SOBOLEV SPACES ON LIE MANIFOLDS AND REGULARITY FOR POLYHEDRAL DOMAINS

BERND AMMANN, ALEXANDRU D. IONESCU, AND VICTOR NISTOR

Abstract. We study some basic analytic questions related to differential operators on Lie manifolds, which are manifolds whose large scale geometry can be described by a a Lie algebra of vector fields on a compactification. We extend to Lie manifolds several classical results on Sobolev spaces, elliptic regularity, and mapping properties of pseudodifferential operators. A tubular neighborhood theorem for Lie submanifolds allows us also to extend to regular open subsets of Lie manifolds the classical results on traces of functions in suitable Sobolev spaces. Our main application is a regularity result on polyhedral domains $\mathbb{P} \subset \mathbb{R}^3$ using the weighted Sobolev spaces $\mathcal{K}_a^m(\mathbb{P})$. In particular, we show that there is no loss of $\mathcal{K}_a^m$–regularity for solutions of strongly elliptic systems with smooth coefficients. For the proof, we identify $\mathcal{K}_a^m(\mathbb{P})$ with the Sobolev spaces on $\mathbb{P}$ associated to the metric $r_{\mathbb{P}}^{-2} g_E$, where g_E is the Euclidean metric and $r_{\mathbb{P}}(x)$ is a smoothing of the Euclidean distance from x to the set of singular points of $\mathbb{P}$. A suitable compactification of the interior of $\mathbb{P}$ then becomes a regular open subset of a Lie manifold. We also obtain the well-posedness of a non-standard boundary value problem on a smooth, bounded domain with boundary $\mathcal{O} \subset \mathbb{R}^n$ using weighted Sobolev spaces, where the weight is the distance to the boundary.

Contents

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Introduction

We study some basic analytic questions on non-compact manifolds. In order to obtain stronger results, we restrict ourselves to "Lie manifolds," a class of manifolds whose large scale geometry is determined by a compactification to a manifold with corners and a Lie algebra of vector fields on this compactification (Definition 1.3). One of the motivations for studying Lie manifolds is the loss of (classical Sobolev) regularity of solutions of elliptic equations on non-smooth domains. To explain this loss of regularity, let us recall first that the Poisson problem

$$\Delta u = f \in H^{m-1}(\Omega), \quad m \in \mathbb{N} \cup \{0\},\ \Omega \subset \mathbb{R}^n \text{ bounded}, \tag{1}$$

has a unique solution $u \in H^{m+1}(\Omega)$, $u = 0$ on $\partial\Omega$, provided that $\partial\Omega$ *is smooth*. In particular, u will be smooth up to the boundary if $\partial\Omega$ and f are smooth (in the following, when dealing with functions defined on an open set, by "smooth," we shall mean "smooth up to the boundary"). See the books of Evans [16], or Taylor [58] for a proof of this basic well-posedness result.

This well-posedness result is especially useful in practice for the numerical approximation of the solution u of Equation (1) [8]. However, in practice, it is only rarely the case that Ω is smooth. The lack of smoothness of the domains interesting in applications has motivated important work on Lipschitz domains, see for instance [23, 40] or [65]. These papers have extended to Lipschitz domains some of the classical results on the Poisson problem on smooth, bounded domains, using the classical Sobolev spaces

$$H^m(\Omega) := \{u,\ \partial^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}.$$

It turns out that, if $\partial\Omega$ is *not* smooth, then the smoothness of f on $\overline{\Omega}$ (i. e., up to the boundary) does not imply that the solution u of Equation (1) is smooth as well on $\overline{\Omega}$. This is the *loss of regularity* for elliptic problems on non-smooth domains mentioned above.

The loss of regularity can be avoided, however, by a conformal blowup of the singular points. This conformal blowup replaces a neighborhood of each connected component of the set of singular boundary points by a complete, but non-compact end. (Here "complete" means complete as a metric space, not geodesically complete.) It can be proved then that the resulting Sobolev spaces are the "Sobolev spaces with weights" considered for instance in [25, 26, 35, 46]. Let $f > 0$ be a smooth function on a domain Ω, we then define the *mth Sobolev space with weight f* by

$$\mathcal{K}_a^m(\Omega; f) := \{u,\ f^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \ |\alpha| \le m\}, \quad m \in \mathbb{N} \cup \{0\},\ a \in \mathbb{R}. \tag{2}$$

Indeed, if $\Omega = \mathbb{P} \subset \mathbb{R}^2$ is a polygon, and if we choose

$$f(x) = \vartheta(x) = \text{ the } \textit{distance to the non-smooth boundary points of } \mathbb{P}, \tag{3}$$

then there is no loss of regularity in the spaces $\mathcal{K}_a^m(\Omega) := \mathcal{K}_a^m(\Omega; \vartheta)$ [26, Theorem 6.6.1]. In this paper, we extend this regularity result to polyhedral domains in three dimensions, Theorem 6.1, with the same choice of the weight (in three dimensions the weight is the distance to the edges). The analogous result in arbitrary dimensions leads to topological difficulties [9, 66].

Our regularity result requires us first to study the weighted Sobolev spaces $\mathcal{K}_a^m(\Omega) := \mathcal{K}_a^m(\Omega; \vartheta)$ where $\vartheta(x)$ is the distance to the set of singular points on the boundary. Our approach to Sobolev spaces on polyhedral domains is to show first that $\mathcal{K}_a^m(\Omega)$ is isomorphic to a Sobolev space on a certain non-compact Riemannian manifold M with smooth boundary. This non-compact manifold M is obtained from our polyhedral domain by replacing the Euclidean metric g_E with

$$r_{\mathbb{P}}^{-2} g_E, \quad r_{\mathbb{P}} \text{ a smoothing of } \vartheta, \tag{4}$$

which blows up at the faces of codimension two or higher, that is, at the set of singular boundary points. (The metric $r_{\mathbb{P}}^{-2} g_E$ is Lipschitz equivalent to $\vartheta^{-2} g_E$, but the latter is not smooth.) The resulting non-compact Riemannian manifold turns out to be a regular open subset in a "Lie manifold." (see Definition 1.3, Subsection 1.6, and Section 6 for the precise definitions). A *Lie manifold* is a compact manifold with corners M together with a $C^\infty(M)$-module $\mathcal{V}$ whose elements are vector fields on M. The space $\mathcal{V}$ must satisfy a number of axioms, in particular, $\mathcal{V}$ is required to be closed under the Lie bracket of vector fields. This property is the origin of the name *Lie* manifold. The $C^\infty(M)$-module $\mathcal{V}$ can be identified with the sections of a vector bundle A over M. Choosing a metric on A defines a complete Riemannian metric on the interior of M. See Section 1 or [4] for details.

The framework of Lie manifolds is quite convenient for the study of Sobolev spaces, and in this paper we establish, among other things, that the main results on the classical Sobolev spaces remain true in the framework of Lie manifolds. The regular open sets of Lie manifolds then play in our framework the role played by smooth, bounded domains in the classical theory.

Let $\mathbb{P} \subset \mathbb{R}^n$ be a polyhedral domain. We are especially interested in describing the spaces $\mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P})$ of restrictions to the boundary of the functions in the weighted Sobolev space $\mathcal{K}_a^m(\mathbb{P};\vartheta) = \mathcal{K}_a^m(\mathbb{P}; r_{\mathbb{P}})$ on $\mathbb{P}$. Using the conformal change of metric of Equation (4), the study of restrictions to the boundary of functions in $\mathcal{K}_a^m(\mathbb{P})$ is reduced to the analogous problem on a suitable regular open subset $\Omega_{\mathbb{P}}$ of some Lie manifold. More precisely, $\mathcal{K}_a^m(\mathbb{P}) = r_{\mathbb{P}}^{a-n/2} H^m(\Omega_{\mathbb{P}})$. A consequence of this is that

$$\mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P}) = \mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P};\vartheta) = r_{\mathbb{P}}^{a-n/2} H^{m-1/2}(\partial\Omega_{\mathbb{P}}). \tag{5}$$

(In what follows, we shall usually simply denote $\mathcal{K}_a^m(\mathbb{P}) := \mathcal{K}_a^m(\mathbb{P};\vartheta) = \mathcal{K}_a^m(\mathbb{P}; r_{\mathbb{P}})$ and $\mathcal{K}_a^m(\partial\mathbb{P}) := \mathcal{K}_a^m(\partial\mathbb{P};\vartheta) = \mathcal{K}_a^m(\partial\mathbb{P}; r_{\mathbb{P}})$, where, we recall, $\vartheta(x)$ is the distance from x to the set of non-smooth boundary points and $r_{\mathbb{P}}$ is a smoothing of ϑ that satisfies $r_{\mathbb{P}}/\vartheta \in [c, C]$, $c, C > 0$.)

Equation (5) is one of the motivations to study Sobolev spaces on Lie manifolds. In addition to the non-compact manifolds that arise from polyhedral domains, other examples of Lie manifolds include the Euclidean spaces $\mathbb{R}^n$, manifolds that are Euclidean at infinity, conformally compact manifolds, manifolds with cylindrical and polycylindrical ends, and asymptotically hyperbolic manifolds. These classes of non-compact manifolds appear in the study of the Yamabe problem [32, 48] on compact manifolds, of the Yamabe problem on asymptotically cylindrical manifolds [2], of analysis on locally symmetric spaces, and of the positive mass theorem [49, 50, 67], an analogue of the positive mass theorem on asymptotically hyperbolic manifolds [6]. Lie manifolds also appear in Mathematical Physics and in Numerical Analysis. Classes of Sobolev spaces on non-compact manifolds have been studied in many papers, of which we mention only a few [15, 18, 27, 30, 34, 36, 39, 37, 38, 51, 52, 53, 63, 64] in addition to the works mentioned before. Our work can also be used to unify some of the various approaches found in these papers.

Let us now review in more detail the contents of this paper. A large part of the technical material in this paper is devoted to the study of Sobolev spaces on Lie manifolds (with or without boundary). If M is a *compact* manifold with corners, we shall denote by ∂M the union of all boundary faces of M and by $M_0 := M \smallsetminus \partial M$ the interior of M. We begin in Section 1 with a review of the definition of a structural Lie algebra of vector fields $\mathcal{V}$ on a manifold with corners M. This Lie algebra of vector fields will provide the derivatives appearing in the definition of the Sobolev spaces. Then we define a Lie manifold as a pair $(M, \mathcal{V})$, where M is a compact manifold with corners and $\mathcal{V}$ is a structural Lie algebra of vector fields that is unrestricted in the interior M_0 of M. We will explain the above mentioned fact that the interior of M carries a complete metric g. This metric is unique up to Lipschitz equivalence (or quasi-isometry). We also introduce in this section Lie manifolds with (true) boundary and, as an example, we discuss the example of a Lie manifold with true boundary corresponding to curvilinear polygonal domains. In Section 2 we discuss Lie submanifolds, and most importantly, the global tubular neighborhood theorem. The proof of this global tubular neighborhood theorem is based on estimates on the second fundamental form of the boundary, which are obtained from the properties of the structural Lie algebra of vector fields. This property distinguishes Lie manifolds from general manifolds with boundary and bounded geometry, for which a global tubular neighborhood is part of the definition. In Section 3, we define the Sobolev spaces $W^{s,p}(M_0)$ on the interior M_0 of a Lie manifold M, where either $s \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$ or $s \in \mathbb{R}$ and $1 < p < \infty$. We first define the spaces $W^{s,p}(M_0)$, $s \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$, by differentiating with respect to vector fields in $\mathcal{V}$. This definition is in the spirit of the standard definition of Sobolev spaces on $\mathbb{R}^n$. Then we prove that there are two alternative, but equivalent ways to define these Sobolev spaces, either by using a suitable class of partitions of unity (as in [54, 55, 62] for example), or as the domains of the powers of the Laplace operator (for $p = 2$). We also consider these spaces on open subsets $\Omega_0 \subset M_0$. The spaces $W^{s,p}(M_0)$, for $s \in \mathbb{R}$, $1 < p < \infty$ are defined by interpolation and duality or, alternatively, using partitions of unity. In Section 4, we discuss regular open subsets $\Omega \subset M$. In the last two sections, several of the classical results on Sobolev spaces on smooth domains were extended to the spaces $W^{s,p}(M_0)$. These results include the density of smooth, compactly supported functions, the Gagliardo-Nirenberg-Sobolev inequalities, the extension theorem, the trace theorem, the characterization of the range of the trace map in the Hilbert space case ($p = 2$), and the Rellich-Kondrachov compactness theorem.

In Section 5 we include as an application a regularity result for strongly elliptic boundary value problems, Theorem 5.1. This theorem gives right away the following result, proved in Section 6, which states that there is no loss of regularity for these problems within weighted Sobolev spaces.

Theorem 0.1. *Let $\mathbb{P} \subset \mathbb{R}^3$ be a polyhedral domain and P be a strongly elliptic, second order differential operator with coefficients in $\mathcal{C}^\infty(\overline{\mathbb{P}})$. Let $u \in \mathcal{K}^1_{a+1}(\mathbb{P})$, $u = 0$ on $\partial\mathbb{P}$, $a \in \mathbb{R}$. If $Pu \in \mathcal{K}^{m-1}_{a-1}(\mathbb{P})$, then $u \in \mathcal{K}^{m+1}_{a+1}(\mathbb{P})$ and there exists $C > 0$ independent of u such that*

$$\|u\|_{\mathcal{K}^{m+1}_{a+1}(\mathbb{P})} \le C\big(\|Pu\|_{\mathcal{K}^{m-1}_{a-1}(\mathbb{P})} + \|u\|_{\mathcal{K}^{0}_{a+1}(\mathbb{P})}\big), \quad m \in \mathbb{N} \cup \{0\}.$$

The same result holds for strongly elliptic systems.

Note that the above theorem does not constitute a Fredholm (or normal solvability) result, because the inclusion $\mathcal{K}^{m+1}_{a+1}(\mathbb{P}) \to \mathcal{K}^{0}_{a+1}(\mathbb{P})$ is *not compact.* See also [25, 26, 35, 46] and the references therein for similar results.

In Section 7, we obtain a "non-standard boundary value problem" on a smooth domain $\mathcal{O}$ in weighted Sobolev spaces with weight given by the distance to the boundary. The boundary conditions are thus replaced by growth conditions. Finally, in the last section, Section 8, we obtain mapping properties for the pseudodifferential calculus $\Psi^\infty_{\mathcal{V}}(M)$ defined in [3] between our weighted Sobolev spaces $\rho^s W^{r,p}(M)$. We also obtain a general elliptic regularity result for elliptic pseudodifferential operators in $\Psi^\infty_{\mathcal{V}}(M)$.

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1. Lie manifolds

As explained in the Introduction, our approach to the study of weighted Sobolev spaces on polyhedral domains is based on their relation to Sobolev spaces on Lie manifolds with true boundary. Before we recall the definition of a Lie manifold and some of their basic properties, we shall first look at the following example, which is one of the main motivations for the theory of Lie manifolds.

