T_{(g,\varphi)}\mathcal{F} \to T_g\mathcal{M} = \Gamma(\bigodot^2 T^*M)$$

restricts to an isomorphism $I_{(g,\varphi)} : \mathcal{H}_{(g,\varphi)} \to \Gamma(\bigodot^2 T^*M)$. In other words we obtain an injective map of Fréchet spaces $L_{(g,\varphi)} : \Gamma(\bigodot^2 T^*M) \to T_{(g,\varphi)}\mathcal{F}$ by postcomposing $(I_{(g,\varphi)})^{-1}$ with the inclusion of $\mathcal{H}_{(g,\varphi)} \subset T_{(g,\varphi)}\mathcal{F}$.

The space $\mathcal{H}_{(g,\varphi)}$ will smoothly depend on (g,φ) and will be compatible with the vector bundle structure of $\mathcal{F}$. We do not require more knowledge about Fréchet manifolds in our article, so we do not introduce this Fréchet structure in more detail.

To describe the horizontal space precisely we give the maps $L_{(g,\varphi)}$: We assume that $h \in T_g\mathcal{M} = \Gamma(\bigodot^2 T^*M)$. Let g_t be a smooth path of metrics such that $g_0 = g$ and $\frac{d}{dt}|_{t=0} g_t = h$. We consider the cylinder $Z = M \times [0,1]$ with the metric $g_t + dt^2$. Then, similar to [6] we can extend the spinor $\varphi_0 := \varphi$, defined on $M \times \{0\}$, uniquely to a spinor $\varphi_\bullet$ on $M \times [0,1]$ satisfying $\nabla_{\partial_t}\varphi_\bullet = 0$ where ∇ denotes the Levi-Civita connection on the cylinder. Note that $\varphi_\bullet$ may be interpreted as a family, parametrized over $[0,1]$, of spinor fields φ_t with respect to the family of metrics g_t. The restriction of $\varphi_\bullet$ to $M \times \{t\}$, denoted by φ_t is a spinor for the metric g_t, and $t \mapsto \varphi_t$ is a smooth path in $\mathcal{F}$. We now define

$$L_{(g,\varphi)}(h) = \frac{d}{dt}\Big|_{t=0} \varphi_t.$$

Lemma 5. *The map $L_{(g,\varphi)}$ is well-defined, linear, and smooth in (g,φ).*

The lemma follows from the construction of the universal spinor bundle given in [31]. The strategy in that paper is as follows: Let $\pi^S : SM \to M$ be the bundle of positive definite symmetric bilinear forms on TM. A complex vector bundle $\pi^{\mathrm{univ}} : \Sigma M \to SM$ is constructed, called the *universal spinor bundle*, which carries a scalar product, a Clifford multiplication with vectors in TM and partial connection on ΣM with respect to $\pi^S : SM \to M$. The Clifford multiplication is given by a bilinear map $\mathrm{cl}_g : T_{\pi^S(g)}M \times \Sigma_g M \to \Sigma_g M$ for every $g \in SM$. Note that g is a scalar product on one single tangent space, namely on $T_{\pi^S(g)}M$, and thus Σ_g is thus a fiber of the vector bundle $\pi^{\mathrm{univ}} : \Sigma M \to SM$. This Clifford multiplication shall satisfy the Clifford relations and shall depend smoothly on g. By "partial connection" we mean that for any section φ of the bundle $\pi^{\mathrm{univ}} : \Sigma M \to SM$, the covariant derivative $\nabla_X \varphi$ is defined for some $X \in T(SM)$ if and only $d\pi^S(X) = 0$ (i.e., X is vertical for π^S). This partial connection comes from the vertical connection defined in [31, Definition 2.10], and allows to define a map $L_{(g,\varphi)}$ as above. In particular this shows that $\frac{d}{dt}|_{t=0}\varphi_t$ does not depend on how we choose g_t, but only its derivative a $t = 0$.

3 Variation of the parallel spinor equation

Let us first recall a result from [4]: For $(g,\varphi) \in \mathcal{F}$ let $\hat{\kappa}_{g,\varphi} : \mathcal{H}_{(g,\varphi)} \to \Gamma(\Sigma^g M \otimes T^*M)$ be the map defined by

$$\hat{\kappa}_{g,\varphi}(h) \quad := \quad \tfrac{1}{4} \sum_{i \neq j} (\nabla^g_{e_i} h)(\,\cdot\,, e_j) e_i \cdot e_j \cdot \varphi. \tag{15}$$

In [4, Sec. 4.3] a similar map was defined, namely $\kappa_{g,\varphi} : T_{(g,\varphi)}\mathcal{F} = \mathcal{H}_{(g,\varphi)} \oplus \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M \otimes T^*M)$ which is related to $\hat{\kappa}_{g,\varphi}$ by the formula $\kappa_{g,\varphi}(h,\psi) = \hat{\kappa}_{g,\varphi}(h) + \nabla^g \psi$. It was shown in [4, Lemma 4.12] that this is related to the differential of the map $K_X : \mathcal{F} \to \mathcal{F}$, $(g,\varphi) \mapsto (g, \nabla^g_X \varphi)$ as follows:

$$(d_{(g,\varphi)} K_X)(h,\psi) = (h, \kappa_{g,\varphi}(h,\psi)(X)) = (h, \hat{\kappa}_{g,\varphi}(h)(X) + \nabla^g_X \psi). \tag{16}$$

We define the Wang map $\mathcal{W}_{g,\varphi}$ as the composition

$$\bigodot^2 T^*M \hookrightarrow \bigotimes^2 T^*M \xrightarrow{\mathrm{cl}^g(\,\cdot\,,\varphi)\otimes \mathrm{id}} \Sigma^g M \otimes T^*M$$

so that

$$\mathcal{W}_{g,\varphi}(h) = (X \mapsto \sum_{j=1}^n h(X, e_j) e_j \cdot \varphi).$$

It is easy to see (c. f. [15, Lemma 2.3, 1.]) that

$$|\mathcal{W}_{g,\varphi}(h)| = |\varphi| \cdot |h|$$

and

$$\nabla_X \mathcal{W}_{g,\varphi}(h) = \mathcal{W}_{g,\varphi}(\nabla_X h) + \mathcal{W}_{g,\nabla_X \varphi}(h).$$

Lemma 6. *If φ is a parallel spinor on (M,g), then the diagram*

$$\begin{array}{ccc} \bigodot^2 T^*M & \xrightarrow{\mathcal{W}_{g,\varphi}} & \Gamma(\Sigma^g M \otimes T^*M) \\ & \searrow^{4\hat{\kappa}_{g,\varphi} + \varphi \otimes \mathrm{div}^g} & \downarrow{\scriptstyle D^{T^*M}} \\ & & \Gamma(\Sigma^g M \otimes T^*M) \end{array}$$

commutes. In other words we have for any $h \in \bigodot^2 T^*M$

$$D^{T^*M} \circ \mathcal{W}_{g,\varphi}(h) = 4\hat{\kappa}_{g,\varphi}(h) + \varphi \otimes \mathrm{div}^g(h),$$

where, for a vector bundle E over M with connection we define the Dirac operator D^E on $\Sigma^g M \otimes E$ by $\mathrm{cl}^g \circ \nabla^{\Sigma^g M \otimes E}$ where $\nabla^{\Sigma^g M \otimes E}$ is the product connection.

Proof. Let $h \in \Gamma(\bigodot^2 T^*M)$. Then $\mathcal{W}_{g,\varphi}(h) = \sum_{j=1}^n h(\,\cdot\,, e_j) e_j \cdot \varphi$. We now assume that $(e_1, \ldots, e_n)$ is a local orthonormal frame satisfying $\nabla e_j|_p = 0$ at $p \in M$ for

$1 \le j \le n$, and we calculate in p

$$\begin{aligned} D^{T^*M}\left(\mathcal{W}_{g,\varphi}(h)\right) &= \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\left(\mathcal{W}_{g,\varphi}(h)\right) \\ &= \sum_{i,j=1}^{n} (\nabla_{e_i} h)(\,\cdot\,, e_j) e_i \cdot e_j \cdot \varphi \\ &= 4\hat{\kappa}_{g,\varphi}(h) + \sum_{i=1}^{n} (\nabla_{e_i} h)(\,\cdot\,, e_i) e_i \cdot e_i \cdot \varphi \\ &= 4\hat{\kappa}_{g,\varphi}(h) + \varphi \otimes \underbrace{\sum_{i=1}^{n} (-\nabla_{e_i} h)(\,\cdot\,, e_i)}_{=\mathrm{div}^g h}. \end{aligned}$$

□

Very similarly we prove for arbitrary sections φ and h:

$$\begin{aligned} D^{T^*M}\left(\mathcal{W}_{g,\varphi}(h)\right)(X) = 4\hat{\kappa}_{g,\varphi}(h)(X) + (\mathrm{div}^g h)(X)\varphi \\ - 2\nabla_{h^\#(X)}\varphi - (h^\#(X)) \cdot D^g \varphi, \end{aligned} \tag{17}$$

where $h^\# \in \mathrm{End}(TM)$ is defined as $h^\#(X) := h(\,\cdot\,, X)^\#$.

The following lemma is a straightforward generalization of a Lemma by McKenzie Wang [39, Lemma 3.3 (d) for $k = 1$], see also [15, Prop. 2.4].

Lemma 7. *For $\varphi \in \Gamma(\Sigma^g M)$ and $h \in \Gamma(\odot^2 T^*M)$ we have*

$$\begin{aligned} (D^{T^*M})^2 \mathcal{W}_{g,\varphi}(h) = \mathcal{W}_{g,\varphi}(\Delta_E h) - 2\sum_{\ell} \mathcal{W}_{g,\nabla_{e_\ell}\varphi}(\nabla_{e_\ell} h) + \mathcal{W}_{g,D^2\varphi}(h) \\ - 2(X \mapsto \sum_{\ell} h^\sharp(e_\ell) \cdot R_{X,e_\ell}\varphi). \end{aligned}$$

In particular, if φ is a parallel spinor, then the diagram

$$\begin{array}{ccc} \Gamma(\odot^2 T^*M) & \xrightarrow{\mathcal{W}_{g,\varphi}} & \Gamma(\Sigma^g M \otimes T^*M) \\ \Big\downarrow{\scriptstyle \Delta_E} & & \Big\downarrow{\scriptstyle (D^{T^*M})^2} \\ \Gamma(\odot^2 T^*M) & \xrightarrow{\mathcal{W}_{g,\varphi}} & \Gamma(\Sigma^g M \otimes T^*M) \end{array}$$

commutes.