Example 1.1. Let us take a closer look at the local structure of the Sobolev space $\mathcal{K}^m_a(\mathbb{P})$ associated to a polygon $\mathbb{P}$ (recall (2)). Consider $\Omega := \{(r,\theta) \,|\, 0 < \theta < \alpha\}$, which models an angle of $\mathbb{P}$. Then the distance to the vertex is simply $\vartheta(x) = r$, and the *weighted Sobolev spaces* associated to Ω, $\mathcal{K}^m_a(\Omega)$, can alternatively be described as

$$(6) \quad \mathcal{K}^m_a(\Omega) = \mathcal{K}^m_a(\Omega;\vartheta) := \{u \in L^2_{loc}(\Omega),\ r^{-a}(r\partial_r)^i\partial_\theta^j u \in L^2(\Omega), \quad i+j \leq m\}.$$

The point of the definition of the spaces $\mathcal{K}^m_a(\Omega)$ was the replacement of the local basis $\{r\partial_x, r\partial_y\}$ with the local basis $\{r\partial_r, \partial_\theta\}$ that is easier to work with on the desingularization $\Sigma(\Omega) := [0,\infty)\times[0,\alpha] \ni (r,\theta)$ of Ω. By further writing $r = e^t$, the vector field $r\partial_r$ becomes ∂_t. Since $dt = r^{-1}dr$, the space $\mathcal{K}^m_1(\Omega)$ then identifies with $H^m(\mathbb{R}_t \times (0,\alpha))$. The weighted Sobolev space $\mathcal{K}^m_1(\Omega)$ has thus become a classical Sobolev space on the cylinder $\mathbb{R}\times(0,\alpha)$, as in [25].

The aim of the following definitions is to define such a desingularisation in general. The desingularisation will carry the structure of a Lie manifold, defined in the next subsection.

We shall introduce a further, related definition, namely the definition of a "Lie submanifolds of a Lie manifold" in Section 4.

1.1. Definition of Lie manifolds.

At first, we want to recall the definition of manifolds with corners. A *manifold with corners* is a closed subset M of a differentiable manifold such that every point $p \in M$ lies in a coordinate chart whose restriction to M is a diffeomorphism to $[0,\infty)^k \times \mathbb{R}^{n-k}$, for some $k = 0, 1, \ldots, n$ depending on p. Obviously, this definition includes the property that the transition map of two different charts are smooth up to the boundary. If $k = 0$ for all $p \in M$, we shall say that M is a *smooth manifold.* If $k \in \{0, 1\}$, we shall say that M is a *smooth manifold with smooth boundary.*

Let M be a *compact* manifold with corners. We shall denote by ∂M the union of all boundary faces of M, that is, ∂M is the union of all points not having a neighborhood diffeomorphic to $\mathbb{R}^n$. Furthermore, we shall write $M_0 := M \smallsetminus \partial M$ for the *interior* of M. In order to avoid confusion, we shall use this notation and terminology only when M is compact. Note that our definition allows ∂M to be a smooth manifold, possibly empty.

As we shall see below, a Lie manifold is described by a Lie algebra of vector fields satisfying certain conditions. We now discuss some of these conditions.

Definition 1.2. A subspace $\mathcal{V} \subseteq \Gamma(M; TM)$ of the Lie algebra of all smooth vector fields on M is said to be a *structural Lie algebra of vector fields on* M provided that the following conditions are satisfied:

(i) $\mathcal{V}$ is closed under the Lie bracket of vector fields;
(ii) every $V \in \mathcal{V}$ is tangent to all boundary hyperfaces of M;
(iii) $\mathcal{C}^\infty(M)\mathcal{V} = \mathcal{V}$; and
(iv) each point $p \in M$ has a neighborhood U_p such that

$$\mathcal{V}_{U_p} := \{X|_{\overline{U}_p} \, | X \in \mathcal{V}\} \simeq C^\infty(\overline{U}_p)^k$$

in the sense of $C^\infty(\overline{U}_p)$-modules.

The condition (iv) in the definition above can be reformulated as follows:

(iv') For every $p \in M$, there exist a neighborhood $U_p \subset M$ of p and vector fields $X_1, X_2, \dots, X_k \in \mathcal{V}$ with the property that, for any $Y \in \mathcal{V}$, there exist functions $f_1, \dots, f_k \in \mathcal{C}^\infty(M)$, uniquely determined on U_p, such that

$$Y = \sum_{j=1}^{k} f_j X_j \quad \text{on } U_p. \tag{7}$$

We now have defined the preliminaries for the following important definition.

Definition 1.3. A *Lie structure at infinity* on a smooth manifold M_0 is a pair $(M, \mathcal{V})$, where M is a compact manifold with interior M_0 and $\mathcal{V} \subset \Gamma(M; TM)$ is a structural Lie algebra of vector fields on M with the following property: If $p \in M_0$, then any local basis of $\mathcal{V}$ in a neighborhood of p is also a local basis of the tangent space to M_0.

It follows from the above definition that the constant k of Equation (7) equals to the dimension n of M_0.

A *manifold with a Lie structure at infinity* (or, simply, a *Lie manifold*) is a manifold M_0 together with a Lie structure at infinity $(M, \mathcal{V})$ on M_0. We shall sometimes denote a Lie manifold as above by $(M_0, M, \mathcal{V})$, or, simply, by $(M, \mathcal{V})$, because M_0 is determined as the interior of M. (In [4], only the term "manifolds with a Lie structure at infinity" was used.)

Example 1.4. If $F \subset TM$ is a sub-bundle of the tangent bundle of a smooth manifold (so M has no boundary) such that $\mathcal{V}_F := \Gamma(M; F)$ is closed under the Lie bracket, then $\mathcal{V}_F$ is a structural Lie algebra of vector fields. Using the Frobenius theorem it is clear that such vector bundles are exactly the tangent bundles of k-dimensional foliations on M, $k = \operatorname{rank} F$. However, $\mathcal{V}_F$ does not define a Lie structure at infinity, unless $F = TM$.

Remark 1.5. We observe that Conditions (iii) and (iv) of Definition 1.2 are equivalent to the condition that $\mathcal{V}$ be a projective $\mathcal{C}^\infty(M)$-module. Thus, by the Serre-Swan theorem [24], there exists a vector bundle $A \to M$, unique up to isomorphism, such that $\mathcal{V} = \Gamma(M; A)$. Since $\mathcal{V}$ consists of vector fields, that is $\mathcal{V} \subset \Gamma(M; TM)$, we also obtain a natural vector bundle morphism $\varrho_M : A \to TM$, called the *anchor map*. The Condition (ii) of Definition 1.3 is then equivalent to the fact that ϱ_M is an isomorphism $A|_{M_0} \simeq TM_0$ on M_0. We will take this isomorphism to be an identification, and thus we can say that A is *an extension* of TM_0 to M (that is, $TM_0 \subset A$).

1.2. **Riemannian metric.** Let $(M_0, M, \mathcal{V})$ be a Lie manifold. By definition, a *Riemannian metric on M_0 compatible with the Lie structure at infinity* $(M, \mathcal{V})$ is a metric g_0 on M_0 such that, for any $p \in M$, we can choose the basis $X_1, \dots, X_k$ in Definition 1.2 (iv') (7) to be orthonormal with respect to this metric everywhere on $U_p \cap M_0$. (Note that this condition is a restriction only for $p \in \partial M := M \smallsetminus M_0$.) Alternatively, we will also say that (M_0, g_0) is a *Riemannian Lie manifold*. Any Lie manifold carries a compatible Riemannian metric, and any two compatible metrics are bi-Lipschitz to each other.

Remark 1.6. Using the language of Remark 1.5, g_0 is a compatible metric on M_0 if, and only if, there exists a metric g on the vector bundle $A \to M$ which restricts to g_0 on $TM_0 \subset A$.

The geometry of a Riemannian manifold (M_0, g_0) with a Lie structure $(M, \mathcal{V})$ at infinity has been studied in [4]. For instance, (M_0, g_0) is necessarily complete and, if $\partial M \neq \emptyset$, it is of infinite volume. Moreover, all the covariant derivatives of the Riemannian curvature tensor are bounded. Under additional mild assumptions, we also know that the injectivity radius is bounded from below by a positive constant, i. e., (M_0, g_0) is of bounded geometry. (A *manifold with bounded geometry* is a Riemannian manifold with positive injectivity radius and with bounded covariant derivatives of the curvature tensor, see [54] and references therein).

On a Riemannian Lie manifold $(M_0, M, \mathcal{V}, g_0)$, the exponential map $\exp : TM_0 \to M_0$ is well-defined for all $X \in TM_0$ and extends to a differentiable map $\exp : A \to M$. A convenient way to introduce the exponential map is via the geodesic spray, as done in [4]. Similarly, any vector field $X \in \mathcal{V} = \Gamma(M; A)$ is integrable and will map any (connected) boundary face of M to itself. The resulting diffeomorphism of M_0 will be denoted ψ_X.

1.3. **Examples.** We include here two examples of Lie manifolds together with compatible Riemannian metrics. The reader can find more examples in [4, 31].

Examples 1.7.

(a) Take $\mathcal{V}_b$ to be the set of all vector fields tangent to all faces of a manifold with corners M. Then $(M, \mathcal{V}_b)$ is a Lie manifold. This generalizes Example 1.1. See also Subsection 1.6 and Section 6. Let $r \geq 0$ to be a smooth function on M that is equal to the distance to the boundary in a neighborhood of ∂M, and is > 0 outside ∂M (i. e., on M_0). Let h be a smooth metric on M, then $g_0 = h + (r^{-1}dr)^2$ is a compatible metric on M_0.

(b) Take $\mathcal{V}_0$ to be the set of all vector fields vanishing on all faces of a manifold with corners M. Then $(M, \mathcal{V}_0)$ is a Lie manifold. If ∂M is a smooth manifold (i. e., if M is a smooth manifold with boundary), then $\mathcal{V}_0 = r\Gamma(M; TM)$, where r is as in (a).

1.4. **$\mathcal{V}$-differential operators.** We are especially interested in the analysis of the differential operators generated using only derivatives in $\mathcal{V}$. Let $\mathrm{Diff}^*_{\mathcal{V}}(M)$ be the algebra of differential operators on M generated by multiplication with functions in $\mathcal{C}^\infty(M)$ and by differentiation with vector fields $X \in \mathcal{V}$. The space of order m differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(M)$ will be denoted $\mathrm{Diff}^m_{\mathcal{V}}(M)$. A differential operator in $\mathrm{Diff}^*_{\mathcal{V}}(M)$ will be called a $\mathcal{V}$-differential operator.

We can define $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to M$, $E, F \subset M \times \mathbb{C}^N$ by

$$\mathrm{Diff}^*_{\mathcal{V}}(M; E, F) := e_F M_N(\mathrm{Diff}^*_{\mathcal{V}}(M)) e_E \,, \tag{8}$$

where $M_N(\mathrm{Diff}^*_{\mathcal{V}}(M))$ is the algebra of $N \times N$-matrices over the ring $\mathrm{Diff}^*_{\mathcal{V}}(M)$, and where $e_E, e_F \in M_N(\mathcal{C}^\infty(M))$ are the projections onto E and, respectively, onto F. It follows that $\mathrm{Diff}^*_{\mathcal{V}}(M; E) := \mathrm{Diff}^*_{\mathcal{V}}(M; E, E)$ is an algebra. It is also closed under taking adjoints of operators in $L^2(M_0)$, where the volume form is defined using a compatible metric g_0 on M_0.

1.5. **Regular open sets.** *We assume from now on that $r_{\mathrm{inj}}(M_0)$, the injectivity radius of (M_0, g_0), is positive.*

One of the main goals of this paper is to prove the results on weighted Sobolev spaces on polyhedral domains that are needed for regularity theorems. We shall do that by reducing the study of weighted Sobolev spaces to the study of Sobolev spaces on "regular open subsets" of Lie manifolds, a class of open sets that plays in the framework of Lie manifolds the role played by domains with smooth boundaries in the framework of bounded, open subsets of $\mathbb{R}^n$. Regular open subsets are defined below in this subsection.

Let $N \subset M$ be a submanifold of codimension one of the Lie manifold $(M, \mathcal{V})$. Note that this implies that N is a closed subset of M. We shall say that N is a *regular* submanifold of $(M, \mathcal{V})$ if we can choose a neighborhood V of N in M and a compatible metric g_0 on M_0 that restricts to a product-type metric on $V \cap M_0 \simeq (\partial N_0) \times (-\varepsilon_0, \varepsilon_0)$, $N_0 = N \smallsetminus \partial N = N \cap M_0$. Such neighborhoods will be called *tubular neighborhoods.*

In Section 2, we shall show that a codimension one manifold is regular if, and only if, it is a tame submanifold of M; this gives an easy, geometric, necessary and sufficient condition for the regularity of a codimension one submanifold of M. This is relevant, since the study of manifolds with boundary and bounded geometry presents some unexpected difficulties [47].

In the following, it will be important to distinguish properly between the boundary of a topological subset, denoted by ∂_{top}, and the boundary in the sense of manifolds with corners, denoted simply by ∂.

Definition 1.8. Let $(M, \mathcal{V})$ be a Lie manifold and $\Omega \subset M$ be an open subset. We shall say that Ω is a *regular open subset* in M if, and only if, Ω is connected, Ω and $\overline{\Omega}$ have the same boundary, $\partial_{\mathrm{top}}\Omega$ (in the sense of subsets of the topological space M), and $\partial_{\mathrm{top}}\Omega$ is a regular submanifold of M.

Let $\Omega \subset M$ be a regular open subset. Then $\overline{\Omega}$ is a compact manifold with corners. The reader should be aware of the important fact that $\partial_{\mathrm{top}}\Omega = \partial_{\mathrm{top}}\overline{\Omega}$ is contained in $\partial\overline{\Omega}$, but in general $\partial\overline{\Omega}$ and $\partial_{\mathrm{top}}\Omega$ are not equal. The set $\partial_{\mathrm{top}}\Omega$ will be called the *true boundary* of $\overline{\Omega}$. Furthermore, we introduce $\partial_\infty\Omega := \partial\overline{\Omega} \cap \partial M$, and call it the *boundary at infinity* of $\overline{\Omega}$. Obviously, one has $\partial\overline{\Omega} = \partial_{\mathrm{top}}\overline{\Omega} \cup \partial_\infty\overline{\Omega}$. The true boundary and the boundary at infinity intersect in a (possibly empty) set of codimension ≥ 2. See Figure 1. We will also use the notation $\partial\Omega_0 := \partial_{\mathrm{top}}\Omega \cap M_0 = \partial\overline{\Omega} \cap M_0$.

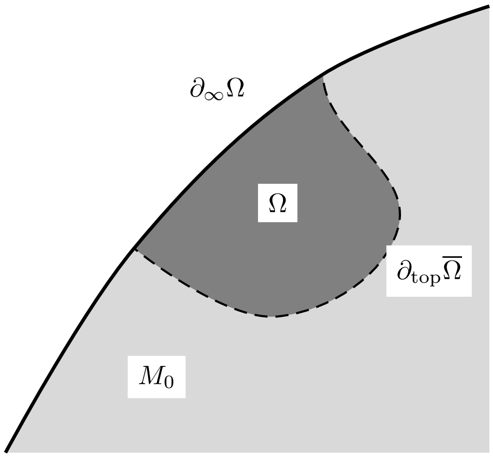

FIGURE 1. A regular open set Ω. Note that the interior of $\partial_\infty\overline{\Omega}$ is contained in Ω, but the true boundary $\partial_{\mathrm{top}}\Omega = \partial_{\mathrm{top}}\overline{\Omega}$ is not contained in Ω

The space of restrictions to Ω or $\overline{\Omega}$ of order m differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(M)$ will be denoted $\mathrm{Diff}^m_{\mathcal{V}}(\Omega)$, respectively $\mathrm{Diff}^m_{\mathcal{V}}(\overline{\Omega})$. Similarly, we shall denote by $\mathcal{V}(\Omega)$ the space of restrictions to $\overline{\Omega}$ of vector fields in $\mathcal{V}$, the structural Lie algebra of vector fields on M.