Proof. We can locally write

$$\mathcal{W}_{g,\varphi}(h) = \sum_{i,j} h_{ij} e_j \cdot \varphi \otimes e_i^*$$

where $\{e_1, \dots e_n\}$ is a local orthonormal frame with respect to g with $\nabla e_i = 0$ at p. We define $\nabla_k h_{ij} := (\nabla_{e_k} h)(e_i, e_j)$, $\nabla^2_{k,l} h_{ij} := (\nabla^2_{e_k,e_l} h)(e_i, e_j)$, and let

$R_{e_k,e_l}h_{ij} = \nabla^2_{k,l}h_{ij} - \nabla^2_{l,k}h_{ij}$ be the associated curvature. By using the Clifford relations, we get on the domain of the frame:

$$\begin{aligned}
D^{T^*M}(\mathcal{W}_{g,\varphi}(h)) &= \sum_{i,j,k} (\nabla_k h_{ij} e_k \cdot e_j \cdot \varphi + h_{ij} e_k \cdot e_j \cdot \nabla_{e_k}\varphi) \otimes e_i^* \\
&= \sum_{i,j,k} (\nabla_k h_{ij} e_k \cdot e_j \cdot \varphi - h_{ij} e_j \cdot e_k \cdot \nabla_{e_k}\varphi - 2h_{ij}\delta_{kj}\nabla_{e_k}\varphi) \otimes e_i^* \\
&= \underbrace{\sum_{i,j,k} \nabla_k h_{ij} e_k \cdot e_j \cdot \varphi \otimes e_i^*}_{=:(\mathrm{A})} + \underbrace{(-\mathcal{W}_{g,D\varphi}(h))}_{=:(\mathrm{B})} + \underbrace{(-2\sum_{i,j} h_{ij}\nabla_{e_j}\varphi \otimes e_i^*)}_{=:(\mathrm{C})}
\end{aligned}$$

Now we apply D^{T^*M} to this equation and we use the notation $\psi = D\varphi$. Then we get from the previous equation at the point p

$$\begin{aligned}
D^{T^*M}((\mathrm{B})) &= -D^{T^*M}(\mathcal{W}_{g,\psi}(h)) \\
&= -\sum_{i,j,k} \nabla_k h_{ij} e_k \cdot e_j \cdot \psi \otimes e_i^* + \mathcal{W}_{g,D\psi}(h) + 2\sum_{i,j} h_{ij}\nabla_{e_j}\psi \otimes e_i^*.
\end{aligned}$$

Moreover,

$$\begin{aligned}
D^{T^*M}((\mathrm{C})) &= -2\sum_{i,j,k} (h_{ij} e_k \cdot \nabla^2_{e_k,e_j}\varphi + \nabla_k h_{ij} e_k \cdot \nabla_{e_j}\varphi) \otimes e_i^* \\
&= -2\sum_{i,j,k} (h_{ij} e_k \cdot \nabla^2_{e_j,e_k}\varphi + h_{ij} e_k \cdot R_{e_k,e_j}\varphi + \nabla_k h_{ij} e_k \cdot \nabla_{e_j}\varphi) \otimes e_i^* \\
&= -2\sum_{i,j} (h_{ij}\nabla_{e_j}\psi + \sum_k h_{ij} e_k \cdot R_{e_k,e_j}\varphi + \sum_k \nabla_k h_{ij} e_k \cdot \nabla_{e_j}\varphi) \otimes e_i^*
\end{aligned}$$

and

$$D^{T^*M}((\mathrm{A})) = \underbrace{\sum_{i,j,k,l} \nabla^2_{lk} h_{ij} e_l \cdot e_k \cdot e_j \cdot \varphi \otimes e_i^*}_{=:(\mathrm{E})} + \underbrace{\sum_{i,j,k,l} \nabla_k h_{ij} e_l \cdot e_k \cdot e_j \cdot \nabla_{e_l}\varphi \otimes e_i^*}_{=:(\mathrm{F})}.$$

By using the relation $e_l \cdot e_k \cdot e_j = e_k \cdot e_j \cdot e_l + 2e_k\delta_{lj} - 2\delta_{kl}e_j$,

$$(\mathrm{F}) = \sum_{i,j,k} (\nabla_k h_{ij} e_k \cdot e_j \cdot \psi + 2\nabla_k h_{ij} e_k \cdot \nabla_{e_j}\varphi - 2\nabla_k h_{ij} e_j \cdot \nabla_{e_k}\varphi) \otimes e_i^*.$$

By adding up, a lot of terms cancel and we are left with

$$\begin{aligned}
(D^{T^*M})^2(\mathcal{W}_{g,\varphi}(h)) = (\mathrm{E}) + \mathcal{W}_{g,D^2\varphi}(h) - 2\sum_{i,j,k} \nabla_k h_{ij} e_j \cdot \nabla_{e_k}\varphi \otimes e_i^* \\
+ 2\sum_{i,j,k} h_{ij} e_k \cdot R_{e_j,e_k}\varphi \otimes e_i^*
\end{aligned}$$

so it remains to consider the term (E). We have

$$\begin{aligned}
(\mathrm{E}) &= -\sum_{i,j,k} \nabla^2_{kk} h_{ij} e_j \cdot \varphi \otimes e_i^* + \sum_{\substack{i,j,k,l \\ k\neq l}} \nabla^2_{lk} h_{ij} e_l \cdot e_k \cdot e_j \cdot \varphi \otimes e_i^* \\
&= \sum_{i,j} (\nabla^*\nabla h)_{ij} e_j \cdot \varphi \otimes e_i^* + \sum_{i,j,k,l} \frac{1}{2} R_{e_k,e_l} h_{ij} e_k \cdot e_l \cdot e_j \cdot \varphi \otimes e_i^*.
\end{aligned}$$

By the formula which expresses the curvature of $T^*M \otimes T^*M$ in terms of the curvature of TM, we obtain $R_{e_k,e_l} h_{ij} = -\sum_r (R_{klir} h_{rj} + R_{kljr} h_{ir})$. Using the relation $e_k \cdot e_l \cdot e_j = e_j \cdot e_k \cdot e_l - 2e_k \delta_{jl} + 2e_l \delta_{jk}$, we get

$$\begin{aligned}
-\frac{1}{2} \sum_{i,j,k,l,r} & R_{klir} h_{rj} e_k \cdot e_l \cdot e_j \varphi \otimes e_i^* \\
&= -\frac{1}{2} \sum_{i,j,k,l,r} R_{klir} h_{rj} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* \\
&\quad + \sum_{i,k,l,r} (R_{klir} h_{rl} e_k - R_{klir} h_{rk} e_l) \cdot \varphi \otimes e_i^* \\
&= -\frac{1}{2} \sum_{i,j,k,l,r} R_{klir} h_{rj} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* - 2 \sum_{i,j} (\mathring{R} h)_{ij} e_j \cdot \varphi \otimes e_i^*
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{2} \sum_{i,j,k,l,r} & R_{kljr} h_{ir} e_k \cdot e_l \cdot e_j \cdot \varphi \otimes e_i^* = \\
&= -\frac{1}{2} \sum_{i,j,k,l,r} R_{kljr} h_{ir} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* \\
&\quad + \sum_{i,k,l,r} (R_{kllr} h_{ir} e_k - R_{klkr} h_{ir} e_l) \cdot \varphi \otimes e_i^* \\
&= \quad -\frac{1}{2} \sum_{i,j,k,l,r} R_{kljr} h_{ir} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* + 2 \sum_{i,j,r} \mathrm{Ric}_{jr}\, h_{ir} e_j \cdot \varphi \otimes e_i^*.
\end{aligned}$$

We get

$$\begin{aligned}
(D^{T^*M})^2 (\mathcal{W}_{g,\varphi}(h)) = \mathcal{W}_{g,\varphi}\Big(\underbrace{\nabla^* \nabla h - 2\mathring{R} h}_{= \Delta_E h} \Big) & - \frac{1}{2} \sum_{i,j,k,l,r} R_{klir} h_{rj} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* \\
& - \frac{1}{2} \sum_{i,j,k,l,r} R_{kljr} h_{ir} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* \\
& + 2 \sum_{i,j,r} \mathrm{Ric}_{jr}\, h_{ir} e_j \cdot \varphi \otimes e_i^* + \mathcal{W}_{g,D^2\varphi}(h) \\
& - 2 \sum_{i,j,k} \nabla_k h_{ij} e_j \cdot \nabla_{e_k} \varphi \otimes e_i^* + 2 \sum_{i,j,k} h_{ij} e_k \cdot R_{e_j,e_k} \varphi \otimes e_i^*.
\end{aligned}$$

From the standard identities

$$R_{e_i,e_j} \varphi = \frac{1}{4} \sum_{k,l} R_{ijkl} e_k \cdot e_l \cdot \varphi, \qquad \sum_i e_i \cdot R_{e_i,e_j} \varphi = \frac{1}{2} \mathrm{Ric}(e_j) \cdot \varphi,$$

we deduce

$$\begin{aligned}
-\frac{1}{2} \sum_{i,j,k,l,r} R_{klir} h_{rj} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* &= -2 \sum_{i,j,r} h_{rj} e_j \cdot R_{e_i,e_r} \varphi \otimes e_i^* \\
-\frac{1}{2} \sum_{i,j,k,l,r} R_{kljr} h_{ir} e_j \cdot e_k \cdot e_l \cdot \varphi \otimes e_i^* &= -2 \sum_{i,j,r} h_{ri} e_j \cdot R_{e_j,e_r} \varphi \otimes e_i^* \\
&= -\sum_{i,j,r} h_{ri}\, \mathrm{Ric}_{jr}\, e_j \cdot \varphi \otimes e_i^* \\
2 \sum_{i,j,k} h_{ij} e_k \cdot R_{e_j,e_k} \varphi \otimes e_i^* &= -\sum_{i,j,k} h_{ij}\, \mathrm{Ric}_{jk}\, e_k \cdot \varphi \otimes e_i^*
\end{aligned}$$

which yields the final result. □

4 The BBGM parallel transport preserves parallel spinors

Proposition 8. *Let $I \subset \mathbb{R}$ be an interval, $(g_s)_{s\in I}$ be a path of Ricci-flat metrics with a divergence free derivative, $0 \in I$, and let φ_0 be a parallel spinor on (M, g_0). Let, for all $s \in I$, φ_s be a spinor on (M, g_s) such that $\frac{d}{ds}\varphi_s = 0$. Then φ_s is a parallel spinor on (M, g_s) for all $s \in I$.*

Here $\frac{d}{ds}\varphi_s = 0$ should be understood as a derivation with respect to the BBGM connection.

In the proof of Proposition 8, we need a statement about continuous dependence of eigenvalues of the Dirac operator on the metric.

Theorem 9. *There exists a family of functions $\lambda_j : \mathcal{M} \to \mathbb{R}$, $j \in \mathbb{Z}$ such that*

- $\{\lambda_j(g)\}_{j\in\mathbb{Z}} = \operatorname{spec}(D_g)$ *for all* $g \in \mathcal{M}$
- *The family is nondecreasing, i.e. $\lambda_i \le \lambda_j$ whenever $i \le j$.*
- $\sup_{j\in\mathbb{Z}} |\operatorname{arsinh}(\lambda_j(g)) - \operatorname{arsinh}(\lambda_j(\tilde{g})| \le C \|g - \tilde{g}\|_{C^1}$

The theorem in the above version is proven in detail in [33, Main Theorem 2], but the statement we need was well-known long before.

To prove the proposition, we will at first prove the same statement under more restrictive assumptions.

Lemma 10. *Let $(g_s)_{s\in I}$ be a path of Ricci-flat metrics with a divergence free derivative, $0 \in I$, and let φ_0 be a parallel spinor on (M, g_0). Assume that $\dim\ker D^{T^*M}$ is constant along I. Let φ_s be a spinor on (M, g_s) such that $\frac{d}{ds}\varphi_s = 0$. Then φ_s is a parallel spinor on (M, g_s) for all $s \in I$.*

Proof of Lemma 10. Without loss of generality we can assume that I is compact. Using Rayleigh-Ritz type arguments, explained e.g. in detail in [33], one can show the existence of continuous functions $I \ni s \mapsto \lambda_i(s) \in \mathbb{R}$, $\lambda_i(s) \le \lambda_{i+1}(s)$ such that $(\lambda_i(s))_{i\in\mathbb{Z}}$ are the eigenvalues (including the correct multiplicity) of D^{T^*M}. If the kernel of D^{T^*M} has constant dimension k, then we can assume $\lambda_0(s) < \lambda_1(s) = \cdots = \lambda_k(s) = 0 < \lambda_{k+1}(s)$. Then for

$$\gamma := \min_{s\in I} \min\{-\lambda_0(s), \lambda_{k+1}(s)\} > 0$$

the operator $D^{T^*M}_{g_s}$ has no eigenvalue in $(-\gamma, 0) \cup (0, \gamma)$. Because of this, there are bounded operators

$$\left(D^{T^*M}_{g_s}\right)^{-1} : L^2_{g_s} \cap \ker(D^{T^*M}_{g_s})^\perp \to L^2_{g_s} \cap \ker(D^{T^*M}_{g_s})^\perp \tag{18}$$

that invert $D^{T^*M}_{g_s}$ on the orthogonal complement of the kernel and that are uniformly bounded by γ^{-1}. By elliptic theory, see e.g. [26, Chapter III §5] we have isomorphisms

$$D^{T^*M}_{g_s} : H^1_{g_s} \cap \ker(D^{T^*M}_{g_s})^\perp \to L^2_{g_s} \cap \ker(D^{T^*M}_{g_s})^\perp. \tag{19}$$

Because of $\mathrm{im}(D^{T^*M}_{g_s}) \subset \ker(D^{T^*M}_{g_s})^\perp$, there is a constant $C_0 > 0$ such that for any $s \in I$ and for any $\sigma \in \Gamma(\Sigma^{g_s} M \otimes T^*M)$ we have

$$\|D^{T^*M}_{g_s}\sigma\|_{H^1_{g_s}} \leq C_0 \|\left(D^{T^*M}_{g_s}\right)^2 \sigma\|_{L^2_{g_s}}. \tag{20}$$

Moreover, $g_s' \in \ker(\Delta_{E,g_s})$ by the facts collected in Subsection 2.1. We calculate using Lemma 7

$$\begin{aligned}(D^{T^*M}_{g_s})^2 \mathcal{W}_{g_s,\varphi_s}(g_s') = &-2\sum_\ell \mathcal{W}_{g_s,\nabla_{e_\ell}\varphi_s}(\nabla_{e_\ell} g_s') + \mathcal{W}_{g_s,D^2\varphi_s}(g_s') \\ &- 2(X \mapsto \sum_\ell (g_s')^\sharp(e_\ell)\cdot R_{X,e_\ell}\varphi_s),\end{aligned}$$

thus pointwise $(D^{T^*M}_{g_s})^2 \mathcal{W}_{g_s,\varphi_s}(g_s')$ is a linear expression in $\nabla\varphi_s$ and its first derivative. Thus

$$\left\|\left(D^{T^*M}_{g_s}\right)^2 \mathcal{W}_{(g_s,\varphi_s)}(g_s')\right\|_{L^2} \leq C\|\nabla\varphi_s\|_{H^1},$$

where C depends on $\sup_{s\in I}\|g_s'\|_{C^1(g_s)}$. This implies, using Eq. (20):

$$\left\|D^{T^*M}_{g_s}\mathcal{W}_{(g_s,\varphi_s)}(g_s')\right\|_{H^1} \leq CC_\gamma\|\nabla\varphi_s\|_{H^1}. \tag{21}$$

We now differentiate $\nabla^{g_s}\varphi_s$ using formulae (16) and (17) and Lemma 6.

$$\begin{aligned}\frac{d}{ds}\nabla^{g_s}\varphi_s &= \hat{\kappa}_{g_s,\varphi_s}(g_s') + \nabla^{g_s}\underbrace{\varphi_s'}_{=0} \\ &= \frac{1}{4}\left(D^{T^*M}_{g_s}\left(\mathcal{W}_{(g_s,\varphi_s)}(g_s')\right) - \varphi_s \otimes \underbrace{\mathrm{div}^{g_s} g_s'}_{=0} + 2\nabla_{(g_s')^\#(\cdot)}\varphi_s + (g_s')^\#(\cdot)\cdot D^{g_s}\varphi_s\right) \\ &= \frac{1}{4}D^{T^*M}_{g_s}\mathcal{W}_{(g_s,\varphi_s)}(g_s') + \frac{1}{2}\nabla_{(g_s')^\#(\cdot)}\varphi_s + \frac{1}{4}(g_s')^\#(\cdot)\cdot D^{g_s}\varphi_s.\end{aligned}$$

This implies, with Eq. 21:

$$\left\|\frac{d}{ds}\nabla^{g_s}\varphi_s\right\|_{H^1(g_s)} \leq \tilde{C}\left(\|\nabla^{g_s}\varphi_s\|_{H^1(g_s)} + \underbrace{\|D^{g_s}\varphi_s\|_{H^1(g_s)}}_{\leq\sqrt{n}\|\nabla^{g_s}\varphi_s\|_{H^1(g_s)}}\right) \leq C_1\|\nabla^{g_s}\varphi_s\|_{H^1(g_s)}$$

where $\tilde{C}$ and C_1 depend on $\sup_{s\in I}\|g_s'\|_{C^1(g_s)}$.

We set $\ell(s) := \|\nabla^{g_s}\varphi_s\|^2_{H^1(g_s)}$. In order to derive $\ell(s)$ one has to be aware that also the evaluating metric depends on s. The dependence on the metrics effects $\ell(s)$ in the metric contractions, the volume element and in the covariant derivatives used to define by $\|\sigma\|^2_{H^1(g)} = \|\sigma\|^2_{L^2(g)} + \|\nabla^g\sigma\|^2_{L^2(g)}$. Due to compactness of I, these effects lead to a number $C_2 > 0$, constant in s, such that

$$|\ell'(s) - 2\langle\frac{d}{ds}\nabla^{g_s}\varphi_s, \nabla^{g_s}\varphi_s\rangle_{H^1(g_s)}| \leq C_2\ell(s)$$

and we get

$$\ell'(s) \leq 2|\langle\frac{d}{ds}\nabla^{g_s}\varphi_s, \nabla^{g_s}\varphi_s\rangle_{H^1(g_s)}| + C_2\ell(s) \leq (2C_1 + C_2)\,\ell(s).$$

As $\ell(0) = 0$ this implies with Grönwall's inequality that $\ell(s)$ vanishes for all s. □

Remark 11. In the proof above the derivative $\frac{d}{ds}\nabla^{g_s}\varphi_s$ should be taken with some care. Here we derive an s-dependent family $\nabla^{g_s}\varphi_s \in \Gamma(T^*M \otimes \Sigma^{g_s}M)$ with respect to the metric variation given by s. This is the BBGM-derivative in the spinorial part. On the cotangential part, one could also use a BBGM-kind of derivative, but this is not what was used. On the cotangential part, we simply used the derivative in the usual sense, i.e. the derivative of a curve in the vector space $\Gamma(T^*M)$.

Proof of Proposition 8. In a first step we prove the Proposition for analytic families g_s, $s \in I$ of Ricci-flat metrics, in other words we assume that the map $I \to \bigodot^2 T^*M$ defined by g_s is analytic.

Then $D_{g_s}^{T^*M}$ is an analytic family of operators. This implies that the eigenvalues $\lambda_i(s)$ of $D_{g_s}^{T^*M}$ can be numbered such that $s \mapsto \lambda_i(s)$ is analytic in s [21, Appendix A]. Due to Theorem 9, only finitely many $\lambda_i(s)$ have zero sets on a compact interval. Let $\mu := \min \dim\ker D_{g_s}^{T^*M}$, and $J := \{s \in I \mid \dim\ker D_{g_s}^{T^*M} > \mu\}$ which is a closed discrete subset due to analyticity. (We conjecture that $J = \varnothing$, but we were unable to prove it.) Let I_0 be a connected component of $I \setminus J$. The previous lemma states that the Bourguignon-Gauduchon parallel transport along $s \mapsto g_s$ maps parallel spinor $\varphi \in \Gamma(\Sigma^{g_s}M)$ to a parallel spinor $P_{s,r}(\varphi) \in \Gamma(\Sigma^{g_r}M)$ for any $s, r \in I_0$. Let $\bar{r} \in \partial I_0$. Then by continuity,

$$P_{s,\bar{r}}(\varphi) = \lim_{r \to \bar{r}} P_{s,r}(\varphi)$$

is a parallel section of $\Gamma(\Sigma^{g_{\bar{r}}}M)$. We use the fact that the dimension of the space of parallel spinors is locally constant (see [3]), thus for any $s \in I_0$ and and $r \in \bar{I}_0$ (including $r \in \partial I_0$) the monomorphism $P_{s,r}$ from parallel spinors in $\Gamma(\Sigma^{g_s}M)$ to parallel spinors in $\Gamma(\Sigma^{g_r}M)$ is an isomorphism. Thus BBGM parallel transport perserves parallel spinors along $\bar{I}_0$, and by an induction argument (using that J is finite in compact intervals) this also holds along I. The proposition is thus proven for analytic families $s \mapsto g_s$ and thus also for piecewise analytic families $s \mapsto g_s$.

Now, as the Riemannian metrics form an open cone in the vector space of bilinear forms, an arbitrary smooth family $s \mapsto g_s$ can be approximated by piecewise analytic paths in the C^1-norm. We claim that the BBGM parallel transport is continuous in this limit. This can be seen most easily in the universal spinor bundle formulation: There, for SM being the bundle of symmetric positive definite bilinear forms, a Clifford bundle $\pi^{\mathrm{univ}} : \Sigma M \to SM$ is constructed such that, for each $g \in \Gamma(SM)$, $\Sigma^g M$ is isomorphic as a Clifford bundle to $g^*\Sigma M$. Furthermore, the vector bundle $\pi^{\mathrm{univ}}\colon \Sigma M \to SM$ carries a vertical (w.r.t. $SM \to M$) covariant derivative ∇ whose parallel transport P is linked to the BBGM parallel transport $\hat{P}$ as follows: Let $g_\bullet : [0, \ell] \to \Gamma(SM)$, $s \mapsto g_s$ be a C^1-curve of Riemannian metrics on M. Let $p \in M$ and let $g|_p : s \mapsto g_s(p)$ and $\sigma \in (\pi^{\mathrm{univ}})^{-1}(g_0(p)) = \Sigma_p^{g_0}M$, then for any $\tilde{\sigma} \in \Gamma(\Sigma^{g_0}M)$ with $\tilde{\sigma}|_p = \sigma$ we have

$$\hat{P}_{g_\bullet}(\tilde{\sigma})\big|_p = P_{g|_p}(\sigma).$$

The parallel transport along a curve with respect to a connection is given by a first order ordinary differential equation (ODE), satisfying the conditions of the theorem of Picard-Lindelöf. Using the universal spinor bundle formalism we argued that the BBGM-parallel transport is given by such an ODE and

that its coefficient functions converge uniformly (i.e. in the C^0-norm) when a path of metrics converges in the C^1-norm to a limit path of metrics.[2] Thus the theorem of Picard-Lindelöf, taking into account that both M and $[0,\ell]$ are compact, implies that the BBGM parallel transport converges uniformly when we approximate a smooth path of metrics $s \mapsto g_s$ by piecewise analytic paths of metrics in the C^1-norm. Thus the BBGM parallel transport also preserves parallel spinors along the smooth family $s \mapsto g_s$. □

5 Construction of solutions to the constraint equations

In this section we want to use the BBGM connection to construct solutions of the constraint equation on suitable manifolds of the form $(M \times I, g_s + ds^2)$ where I is an interval and where M is an $m = (n-1)$-dimensional manifold.