Let $F \subset \partial\Omega$ be any boundary hyperface of $\overline{\Omega}$ of codimension 1. Such a face is either contained in $\partial_{\mathrm{top}}\overline{\Omega}$ or in $\partial_\infty\overline{\Omega}$. If $F \subset \partial_\infty\Omega$, then the restrictions of all vector fields in $\mathcal{V}$ to F are tangent to F. However, if $F \subset \partial_{\mathrm{top}}\overline{\Omega}$ the regularity of the boundary implies that there are vector fields in $\mathcal{V}$ whose restriction to F is not tangent to F. In particular, the true boundary $\partial_{\mathrm{top}}\overline{\Omega}$ of $\overline{\Omega}$ is uniquely determined by $(\overline{\Omega}, \mathcal{V}(\Omega))$, and hence so is $\Omega = \overline{\Omega} \smallsetminus \partial_{\mathrm{top}}\overline{\Omega}$. We therefore obtain a one-to-one correspondence between Lie manifolds with true boundary and regular open subsets (of some Lie manifold M).

Assume we are given Ω, $\overline{\Omega}$ (the closure in M), and $\mathcal{V}(\Omega)$, with Ω a regular open subset of some Lie manifold $(M, \mathcal{V})$. In the cases of interest, for example if $\partial_{\mathrm{top}}\overline{\Omega}$ is a tame submanifold of M (see Subsection 2.3 for the definition of tame submanifolds), we can replace the Lie manifold $(M, \mathcal{V})$ in which Ω is a regular open set with a Lie manifold $(N, \mathcal{W})$ canonically associated to $(\Omega, \overline{\Omega}, \mathcal{V}(\Omega))$ as follows. Let N be obtained by gluing two copies of $\overline{\Omega}$ along $\partial_{\mathrm{top}}\overline{\Omega}$, the so-called *double* of $\overline{\Omega}$, also denoted $\overline{\Omega}^{db} = N$. A smooth vector field X on $\overline{\Omega}^{db}$ will be in $\mathcal{W}$, the structural Lie algebra of vector fields $\mathcal{W}$ on $\overline{\Omega}^{db}$ if, and only if, its restriction to each copy of $\overline{\Omega}$ is in $\mathcal{V}(\Omega)$. Then Ω will be a regular open set of the Lie manifold $(N, \mathcal{W})$. For this reason, the pair $(\overline{\Omega}, \mathcal{V}(\Omega))$ will be called a *Lie manifold with true boundary.* In particular, the true boundary of a Lie manifold with true boundary is a tame submanifold of the double. The fact that the double is a Lie manifold is justified in Remark 2.10.

1.6. **Curvilinear polygonal domains.** We conclude this section with a discussion of a curvilinear polygonal domain $\mathbb{P}$, an example that generalizes Example 1.1 and is one of the main motivations for considering Lie manifolds. To study function spaces on $\mathbb{P}$, we shall introduce a "desingularization" $(\Sigma(\mathbb{P}), \kappa)$ of $\mathbb{P}$ (or, rather, of $\overline{\mathbb{P}}$), where $\Sigma(\mathbb{P})$ is a compact manifold with corners and $\kappa : \Sigma(\mathbb{P}) \to \overline{\mathbb{P}}$ is a continuous map that is a diffeomorphism from the interior of $\Sigma(\mathbb{P})$ to $\mathbb{P}$ and maps the boundary of $\Sigma(\mathbb{P})$ onto the boundary of $\mathbb{P}$.

Let us denote by B^k the open unit ball in $\mathbb{R}^k$.

Definition 1.9. An open, connected subset $\mathbb{P} \subset M$ of a two dimensional manifold M will be called a *curvilinear polygonal domain* if, by definition, $\overline{\mathbb{P}}$ is compact and for every point $p \in \partial\mathbb{P}$ there exists a diffeomorphism $\phi_p : V_p \to B^2$, $\phi_p(p) = 0$, defined on a neighborhood $V_p \subset M$ such that

$$\phi_j(V_p \cap \mathbb{P}) = \{(r\cos\theta, r\sin\theta),\ 0 < r < 1,\ 0 < \theta < \alpha_p\}, \quad \alpha_p \in (0, 2\pi). \tag{9}$$

A point $p \in \partial\mathbb{P}$ for which $\alpha_p \neq \pi$ will be called a *vertex of* $\mathbb{P}$. The other points of $\partial\mathbb{P}$ will be called *smooth boundary points*. It follows that every curvilinear polygonal domain has finitely many vertices and its boundary consists of a finite union of smooth curves γ_j (called the *edges* of $\mathbb{P}$) which have no other common points except the vertices. Moreover, every vertex belongs to exactly two edges.

Let $\{P_1, P_2, \ldots, P_k\} \subset \overline{\mathbb{P}}$ be the vertices of $\mathbb{P}$. The cases $k = 0$ and $k = 1$ are also allowed. Let $V_j := V_{P_j}$ and $\phi_j := \phi_{P_j} : V_j \to B^2$ be the diffeomorphisms defined by Equation (9). Let $(r, \theta) : \mathbb{R}^2 \smallsetminus \{(0,0)\} \to (0, \infty) \times [0, 2\pi)$ be the polar coordinates. We can assume that the sets V_j are disjoint and define $r_j(x) = r(\phi_j(x))$ and $\theta_j(x) = \theta(\phi_j(x))$.

The *desingularization* $\Sigma(\mathbb{P})$ of $\mathbb{P}$ will replace each of the vertices P_j, $j = 1, \ldots, k$ of $\mathbb{P}$ with a segment of length $\alpha_j = \alpha_{P_j} > 0$. Assume that $\mathbb{P} \subset \mathbb{R}^2$. We can realize $\Sigma(\mathbb{P})$ in $\mathbb{R}^3$ as follows. Let ψ_j be smooth functions supported on V_j with $\psi_j = 1$ in a neighborhood of P_j.

$$\Phi : \overline{\mathbb{P}} \smallsetminus \{P_1, P_2, \ldots, P_k\} \to \mathbb{R}^2 \times \mathbb{R}, \quad \Phi(p) = \big(p\,,\ \sum_j \psi_j(p)\theta_j(p)\big).$$

Then $\Sigma(\mathbb{P})$ is (up to a diffeomorphism) the closure of $\Phi(\mathbb{P})$ in $\mathbb{R}^3$. The desingularization map is $\kappa(p, z) = p$.

The structural Lie algebra of vector fields $\mathcal{V}(\mathbb{P})$ on $\Sigma(\mathbb{P})$ is given by (the lifts of) the smooth vector fields X on $\overline{\mathbb{P}} \smallsetminus \{P_1, P_2, \ldots, P_k\}$ that, on V_j, can be written as

$$X = a_r(r_j, \theta_j) r_j \partial_{r_j} + a_\theta(r_j, \theta_j)\partial_{\theta_j},$$

with a_r and a_θ smooth functions of (r_j, θ_j), $r_j \geq 0$. Then $(\Sigma(\mathbb{P}), \mathcal{V}(\mathbb{P}))$ is a Lie manifold with true boundary.

To define the structural Lie algebra of vector fields on $\Sigma(\mathbb{P})$, we now choose a smooth function $r_{\mathbb{P}} : \mathbb{P} \to [0, \infty)$ with the following properties

(i) $r_{\mathbb{P}}$ is continuous on $\overline{\mathbb{P}}$,
(ii) $r_{\mathbb{P}}$ is smooth on $\mathbb{P}$,
(iii) $r_{\mathbb{P}}(x) > 0$ on $\overline{\mathbb{P}} \smallsetminus \{P_1, P_2, \ldots, P_k\}$,
(iv) $r_{\mathbb{P}}(x) = r_j(x)$ if $x \in V_j$.

Note that $r_{\mathbb{P}}$ lifts to a *smooth positive function* on $\Sigma(\mathbb{P})$. Of course, $r_{\mathbb{P}}$ is determined only up to a smooth positive function ψ on $\Sigma(\mathbb{P})$ that equals to 1 in a neighborhood of the vertices.

Definition 1.10. A function of the form $\psi r_{\mathbb{P}}$, with $\psi \in \mathcal{C}^\infty(\Sigma(\mathbb{P}))$, $\psi > 0$ will be called a *canonical weight function of* $\mathbb{P}$.

In what follows, we can replace $r_{\mathbb{P}}$ with any canonical weight function. Canonical weight functions will play an important role again in Section 6. Canonical weights are example of "admissible weights," which will be used to define weighted Sobolev spaces.

Then an alternative definition of $\mathcal{V}(\mathbb{P})$ is

$$\mathcal{V}(\mathbb{P}) := \{\, r_{\mathbb{P}}\,(\psi_1\partial_1 + \psi_2\partial_2)\,\}, \quad \psi_1, \psi_2 \in \mathcal{C}^\infty(\Sigma(\mathbb{P})). \tag{10}$$

Here ∂_1 denotes the vector field corresponding to the derivative with respect to the first component. The vector field ∂_2 is defined analogously. In particular,

$$r_{\mathbb{P}}(\partial_j r_{\mathbb{P}}) = r_{\mathbb{P}}\frac{\partial r_{\mathbb{P}}}{\partial x_j} \in \mathcal{C}^\infty(\Sigma(\mathbb{P})), \tag{11}$$

which is useful in establishing that $\mathcal{V}(\mathbb{P})$ is a Lie algebra. Also, let us notice that both $\{r_{\mathbb{P}}\partial_1, r_{\mathbb{P}}\partial_2\}$ and $\{r_{\mathbb{P}}\partial_{r_{\mathbb{P}}}, \partial_\theta\}$ are local bases for $\mathcal{V}(\mathbb{P})$ on V_j. The transition functions lift to smooth functions on $\Sigma(\mathbb{P})$ defined in a neighborhood of $\kappa^{-1}(P_j)$, but cannot be extended to smooth functions defined in a neighborhood of P_j in $\overline{\mathbb{P}}$.

Then $\partial_{\mathrm{top}}\Sigma(\mathbb{P})$, the true boundary of $\Sigma(\mathbb{P})$, consists of the disjoint union of the edges of $\mathbb{P}$ (note that the interiors of these edges have disjoint closures in $\Sigma(\mathbb{P})$). Anticipating the definition of a Lie submanifold in Section 2, let us notice that $\partial_{\mathrm{top}}\Sigma(\mathbb{P})$ is a Lie submanifold, where the Lie structure consists of the vector fields on the edges that vanish at the end points of the edges.

The function ϑ used to define the Sobolev spaces $\mathcal{K}^m_a(\mathbb{P}) := \mathcal{K}^m_a(\mathbb{P};\vartheta)$ in Equation (2) is closely related to the function $r_{\mathbb{P}}$. Indeed, $\vartheta(x)$ is the distance from x to the vertices of $\mathbb{P}$. Therefore $\vartheta/r_{\mathbb{P}}$ will extend to a continuous, nowhere vanishing function on $\Sigma(\mathbb{P})$, which shows that

$$\mathcal{K}^m_a(\mathbb{P};\vartheta) = \mathcal{K}^m_a(\mathbb{P};r_{\mathbb{P}}). \tag{12}$$

If P is an order m differential operator with smooth coefficients on $\mathbb{R}^2$ and $\mathbb{P} \subset \mathbb{R}^2$ is a polygonal domain, then $r^m_{\mathbb{P}}P \in \mathrm{Diff}^m_{\mathcal{V}}(\Sigma(\mathbb{P}))$, by Equation (10). However, in general, $r^m_{\mathbb{P}}P$ will not define a smooth differential operator on $\overline{\mathbb{P}}$.

2. Submanifolds

In this section we introduce various classes of submanifolds of a Lie manifold. Some of these classes were already mentioned in the previous sections.

2.1. **General submanifolds.**

We first introduce the most general class of submanifolds of a Lie manifold.

We first fix some notation. Let $(M_0, M, \mathcal{V})$ and $(N_0, N, \mathcal{W})$ be Lie manifolds. We know that there exist vector bundles $A \to M$ and $B \to N$ such that $\mathcal{V} \simeq \Gamma(M;A)$ and $\mathcal{W} \simeq \Gamma(N;B)$, see Remark 1.5. We can assume that $\mathcal{V} = \Gamma(M;A)$ and $\mathcal{W} = \Gamma(N;B)$ and write (M,A) and (N,B) instead of $(M_0, M, \mathcal{V})$ and $(N_0, N, \mathcal{W})$.

Definition 2.1. Let (M,A) be a Lie manifold with anchor map $\varrho_M : A \to TM$. A Lie manifold (N,B) is called a *Lie submanifold* of (M,A) if

(i) N is a closed submanifold of M (possibly with corners, no transversality at the boundary required),
(ii) $\partial N = N \cap \partial M$ (that is, $N_0 \subset M_0$, $\partial N \subset \partial M$), and

(iii) B is a sub vector bundle of $A|_N$, and
(iv) the restriction of ϱ_M to B is the anchor map of $B \to N$.

Remark 2.2. An alternative form of Condition (iv) of the above definition is

$$\begin{aligned}\mathcal{W} = \Gamma(N;B) &= \{X|_N \mid X \in \Gamma(M;A) \text{ and } X|_N \text{ tangent to } N\} \\ &= \{X \in \Gamma(N; A|_N) \mid \varrho_M \circ X \in \Gamma(N;TN)\}.\end{aligned} \tag{13}$$

We have the following simple corollary that justifies Condition (iv) of Definition 2.1.

Corollary 2.3. *Let g_0 be a metric on M_0 compatible with the Lie structure at infinity on M_0. Then the restriction of g_0 to N_0 is compatible with the Lie structure at infinity on N_0.*

Proof. Let g be a metric on A whose restriction to TM_0 defines the metric g_0. Then g restricts to a metric h on B, which in turn defines a metric h_0 on N_0. By definition, h_0 is the restriction of g_0 to N_0. □

We thus see that any submanifold (in the sense of the above definition) of a Riemannian Lie manifold is itself a Riemannian Lie manifold.

2.2. **Second fundamental form.** We define the *A-normal bundle* of the Lie submanifold (N, B) of the Lie manifold (M, A) as $\nu^A = (A|_N)/B$ which is a bundle over N. Then the anchor map ϱ_M defines a map $\nu^A \to (TM|_N)/TN$, called the *anchor map of ν^A*, which is an isomorphism over N_0.

We denote the Levi-Civita-connection on A by ∇^A and the Levi-Civita connection on B by ∇^B [4]. Let $X, Y, Z \in \mathcal{W} = \Gamma(N;B)$ and $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{V} = \Gamma(M;A)$ be such that $X = \tilde{X}|_N$, $Y = \tilde{Y}|_N$, $Z = \tilde{Z}|_N$. Then $\nabla^A_{\tilde{X}}\tilde{Y}|_N$ depends only on $X, Y \in \mathcal{W} = \Gamma(N;B)$ and will be denoted $\nabla^A_X Y$ in what follows. Furthermore, the Koszul formula gives

$$\begin{aligned}2g(\tilde{Z}, \nabla^A_{\tilde{Y}}\tilde{X}) =& \partial_{\varrho_M(\tilde{X})} g(\tilde{Y}, \tilde{Z}) + \partial_{\varrho_M(\tilde{Y})} g(\tilde{Z}, \tilde{X}) - \partial_{\varrho_M(\tilde{Z})} g(\tilde{X}, \tilde{Y}) \\ &- g([\tilde{X}, \tilde{Z}], \tilde{Y}) - g([\tilde{Y}, \tilde{Z}], \tilde{X}) - g([\tilde{X}, \tilde{Y}], \tilde{Z}), \\ 2g(Z, \nabla^B_Y X) =& \partial_{\varrho_M(X)} g(Y, Z) + \partial_{\varrho_M(Y)} g(Z, X) - \partial_{\varrho_M(Z)} g(X, Y) \\ &- g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z).\end{aligned}$$

As this holds for arbitrary sections Z of $\Gamma(N;B)$ with extensions $\tilde{Z}$ on $\Gamma(M;A)$, we see that $\nabla^B_X Y$ is the tangential part of $\nabla^A_X Y|_N$.

The normal part of ∇^A then gives rise to the *second fundamental form* II defined as

$$\mathrm{II} : \mathcal{W} \times \mathcal{W} \to \Gamma(\nu^A), \quad \mathrm{II}(X, Y) := \nabla^A_X Y - \nabla^B_X Y.$$

The Levi-Civita connections ∇^A and ∇^B are torsion free, and hence II is symmetric because

$$\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = [\tilde{X}, \tilde{Y}]|_N - [X, Y] = 0.$$

A direct computation reveals also that $\mathrm{II}(X, Y)$ is tensorial in X, and hence, because of the symmetry, it is also tensorial in Y. ("Tensorial" here means $\mathrm{II}(fX, Y) = f\mathrm{II}(X, Y) = \mathrm{II}(X, fY)$, as usual.) Therefore the second fundamental form is a vector bundle morphism $\mathrm{II} : B \otimes B \to \nu^A$, and the endomorphism at $p \in M$ is denoted by $\mathrm{II}_p : B_p \otimes B_p \to A_p$. It then follows from the compactness of N that

$$\|\mathrm{II}_p(X_p, Y_p)\| \leq C\|X_p\| \, \|Y_p\|,$$

with a constant C independent of $p \in N$. Clearly, on the interior $N_0 \subset M_0$ the second fundamental form coincides with the classical second fundamental form.

Corollary 2.4. *Let (N, B) be a submanifold of (M, A) with a compatible metric. Then the (classical) second fundamental form of N_0 in M_0 is uniformly bounded.*

2.3. **Tame submanifolds.** We now introduce tame manifolds. Our main interest in tame manifolds is the global tubular neighborhood theorem, Theorem 2.7, which asserts that a tame submanifold of a Lie manifold has a tubular neighborhood in a strong sense. In particular, we will obtain that a tame submanifold of codimension one is regular. This is interesting because being tame is an algebraic condition that can be easily verified by looking at the structural Lie algebras of vector fields. On the other hand, being a regular submanifold is an analytic condition on the metric that may be difficult to check directly.

Definition 2.5. Let (N, B) be a Lie submanifold of the Lie manifold (M, A) with anchor map $\varrho_M : A \to TM$. Then (N, B) is called a *tame submanifold of* M if T_pN and $\varrho_M(A_p)$ span T_pM for all $p \in \partial N$.

Let (N, B) be a tame submanifold of the Lie manifold (M, A). Then the anchor map $\varrho_M : A \to TM$ defines an isomorphism from A_p/B_p to T_pM/T_pN for any $p \in N$. In particular, the anchor map ϱ_M maps $B^\perp$, the orthogonal complement of B in A, injectively into $\varrho_M(A) \subset TM$. For any boundary face F and $p \in F$ we have $\varrho_M(A_p) \subset T_pF$. Hence, for any $p \in N \cap F$, the space T_pM is spanned by T_pN and T_pF. As a consequence, $N \cap F$ is a submanifold of F of codimension $\dim M - \dim N$. The codimension of $N \cap F$ in F is therefore independent of F, in particular independent of the dimension of F.

Examples 2.6.
(1) Let M be any compact manifold (without boundary). Fix a $p \in M$. Let (N, B) be a manifold with a Lie structure at infinity. Then $(N_0 \times \{p\}, N \times \{p\}, B)$ is a tame submanifold of $(N_0 \times M, N \times M, B \times TM)$.
(2) If $\partial N \neq \emptyset$, the diagonal N is a submanifold of $N \times N$, but not a tame submanifold.
(3) Let N be a submanifold with corners of M such that N is transverse to all faces of M. We endow these manifolds with the b-structure at infinity $\mathcal{V}_b$ (see Example 1.7 (i)). Then $(N, \mathcal{V}_b)$ is a tame Lie submanifold of $(M, \mathcal{V}_b)$.
(4) A regular submanifold (see section 1) is a also a tame submanifold.

We now prove the main theorem of this section. Note that this theorem is not true for a general manifold of bounded geometry with boundary (for a manifold with bounded geometry and boundary, the existence of a global tubular neighborhood of the boundary is part of the definition, see [47]).

Theorem 2.7 (Global tubular neighborhood theorem)**.** *Let (N, B) be a tame submanifold of the Lie manifold (M, A). For $\epsilon > 0$, let $(\nu^A)_\epsilon$ be the set of all vectors normal to N of length smaller than ϵ. If $\epsilon > 0$ is sufficiently small, then the normal exponential map $\exp^\nu$ defines a diffeomorphism from $(\nu^A)_\epsilon$ to an open neighborhood V_ϵ of N in M. Moreover, dist$(\exp^\nu(X), N) = |X|$ for $|X| < \epsilon$.*

Proof. Recall from [4] that the exponential map $\exp : TM_0 \to M_0$ extends to a map $\exp : A \to M$. The definition of the normal exponential function $\exp^\nu$ is obtained by identifying the quotient bundle ν^A with $B^\perp$, as discussed earlier. This gives

$$\exp^\nu : (\nu^A)_\epsilon \to M.$$

The differential $d\exp^\nu$ at $0_p \in \nu^A_p$, $p \in N$ is the restriction of the anchor map to $B^\perp \cong \nu^A$, hence any point $p \in N$ has a neighborhood $U(p)$ and $\tau_p > 0$ such that

$$\exp^\nu : (\nu^A)_{\tau_p}|_{U_p} \to M \tag{14}$$

is a diffeomorphism onto its image. By compactness $\tau_p \geq \tau > 0$. Hence, $\exp^\nu$ is a local diffeomorphism of $(\nu^A)_\tau$ to a neighborhood of N in M. It remains to show that it is injective for small $\epsilon \in (0, \tau)$.

Let us assume now that there is no $\epsilon > 0$ such that the theorem holds. Then there are sequences $X_i, Y_i \in \nu^A$, $i \in \mathbb{N}$, $X_i \neq Y_i$ such that $\exp^\nu X_i = \exp^\nu Y_i$ with $|X_i|, |Y_i| \to 0$ for $i \to \infty$. After taking a subsequence we can assume that the basepoints p_i of X_i converge to p_∞ and the basepoints q_i of Y_i converge to q_∞. As the distance in M of p_i and q_i converges to 0, we conclude that $p_\infty = q_\infty$. However, $\exp^\nu$ is a diffeomorphism from $(\nu^A)_\tau|_{U(p_\infty)}$ into a neighborhood of $U(p_\infty)$. Hence, we see that $X_i = Y_i$ for large i, which contradicts the assumptions. □

We now prove that every tame codimension one Lie submanifold is regular.

Proposition 2.8. *Let (N, B) be a tame submanifold of codimension one of (M, A). We fix a unit length section X of ν^A. Theorem 2.7 states that*

$$\begin{array}{rcl} \exp^\nu : (\nu^A)_\epsilon \cong N \times (-\epsilon, \epsilon) & \to & \{x \,|\, d(x, N) < \epsilon\} =: V_\epsilon \\ (p, t) & \mapsto & \exp\big(tX(p)\big) \end{array}$$

is a diffeomorphism for small $\epsilon > 0$. Then M_0 carries a compatible metric g_0 such that $(\exp^\nu)^* g_0$ is a product metric, i. e., $(\exp^\nu)^* g_0 = g_N + dt^2$ on $N \times (-\epsilon/2, \epsilon/2)$.

Proof. Choose any compatible metric g_1 on M_0. Let g_2 be a metric on U_ϵ such that $(\exp^\nu)^* g_2 = g_1|_N + dt^2$ on $N \times (-\epsilon, \epsilon)$. Let $d(x) := dist(x, N)$. Then

$$g_0 = (\chi \circ d)\, g_1 + (1 - \chi \circ d)\, g_2,$$

has the desired properties, where the cut-off function $\chi : \mathbb{R} \to [0, 1]$ is 1 on $(-\epsilon/2, \epsilon/2)$ and has support in $(-\epsilon, \epsilon)$, and satisfies $\chi(-t) = \chi(t)$. □

The above definition shows that any tame submanifold of codimension 1 is a regular submanifold. Hence, the concept of a tame submanifold of codimension 1 is the same as that of a regular submanifolds. We hence obtain a new criterion for deciding that a given domain in a Lie manifold is regular.

Proposition 2.9. *Assume the same conditions as the previous proposition. Then $d\exp^\nu\left(\frac{\partial}{\partial t}\right)$ defines a smooth vector field on $V_{\epsilon/2}$. This vector field can be extended smoothly to a vector field Y in $\mathcal{V}$. The restriction of A to $V_{\epsilon/2}$ splits in the sense of smooth vector bundles as $A = A_1 \oplus A_2$ where $A_1|_N = \nu^A$ and $A_2|_N = B$. This splitting is parallel in the direction of Y with respect to the Levi-Civita connection of the product metric g_0, i.e. if Z is a section of A_i, then $\nabla_Y Z$ is a section of A_i as well.*

Proof. Because of the injectivity of the normal exponential map, the vector field $Y_1 := d\exp^\nu\left(\frac{\partial}{\partial t}\right)$ is well-defined, and the diffeomorphism property implies smoothness on V_ϵ. At first, we want to argue that $Y_1 \in \mathcal{V}(V_\epsilon)$. Let $\pi : S(A) \to M$ be the bundle of unit length vectors in A. Recall from [4], section 1.2 that $S(A)$ is naturally a Lie manifold, whose Lie structure is given by the *thick pullback* $\pi^{\#}(A)$ of A. Now the flow lines of Y_1 are geodesics, which yield in coordinates solutions to a second order ODE in t. In [4], section 3.4 this ODE was studied on Lie manifolds. The solutions are integral lines of the geodesic spray $\sigma : S(A) \to f^{\#}(A)$. As the integral lines of this flow stay in $S(A) \subset A$ and as they depend smoothly on the initial data and on t, we see that Y_1 is a smooth section of constant length one of $A|_{V_\epsilon}$.

Multiplying with a suitable cutoff-function with support in V_ϵ one sees that we obtain the desired extension $Y \in \mathcal{V}$. Using parallel transport in the direction of Y, the splitting $A|_N = \nu^A \oplus TN$ extends to a small neighborhood of N. This splitting is clearly parallel in the direction of Y. □

Remark 2.10. Let $N \subset M$ be a tame submanifold of the Lie manifold $(M, \mathcal{V})$ and $Y \in \mathcal{V}$ as above. If Y has length one in a neighborhood of N and is orthogonal to N, then $V := \bigcup_{|t|<\epsilon} \phi_t(N)$ will be a tubular neighborhood of N. According to the previous proposition the restriction of $A \to M$ to V has a natural product type decomposition. This justifies, in particular, that the double of a Lie manifold with boundary is again a Lie manifold, and that the Lie structure defined on the double satisfies the natural compatibility conditions with the Lie structure on a Lie manifold with boundary.

3. Sobolev spaces

In this section we study Sobolev spaces on Lie manifolds without boundary. These results will then be used to study Sobolev spaces on Lie manifolds with true boundary, which in turn, will be used to study weighted Sobolev spaces on polyhedral domains. The goal is to extend to these classes of Sobolev spaces the main results on Sobolev spaces on smooth domains.

Conventions. *Throughout the rest of this paper, $(M_0, M, \mathcal{V})$ will be a fixed Lie manifold. We also fix a compatible metric g on M_0, i. e., a metric compatible with the Lie structure at infinity on M_0, see Subsection 1.2. To simplify notation we denote the compatible metric by g instead of the previously used g_0. By Ω we shall denote an open subset of M and $\Omega_0 = \Omega \cap M_0$. The letters C and c will be used to denote possibly different constants that may depend only on (M_0, g) and its Lie structure at infinity $(M, \mathcal{V})$.*

We shall denote the volume form (or measure) on M_0 associated to g by $d\,\mathrm{vol}_g(x)$ or simply by dx, when there is no danger of confusion. Also, we shall denote by $L^p(\Omega_0)$ the resulting L^p-space on Ω_0 (i. e., defined with respect to the volume form dx). These spaces are independent of the choice of the compatible metric g on M_0, but their norms, denoted by $\|\cdot\|_{L^p}$, do depend upon this choice, although this is not reflected in the notation. Also, we shall use the fixed metric g on M_0 to trivialize all density bundles. Then the space $\mathcal{D}'(\Omega_0)$ of distributions on Ω_0 is defined, as usual, as the dual of $\mathcal{C}^\infty_{\mathrm{c}}(\Omega_0)$. The spaces $L^p(\Omega_0)$ identify with spaces of distributions on Ω_0 via the pairing

$$\langle u, \phi \rangle = \int_{\Omega_0} u(x)\phi(x)dx, \qquad \text{where } \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega_0) \text{ and } u \in L^p(\Omega_0).$$

3.1. **Definition of Sobolev spaces using vector fields and connections.** We shall define the Sobolev spaces $W^{s,p}(\Omega_0)$ in the following two cases:

- $s \in \mathbb{N} \cup \{0\}$, $1 \leq p \leq \infty$, and arbitrary open sets Ω_0 or
- $s \in \mathbb{R}$, $1 < p < \infty$, and $\Omega_0 = M_0$.

We shall denote $W^{s,p}(\Omega) = W^{s,p}(\Omega_0)$ and $W^{s,p}(M) = W^{s,p}(M_0)$. If Ω is a regular open set, then $W^{s,p}(\overline{\Omega}) = W^{s,p}(\Omega_0)$. In the case $p = 2$, we shall often write H^s instead of $W^{s,2}$. We shall give several definitions for the spaces $W^{s,p}(\Omega_0)$ and show their equivalence. This will be crucial in establishing the equivalence of various definitions of weighted Sobolev spaces on polyhedral domains. The first definition is in terms of the Levi-Civita connection ∇ on TM_0. We shall denote also by ∇ the induced connections on tensors (i. e., on tensor products of TM_0 and T^*M_0).

Definition 3.1 (∇-definition of Sobolev spaces)**.** The Sobolev space $W^{k,p}(\Omega_0)$, $k \in \mathbb{N} \cup \{0\}$, is defined as the space of distributions u on $\Omega_0 \subset M_0$ such that

$$\|u\|^p_{\nabla,W^{k,p}} := \sum_{l=1}^{k} \int_{\Omega_0} |\nabla^l u(x)|^p dx < \infty\,, \quad 1 \leq p < \infty. \tag{15}$$

For $p = \infty$ we change this definition in the obvious way, namely we require that,

$$\|u\|_{\nabla,W^{k,\infty}} := \sup |\nabla^l u(x)| < \infty\,, \quad 0 \leq l \leq k. \tag{16}$$

We introduce an alternative definition of Sobolev spaces.