Assume that g_s is a family of Ricci-flat metrics with divergence-free derivative and that for some s_0 there is a non-trivial parallel spinor φ_{s_0} on (M, g_{s_0}). By rescaling we can achieve that its norm is 1 in every point. We shift it in the s-direction parallely with the BBGM parallel transport. By Prop. 8 we obtain a family φ_s of g_s-parallel spinors of constant norm 1 on M, for every $s \in I$. This yields a fiberwise parallel section of $\Sigma_M^{(\#)} N$. Recall $\nu = \frac{\partial}{\partial s}$.

In the following we assume that $f : I \to \mathbb{R}$ is a given smooth function and choose $s_0 \in I$. We define $F(s) := \exp\left(-\frac{1}{2}\int_{s_0}^{s} f(\sigma)\, d\sigma\right)$, i.e. $-\frac{1}{2} f(s) = F'(s)/F(s)$.

Case n is odd, i.e. $m = \dim M$ even. By possibly changing the orientation and using Proposition 25 in Appendix C we can assume that all $\varphi_s \in \Gamma(\Sigma^{g_s} M)$ have the positive parity for the splitting given by the volume element of M, i.e. we can assume $\omega_{\mathbb{C}}^M \cdot \varphi_s = \varphi_s$. For every $(x,s) \in N = M \times I$ we use the map J from Section 2.3 to get identifications $\Sigma_x M \cong \Sigma_{M(x,s)} N \cong \Sigma_{(x,s)} N$. Again the Clifford multiplication for (M, g_s) will be denoted by $\cdot$, and for the one for (N, h), $h = g_s + ds^2$ we will use $\star$. From Equation (10) we get $\nu \star \varphi = i\varphi$. Using Lemma 4 we obtain

$$\begin{aligned} \nabla_\nu^N \varphi &= 0, \\ \nabla_X^N \varphi &= \frac{1}{2} W(X) \star \nu \star \varphi = \frac{i}{2} W(X) \star \varphi. \end{aligned}$$

We set

$$\Psi(x,s) := F(s)\varphi_s(x). \tag{22}$$

Assuming that X is tangent to M, we obtain

$$\begin{aligned} \nu \star \Psi &= i\Psi, \\ \nabla_\nu^N \Psi &= \frac{F'(s)}{F(s)}\Psi = -\frac{1}{2} f(s)\Psi = \frac{i}{2} f(s) \nu \star \Psi, \\ \nabla_X^N \Psi &= \frac{1}{2} W(X) \star \nu \star \Psi = \frac{i}{2} W(X) \star \Psi. \end{aligned}$$

[2] To be precise: we only need control of the norm

$$\|g_\bullet\| := \max_{s \in [0,\ell]} \left(\|g_s\|_{C^0(M)} + \|\frac{d}{ds} g_s\|_{C^0(M)} \right).$$

Thus Ψ is an imaginary $\overline{W}$-Killing spinor with $\overline{W} = W + f\nu^b \otimes \nu$, i.e. it satisfies (3).

The first relation and the defining equation (22) also imply $-i\langle \nu \star \Psi, \Psi\rangle = F^2$. For X tangent to M, the real part of $\langle X \star \Psi, \Psi\rangle$ vanishes and

$$\mathrm{Im}\langle X \star \Psi, \Psi\rangle = -\mathrm{Re}\langle X \star i\Psi, \Psi\rangle = -\mathrm{Re}\langle X \star \nu \star \Psi, \Psi\rangle \overset{(*)}{=} \mathrm{Re}\langle \Psi, X \star \nu \star \Psi\rangle,$$

where in $(*)$ we used the skew-symmetry of Clifford multiplication with vectors twice and $X \star \nu = -\nu \star X$. On the other hand one $\mathrm{Re}\langle X \star \nu \star \Psi, \Phi\rangle = \mathrm{Re}\langle \Phi, X \star \nu \star \Psi\rangle$, which in particular holds for $\Phi := \Psi$. We thus get $\mathrm{Im}\langle X \star \Psi, \Psi\rangle = 0$.

Thus $-i\langle X \star \Psi, \Psi\rangle = F^2 h(\nu, X)$ for every vector X and consequently, the Dirac current U_Ψ of Ψ, defined by $h(U_\Psi, X) = -i\langle X \star \Psi, \Psi\rangle$ for every vector X (see (1)) is $U_\Psi = F^2\nu$. We obtain

$$U_\Psi \star \Psi = iF^2\Psi,$$

which is the constraint equation (4) for $u_\Psi := \sqrt{h(U_\Psi, U_\Psi)} \equiv F^2$. Note that here we used (5) as definition for u_ψ. We thus have obtained solutions of the constraint equations.

Case n even, i.e. $m = \dim M$ odd. We use the map $J^{(\#),+}$ defined in Section 2.3 to view $\Sigma_x M \cong \Sigma_{M(x,s)} N$ as a subbundle of $\Sigma_{(x,s)} N$. In particular, $(x, s) \mapsto \varphi_s(x)$ then yields a section of constant length 1 of the bundle $\Sigma N \to N$.

Because of equations (12) and (13) we have $\nu \star \varphi = i\varphi$. Using Lemma 4 we obtain

$$\begin{aligned} \nabla^N_\nu \varphi &= 0 \\ \nabla^N_X \varphi &= \frac{1}{2} W(X) \star \nu \star \varphi = \frac{i}{2} W(X) \star \varphi \end{aligned}$$

We set

$$\Psi(x, s) := F(s)\varphi_s(x). \tag{23}$$

Then for X tangent to M,

$$\begin{aligned} \nu \star \Psi &= i\Psi, \\ \nabla^N_\nu \Psi &= \frac{F'(s)}{F(s)}\Psi = -\frac{1}{2} f(s)\Psi = \frac{i}{2} f(s)\nu \star \Psi, \\ \nabla^N_X \Psi &= \frac{1}{2} W(X) \star \nu \star \Psi = \frac{i}{2} W(X) \star \Psi. \end{aligned}$$

Thus, Ψ is an imaginary $\overline{W}$-Killing spinor with $\overline{W} = W + f\nu^b \otimes \nu$, i.e. it satisfies (3). With the same arguments as in the other case, we can prove that (1), (2), (4), and (5) are satisfied, i.e. we have found a solution to the constraint equations.

Example 12. *Let $M = T^{n-1}$, g be a flat metric on M and φ a parallel spinor on it.*

- *Let $g_s \equiv g$ with $s \in I$. The BBGM parallel transport leaves φ invariant and we obtain initial data to the Cauchy problem on the metric $g_s + ds^2$ on either $T^{n-1} \times \mathbb{R}$ or $T^{n-1} \times S^1 = T^n$. The Minkowski metric together with a parallel spinor (or in the S^1 case a $\mathbb{Z}$-quotient of it) is then a solution of the associated Cauchy problem.*

- *Let $g_s = e^{2s}g$ with $s \in \mathbb{R}$. For the submanifolds $M \times \{s\} \subset N$ we have $W = -\mathrm{id}|_{TM}$ for alle $s \in \mathbb{R}$. We take the function $f(s) = -1$, i.e. $F(s) := \exp(s/2)$ and $\overline{W} = -\mathrm{id}|_{TN}$. The metric we obtain is now the hyperbolic metric $h = e^{2s}g + ds^2$ on $T^{n-1} \times \mathbb{R}$ together with an imaginary Killing spinor with Killing constant $-i/2$. The Lorentzian cone $(T^{n-1} \times \mathbb{R} \times \mathbb{R}_{>0}, r^2h - dr^2)$, where $r \in \mathbb{R}_{>0}$, together with a parallel spinor solves solves the associated Cauchy problem. Note that this cone is the quotient by a $\mathbb{Z}^{n-1}$-action of $I_+(0) \subset \mathbb{R}^{n,1}$, defined as the set of all future-oriented time-like vectors in the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n,1}$.*

Remark 13. In this example we have seen two different ways of reconstructing Lorentzian Ricci-flat metrics on quotients of subsets of Minkowski space together with a parallel spinor. However, our construction allows many more interesting examples.

Example 14.

- *Any imaginary Killing spinor on a complete, connected Riemannian spin manifold arises this way. This was proven by Baum, Friedrich, Grunewald and Kath in [7, Chap. 7], more precisely in Theorem 1 on page 160 and Cor. 1 on page 167 [7, Chap. 7].*
- *This was generalized by Rademacher [34], see also [18, Theorem A.4.5]. Rademacher proved that any generalized imaginary Killing spinor with $W = \alpha\,\mathrm{id}$, $\alpha \in C^\infty(N, \mathbb{R})$ arises by our construction.*
- *Our construction generalizes previous constructions of imaginary Killing spinors, as e.g. [19, Prop. 4.6 and Cor. 4.7].*

6 From curves in the moduli space to initial data sets

Let $\mathrm{Diff}_0(M)$ be the identity component of the diffeomorphism group of M, acting on the space $\mathcal{M}_\parallel(M)$ of structured Ricci-flat metrics by pullback (as usual, a semi-Riemannian manifold (M, g) is called structured iff its semi-Riemannian universal covering $(\tilde{M}, \tilde{g})$ admits a parallel spinor). Furthermore, let $\mathcal{M}od_\parallel(M) := \mathcal{M}_\parallel(M)/\mathrm{Diff}_0(M)$ be the associated premoduli space. It was shown in [3] (using previous work about the special case of simply connected manifolds with irreducible holonomy) that $\mathcal{M}od_\parallel(M)$ "naturally" carries the structure of a finite-dimensional smooth manifold. The smooth structure on $\mathcal{M}od_\parallel(M)$ can be described by the following properties

- $\mathcal{M}od_\parallel(M)$ carries the quotient topology, if we equip $\mathcal{M}_\parallel(M)$ and $\mathrm{Diff}_0(M)$ with the standard Fréchet topology,
- any smooth family $N \to \mathcal{M}_\parallel(M)$, $y \mapsto g_y$ yields a smooth map $N \to \mathcal{M}od_\parallel(M)$, $y \mapsto [g_y]$,
- in the case $N = (a, b)$ one has $\frac{d}{ds}|_{s=s_0}[g_s] = 0$ if and only if $\frac{d}{ds}|_{s=s_0} g_s$ is tangent to the $\mathrm{Diff}_0(M)$-orbit through g_{s_0}.

If I is an interval, then any smooth curve $I \to \mathcal{M}od_\parallel(M)$ can be written as $s \mapsto [g_s]$ with a family of metrics $\operatorname{div}^{g_{s_0}}\left(\frac{d}{ds}|_{s=s_0} g_s\right) = 0$, see Appendix D.

The main results of this article is a procedure to construct initial data for the constraint equations for Lorentzian metrics with parallel spinors.

Main Construction 15 (Initial data on an open manifold)**.** *Let I be an interval. For any smooth curve $I \to \mathcal{M}od_\parallel(M)$ $s \mapsto [g_s]$ and every positive smooth function $F : I \to \mathbb{R}_{>0}$ we obtain a solution of the initial data equations* (3) *and* (4) *on $M \times I$ with norm $F(t)$ at any $(x,t) \in M \times I$.*

Here the metric on $M \times I$ is given by $g_s + ds^2$ provided that g_s is chosen such that $\operatorname{div}^{g_{s_0}} \frac{d}{ds}|_{s=s_0} g_s = 0$ and the spinor is given by (22) resp. (23). In particular the spinor can be normalized such that it has norm $F(t)$ at any $(x,t) \in M \times I$. As derived in the preceding section, (3) are then satisfied, as well as (1), (2), (4), and (5).

The situation is slightly more complicated if we want to obtain solutions of the constraint equations on a closed manifold. We start with a closed curve $S^1 \cong \mathbb{R}/L\mathbb{Z} \to \mathcal{M}od_\parallel(M)$. If we identify $\mathbb{R}/L\mathbb{Z}$ with $[0,L]/0 \sim L$, then every curve $\mathbb{R}/L\mathbb{Z} \to \mathcal{M}od_\parallel(M)$ can be written as $[0,L] \to \mathcal{M}od_\parallel(M)$, $s \mapsto [g_s]$ with $\operatorname{div}\left(\frac{d}{ds} g_s\right) = 0$, but in general we will have $g_0 \neq g_L$ although (M, g_0) and (M, g_L) are isometric with respect to an isometry $\zeta \in \operatorname{Diff}_0(M)$. We then glue $(M \times \{0\}, g_0)$ with $(M \times \{L\}, g_L)$ isometrically using the diffeomorphism $\zeta \in \operatorname{Diff}_0(M)$. This yields a closed Riemannian manifold (N, h) diffeomorphic to $M \times S^1$. In order to equip it with a spin structure we have to lift $d\zeta^{\otimes n} : P_{\mathrm{SO}}(M, g_0) \to P_{\mathrm{SO}}(M, g_L)$ to a map between the corresponding spin structures $\zeta_\# : P_{\mathrm{Spin}}(M, g_0) \to P_{\mathrm{Spin}}(M, g_L)$. This yields a spin structure and a spinor bundle on N. Let $F : S^1 \to \mathbb{R}_{>0}$ be given. For any parallel spinor φ_0 on (M, g_0) equation (22) resp. equation (23) yields a generalized imaginary Killing spinor Ψ on $M \times [0, L]$ as in Main Construction 15. The gluing described above allows to view $\varphi_L := \Psi_{M \times \{L\}}$ as a parallel spinor on (M, g_0). However, in general φ_L will differ from φ_0. Let $\Gamma_\parallel(\Sigma_{g_0} M)$ denote the space of parallel spinors on (M, g_0). Then $\varphi_0 \mapsto \varphi_L$ yields a unitary map $P : \Gamma_\parallel(\Sigma_{g_0} M) \to \Gamma_\parallel(\Sigma_{g_0} M)$. The map P does neither depend on F nor on the parametrization of the curve $s \mapsto [g_s]$. We say that (M, g_s, φ_0) satisfies the *fitting condition* if $P(\varphi_0) = \varphi_0$ for a suitable choice of spin structure on N. The fitting condition is always satisfied in the following cases:

(1) (M, g_0) is a 7-dimensional manifold with holonomy G_2

(2) (M, g_0) is an 8-dimensional manifold with holonomy $\mathrm{Spin}(7)$

(3) (M, g_0) is a Riemannian product of manifolds of that kind and of at most one factor diffeomorphic to S^1.

(4) finite quotients of such manifolds

As this statement is not within the core of this article, we only sketch the proof. In the first two cases the spinor bundle $\Sigma_g M$ is the complexification of the real spinor bundle $\Sigma_g^{\mathbb{R}} M$, and thus $\Gamma_\parallel(\Sigma_g M) = \Gamma_\parallel(\Sigma_g^{\mathbb{R}} M) \otimes \mathbb{C}$. The real spinor representations of G_2 and $\mathrm{Spin}(7)$ on $\Sigma_n^{\mathbb{R}}$ have a 1-dimensional invariant subrepresentation, thus $\dim_{\mathbb{R}} \Gamma_\parallel(\Sigma_g^{\mathbb{R}} M) = 1$. Thus P is either $+\mathrm{id}$ or $-\mathrm{id}$, and

the +-sign can be achieved by a suitable choice of the lift $\zeta_\#$. On S^1 it follows from a direct calculation.

The map P behaves "well" under taking products and finite quotients, thus the other two statements follow as well.

For manifolds with a least one factor of holonomy $\mathrm{SU}(k)$ or $\mathrm{Sp}(k)$, or also for tori of dimension > 1, however, we expect that generically the fitting condition does not hold. In this case, we expect that the space of closed paths $s \to [g_s]$ for wich P has finite order (in the sense $\exists \ell \in \mathbb{N} : P^\ell = \mathrm{id}$) is dense in the space of all closed paths $s \to [g_s]$ with respect to the C^∞-topology. This is in fact a consequence of work in progress by Bernd Ammann, Klaus Kröncke and Hartmut Weiß.

Then passing to an ℓ-fold cover of N obtained from running along the path $s \to [g_s]$ not just once, but ℓ times, we obtain a solution of the constraint equation on $M \times S^1$.

Main Construction 16 (Initial data on a closed manifold)**.** *Let $L > 0$. Let $\mathbb{R}/L\mathbb{Z} \to \mathcal{M}od_\Vdash(M)$, $[s] \mapsto [g_s]$ be a smooth path and let $\varphi_0 \in \Gamma_\Vdash(\Sigma^{g_0} M)$ be given such that (M, g_s, φ_0) satisfies the fitting condition. Then for any function $F : S^1 \to \mathbb{R}_{>0}$ we obtain a solution of the initial data equations* (3) *and* (4) *on $M \times S^1$.*

It also seems interesting to us to allow a slight generalization of our initial problem, by considering spin^c spinors with a flat associated line bundle instead of spinors in the usual sense. Assume that $\vartheta \in \mathbb{C}$ has norm 1. Identifying $(v, t) \in \mathbb{C}\times\mathbb{R}$ with $(\vartheta v, t+L)$ yields a complex line bundle L_ϑ over $S^1 = \mathbb{R}/L\mathbb{Z}$. On L_ϑ we choose the connection such that local sections with constant v are parallel. By pull back we obtain complex line bundles with flat, metric connections on $N = M \times S^1$ and on $\overline{N} = N \times (-\epsilon, \epsilon)$. These line bundles will also be denoted by L_ϑ. The bundle $\Sigma_h N \otimes L_\vartheta$ resp. $\Sigma_{\bar{h}}\overline{N} \otimes L_\vartheta$ is then a spin^c-spinor bundle with flat associated bundle L_ϑ. The objects tensored by L_ϑ will be called L_ϑ-twisted. All the results of this article immediately generalize to L_ϑ-twisted spinors. We ask for L_ϑ-twisted parallel spinors, i.e. parallel sections of $\Sigma_{\bar{h}}\overline{N} \otimes L_\vartheta$ instead of parallel spinors in the usual sense. This leads to L_ϑ-twisted constraint equations, and the L_ϑ-twisted Cauchy problem can be solved the same way as the untwisted.

Let $P \in \mathrm{U}\big(\Gamma_\Vdash(\Sigma_{g_0} M)\big)$ be as above. As P is unitary, there is a basis of $\Gamma_\Vdash(\Sigma_{g_0} M)$ consisting of eigenvectors of P for complex eigenvalues of norm 1.

Main Construction 17 (Spin^c-version)**.** *Let $L > 0$. Let $\mathbb{R}/L\mathbb{Z} \to \mathcal{M}od_\Vdash(M)$, $[s] \mapsto [g_s]$ be a smooth path. Let $\varphi_0 \in \Gamma_\Vdash(\Sigma_{g_0} M)$ be an eigenvalue of P to the eigenvalue ϑ. Then for any function $F : S^1 \to \mathbb{R}_{>0}$ we obtain a solution of the L_ϑ-twisted version of the constraint equations* (3) *and* (4) *on $M \times S^1$.*

A Independence of the constraint equations

In this appendix we want to show that the constraint equations (1)–(5) presented in the introduction are not independent equations. We will show that all of them follow from (3) and a rewritten version of (4). In particular, we will see that for a generalized imaginary Killing spinor φ equation (4) implies (2), unless the vector field U_φ vanishes everywhere. In the introduction (1)–(5) are a mixture

of definitions and relations. Let us rewrite them in a form which is more suitable to clarify their dependences.

We assume that (N, h) is a connected Riemannian spin manifold. Let $\Sigma N \to N$ be the associated spinor bundle. Compared to the introduction we slightly simplify our notation: we write $\cdot$ here for the Clifford multiplication instead of writing $\star$ which was used in the introduction in order to distinguish it from other Clifford multiplications.

In the following we have a spinor φ, i.e. a smooth section $\varphi \in \Gamma(\Sigma N)$, a real-valued smooth function $u \in C^\infty(N)$, a vector field $U \in \Gamma(TN)$ and an endomorphism $W \in \Gamma(TN)$. Be aware that we simplify again the notation, by writing W for the endomorphism which was called $\overline{W}$ in the introduction. The equations (1)–(5) turn into

$$h(U,X) = -i\langle X \cdot \varphi, \varphi\rangle, \qquad \forall X \in TN,, \tag{24}$$

$$i\varphi(p) \in \{V \cdot \varphi \mid V \in T_pN\}, \qquad \forall p \in N, \tag{25}$$

$$\nabla^N_X \varphi = \frac{i}{2} W(X) \cdot \varphi, \qquad \forall X \in TN, \tag{26}$$

$$U \cdot \varphi = iu\,\varphi, \tag{27}$$

$$u^2 = h(U,U). \tag{28}$$

We want to discuss the independence of the constraint equations (24) to (28). We start with some elementary lemmata.

Lemma 18. *If* (27) *is satisfied for U, u and φ defined in some $p \in N$. Then we have (in this $p \in N$):*

$$\varphi = 0 \text{ or } h(U,U) = u^2$$

Proof.

$$h(U,U)\,\langle\varphi,\varphi\rangle = \langle -U \cdot U \cdot \varphi, \varphi\rangle = \langle U \cdot \varphi, U \cdot \varphi\rangle = \langle iu\,\varphi, iu\,\varphi\rangle = u^2\,\langle\varphi,\varphi\rangle.$$

□

Lemma 19. *Let $p \in N$ and let* (27) *be satisfied for $U \in T_pN$, $u \in \mathbb{R}$ and $\varphi \in \Sigma_g N_p$. We assume that U_φ is defined by* (1)*, i.e.* (24) *holds for U_φ instead of U at the point p. Then U and U_φ are linearly dependent.*

In that sense (27) implies (24) up to a constant. Obviously for globally defined U, u and φ, the proportionality factor does not have to be constant. We obtain $\lambda_1 U = \lambda_2 U_\varphi$ for some nowhere vanishing function $\lambda : M \to \mathbb{R}^2$.