Definition 3.2 (vector fields definition of Sobolev spaces)**.** Let again $k \in \mathbb{N} \cup \{0\}$. Choose a finite set of vector fields $\mathcal{X}$ such that $\mathcal{C}^\infty(M)\mathcal{X} = \mathcal{V}$. This condition is equivalent to the fact that the set $\{X(p), X \in \mathcal{X}\}$ generates A_p linearly, for any $p \in M$. Then the system $\mathcal{X}$ provides us with the norm

$$\|u\|^p_{\mathcal{X},W^{k,p}} := \sum \|X_1 X_2 \ldots X_l u\|^p_{L^p}\,, \quad 1 \leq p < \infty, \tag{17}$$

the sum being over all possible choices of $0 \leq l \leq k$ and all possible choices of not necessarily distinct vector fields $X_1, X_2, \ldots, X_l \in \mathcal{X}$. For $p = \infty$, we change this definition in the obvious way:

$$\|u\|_{\mathcal{X},W^{k,\infty}} := \max \|X_1 X_2 \ldots X_l u\|_{L^\infty}\,, \tag{18}$$

the maximum being taken over the same family of vector fields.

In particular,

$$W^{k,p}(\Omega_0) = \{u \in L^p(\Omega_0),\ Pu \in L^p(\Omega_0),\ \text{for all } P \in \mathrm{Diff}^k_{\mathcal{V}}(M)\} \tag{19}$$

Sometimes, when we want to stress the Lie structure $\mathcal{V}$ on M, we shall write $W^{k,p}(\Omega_0; M, \mathcal{V}) := W^{k,p}(\Omega_0)$.

Example 3.3. Let $\mathbb{P}$ be a curvilinear polygonal domain in the plane and let $\Sigma(\mathbb{P})^{db}$ be the "double" of $\Sigma(\mathbb{P})$, which is a Lie manifold without boundary (see Subsection 1.6). Then $\mathbb{P}$ identifies with a regular open subset of $\Sigma(\mathbb{P})^{db}$, and we have

$$\mathcal{K}^m_1(\mathbb{P}) = W^{m,2}(\mathbb{P}) = W^{m,2}(\mathbb{P}; \Sigma(\mathbb{P})^{db}, \mathcal{V}(\mathbb{P})).$$

The following proposition shows that the second definition yields equivalent norms.

Proposition 3.4. *The norms $\|\cdot\|_{\mathcal{X},W^{k,p}}$ and $\|\cdot\|_{\nabla,W^{k,p}}$ are equivalent for any choice of the compatible metric g on M_0 and any choice of a system of the finite set $\mathcal{X}$ such that $\mathcal{C}^\infty(M)\mathcal{X} = \mathcal{V}$. The spaces $W^{k,p}(\Omega_0)$ are complete Banach spaces in the resulting topology. Moreover, $H^k(\Omega_0) := W^{k,2}(\Omega_0)$ is a Hilbert space.*

Proof. As all compatible metrics g are bi-Lipschitz to each others, the equivalence classes of the $\|\cdot\|_{\mathcal{X},W^{k,p}}$-norms are independent of the choice of g. We will show that for any choice $\mathcal{X}$ and g, $\|\cdot\|_{\mathcal{X},W^{k,p}}$ and $\|\cdot\|_{\nabla,W^{k,p}}$ are equivalent. It is clear that then the equivalence class of $\|\cdot\|_{\mathcal{X},W^{k,p}}$ is independent of the choice of $\mathcal{X}$, and the equivalence class of $\|\cdot\|_{\nabla,W^{k,p}}$ is independent of the choice of g.

We argue by induction in k. The equivalence is clear for $k = 0$. We assume now that the $W^{l,p}$-norms are already equivalent for $l = 0,\dots,k-1$. Observe that if $X, Y \in \mathcal{V}$, then the Koszul formula implies $\nabla_X Y \in \mathcal{V}$ [4]. To simplify notation, we define inductively $\mathcal{X}^0 := \mathcal{X}$, and $\mathcal{X}^{i+1} = \mathcal{X}^i \cup \{\nabla_X Y \mid X, Y \in \mathcal{X}^i\}$.

By definition any $V \in \Gamma(M; T^*M^{\otimes k})$ satisfies $(\nabla\nabla V)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y} V$. This implies for $X_1,\dots,X_k \in \mathcal{X}$

$$(\underbrace{\nabla\dots\nabla f}_{k\text{-times}})(X_1,\dots,X_k) = X_1\dots X_k f + \sum_{l=0}^{k-1}\sum_{Y_j\in\mathcal{X}^{k-l}} a_{Y_1,\dots,Y_l} Y_1\dots Y_l\, f,$$

for appropriate choices of $a_{Y_1,\dots,Y_l} \in \mathbb{N}\cup\{0\}$. Hence,

$$\|(\underbrace{\nabla\dots\nabla f}_{k\text{-times}})\|_{L^p} \le C\sum \|\nabla\dots\nabla f(X_1,\dots,X_k)\|_{L^p} \le C\|f\|_{\mathcal{X},W^{k,p}}.$$

By induction, we know that $\|Y_1,\dots,Y_l f\|_{L^p} \le C\|f\|_{\nabla,W^{l,p}}$ for $Y_i \in \mathcal{X}^{k-l}$, $0 \le l \le k-1$, and hence

$$\|X_1\dots X_k f\|_{L^p} \le \underbrace{\|\nabla\dots\nabla f\|_{L^p}\|X_1\|_{L^\infty}\cdots\|X_k\|_{L^\infty}}_{\le C\|f\|_{\nabla,W^{k,p}}} + \underbrace{\sum_{l=0}^{k-1}\sum_{Y_1,\dots,Y_l\in\mathcal{X}^{k-l}} a_{Y_1,\dots,Y_l} Y_1\dots Y_l\, f}_{\le C\|f\|_{\nabla,W^{k-1,p}}}.$$

This implies the equivalence of the norms.

The proof of completeness is standard, see for example [16, 60]. ☐

We shall also need the following simple observation.

Lemma 3.5. *Let $\Omega' \subset \Omega \subset M$ be open subsets, $\Omega_0 = \Omega \cap M_0$, and $\Omega_0' = \Omega' \cap M_0$, $\Omega' \ne \emptyset$. The restriction then defines continuous operators $W^{s,p}(\Omega_0) \to W^{s,p}(\Omega_0')$. If the various choices $(\mathcal{X}, g, x_j)$ are done in the same way on Ω and Ω', then the restriction operator has norm* 1.

3.2. **Definition of Sobolev spaces using partitions of unity.** Yet another description of the spaces $W^{k,p}(\Omega_0)$ can be obtained by using suitable partitions of unity as in [54, Lemma 1.3], whose definition we now recall. See also [13, 18, 51, 52, 55, 62].

Lemma 3.6. *For any $0 < \epsilon < r_{\mathrm{inj}}(M_0)/6$ there is a sequence of points $\{x_j\} \subset M_0$, and a partition of unity $\phi_j \in \mathcal{C}_c^\infty(M_0)$, such that, for some N large enough depending only on the dimension of M_0), we have*

(i) $\mathrm{supp}(\phi_j) \subset B(x_j, 2\epsilon)$;
(ii) $\|\nabla^k\phi_j\|_{L^\infty(M_0)} \le C_{k,\epsilon}$, *with $C_{k,\epsilon}$ independent of j; and*

(iii) *the sets $B(x_j, \epsilon/N)$ are disjoint, the sets $B(x_j, \epsilon)$ form a covering of M_0, and the sets $B(x_j, 4\epsilon)$ form a covering of M_0 of finite multiplicity, i. e.,*

$$\sup_{y \in M_0} \#\{x_j \mid y \in B(x_j, 4\epsilon)\} < \infty.$$

Fix $\epsilon \in (0, r_{\mathrm{inj}}(M_0)/6)$. Let $\psi_j : B(x_j, 4\epsilon) \to B_{\mathbb{R}^n}(0, 4\epsilon)$ normal coordinates around x_j (defined using the exponential map $\exp_{x_j} : T_{x_j}M_0 \to M_0$). The uniform bounds on the Riemann tensor R and its derivatives $\nabla^k R$ imply uniform bounds on $\nabla^k d\exp_{x_j}$, which directly implies that all derivatives of ψ_j are uniformly bounded.

Proposition 3.7. *Let ϕ_i and ψ_i be as in the two paragraphs above. Let $U_j = \psi_j(\Omega_0 \cap B(x_j, 2\epsilon)) \subset \mathbb{R}^n$. We define*

$$\nu_{k,\infty}(u) := \sup_j \|(\phi_j u) \circ \psi_j^{-1}\|_{W^{k,\infty}(U_j)}$$

and, for $1 \leq p < \infty$,

$$\nu_{k,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{k,p}(U_j)}.$$

Then $u \in W^{k,p}(\Omega_0)$ if, and only if, $\nu_{k,p}(u) < \infty$. Moreover, $\nu_{k,p}(u)$ defines an equivalent norm on $W^{k,p}(\Omega_0)$.

Proof. We shall assume $p < \infty$, for simplicity of notation. The case $p = \infty$ is completely similar. Consider then $\mu(u)^p = \sum_j \|\phi_j u\|^p_{W^{k,p}(\Omega_0)}$. Then there exists $C_{k,\varepsilon} > 0$ such that

$$C_{k,\varepsilon}^{-1}\|u\|_{W^{k,p}(\Omega_0)} \leq \mu(u) \leq C_{k,\varepsilon}\|u\|_{W^{k,p}(\Omega_0)}, \tag{20}$$

for all $u \in W^{k,p}(\Omega_0)$, by Lemma 3.6 (i. e., the norms are equivalent). The fact that all derivatives of $\exp_{x_j}$ are bounded uniformly in j further shows that μ and $\nu_{k,p}$ are also equivalent. □

The proposition gives rise to a third, equivalent definition of Sobolev spaces. This definition is similar to the ones in [54, 55, 62, 61] and can be used to define the spaces $W^{s,p}(\Omega_0)$, for any $s \in \mathbb{R}$, $1 < p < \infty$, and $\Omega_0 = M_0$. The cases $p = 1$ or $p = \infty$ are more delicate and we shall not discuss them here.

Recall that the spaces $W^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$ are defined using the powers of $1 + \Delta$, see [56, Chapter V] or [60, Section 13.6].

Definition 3.8 (Partition of unity definition of Sobolev spaces)**.** Let $s \in \mathbb{R}$, and $1 < p < \infty$. Then we define

$$\|u\|^p_{W^{s,p}(M_0)} := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{21}$$

By Proposition 3.7, this norm is equivalent to our previous norm on $W^{s,p}(M_0)$ when s is a nonnegative integer.

Proposition 3.9. *The space $\mathcal{C}_c^\infty(M_0)$ is dense in $W^{s,p}(M_0)$, for $1 < p < \infty$ and $s \in \mathbb{R}$, or $1 \leq p < \infty$ and $s \in \mathbb{N} \cup \{0\}$.*

Proof. For $s \in \mathbb{N} \cup \{0\}$, the result is true for any manifold with bounded geometry, see [7, Theorem 2] or [19, Theorem 2.8], or [20]. For $\Omega_0 = M_0$, $s \in \mathbb{R}$, and $1 < p < \infty$, the definition of the norm on $W^{s,p}(M_0)$ allows us to reduce right away the proof to the case of $\mathbb{R}^n$, by ignoring enough terms in the sum defining the norm

(21). (We also use a cut-off function $0 \leq \chi \leq 1$, $\chi \in \mathcal{C}_c^\infty(B_{\mathbb{R}^n}(0, 4\epsilon))$, $\chi = 1$ on $B_{\mathbb{R}^n}(0, 4\epsilon)$.) □

We now give a characterization of the spaces $W^{s,p}(M_0)$ using interpolation, $s \in \mathbb{R}$. Let $k \in \mathbb{N} \cup \{0\}$ and let $\widetilde{W}^{-k,p}(M_0)$ be the set of distributions on M_0 that extend by continuity to linear functionals on $W^{k,q}(M_0)$, $p^{-1} + q^{-1} = 1$, using Proposition 3.9. That is, let $\widetilde{W}^{-k,p}(M_0)$ be the set of distributions on M_0 that define continuous linear functionals on $W^{k,q}(M_0)$, $p^{-1} + q^{-1} = 1$. We let

$$\widetilde{W}^{\theta k,k,p}(M_0) := [\widetilde{W}^{0,p}(M_0), W^{k,p}(M_0)]_\theta\,, \quad 0 \leq \theta \leq 1\,,$$

be the complex interpolation spaces. Similarly, we define

$$\widetilde{W}^{-\theta k,k,p}(M_0) = [\widetilde{W}^{0,p}(M_0), W^{-k,p}(M_0)]_\theta.$$

(See [12] or [58, Chapter 4] for the definition of the complex interpolation spaces.)

The following proposition is an analogue of Proposition 3.7. Its main role is to give an intrinsic definition of the spaces $W^{s,p}(M_0)$, a definition that is independent of choices.

Proposition 3.10. *Let $1 < p < \infty$ and $k > |s|$. Then we have a topological equality $\widetilde{W}^{s,k,p}(M_0) = W^{s,p}(M_0)$. In particular, the spaces $W^{s,p}(M_0)$, $s \in \mathbb{R}$, do not depend on the choice of the covering $B(x_j, \epsilon)$ and of the subordinated partition of unity and we have*

$$[W^{s,p}(M_0), W^{0,p}(M_0)]_\theta = W^{\theta s,p}(M_0)\,, \quad 0 \leq \theta \leq 1\,.$$

Moreover, the pairing between functions and distributions defines an isomorphism $W^{s,p}(M_0)^* \simeq W^{-s,q}(M_0)$, where $1/p + 1/q = 1$.

Proof. This proposition is known if $M_0 = \mathbb{R}^n$ with the usual metric [60][Equation (6.5), page 23]. In particular, $\widetilde{W}^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. As in the proof of Proposition 3.7 one shows that the quantity

$$\nu_{s,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{\tilde{W}^{s,p}(\mathbb{R}^n)}, \tag{22}$$

is equivalent to the norm on $\tilde{W}^{s,p}(M_0)$. This implies $\tilde{W}^{s,p}(M_0) = W^{s,p}(M_0)$.

Choose k large. Then we have

$$\begin{aligned}[W^{s,p}(M_0), W^{0,p}(M_0)]_\theta &= [W^{s,k,p}(M_0), W^{0,k,p}(M_0)]_\theta \\ &= W^{\theta s,k,p}(M_0) = W^{\theta s,p}(M_0).\end{aligned}$$

The last part follows from the compatibility of interpolation with taking duals. This completes the proof. □

The above proposition provides us with several corollaries. First, from the interpolation properties of the spaces $W^{s,p}(M_0)$, we obtain the following corollary.

Corollary 3.11. *Let $\phi \in W^{k,\infty}(M_0)$, $k \in \mathbb{N} \cup \{0\}$, $p \in (1, \infty)$, and $s \in \mathbb{R}$ with $k \geq |s|$. Then multiplication by ϕ defines a bounded operator on $W^{s,p}(M_0)$ of norm at most $C_k\|\phi\|_{W^{k,\infty}(M_0)}$. Similarly, any differential operator $P \in \mathrm{Diff}^m_{\mathcal{V}}(M)$ defines continuous maps $P : W^{s,p}(M_0) \to W^{s-m,p}(M_0)$.*

Proof. For $s \in \mathbb{N} \cup \{0\}$, this follows from the definition of the norm on $W^{k,\infty}(M_0)$ and from the definition of $\mathrm{Diff}^m_{\mathcal{V}}(M)$ as the linear span of differential operators of the form $fX_1 \dots X_k$, ($f \in \mathcal{C}^\infty(M) \subset W^{k,\infty}$, $X_j \in \mathcal{V}$, and $0 \le k \le m$), and from the definition of the spaces $W^{k,p}(\Omega_0)$.

For $s \le m$, the statement follows by duality. For the other values of s, the result follows by interpolation. □

Next, recall that an isomorphism $\phi : M \to M'$ of the Lie manifolds $(M_0, M, \mathcal{V})$ and $(M_0', M', \mathcal{V}')$ is defined to be a diffeomorphism such that $\phi_*(\mathcal{V}) = \mathcal{V}'$. We then have the following invariance property of the Sobolev spaces that we have introduced.

Corollary 3.12. *Let $\phi : M \to M'$ be an isomorphism of Lie manifolds, $\Omega_0 \subset M_0$ be an open subset and $\Omega' = \phi(\Omega)$. Let $p \in [1,\infty]$ if $s \in \mathbb{N} \cup \{0\}$, and $p \in (1,\infty)$ if $s \notin \mathbb{N} \cup \{0\}$. Then $f \to f \circ \phi$ extends to an isomorphism $\phi^* : W^{s,p}(\Omega') \to W^{s,p}(\Omega)$ of Banach spaces.*

Proof. For $s \in \mathbb{N} \cup \{0\}$, this follows right away from definitions and Proposition 3.4. For $-s \in \mathbb{N} \cup \{0\}$, this follows by duality, Proposition (3.10). For the other values of s, the result follows from the same proposition, by interpolation. □

Recall now that M_0 is complete [4]. Hence the Laplace operator $\Delta = \nabla^* \nabla$ is essentially self-adjoint on $\mathcal{C}^\infty_{\mathrm{c}}(M_0)$ by [17, 45]. We shall define then $(1+\Delta)^{s/2}$ using the spectral theorem.

Proposition 3.13. *The space $H^s(M_0) := W^{s,2}(M_0)$, $s \ge 0$, identifies with the domain of $(1+\Delta)^{s/2}$, if we endow the latter with the graph topology.*

Proof. For $s \in \mathbb{N} \cup \{0\}$, the result is true for any manifold of bounded geometry, by [7, Proposition 3]. For $s \in \mathbb{R}$, the result follows from interpolation, because the interpolation spaces are compatible with powers of operators (see, for example, the chapter on Sobolev spaces in Taylor's book [58]). □

The well known Gagliardo–Nirenberg–Sobolev inequality [7, 16, 19] holds also in our setting.

Proposition 3.14. *Denote by n the dimension of M_0. Assume that $1/p = 1/q - m/n$, $1 < q \le p < \infty$, where $m \ge 0$. Then $W^{s,q}(M_0)$ is continuously embedded in $W^{s-m,p}(M_0)$.*

Proof. If s and m are integers, $s \ge m \ge 0$, the statement of the proposition is true for manifolds with bounded geometry, [7, Theorem 7] or [19, Corollary 3.1.9]. By duality (see Proposition 3.10), we obtain the same result when $s \le 0$, $s \in \mathbb{Z}$. Then, for integer s, m, $0 < s < m$ we obtain the corresponding embedding by composition $W^{s,q}(M_0) \to W^{0,r}(M_0) \to W^{s-m,p}(M_0)$, with $1/r = 1/q - s/n$. This proves the result for integral values of s. For non-integral values of s, the result follows by interpolation using again Proposition 3.10. □

The Rellich-Kondrachov's theorem on the compactness of the embeddings of Proposition 3.14 for $1/p > 1/q - m/n$ is true if M_0 is compact [7, Theorem 9]. This happens precisely when $M = M_0$, which is a trivial case of a manifold with a Lie structure at infinity. On the other hand, it is easily seen (and well known) that this compactness cannot be true for M_0 non-compact. We will nevertheless obtain compactness in the next section by using Sobolev spaces with weights, see Theorem 4.6.

4. Sobolev spaces on regular open subsets

Let $\Omega \subset M$ be an open subset. Recall that Ω is a regular open subset in M if, and only if, Ω and $\overline{\Omega}$ have the same boundary in M, denoted $\partial_{\text{top}}\overline{\Omega}$,, and if $\partial_{\text{top}}\overline{\Omega}$ is a regular submanifold of M. Let $\Omega_0 = \Omega \cap M_0$. Then $\partial\Omega_0 := (\partial\Omega) \cap M_0 = \partial_{\text{top}}\overline{\Omega} \cap M_0$ is a smooth submanifold of codimension one of M_0 (see Figure 1). We shall denote $W^{s,p}(\overline{\Omega}) = W^{s,p}(\Omega) = W^{s,p}(\Omega_0)$. Throughout this section Ω will denote a regular open subset of M.

We have the following analogue of the classical extension theorem.

Theorem 4.1. *Let $\Omega \subset M$ be a regular open subset. Then there exists a linear operator E mapping measurable functions on Ω_0 to measurable functions on M_0 with the properties:*

(i) *E maps $W^{k,p}(\Omega_0)$ continuously into $W^{k,p}(M_0)$ for every $p \in [1, \infty]$ and every integer $k \geq 0$, and*
(ii) *$Eu|_{\Omega_0} = u$.*

Proof. Since $\partial\Omega_0$ is a regular submanifold we can fix a compatible metric g on M_0 and a tubular neighborhood V_0 of $\partial\Omega_0$ such that $V_0 \simeq (\partial\Omega_0) \times (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$. Let $\varepsilon = \min(\varepsilon_0, r_{\text{inj}}(M_0))/20$, where $r_{\text{inj}}(M_0) > 0$ is the injectivity radius of M_0. By Zorn's lemma and the fact that M_0 has bounded geometry we can choose a maximal, countable set of disjoint balls $B(x_i, \varepsilon)$, $i \in I$. Since this family of balls is maximal we have $M_0 = \cup_i B(x_i, 2\varepsilon)$. For each i we fix a smooth function η_i supported in $B(x_i, 3\varepsilon)$ and equal to 1 in $B(x_i, 2\varepsilon)$. This can be done easily in local coordinates around the point x_i; since the metric g is induced by a metric g on A we may also assume that all derivatives of order up to k of η_i are bounded by a constant $C_{k,\varepsilon}$ independent of i. We then set $\tilde{\eta}_i := \big(\sum_{j\in I} \eta_j^2\big)^{-1/2}\eta_i$. Then $\sum_{i\in I} \tilde{\eta}_i^2 = 1$, $\tilde{\eta}_i$ equals 1 on $B(x_i, \epsilon)$ and is supported in $B(x_i, 3\epsilon)$.

Following [56, Ch. 6] we also define two smooth cutoff functions adapted to the set Ω_0. We start with a function $\psi : \mathbb{R} \to [0, 1]$ which is equal to 1 on $[-3, 3]$ and which has support in $[-6, 6]$

Let $\varphi = (\varphi_1, \varphi_2)$ denote the isomorphism between V_0 and $\partial\Omega_0 \times (-\varepsilon_0, \varepsilon_0)$, where $\varphi_1 : V_0 \to \partial\Omega_0$ and $\varphi_2 : V_0 \to (-\varepsilon_0, \varepsilon_0)$. We define

$$\Lambda_+(x) := \begin{cases} 0 & \text{if } x \in M_0 \setminus V_0 \\ \psi(\varphi_2(x)/\varepsilon) & \text{if } x \in V_0, \end{cases}$$

and $\Lambda_-(x) := 1 - \Lambda_+(x)$. Clearly Λ_+ and Λ_- are smooth functions on M_0 and $\Lambda_+(x) + \Lambda_-(x) = 1$. Obviously, Λ_+ is supported in a neighborhood of $\partial\Omega_0$ and Λ_- is supported in the complement of a neighborhood of $\partial\Omega_0$.

Let $\partial\Omega_0 = A_1 \cup A_2 \cup \ldots$ denote the decomposition of $\partial\Omega_0$ into connected components. Let $V_0 = B_1 \cup B_2 \cup \ldots$ denote the corresponding decomposition of V_0 into connected components, namely, $B_j = \varphi^{-1}(A_j \times (-\varepsilon_0, \varepsilon_0))$. Since $\partial\Omega_0 = \partial\overline{\Omega}_0$, we have $\varphi(\Omega_0 \cap B_j) = A_j \times (-\varepsilon_0, 0)$ or $\varphi(\Omega_0 \cap B_j) = A_j \times (0, \varepsilon_0)$. Thus, if necessary, we may change the sign of φ on some of the connected components of V_0 in such a way that

$$\varphi(\Omega_0 \cap V_0) = \partial\Omega_0 \times (0, \varepsilon_0).$$

Let ψ_0 denote a fixed smooth function, $\psi_0 : \mathbb{R} \to [0,1]$, $\psi_0(t) = 1$ if $t \geq -\varepsilon$ and $\psi_0(t) = 0$ if $t \leq -2\varepsilon$, and let

$$\Lambda_0(x) = \begin{cases} 1 & \text{if } x \in \Omega_0 \setminus V_0 \\ 0 & \text{if } x \in M_0 \setminus (\Omega_0 \cup V_0) \\ \psi_0(\varphi_2(x)) & \text{if } x \in V_0. \end{cases}$$

We look now at the points x_i defined in the first paragraph of the proof. Let $J_1 = \{i \in I : d(x_i, \partial\Omega_0) \leq 10\varepsilon\}$ and $J_2 = \{i \in I : d(x_i, \partial\Omega_0) > 10\varepsilon\}$. For every point x_i, $i \in J_1$, there is a point $y_i \in \partial\Omega_0$ with the property that $B(x_i, 4\varepsilon) \subset B(y_i, 15\varepsilon)$. Let $B_{\partial\Omega_0}(y_i, 15\varepsilon)$ denote the ball in $\partial\Omega_0$ of center y_i and radius 15ε (with respect to the induced metric on $\partial\Omega_0$). Let $h_i : B_{\partial\Omega_0}(y_i, 15\varepsilon) \to B_{\mathbb{R}^{n-1}}(0, 15\varepsilon)$ denote the normal system of coordinates around the point y_i. Finally let $g_i : B_{\mathbb{R}^{n-1}}(0, 15\varepsilon) \times (-15\varepsilon, 15\varepsilon) \to V_0$ denote the map $g_i(v,t) = \varphi^{-1}(h_i^{-1}(v), t)$.

Let $E_{\mathbb{R}^n}$ denote the extension operator that maps $W^{k,p}(\mathbb{R}^n_+)$ to $W^{k,p}(\mathbb{R}^n)$ continuously, where $\mathbb{R}^n_+$ denotes the half-space $\{x : x_n > 0\}$. Clearly, $E_{\mathbb{R}^n} u|_{\mathbb{R}^n_+} = u$. The existence of this extension operator is a classical fact, for instance, see [56, Chapter 6]. For any $u \in W^{k,p}(\Omega_0)$ and $i \in J_1$ the function $(\tilde{\eta}_i u) \circ g_i$ is well defined on $\mathbb{R}^n_+$ simply by setting it equal to 0 outside the set $B_{\mathbb{R}^{n-1}}(0, 15\varepsilon) \times (0, 15\varepsilon)$. Clearly, $(\tilde{\eta}_i u) \circ g_i \in W^{k,p}(\mathbb{R}^n_+)$. We define the extension Eu by the formula

$$Eu(x) = \Lambda_0(x)\Lambda_-(x)u(x) + \Lambda_0(x)\Lambda_+(x) \sum_{i \in J_1} \tilde{\eta}_i(x) \Big(E_{\mathbb{R}^n}[(\tilde{\eta}_i u) \circ g_i] \Big)(g_i^{-1} x) \,. \tag{23}$$

Notice that for all $i \in J_2$, the function $\tilde{\eta}_i$ vanishes on the support of Λ_+, and hence

$$\sum_{i \in J_1} \tilde{\eta}_i^2(x) = \sum_{i \in I} \tilde{\eta}_i^2(x) = 1 \ \text{ in } \operatorname{supp} \Lambda_+. \tag{24}$$

This formula implies $Eu|_{\Omega_0} = u$. It remains to verify that

$$\|Eu\|_{W^{k,p}(M_0)} \leq C_k \|u\|_{W^{k,p}(\Omega_0)}.$$

This follows as in [56] using (24), the fact that the extension $E_{\mathbb{R}^n}$ satisfies the same bound, and the definition of the Sobolev spaces using partitions of unity (Proposition 3.7). □

Let Ω be a regular open subset of M and $\Omega_0 = \Omega \cap M$, as before. We shall denote by $\overline{\Omega}_0$ the closure of Ω_0 in M_0.

Theorem 4.2. *The space $\mathcal{C}_c^\infty(\overline{\Omega}_0)$ is dense in $W^{k,p}(\Omega_0)$, for $1 \leq p < \infty$.*

Proof. For any $u \in W^{k,p}(\Omega_0)$ let Eu denote its extension from Theorem 4.1, $Eu \in W^{k,p}(M_0)$. By Proposition 3.9, there is a sequence of functions $f_j \in C_c^\infty(M_0)$ with the property that

$$\lim_{j \to \infty} f_j = Eu \text{ in } W^{k,p}(M_0).$$

Thus $\lim_{j \to \infty} f_j|_{\Omega_0} = u$ in $W^{k,p}(\Omega_0)$, as desired. □

Theorem 4.3. *The restriction map $\mathcal{C}_c^\infty(\overline{\Omega}_0) \to \mathcal{C}_c^\infty(\partial\Omega_0)$ extends to a continuous map $T : W^{k,p}(\Omega_0) \to W^{k-1,p}(\partial\Omega_0)$, for $1 \leq p \leq \infty$.*

Proof. The case $p = \infty$ is obvious. In the case $1 \leq p < \infty$, we shall assume that the compatible metric on M_0 restricts to a product type metric on V_0, our distinguished tubular neighborhood of $\partial\Omega_0$. As the curvature of M_0 and the second fundamental form of $\partial\Omega_0$ in M_0 are bounded (see Corollary 2.4), there is an $\epsilon_1 > 0$ such that, in normal coordinates, the hypersurface $\partial\Omega_0$ is the graph of a function on balls of radius $\leq \epsilon_1$.

We use the definitions of the Sobolev spaces using partitions of unity, Proposition 3.7 and Lemma 3.6 with $\varepsilon = \min(\epsilon_1, \epsilon_0, r_{\mathrm{inj}}(M_0))/10$. Let $B(x_j, 2\varepsilon)$ denote the balls in the cover of M_0 in Lemma 3.6, let $\psi_j : B(\epsilon, x_j) \to B(\epsilon, 0)$ denote normal coordinates based in x_j, and let $1 = \sum_j \phi_j$ be a corresponding partition of unity. Then $\widetilde{\phi}_j = \phi_j|_{\partial\Omega_0}$ form a partition of unity on $\partial\Omega_0$.