Proof. W.l.o.g. $U \neq 0$, $\varphi \neq 0$ at $p \in N$. By Lemma 18 it follows that $u(p) \neq 0$. We calculate for $X \perp U$:

$$\begin{aligned} 0 &= -2h(X,U)\,\langle\varphi,\varphi\rangle = \langle X \cdot U \cdot \varphi, \varphi\rangle + \langle U \cdot X \cdot \varphi, \varphi\rangle \\ &= \langle X \cdot U \cdot \varphi, \varphi\rangle + \langle \varphi, X \cdot U \cdot \varphi\rangle = 2\,\mathrm{Re}\,\langle X \cdot U \cdot \varphi, \varphi\rangle. \end{aligned}$$

Furthermore

$$\langle X \cdot U \cdot \varphi, \varphi\rangle = \langle iu\,X \cdot \varphi, \varphi\rangle = iu\,\langle X \cdot \varphi, \varphi\rangle = -u\,h(U_\varphi, X).$$

This implies $h(U_\varphi, X) = 0$. □

Lemma 20 (In [9, Lemma 5 in Sec. 5.2]). *Assume U and φ satisfy* (24). *Then* (25) *is equivalent to $h(U,U) = \|\varphi\|^4$.*

Proof. We prove the statement in each $p \in N$, so we consider $U \in T_pN$ and $\varphi \in \Sigma_pN$. W.l.o.g. $\varphi \neq 0$. If (e_j) is an orthonormal basis of T_pN, then (ρ_j) with $\rho_j := \frac{1}{\|\varphi\|} e_j \cdot \varphi$ is an orthonormal basis of $E_\varphi := \{V \cdot \varphi \mid V \in T_pN\}$. We write

$$i\varphi = W \cdot \varphi + \psi$$

for some $W \in T_pN$ and $\psi \perp \{V \cdot \varphi \mid V \in T_pN\}$. Thus

$$\langle i\varphi, \rho_j \rangle = \frac{i}{\|\varphi\|} \langle \varphi, e_j \cdot \varphi \rangle = \frac{1}{\|\varphi\|} h(U, e_j).$$

We conclude

$$h(U,U) = \sum_{j=1}^{n} h(U,e_j)^2 = \|\varphi\|^2 \sum_{j=1}^{n} \langle i\varphi, \rho_j \rangle^2 = \|\varphi\|^2 \left(\|i\varphi\|^2 - \|\psi\|^2 \right).$$

This implies that $h(U,U) = \|\varphi\|^4$ if and only if $\psi = 0$. □

Now let $W \in \mathrm{End}(TN)$, not necessarily symmetric. Recall that our notation is slightly simplified if compared to the introduction: the $\overline{W}$ of the introduction is W in this appendix.

Proposition 21 (Dichotomy Proposition). *Let (N,h) be a connected Riemannian manifold and let $W \in \Gamma(\mathrm{End}(TN))$ be a field of endomorphisms. We assume that $U = U_\varphi$ for φ satisfy* (24), (26) *and* (27) *for some $u \in C^\infty(N)$.*

Then $W^T(U_\varphi) = -\mathrm{grad}\langle \varphi, \varphi \rangle$. If $U_\varphi \equiv 0$, then $\|\varphi\|$ is constant. If $U_\varphi \not\equiv 0$, then U_φ and φ vanish nowhere and $u = \|U_\varphi\| = \langle \varphi, \varphi \rangle = \|\varphi\|^2$.

Here W^T denotes the endomorphism in $\mathrm{End}(TN)$ adjoint to W.

Let us compare this propositition to a similar statement by H. Baum and Th. Leistner. In the case that W is symmetric, it yields a criterion implying (27).

Lemma 22 ([9, Lemma 5 in Sec. 5.2]). *Let (N,h) be a connected Riemannian manifold with a non-zero spinor field $\varphi \in \Gamma(\Sigma N)$ and a field of symmetric endomorphisms $W \in \Gamma(\mathrm{End}(TN))$ satisfying* (26), *and let $U = U_\varphi$ be defined by* (24). *Then we have $\nabla U_\varphi = -\|\varphi\|^2 W$. Furthermore $q_\varphi := \|\varphi\|^4 - h(U_\varphi, U_\varphi)$ is non-negative and constant. Moreover, if we define ψ as in the proof of Lemma 20, then $q_\varphi = \|\varphi\|^2 \cdot \|\psi\|^2$. If $q_\varphi = 0$, then* (27) *holds for $u := \|U_\varphi\| = \|\varphi\|^2$.*

Note that in the case of an imaginary Killing spinor, i.e. $W = \mu\,\mathrm{id}$, then q_φ is related to the constant Q_φ defined in [7] for any twistor spinor by $Q_\varphi = n^2\mu^2 q_\varphi$. Note that imaginary Killing spinors are both twistor spinors and generalized imaginary Killing spinors, but there are generalized imaginary Killing spinors, which are not twistor spinors and vice versa. According to [7, Chap. 7] an imaginary Killing spinor is of type I, if and only if $Q_\varphi = 0$. Otherwise it is of type II. Any complete Riemannian manifold carrying a type II imaginary Killing spinor (with $\mu \neq 0$) is homothetic to the hyperbolic space [7, Sec. 7.2]. If it is of type I, then it arise from a warped product construction as in our Section 5, see [7, Sec. 7.3].

Proof of Proposition 21. First we compute

$$\begin{aligned} X\langle\varphi,\varphi\rangle = \langle\nabla_X\varphi,\varphi\rangle + \langle\varphi,\nabla_X\varphi\rangle &= \frac{i}{2}\left(\langle W(X)\cdot\varphi,\varphi\rangle - \langle\varphi,W(X)\cdot\varphi\rangle\right) \\ &= i\langle W(X)\cdot\varphi,\varphi\rangle = -h(U_\varphi,W(X)) \\ &= -h(W^T(U_\varphi),X). \end{aligned}$$

With $X\langle\varphi,\varphi\rangle = h(\mathrm{grad}\langle\varphi,\varphi\rangle,X)$ we obtain $\mathrm{grad}\langle\varphi,\varphi\rangle = -W^T(U_\varphi)$. Obviously this implies that $\|\varphi\|$ is constant in the case $U_\varphi \equiv 0$.

We now consider the case $U_\varphi \not\equiv 0$. Note that Equation (26) is equivalent to saying that φ is a parallel section for the connection $\overline{\nabla}_X\varphi := \nabla^N_X\varphi - \frac{i}{2}W(X)\cdot\varphi$. This implies: If we have some $p \in M$ with $\varphi(p) = 0$, then φ has to vanish on all of N, and thus $U_\varphi \equiv 0$, the case already solved. So let us assume $\varphi(p) \neq 0$ for all $p \in M$. Note that (27) implies

$$\|U_\varphi\| = \frac{\|U_\varphi\cdot\varphi\|}{\|\varphi\|} = \frac{\|iu\varphi\|}{\|\varphi\|} = |u|.$$

We calculate

$$\begin{aligned} u^2\langle\varphi,\varphi\rangle = \langle iu\,\varphi, iu\,\varphi\rangle &= \langle U_\varphi\cdot\varphi, U_\varphi\cdot\varphi\rangle \\ &= h(U_\varphi,U_\varphi)\langle\varphi,\varphi\rangle \\ &= -i\langle U_\varphi\cdot\varphi,\varphi\rangle\langle\varphi,\varphi\rangle \\ &= -i\langle iu\,\varphi,\varphi\rangle\langle\varphi,\varphi\rangle \\ &= u\langle\varphi,\varphi\rangle^2, \end{aligned}$$

i.e. at each point $p \in N$ we have $u(p) = 0$ or $u(p) = \langle\varphi(p),\varphi(p)\rangle = \|\varphi(p)\|^2$. The sets $\{p \in N \mid u(p) = 0\}$ and $\{p \in N \mid u(p) = \|\varphi(p)\|^2\}$ are closed and disjoint, thus the connectedness of N implies that either $u \equiv 0$ or $u \equiv \langle\varphi,\varphi\rangle$. In the case $u \equiv 0$ we obtain $U_\varphi \equiv 0$, and we are again back in the case already solved. So we conclude $u \equiv \langle\varphi,\varphi\rangle = \|\varphi\|^2 > 0$ and thus $u = \|U_\varphi\|$. So everything is proven. □

Now we discuss our main case of interest, i.e. that φ is a generalized imaginary Killing spinor which is by definition a solution of $\nabla_X\varphi = \frac{i}{2}W(X)\cdot\varphi\ \forall X \in TN$ with $W \in \mathrm{End}(TN)$ symmetric. We assume N to be connected and $\varphi \not\equiv 0$ which implies as we have seen that φ vanishes nowhere. According to Lemma 22 we obtain the equation $\nabla U_\varphi = -\|\varphi\|^2 W$ and the fact that $q_\varphi := \|\varphi\|^4 - \|U_\varphi\|^2$ is a non-negative constant. In the case $q_\varphi = 0$ (denoted by "type I" in [9, Sec. 5.2]) we know further that (27) holds for $u = \|U_\varphi\|$.

Corollary 23. *Let (N,h) be a connected Riemannian spin manifold. Let φ be a generalized imaginary Killing spinor, i.e. a solution of* (26) *for a field of symmetric endomorphisms W. Let again $U = U_\varphi$ be defined by* (24)*. We assume $W \not\equiv 0$, thus $\nabla U_\varphi \not\equiv 0$ and hence $U_\varphi \not\equiv 0$. Then the following are equivalent:*

(a) *φ is of type I,*

(b) *φ satisfies $\|U_\varphi\| = \|\varphi\|^2$,*

(c) *φ satisfies equation* (2) *(or equivalently* (25)*),*

(d) *Condition* (27) *holds for* $u := \|U_\varphi\|$.

Proof.
(a)$\Longleftrightarrow$ (b) holds by definition of "type I".
(b)$\Longleftrightarrow$ (c) is stated in Lemma 20.
(b)$\Longrightarrow$ (d) is part of the statement of Lemma 22.
(d)$\Longrightarrow$ (b) is part of the statement of Proposition 21. □

We have seen, in particular, that for a generalized imaginary Killing spinor φ equation (27) implies (25) for $U = U_\varphi$ defined by (24), unless the vector field U_φ and the endomorphism W vanish everywhere.

B More on Hypersurfaces

In this appendix we prove Lemma 4, i.e. formula (14).

It is known since long that one cannot restrict spinors to a hypersurface in a way preserving the connection. The difference of the connections depends on the second fundamental form or equivalently the Weingarten map. This effect is in some applications very helpful, e.g. in the case of surfaces in Euclidean space, where it leads to the spinorial version of the Weierstrass representation, see [17] for a good presentation or see [25] for an earlier, up to branching point aspects complete, but less conceptual publication, based on [24]. How to restrict spinors to hypersurfaces and the effect on the connection was already discussed in mathematical physics in the Riemannian [37, 38] and Lorentzian [40] context, and in spectral theory [5].

As different convention are used in the literature and as we follow, similar to [1, Prop. 5.3.1], another convention than the well-written exposition [6] we want to give a detailed proof of Lemma 4 in this appendix.

As in Subsection 2.3 and Appendix B we assume that (N, h) is an n-dimensional Riemannian spin manifold, $N = M \times (a, b)$, $h = g_s + ds^2$ for a family of metrics (g_s), $s \in (a, b)$ on M. Let ∇^{M,g_s} be the Levi-Civity connection of (M, g_s) and ∇^N the one of (N, h). We write $\nu := \partial/\partial s$ for the unit normal vector field of $M \times \{s\}$ in N. For $X \in TM \cong T(M \times \{s\}) \subset TN$, and $Y \in \Gamma(TM)$ we have

$$\nabla^N_X Y = \nabla^{M,g_s}_X Y + \mathrm{II}(X, Y), \qquad g_s(W(X), Y)\nu = \mathrm{II}(X, Y),$$

and we have, see e.g. [6, (4.1)] $g_s(W(X), Y) = -\frac{1}{2}(d/ds)\, g_s$. Note that the second fundamental form II has values in the normal bundle. Again, we use $\star$ for the Clifford multipication on N.