Start with a function $u \in W^{k,p}(\Omega_0)$ and let $u_j = (u\phi_j) \circ \psi_j^{-1}$, $u_j \in W^{k,p}(\psi_j(\Omega_0 \cap B(x_j, 4\varepsilon)))$. In addition $u_j \equiv 0$ outside the set $\psi_j(\Omega_0 \cap B(x_j, 2\varepsilon))$. If $B(x_j, 4\varepsilon) \cap \partial\Omega_0 = \emptyset$ let $\widetilde{T}(u_j) = 0$. Otherwise notice that $B(x_j, 4\varepsilon)$ is included in V_0, the tubular neighborhood of $\partial\Omega_0$, thus the set $\psi_j(\partial\Omega_0 \cap B(x_j, 4\varepsilon))$ is the intersection of a graph and the ball $B_{\mathbb{R}^n}(0, 4\varepsilon)$. We can then let $\widetilde{T}(u_j)$ denote the Euclidean restriction of u_j to $\psi_j(\partial\Omega_0 \cap B(x_j, 4\varepsilon))$ (see [16, Section 5.5]). Clearly $\widetilde{T}(u_j)$ is supported in $\psi_j(\partial\Omega_0 \cap B(x_j, 2\varepsilon))$ and

$$\|\widetilde{T}(u_j) \circ \widetilde{\psi}_j\|_{W^{k-1,p}(\partial\Omega_0)} \leq C\|u_j\|_{W^{k,p}(\psi_j(\Omega_0 \cap B(x_j, 4\varepsilon)))},$$

where $\widetilde{\psi}_j = \psi_j|_{\Omega_0}$ and the constant C is independent of j (recall that $\psi_j(\partial\Omega_0 \cap B(x_j, 4\varepsilon))$ is the intersection of a hyperplane and the ball $B_{\mathbb{R}^n}(0, 4\varepsilon)$). Let

$$Tu = \sum_j \widetilde{T}(u_j) \circ \widetilde{\psi}_j.$$

Since the sum is uniformly locally finite, Tu is well-defined and we have

$$\begin{aligned}\|Tu\|^p_{W^{k-1,p}(\partial\Omega_0)} &\leq C \sum_j \|\widetilde{T}(u_j) \circ \widetilde{\psi}_j\|^p_{W^{k-1,p}(\partial\Omega_0)} \\ &\leq C \sum_j \|u_j\|^p_{W^{k,p}(\psi_j(\Omega_0 \cap B(x_j, 4\varepsilon)))} \leq C\|u\|^p_{W^{k,p}(\Omega_0)},\end{aligned}$$

with constants C independent of u. The fact that $Tu|_{\mathcal{C}_c^\infty(\Omega_0)}$ is indeed the restriction operator follows immediately from the definition. □

We shall see that if $p = 2$, we get a surjective map $W^{s,2}(\Omega_0) \to W^{s-1/2,2}(\partial\Omega_0)$ (Theorem 4.7).

In the following, ∂_ν denotes derivative in the normal direction of the hypersurface $\partial\Omega_0 \subset M_0$.

Theorem 4.4. *The closure of $\mathcal{C}_c^\infty(\Omega_0)$ in $W^{k,p}(\Omega_0)$ is the intersection of the kernels of $T \circ \partial_\nu^j : W^{k,p}(\Omega_0) \to W^{k-j-1,p}(\Omega_0)$, $0 \leq j \leq k-1$, $1 \leq p < \infty$.*

Proof. The proof is reduced to the Euclidean case [1, 16, 33, 58] following the same pattern of reasoning as in the previous theorem. □

The Gagliardo–Nirenberg–Sobolev theorem holds also for manifolds with boundary.

Theorem 4.5. *Denote by n the dimension of M and let $\Omega \subset M$ be a regular open subset in M. Assume that $1/p = 1/q - m/n > 0$, $1 \leq q < \infty$, where $m \leq k$ is an integer. Then $W^{k,q}(\Omega_0)$ is continuously embedded in $W^{k-m,p}(\Omega_0)$.*

Proof. This can be proved using Proposition 3.14 and Theorem 4.1. Indeed, denote by

$$j : W^{k,q}(M_0) \to W^{k-m,p}(M_0)$$

the continuous inclusion of Proposition 3.14. Also, denote by r the restriction maps $W^{k,p}(M_0) \to W^{k,p}(\Omega_0)$. Then the maps

$$W^{k,q}(\Omega_0) \xrightarrow{E} W^{k,q}(M_0) \xrightarrow{j} W^{k-m,p}(M_0) \xrightarrow{r} W^{k-m,p}(\Omega_0)$$

are well defined and continuous. Their composition is the inclusion of $W^{k,q}(\Omega_0)$ into $W^{k-m,p}(\Omega_0)$. This completes the proof. □

For the proof of a variant of Rellich–Kondrachov's compactness theorem, we shall need Sobolev spaces with weights. Let $\Omega \subset M$ be a regular open subset. Let $a_H \in \mathbb{R}$ be a parameter associated to each boundary hyperface (i. e., face of codimension one) of the manifold with corners $\overline{\Omega}$. Fix for any boundary hyperface $H \subset \overline{\Omega}$ a defining function ρ_H, that is a function $\rho_H \geq 0$ such that $H = \{\rho_H = 0\}$ and $d\rho_H \neq 0$ on H. Let

$$\rho = \prod \rho_H^{a_H}, \tag{25}$$

the product being taken over all boundary hyperfaces of $\overline{\Omega}$. A function of the form $\psi\rho$, with $\psi > 0$, ψ smooth on $\overline{\Omega}$, and ρ as in Equation (25) will be called an *admissible weight* of $\overline{\Omega}$ (or simply an admissible weight when Ω is understood). We define then the weighted Sobolev space $W^{k,p}(\Omega_0)$ by

$$\rho W^{k,p}(\Omega_0) := \{\rho u, u \in W^{k,p}(\Omega_0)\}, \tag{26}$$

with the norm $\|\rho^s u\|_{\rho^s W^{k,p}(\Omega_0)} := \|u\|_{W^{k,p}(\Omega_0)}$.

Note that in the definition of an admissible weight of $\overline{\Omega}$, for a regular open subset $\Omega \subset M$ of the Lie manifold $(M, \mathcal{V})$, we allow also powers of the defining functions of the boundary hyperfaces contained in $\partial\Omega = \partial_{\mathrm{top}}\overline{\Omega}$, the true boundary of $\overline{\Omega}$. In the next compactness theorem, however, we shall allow only the powers of the defining functions of M, or, which is the same thing, only powers of the defining functions of the boundary hyperfaces of $\overline{\Omega}$ whose union is $\partial_\infty\overline{\Omega}$ (see Figure 1).

Theorem 4.6. *Denote by n the dimension of M and let $\Omega \subset M$ be a regular open subset, $\Omega_0 = \Omega \cap M_0$. Assume that $1/p > 1/q - m/n > 0$, $1 \leq q < \infty$, where $m \in \{1, \ldots, k\}$ is an integer, and that $s > s'$ are real parameters. Then $\rho^s W^{k,q}(\Omega_0)$ is compactly embedded in $\rho^{s'} W^{k-m,p}(\Omega_0)$ for any admissible weight $\rho := \prod_H \rho_H^{a_H}$ of M such that $a_H > 0$ for any boundary hyperface H of M.*

Proof. The same argument as that in the proof of Theorem 4.5 allows us to assume that $\Omega_0 = M_0$. The norms are chosen such that $W^{k,p}(\Omega_0) \ni u \mapsto \rho^s u \in \rho^s W^{k,p}(\Omega_0)$ is an isometry. Thus, it is enough to prove that $\rho^s : W^{k,q}(\Omega_0) \to W^{k-m,p}(\Omega_0)$, $s > 0$, is a compact operator.

For any defining function ρ_H and any $X \in \mathcal{V}$, we have that $X(\rho_H)$ vanishes on H, since X is tangent to H. We obtain that $X(\rho^s) = \rho^s f_X$, for some $f_X \in \mathcal{C}^\infty(M)$. Then, by induction, $X_1 X_2 \ldots X_k(\rho^s) = \rho^s g$, for some $g \in \mathcal{C}^\infty(M)$.

Let $\chi \in \mathcal{C}^\infty([0,\infty)$ be equal to 0 on $[0, 1/2]$, equal to 1 on $[1,\infty)$, and non-negative everywhere. Define $\phi_\epsilon = \chi(\epsilon^{-1}\rho^s)$. Then

$$\|X_1 X_2 \ldots X_k\big(\rho^s\phi_\epsilon - \rho^s\big)\|_{L^\infty} \to 0\,, \quad \text{as } \epsilon \to 0,$$

for any $X_1, X_2, \ldots, X_k \in \mathcal{V}$. Corollary 3.11 then shows that $\rho^s \phi_\epsilon \mapsto \rho^s$ in the norm of bounded operators on $W^{s,p}(\Omega_0)$. But multiplication by $\rho^s \phi_\epsilon$ is a compact operator, by the Rellich-Kondrachov's theorem for compact manifolds with boundary [7, Theorem 9]. This completes the proof. $\square$

We end with the following generalization of the classical restriction theorem for the Hilbertian Sobolev spaces $H^s(M_0) := W^{s,2}(M_0)$.

Theorem 4.7. *Let $N_0 \subset M_0$ be a tame submanifold of codimension k of the Lie manifold $(M_0, M, \mathcal{V})$. Restriction of smooth functions extends to a bounded, surjective map*

$$H^s(M_0) \to H^{s-k/2}(N_0),$$

for any $s > k/2$. In particular, $H^s(\Omega_0) \to H^{s-1/2}(\partial\Omega_0)$ is continuous and surjective.

Proof. Let $B \to N$ be the vector bundle defining the Lie structure at infinity (N, B) on N_0 and $A \to M$ be the vector bundle defining the Lie structure at infinity (M, A) on M_0. (See Section 2 for further explanation of this notation.) The existence of tubular neighborhoods, Theorem 2.7, and a partition of unity argument, allows us to assume that $M = N \times S^1$ and that $A = B \times TS^1$ (external product). Since the Sobolev spaces $H^s(M_0)$ and $H^{s-1/2}(N_0)$ do not depend on the metric on A and B, we can assume that the circle S^1 is given the invariant metric making it of length 2π and that M_0 is given the product metric. The rest of the proof now is independent of the way we obtain the product metric on M_0.

Let S^1 be the unit circle in the plane. Let us denote by Δ_M, Δ_N, and Δ_{S^1} the Laplace operators on M_0, N_0, and S^1, respectively. Then $\Delta_M = \Delta_N + \Delta_{S^1}$ and $\Delta_{S^1} = -\partial^2/\partial\theta^2$ has spectrum $\{4\pi^2 n^2 \,|\, n \in \mathbb{N} \cup \{0\}\}$. We can decompose $L^2(N_0 \times S^1)$ according to the eigenvalues $n \in \mathbb{Z}$ of $-\frac{1}{2\pi\imath}\partial_\theta$:

$$L^2(N_0 \times S^1) \simeq \bigoplus_{n \in \mathbb{Z}} L^2(N_0 \times S^1)_n \simeq \bigoplus_{n \in \mathbb{Z}} L^2(N_0),$$

where the isomorphism $L^2(N_0 \times S^1)_n \simeq L^2(N_0)$ is obtained by restricting to $N_0 = N_0 \times \{1\}$, $1 \in S^1$. We use this isomorphism to identify the above spaces in what follows.

Let $\xi \in L^2(N_0 \times S^1)$. Then ξ identifies with a sequence (ξ_n) under the above isomorphism. By Proposition 3.13, we have that $\xi \in H^s(N_0 \times S^1)$ if, and only if, $(1 + \Delta_M)^{s/2}\xi = \sum_n \big((1 + n^2 + \Delta_N)^{s/2}\xi_n\big) \in \bigoplus_{n \in \mathbb{Z}} L^2(N_0) \simeq L^2(N_0 \times S^1)$. The restriction of ξ to N_0 is then given by $\sum_n \xi_n$. We want to show that $\sum_n \xi_n \in H^{s-1/2}(N_0)$, which is equivalent to $(1 + \Delta_N)^{s/2-1/4}\big(\sum \xi_n\big) \in L^2(N_0)$.

The spectral spaces of Δ_N corresponding to $[m, m+1) \subset \mathbb{R}$, $m \in \mathbb{N} \cup \{0\}$ give an orthogonal direct sum decomposition of $L^2(N_0)$.

We decompose $\xi_n = \sum_m \xi_{mn}$, with ξ_{mn} in the spectral space corresponding to $[m, m+1)$ of Δ_N. Note that ξ_{mn} is orthogonal to $\xi_{m'n}$ for $m \neq m'$. Denote $h = (1 + m^2)^{-1/2}$, $f(t) = (1 + t^2)^{-s}$, and $C = 1 + \int_{\mathbb{R}} f(t)dt$. Then an application of the Cauchy–Schwartz inequality gives

$$
\begin{aligned}
(27)\quad & (1+m^2)^{s-1/2}\Big(\sum_n \|\xi_{mn}\|\Big)^2 \\
& \leq (1+m^2)^{s-1/2}\Big(\sum_n (1+n^2+m^2)^{-s}\Big)\sum_n \|(1+n^2+m^2)^{s/2}\xi_{mn}\|^2 \\
& \leq h\Big(\sum_n f(nh)\Big)\sum_n \|(1+n^2+m^2)^{s/2}\xi_{mn}\|^2 \leq C_s \sum_n \|(1+n^2+m^2)^{s/2}\xi_{mn}\|^2.
\end{aligned}
$$

The constant C_s is independent of m (but depends on s). We sum over m and obtain

$$
\begin{aligned}
(28)\quad & \|\sum_n (1+\Delta_N)^{s/2-1/4}\xi_n\|^2 = \sum_m \|\sum_n (1+\Delta_N)^{s/2-1/4}\xi_{nm}\|^2 \\
& \leq \sum_m (1+(m+1)^2)^{s-1/2}\Big(\sum_n \|\xi_{nm}\|\Big)^2 \leq 2^s \sum_m (1+m^2)^{s-1/2}\Big(\sum_n \|\xi_{nm}\|\Big)^2 \\
& \leq 2^s C_s \sum_{n,m} \|(1+n^2+m^2)^{s/2}\xi_{nm}\|^2 \leq 2^s C_s \sum_{n,m} \|(1+n^2+\Delta_N)^{s/2}\xi_{nm}\|^2 \\
& = 2^s C_s \sum_n \|(1+n^2+\Delta_N)^{s/2}\xi_n\|^2,
\end{aligned}
$$

with the same constant C_s as in Equation (27). This shows that $\zeta := \sum_n \xi_n \in H^{s-1/2}(N_0)$ if $\xi = (\xi_n) \in \bigoplus_n L^2(N_0) \simeq L^2(N_0 \times S^1)$ is a *finite* sequence such that $\|\xi\|_{H^s} := \sum_n \|(1+n^2+\Delta_N)^{s/2}\xi_n\|^2_{L^2(N_0)} < \infty$, and that ζ depends continuously on $\xi \in H^s(N_0 \times S^1)$. This completes the proof. $\square$

We finally obtain the following consequences for a curvilinear polygonal domain $\mathbb{P}$ (see Subsection 1.6). First, recall that the distance $\vartheta(x)$ from x to the vertices of a curvilinear polygon $\mathbb{P}$ and $r_{\mathbb{P}}$ have bounded quotients, and hence define the same weighted Sobolev spaces (Equation (12)). Moreover, the function $r_{\mathbb{P}}$ is an admissible weight. Recall that $\mathbb{P}$ has a compactification $\Sigma(\mathbb{P})$ that is a Lie manifold with boundary (that is, the closure of a regular open subset of a Lie manifold M). Let us write $W^{m,p}(\Sigma(\mathbb{P})) := W^{m,p}(\mathbb{P})$ the Sobolev spaces defined by the structural Lie algebra of vector fields on $\Sigma(\mathbb{P})$. Then

$$
(29)\qquad \mathcal{K}^m_a(\mathbb{P};\vartheta) = r_\Omega^{a-1}\mathcal{K}^m_1(\mathbb{P};r_{\mathbb{P}}) = r_{\mathbb{P}}^{a-1}W^{m,2}(\Sigma(\mathbb{P})).
$$

This identifies the weighted Sobolev spaces on $\mathbb{P}$ with a weighted Sobolev space of the form $\rho W^{k,p}(\Omega_0)$.