For concrete calculations we choose an open subset $U \subset M \times (a, b) = N$ and for every $(x, s) \in U$ we choose a positively oriented g_s-orthonormal basis $q_M = (e_1, \dots, e_{n-1})$ of $T_x M$, smoothly depending on x and s. Thus q_M is a local section of $P_{\mathrm{SO},M}(N)$. In other words: q_M is a frame of the vertical bundle of $N \to (a, b)$. Furthermore $q_N := (e_1, \dots, e_n)$, $e_n := \nu$ is a frame for (N, h), i.e. a local section of $P_{\mathrm{SO}}(N)$. We define the associated Christoffel symbols by

$$\nabla^{M,g_s}_{e_i} e_j = \sum_{k=1}^{n-1} \Gamma^{M\,k}_{ij} e_k, \qquad \nabla^N_{e_i} e_j = \sum_{k=1}^{n} \Gamma^{N\,k}_{ij} e_k.$$

Now let $\tilde{q}_M$ resp. $\tilde{q}_N$ be a spinorial lift of q_M resp. q_N, i.e. a local section of $P_{\mathrm{Spin},M}(N)$ resp. $P_{\mathrm{Spin}}(N)$, such that postcomposing $\tilde{q}_M$ resp. $\tilde{q}_N$ with

$P_{\mathrm{Spin},M}(N) \to P_{\mathrm{SO},M}(N)$ resp. $P_{\mathrm{Spin}}(N) \to P_{\mathrm{SO}}(N)$ yields q_M resp. q_N. On U we can write a spinor Φ, i.e. a section of $\Phi \in \Gamma(\Sigma^{(\#)}N)$ as $\Phi = [\tilde{q}_N, \sigma]$, with $\sigma : U \to \Sigma_n^{(\#)}$. We also may view as $\Sigma^{(\#)}N$ as an associated bundle to $P_{\mathrm{SO},M}(N)$, and with proper identifications we get $[\tilde{q}_N, \sigma] = [\tilde{q}_M, \sigma]$. The connection ∇^N defines the standard Levi-Civita connection on ΣN, again denoted by ∇^N. On the other hand, ∇^{M,g_s} defines a connection on $\Sigma N|_{M\times\{s\}}$.

Proposition 24. *For any $i = 1, \dots, n-1$ and $\Phi \in \Gamma(\Sigma^{(\#)}N)$ we have*

$$\nabla^N_{e_i}\Phi = \nabla^{M,g_s}_{e_i}\Phi + \frac{1}{2}\sum_{j=1}^{n-1} e_j \star \big(\mathrm{II}(e_i,e_j)\big) \star \Phi.$$

Proof. For any $i, j, k \in \{1, \dots, n-1\}$ we have

$$\Gamma^{N\,k}_{ij} = \Gamma^{M\,k}_{ij}, \qquad \Gamma^{N\,n}_{ij}\, e_n = -\Gamma^{N\,j}_{in}\, e_n = \mathrm{II}(e_i,e_j), \qquad \Gamma^{N\,n}_{in} = 0.$$

Let $(E_1, \dots, E_n)$ denote the canonical basis of $\mathbb{R}^n$. Writing the connection in local coordinates we obtain for $i = 1, 2, \dots, n-1$

$$\begin{aligned}
\big(\nabla^N_{e_i}[\tilde{q}_M,\sigma]\big) &= \left[\tilde{q}_M, \partial_{e_i}\sigma + \frac{1}{4}\sum_{j,k=1}^{n}\Gamma^{N\,k}_{ij} E_j \star E_k \star \sigma\right] \\
&= \underbrace{\left[\tilde{q}_M, \partial_{e_i}\sigma + \frac{1}{4}\sum_{j,k=1}^{n-1}\Gamma^{N\,k}_{ij} E_j \star E_k \star \sigma\right]}_{\nabla^{M,g_s}_{e_i}[\tilde{q}_M,\sigma]} + \underbrace{\left[\tilde{q}_M, \frac{1}{4}\sum_{j=1}^{n-1}\Gamma^{N\,n}_{ij} E_j \star E_n \star \sigma\right]}_{\frac{1}{4}\sum_{j=1}^{n-1} e_j\cdot\left(\mathrm{II}(e_i,e_j)\right)\cdot[\tilde{q}_M,\sigma]} \\
&\quad + \underbrace{\left[\tilde{q}_M, \frac{1}{4}\sum_{k=1}^{n-1}\Gamma^{N\,k}_{in} E_n \star E_k \star \sigma\right]}_{\frac{1}{4}\sum_{j=1}^{n-1} e_j\star\left(\mathrm{II}(e_i,e_j)\right)\star[\tilde{q}_M,\sigma]} \\
&= \nabla^{M,g_s}_{e_i}[\tilde{q}_M,\sigma] + \frac{1}{2}\sum_{j=1}^{n-1} e_j \star \big(\mathrm{II}(e_i,e_j)\big) \star [\tilde{q}_M,\sigma]. \quad \square
\end{aligned}$$

Obviously we have the following calculation in the Clifford algebra

$$\sum_{j=1}^{n-1} e_j \star \mathrm{II}(e_i,e_j) = \sum_{j=1}^{n-1} e_j \star g_s(W(e_i),e_j)\nu = W(e_i)\star\nu,$$

and thus we obtain for $X \in TM$ and $\Phi \in \Gamma(\Sigma^{(\#)}N)$ the equation

$$\nabla^N_X\Phi = \nabla^{M,g_s}_X\Phi + \frac{1}{2}W(X)\star\nu\star\Phi.$$

As the maps $J^{(\#),(\pm)}$ in Lemma 4 are constructed from an algebraic map using the associated bundle construction, they are ∇^{M,g_s}-parallel, i.e.

$$\nabla^{M,g_s}_X J^{(\#),(\pm)}\varphi = J^{(\#),(\pm)}(\nabla^{M,g_s}_X\varphi)$$

for all $X \in TM$. Now Lemma 4 follows immediately by setting $\Phi := J^{(\#),(\pm)}(\varphi)$.

C Change of orientation

In this appendix we will prove the following proposition.

Proposition 25. *Let (M,g) be a Riemannian manifold with an orientation $\mathcal{O}$, $m = \dim M$. Let $P_{\mathrm{Spin}}(M,g,\mathcal{O}) \to P_{\mathrm{SO}}(M,g,\mathcal{O})$ be a spin structure on $(M,g,\mathcal{O})$ with associated spinor bundle $\Sigma(M,g,\mathcal{O}) := P_{\mathrm{Spin}}(M,g,\mathcal{O}) \times_{\sigma_m} \Sigma_m$ resp. $\Sigma^{\#}(M,g,\mathcal{O}) := P_{\mathrm{Spin}}(M,g,\mathcal{O}) \times_{\sigma_m} \Sigma_m^{\#}$.*
Then there is a spin structure $P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \to P_{\mathrm{SO}}(M,g,-\mathcal{O})$ on $(M,g,-\mathcal{O})$ with the following properties:
Case m even: let $\Sigma(M,g,-\mathcal{O}) := P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \times_{\sigma_m} \Sigma_m$ be the associated spinor bundle, then there is a parallel isometric, complex linear bundle isomorphism $\Psi : \Sigma(M,g,-\mathcal{O}) \to \Sigma(M,g,\mathcal{O})$, commuting with Clifford multiplication. This map Ψ maps $(\Sigma(M,g,-\mathcal{O}))^+$ to $(\Sigma(M,g,\mathcal{O}))^-$ and $(\Sigma(M,g,-\mathcal{O}))^-$ to $(\Sigma(M,g,\mathcal{O}))^+$.
Case m odd: let $\Sigma^{(\#)}(M,g,-\mathcal{O}) := P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \times_{\tilde{\sigma}_m} \Sigma_m^{(\#)}$ be one of the associated spinor bundles. Then there is a parallel isometric, complex linear bundle isomorphism $\Psi : \Sigma(M,g,-\mathcal{O}) \to \Sigma^{\#}(M,g,\mathcal{O})$, commuting with Clifford multiplication. The same statement holds if we exchange the role of Σ_m and $\Sigma_m^{\#}$ in the definitions of $\Sigma(M,g,\pm\mathcal{O})$.

We define a map $\rho : P_{\mathrm{SO}}(M,g,\mathcal{O}-) \to P_{\mathrm{SO}}(M,g,\mathcal{O})$ by $\rho\big((e_1,e_2,\dots,e_m)\big) := (-e_1,e_2,\dots,e_m)$ for any $(-\mathcal{O})$-oriented orthonormal frame $\mathcal{E} := (e_1,\dots,e_m)$.

As a covering of smooth manifolds we define $P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \to P_{\mathrm{SO}}(M,g,-\mathcal{O})$ as the pullback of $P_{\mathrm{Spin}}(M,g,\mathcal{O}) \to P_{\mathrm{SO}}(M,g,\mathcal{O})$ by the diffeomorphism ρ. Let $\tilde{\rho} : P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \to P_{\mathrm{Spin}}(M,g,\mathcal{O})$ be the diffeomorphism defined by the following pull-back square:

$$\begin{array}{ccc} P_{\mathrm{Spin}}(M,g,-\mathcal{O}) & \xrightarrow{\tilde{\rho}} & P_{\mathrm{Spin}}(M,g,\mathcal{O}) \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}}(M,g,-\mathcal{O}) & \xrightarrow{\rho} & P_{\mathrm{SO}}(M,g,\mathcal{O}) \end{array}$$

However in order to get an appropriate structure as $\mathrm{Spin}(m)$-principal bundle on $P_{\mathrm{Spin}}(M,g,-\mathcal{O})$, some care is necessary, as ρ is not $\mathrm{SO}(m)$-equivariant. If we define $J := \mathrm{diag}(-1,1,1,1,...1)$ and abbreviate $\mathcal{E} := (e_1,\dots,e_m)$, then we have $\rho(\mathcal{E}) = \mathcal{E}\cdot J$ and thus $\rho(\mathcal{E}A) = \rho(\mathcal{E})J^{-1}AJ$.

Conjugation with $J := J^{-1}$ is a Lie group automorphism of $\mathrm{SO}(m)$ and lifts to $\mathrm{Spin}(m) \subset \mathrm{Cl}_m$, as conjugation with $E_1 := (1,0...,0)$ in the Clifford algebra sense. For any $\widetilde{\mathcal{E}} \in P_{\mathrm{Spin}}(M,g,-\mathcal{O})$ and $B \in \mathrm{Spin}(m)$ we define

$$\widetilde{\mathcal{E}}B := \tilde{\rho}^{-1}\Big(\tilde{\rho}(\widetilde{\mathcal{E}})\big(E_1 B(-E_1)\big)\Big).$$

This definition turns $P_{\mathrm{Spin}}(M,g,-\mathcal{O})$ into a $\mathrm{Spin}(m)$-principal bundle and the map $P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \to P_{\mathrm{SO}}(M,g,-\mathcal{O})$ is then $\mathrm{Spin}(m) \to \mathrm{SO}(m)$-equivariant.

Let as before $\sigma : \mathrm{Cl}_m \to \mathrm{End}(\Sigma_m^{(\#)})$ be an irreducible representation of the Clifford algebra, and let $\Sigma^{(\#)}(M,g,\mathcal{O}) := P_{\mathrm{Spin}}(M,g,\mathcal{O}) \times_{\sigma} \Sigma_m^{(\#)}$ with the obvious modifications for $-\mathcal{O}$.

Lemma 26 (Lift to the spinor bundle)**.** *The map*

$$P_{\mathrm{Spin}}(M,g,-\mathcal{O})\times\Sigma_m^{(\#)}\ni(\widetilde{\mathcal{E}},\varphi)\mapsto(\tilde{\rho}(\widetilde{\mathcal{E}}),\sigma(E_1)\varphi)\in P_{\mathrm{Spin}}(M,g,\mathcal{O})\times\Sigma_m^{(\#)}$$

is compatible with the equivalence relation given by σ. Thus it descends to a map

$$\rho_\#:\Sigma^{(\#)}(M,g,-\mathcal{O})=P_{\mathrm{Spin}}(M,g,-\mathcal{O})\times_\sigma\Sigma_m^{(\#)}\to\Sigma^{(\#)}(M,g,\mathcal{O}).$$

Proof. $(\widetilde{\mathcal{E}}B,\sigma(B^{-1})\varphi)$ is mapped to

$$\Big(\tilde{\rho}(\widetilde{\mathcal{E}}B),\sigma(E_1)\sigma(B^{-1})\varphi\Big)=\Big(\tilde{\rho}(\widetilde{\mathcal{E}})\big(E_1B(-E_1)\big),\sigma\big((E_1B(-E_1))^{-1}\big)\sigma(E_1)\varphi\Big).$$

This pair is equivalent to $\big(\tilde{\rho}(\widetilde{\mathcal{E}}),\sigma(E_1)\varphi\big)$ which is the image of $\big(\widetilde{\mathcal{E}},\varphi\big)$. □

Remark 27. For future publications it might be helpful here to briefly discuss, what happens if apply this change of orientation twice. Obviously, by replacing $\mathcal{O}$ by $-\mathcal{O}$ we get another map $\rho_\#:\Sigma^{(\#)}(M,g,\mathcal{O})\to\Sigma^{(\#)}(M,g,-\mathcal{O})$. It is easy to check that we then have $(\rho_\#)^2=-\mathrm{id}$.

In the following sections of an associated vector bundle $V=P\times_\rho W$ — where P is a principal G-bundle and where W is a G-representation — are written as an equivalence class $[A,w]$ of the pair (A,w) with respect to the action of $\rho:G\to\mathrm{GL}(W)$. Here A is a local section of P and w a locally defined function $M\to W$.

Lemma 28 (Compatibility with the Clifford action)**.**

$$X\cdot\rho_\#(\varphi)=-\rho_\#(X\cdot\varphi)$$

for $X\in T_pM$, $\varphi\in\Sigma_pM$.

In particular, this lemma implies that although $\rho_\#$ yields an isomorphism between spinor bundles for different orientations, it does not yet have the properties that we request for Ψ.

Proof. We view TM as an associated bundle to $P_{\mathrm{Spin}}(M,g,\pm\mathcal{O})$ where either sign yields a possible description. In these descriptions $([\widetilde{\mathcal{E}},v])$ and $[\tilde{\rho}(\widetilde{\mathcal{E}}),Jv]$ represent the same vector. Thus

$$\begin{aligned}[\widetilde{\mathcal{E}},v]\cdot\rho_\#([\widetilde{\mathcal{E}},\varphi])&=[\tilde{\rho}(\widetilde{\mathcal{E}}),Jv)]\cdot[\tilde{\rho}(\widetilde{\mathcal{E}}),\sigma(E_1)\varphi]\\&=[\tilde{\rho}(\widetilde{\mathcal{E}}),\sigma(Jv)\sigma(E_1)\varphi]\\&=[\tilde{\rho}(\widetilde{\mathcal{E}}),-\sigma(E_1)\sigma(v)\varphi]\\&=-\rho_\#([\widetilde{\mathcal{E}},v]\cdot[\widetilde{\mathcal{E}},\varphi]).\end{aligned}$$

Here we used that $(Jv)\cdot E_1=-E_1\cdot v$ in Cl_m. □

Lemma 29. *Let $X\in T_pM$, $\varphi\in\Gamma(\Sigma(M,g,-\mathcal{O}))$. Then*

$$\nabla_X\rho_\#(\varphi)=\rho_\#(\nabla_X\varphi).\tag{29}$$

Proof. The differential of $\rho : P_{\mathrm{SO}}(M,g,-\mathcal{O}) \to P_{\mathrm{SO}}(M,g,\mathcal{O})$ maps $TP_{\mathrm{SO}}(M,g,-\mathcal{O})$ to $TP_{\mathrm{SO}}(M,g,\mathcal{O})$. The connection 1-form $\omega : \mathrm{SO}(M) \to \mathfrak{so}(n)$ then pulls back according to

$$\omega((d\rho)(Y)) = J\omega(Y)J$$

for $Y \in T_{\mathcal{E}}P_{\mathrm{SO}}(M,g,-\mathcal{O})$, a lift of $X \in TM$ under the projection $P_{\mathrm{SO}}(M,g,-\mathcal{O}) \to M$. We lift this to a connection 1-form $\tilde{\omega} : P_{\mathrm{Spin}}(M,g,-\mathcal{O}) \to \mathfrak{spin}(m) \subset \mathrm{Cl}_m$ which thus transforms as

$$\tilde{\omega}((d\tilde{\rho})(\tilde{Y})) = -E_1\omega(\tilde{Y})E_1$$

where $\tilde{Y} \in TP_{\mathrm{Spin}}(M,g,-\mathcal{O})$ is a lift of Y. And this induces the relation (29). □

Note that obviously $\rho_\#$ is isometric in each fiber.

Now finally in order to get a map Ψ as in the proposition, we will compose $\rho_\#$ with a further bundle isomorphism. We define $\hat{\sigma}(X) := -\sigma(X)$ for any $X \in \mathbb{R}^m$. Obviously, $\hat{\sigma}$ satisfies the Clifford relations, and thus $(\Sigma_m^{(\#)}, \hat{\sigma})$ describes a representation of Cl_m which is due to its dimension irreducible. The classification of such representations implies that this representation is isomorphic to either (Σ_m, σ) or (possibly in the case m odd) $(\Sigma_m^\#, \sigma)$. If m is even, then we obtain a complex vector space isomorphism $K \in \mathrm{End}(\Sigma_m)$ satisfying

$$K(\sigma(X)\varphi) = -\sigma(X)K(X) \tag{30}$$

for all $X \in \mathbb{R}^m$ and $\varphi \in \Sigma_m$. We can choose K to be isometric. Similarly, by checking the effect of Clifford multiplication by the volume element we obtain for m odd isometric complex isomorphisms $K : \Sigma_m \to \Sigma_m^\#$ and $K : \Sigma_m^\# \to \Sigma_m$ satisfying (30) for $X \in \mathbb{R}^m$ and $\varphi \in \Sigma_m$ resp. $\varphi \in \Sigma_m^\#$. The associated map defined by K defines parallel, fiberwise isometric complex linear isomorphisms of vector bundles over id_M

$$K : \Sigma(M,g,\mathcal{O}) \to \Sigma(M,g,\mathcal{O})$$

for m even and

$$K : \Sigma(M,g,\mathcal{O}) \to \Sigma^\#(M,g,\mathcal{O}), \quad K : \Sigma^\#(M,g,\mathcal{O}) \to \Sigma(M,g,\mathcal{O})$$

for m odd, satisfying (30) for $X \in T_pM$ and $\varphi \in \Sigma^{(\#)}(M,g,\mathcal{O})$.

The composition $\Psi := K \circ \rho_\#$ now satisfies all properties requested in the proposition.

Remark 30. Let m be even. Recall that a spinor φ is called positive resp. negative if $\omega_{\mathbb{C}}\varphi = \varphi$ resp. $\omega_{\mathbb{C}}\varphi = -\varphi$. It is easy to check that $\rho_\#$ and Ψ map positive spinors to negative ones and vice versa, while K preserves positivity and negativity of spinors.

The proof of Proposition 25 is thus complete.

D Making paths of metrics divergence free

In this appendix we show the following well-known lemma.

Lemma 31. *Let g_s, $s \in [0,\ell]$ be a path of Riemannian metrics on a closed manifold M. We assume that the dimension of the space of Killing vector fields of (M, g_s) does not depend on s. Then there exists a family of diffeomorphisms $\varphi_s : M \to M$, depending smoothly on $s \in [0,\ell]$, $\varphi_0 = \mathrm{id}_M$, such that $\tilde{g}_s := \varphi_s^* g_s$ satisfies for all $s \in [0,\ell]$:*

$$\mathrm{div}^{\tilde{g}_s} \frac{d}{ds}\tilde{g}_s = 0. \tag{31}$$

Proof. We make the following ansatz. Let $X_s \in \Gamma(TM)$ be a vector field smoothly depending on the parameter $s \in [0,\ell]$. Let φ_s be the flow generated by X_s, i.e.

$$\frac{d}{ds}\varphi_s(x) = X_s\big|_{\varphi_s(x)}.$$

For this we define $h_s := \frac{d}{ds} g_s$. Then

$$\frac{d}{ds}\tilde{g}_s = \varphi_s^* \left(\mathcal{L}_{X_s} g_s + h_s\right),$$

where $\mathcal{L}_{X_s}$ denotes the Lie derivative in the direction of X_s. Thus (31) is equivalent to

$$\mathrm{div}^{g_s} \left(\mathcal{L}_{X_s} g_s + h_s\right) = 0.$$

Now let $(\mathrm{div}^{g_s})^* : \Omega^1(M) \to \Gamma(T^*M \odot T^*M)$ be the adjoint of div^{g_s}. Then, see [12, Lemma 1.60],

$$\mathcal{L}_X g = -2(\mathrm{div}^g)^*(X^b)$$

for all $X \in \Gamma(TM)$ and Riemannian metrics g, where $X^b := g(X, \cdot)$. We thus see that (31) is in fact equivalent to

$$2\,\mathrm{div}^{g_s}(\mathrm{div}^{g_s})^*\alpha_s = \mathrm{div}^{g_s} h_s. \tag{32}$$

where we set $\alpha_s := g_s(X_s, \cdot)$. By calculation the principal symbol of $P_s := \mathrm{div}^{g_s}(\mathrm{div}^{g_s})^*$ one sees that P_s is a self-adjoint elliptic operator, thus has discrete (non-negative) spectrum. We have $\ker(P_s) = \ker((\mathrm{div}^{g_s})^*)$. Thus again by [12, Lemma 1.60] the kernel of P_s is the space of all Killing vector fields of (M, g_s). We furthermore have

$$\mathrm{im}(P_s) = \ker(P_s)^\perp = \ker(\mathrm{div}^{g_s})^*)^\perp = \mathrm{im}(\mathrm{div}^{g_s}).$$

thus (32) has a unique solution α_s that is L^2-orthogonal to any Killing vector field. By assumption, the dimension of $\ker(P_s)$ is constant. Thus, the spaces $\mathrm{im}(P_s)$ form a smooth family of isomorphic vector spaces and we have a smooth family of isomorphisms P_s on $\mathrm{im}(P_s)$. Thus, α_s and hence also X_s and φ_s depend smoothly on s. This solves the problem. □

Note that we apply the above theorem to a family of Ricci-flat metrics. On closed Ricci-flat Riemannian manifolds every Killing vector field is parallel. Furthermore X is then parallel if and only if X^b is harmonic. Thus the dimension of the space of Killing vector fields is the first Betti-number and thus independent of s.

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