Motivated by Equation (29), we now define

$$
(30)\qquad \mathcal{K}^m_a(\partial\mathbb{P}) = \mathcal{K}^m_a(\partial\mathbb{P};\vartheta) = \mathcal{K}^m_a(\partial\mathbb{P};r_{\mathbb{P}}) = r_{\mathbb{P}}^{a-1/2}W^{m,2}(\partial\mathbb{P}).
$$

More precisely, let us notice that we can identify each edge with $[0,1]$. Then $\mathcal{K}^m_a(\partial\mathbb{P})$ consists of the functions $f : \partial\mathbb{P} \to \mathbb{C}$ that, on each edge, are such that $t^k(1-t)^k f^{(k)} \in L^2([0,1])$, $0 \leq k \leq m$ (here we identify that edge with $[0,1]$). This last condition is equivalent to $[t(1-t)\partial_t]^k f \in L^2([0,1])$, $0 \leq k \leq m$.

Proposition 4.8. *Let $\mathbb{P} \subset \mathbb{R}^2$ be a curvilinear polygonal domain and P be a differential operator of order m with coefficients in $\mathcal{C}^\infty(\overline{\mathbb{P}})$. Then $P_\lambda := r_{\mathbb{P}}^\lambda P r_{\mathbb{P}}^{-\lambda}$ defines a continuous family of bounded maps $P_\lambda : \mathcal{K}^s_a(\mathbb{P}) \to \mathcal{K}^{s-m}_{a-m}(\mathbb{P})$, for any $s, a \in \mathbb{R}$. Let $\mathbb{P}'$ be $\overline{\mathbb{P}}$ with the vertices removed. Then $\mathcal{C}^\infty_c(\mathbb{P}')$ is dense in $\mathcal{K}^m_a(\mathbb{P})$.*

Also, the restriction to the boundary extends to a continuous, surjective trace map $\mathcal{K}_a^s(\mathbb{P}) \to \mathcal{K}_{a-1/2}^{s-1/2}(\partial\mathbb{P})$. *If* $s = 1$, *then the kernel of the trace map is the closure of* $\mathcal{C}_c^\infty(\mathbb{P})$ *in* $\mathcal{K}_a^1(\mathbb{P})$.

The above proposition, except maybe for the description of the restrictions to the boundary, is well known in two dimensions. It will serve as a model for the results in three dimensions that we present in the last section.

5. A regularity result

We include in this section an application to the regularity of boundary value problems, Theorem 5.1. Its proof is reduced to the Euclidean case using a partition of unity argument and the tubular neighborhood theorem 2.7, both of which require some non-trivial input from differential geometry.

Let us introduce some notation first that will be also useful in the following. Let $\exp : TM_0 \longrightarrow M_0 \times M_0$ be given by $\exp(v) := (x, \exp_x(v))$, $v \in T_xM_0$. If E is a real vector bundle with a metric, we shall denote by $(E)_r$ the set of all vectors v of E with $|v| < r$. Let $(M_0^2)_r := \{(x,y), x, y \in M_0, d(x,y) < r\}$. Then the exponential map defines a diffeomorphism $\exp : (TM_0)_r \to (M_0^2)_r$. We shall also need the admissible weight function ρ defined in Equation (25) and the weighted Sobolev spaces $\rho^s W^{k,p}(\Omega_0) := \{\rho^s u, u \in W^{k,p}(\Omega_0)\}$ introduced in Equation 26.

Recall [58], Chapter 5, Equation (11.79), that a differential operator P of order m is called *strongly elliptic* if there exists $C > 0$ such that $Re\big(\sigma^{(m)}(P)(\xi)\big) \geq C\|\xi\|^m$ for all ξ.

Theorem 5.1. *Let* $\Omega \subset M$ *be a regular open subset of the Lie manifold* $(M, \mathcal{V})$. *Let* $P \in \mathrm{Diff}^2_{\mathcal{V}}(M)$ *be an order* 2 *strongly elliptic operator on* M_0 *generated by* $\mathcal{V}$ *and* $s \in \mathbb{R}$, $t \in \mathbb{Z}$, $1 < p < \infty$. *Then there exists* $C > 0$ *such that, for any* $u \in \rho^s W^{1,p}(\Omega_0)$, $u|_{\partial\Omega_0} = 0$, *we have*

$$\|u\|_{\rho^s W^{t+2,p}(\Omega_0)} \leq C(\|Pu\|_{\rho^s W^{t,p}(\Omega_0)} + \|u\|_{\rho^s L^p(\Omega_0)}).$$

In particular, let $u \in \rho^s W^{1,p}(\Omega_0)$ *be such that* $Pu \in \rho^s W^{t,p}(\Omega_0)$, *and* $u|_{\partial\Omega_0} = 0$, *then* $u \in \rho^s W^{t+2,p}(\Omega_0)$.

Proof. Note that, locally, this is a well known statement. In particular, $\phi u \in W^{t+2,p}(\Omega_0)$, for any $\phi \in \mathcal{C}_c^\infty(M_0)$. The result will follow then if we prove that

$$\|u\|_{\rho^s W^{t+2,p}(M_0)} \leq C(\|Pu\|_{\rho^s W^{t,p}(M_0)} + \|u\|_{\rho^s L^p(M_0)}) \tag{31}$$

for any $u \in W^{t+2,p}_{\mathrm{loc}}(\Omega_0)$. Here, of course, $\|u\|_{\rho^s L^p(M_0)} = \|\rho^{-s}u\|_{L^p(M_0)}$ (see Equation (26)).

Let $r < r_{\mathrm{inj}}(M_0)$ and let $\exp : (TM_0)_r \to (M_0^2)_r$ be the exponential map. The statement is trivially true for $t \leq -2$, so we will assume $t \geq -1$ in what follows. Also, we will assume first that $s = 0$. The general case will be reduced to this one at the end. Assume first that $\Omega_0 = M_0$.

Let P_x be the differential operators defined on $B_{T_xM_0}(0,r)$ obtained from P by the local diffeomorphism $\exp : B_{T_xM_0}(0,r) \to M_0$. We claim that there exists a constant $C > 0$, independent of $x \in M_0$ such that

$$\|u\|^p_{W^{t+2,p}(T_xM_0)} \leq C\big(\|P_xu\|^p_{W^{t,p}(T_xM_0)} + \|u\|^p_{L^p(T_xM_0)}\big), \tag{32}$$

for any function $u \in \mathcal{C}_c^\infty(B_{T_xM_0}(0,r))$. This is seen as follows. We can find a constant $C_x > 0$ with this property for any $x \in M_0$ by the ellipticity of P_x. (For

$p = 2$, a complete proof can be found in [58], Propositions 11.10 and 11.16. For general p, the result can be proved as [16], Theorem 1 in subsection 5.8.1, page 275.) Choose C_x to be the least such constant. Let $\pi : A \to M$ be the extension of the tangent bundle of M_0, see Remark 1.5 and let $A_x = \pi^{-1}(x)$. The family P_x, $x \in M_0$, extends to a family P_x, $x \in M$, that is smooth in x. The smoothness of the family P_x in $x \in M$ shows that C_x is upper semi-continuous (i. e., the set $\{C_x < \eta\}$ is open for any x). Since M is compact, C_x will attain its maximum, which therefore must be positive. Let C be that maximum value.

Let now ϕ_j be the partition of unity and ψ_j be the diffeomorphisms appearing in Equation (22), for some $0 < \epsilon < r/6$. In particular, the partition of unity ϕ_j satisfies the conditions of Lemma 3.6, which implies that $\mathrm{supp}(\phi_j) \subset B(x_j, 2\epsilon)$ and the sets $B(x_j, 4\epsilon)$ form a covering of M_0 of finite multiplicity. Let $\eta_j = 1$ on the support of ϕ_j, $\mathrm{supp}(\eta_j) \subset B(x_j, 4\epsilon)$. We then have

$$\begin{aligned}
\nu_{t+2,p}(u)^p &:= \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{t+2,p}(\mathbb{R}^n)} \\
&\leq C \sum_j \Big(\|P_x(\phi_j u)\|^p_{W^{t,p}(T_x M_0)} + \|\phi_j u\|^p_{L^p(T_x M_0)} \Big) \\
&\leq C \sum_j \Big(\|\phi_j P_x u\|^p_{W^{t,p}(T_x M_0)} + \|[P_x, \phi_j] u\|^p_{W^{t,p}(T_x M_0)} + \|\phi_j u\|^p_{L^p(T_x M_0)} \Big) \\
&\leq C \sum_j \Big(\|\phi_j P_x u\|^p_{W^{t,p}(T_x M_0)} + \|\eta_j u\|^p_{W^{t+1,p}(T_x M_0)} + \|\phi_j u\|^p_{L^p(T_x M_0)} \Big) \\
&\leq C\big(\nu_{t,p}(Pu)^p + \nu_{t+1}(u)^p\big).
\end{aligned}$$

The equivalence of the norm $\nu_{s,p}$ with the standard norm on $W^{s,p}(M_0)$ (Propositions 3.7 and 3.10) shows that $\|u\|_{W^{t+2,p}(M_0)} \leq C(\|Pu\|_{W^{t,p}(M_0)} + \|u\|_{W^{t+1,p}(M_0)})$, for any $t \geq -1$. This is known to imply

$$\|u\|_{W^{t+2,p}(M_0)} \leq C(\|Pu\|_{W^{t,p}(M_0)} + \|u\|_{L^p(M_0)}) \tag{33}$$

by a boot-strap procedure, for any $t \geq -1$. This proves our statement if $s = 0$ and $\Omega_0 = M_0$.

The case of arbitrary domains Ω_0 follows in exactly the same way, but using a product type metric in a neighborhood of $\partial_{\mathrm{top}}\Omega_0$ and the analogue of Equation (32) for a half-space, which shows that Equation (31) continues to hold for M_0 replaced with Ω_0.

The case of arbitrary $s \in \mathbb{R}$ is obtained by applying Equation (33) to the elliptic operator $\rho^{-s} P \rho^s \in \mathrm{Diff}^2_{\mathcal{V}}(M)$ and to the function $\rho^{-s} u \in W^{k,p}(\Omega_0)$, which then gives Equation (31) right away. □

For $p = 2$, by combining the above theorem with Theorem 4.7, we obtain the following corollary.

Corollary 5.2. *We keep the assumptions of Theorem 5.1. Let $u \in \rho^s H^1(\Omega_0)$ be such that $Pu \in \rho^s H^t(\Omega_0)$ and $u|_{\partial\Omega_0} \in \rho^s H^{t+3/2}(\Omega_0)$, $s \in \mathbb{R}$, $t \in \mathbb{Z}$. Then $u \in \rho^s H^{t+2}(\Omega_0)$ and*

$$\|u\|_{\rho^s H^{t+2}(\Omega_0)} \leq C(\|Pu\|_{\rho^s H^t(\Omega_0)} + \|u\|_{\rho^s L^2(\Omega_0)} + \|u|_{\partial\Omega_0}\|_{\rho^s H^{t+3/2}(\Omega_0)}). \tag{34}$$

Proof. For $u|_{\partial\Omega_0} = 0$, the result follows from Theorem 5.1. In general, choose a suitable $v \in H^{t+2}(\Omega_0)$ such that $v|_{\partial\Omega_0} = u|_{\partial\Omega_0}$, which is possible by Theorem 4.7. Then we use our result for $u - v$. □

6. Polyhedral domains in three dimensions

We now include an application of our results to polyhedral domains $\mathbb{P} \subset \mathbb{R}^3$. A *polyhedral domain in* $\mathbb{P} \subset \mathbb{R}^3$ is a bounded, connected open set such that $\partial\mathbb{P} = \partial\overline{\mathbb{P}} = \bigcup \overline{D}_j$ where

- each D_j is a polygonal domain with straight edges contained in an affine 2-dimensional subspace of $\mathbb{R}^3$
- each edge is contained in exactly two closures of polygonal domains $\overline{D}_j$.

(See Subsection 1.6 for the definition of a polygonal domain.)

The vertices of the polygonal domains D_j will form the vertices of $\mathbb{P}$. The edges of the polygonal domains D_j will form the edges of $\mathbb{P}$. For each vertex P of $\mathbb{P}$, we choose a small open ball V_P centered in P. We assume that the neighborhoods V_P are chosen to be disjoint. For each vertex P, there exists a unique closed polyhedral cone C_P with vertex at P, such that $\overline{\mathbb{P}} \cap V_P = C_P \cap V_P$. Then $\mathbb{P} \subset \bigcup C_P$.

We now proceed to define canonical weight functions of $\mathbb{P}$ in analogy with the definition of canonical weights of curvilinear polygonal domains, Definition 1.10. We want to define first a continuous function $r_{\mathbb{P}} : \overline{\Omega} \to [0, \infty)$ that is positive and differentiable outside the edges. Let $\vartheta(x)$ be the distance from x to the edges of $\mathbb{P}$, as before. We want $r_{\mathbb{P}}(x) = \vartheta(x)$ close to the edges but far from the vertices and we want the quotients $r_{\mathbb{P}}(x)/\vartheta(x)$ and $\vartheta(x)/r_{\mathbb{P}}(x)$ to extend to continuous functions on $\overline{\Omega}$. Using a smooth partition of unity, in order to define $r_{\mathbb{P}}$, we need to define it close to the vertices.

Let us then denote by $\{P_k\}$ the set of vertices of $\mathbb{P}$. Choose a continuous function $r : \overline{\mathbb{P}} \to [0, \infty)$ such that $r(x)$ is the distance from x to the vertex P if $x \in V_P \cap \mathbb{P}$, and such that $r(x)$ is differentiable and positive on $\overline{\mathbb{P}} \smallsetminus \{P_k\}$. Let S^2 be the unit sphere centered at P and let r_P be a canonical weight associated to the curvilinear polygon $C_P \cap S^2$ (see Definition 1.10). We extend this function to C_P to be constant along the rays, except at P, where $r_P(P) = 0$. Finally, we let $r_{\mathbb{P}}(x) = r(x) r_P(x)$, for x close to P. Then a *canonical weight of* $\mathbb{P}$ is any function of the form $\psi r_{\mathbb{P}}$, where ψ is a smooth, nowhere vanishing function on $\overline{\mathbb{P}}$.

For any canonical weight $r_{\mathbb{P}}$, we then we have the following analogue of Equation (12)

$$\mathcal{K}_a^m(\mathbb{P}) := \mathcal{K}_a^m(\mathbb{P}; \vartheta) = \mathcal{K}_a^m(\mathbb{P}; r_{\mathbb{P}}). \tag{35}$$

Let us define, for every vertex P of $\mathbb{P}$, a spherical coordinate map $\Theta_P : \mathbb{P} \smallsetminus \{P\} \to S^2$ by $\Theta_P(x) = |x - P|^{-1}(x - P)$. Then, for each edge $e = [AB]$ of $\mathbb{P}$ joining the vertices A and B, we define a generalized cylindrical coordinate system (r_e, θ_e, z_e) to satisfy the following properties:

(i) $r_e(x)$ be the distance from x to the line containing e.
(ii) A as the origin (i. e., $r_e(A) = z_e(A) = 0$),
(iii) $\theta_e = 0$ on one of the two faces containing e, and
(iv) $z_e \geq 0$ on the edge e.

Let $\psi : S^2 \to [0, 1]$ be a smooth function on the unit sphere that is equal to 1 in a neighborhood of $(0, 0, 1) = \{\phi = 0\} \cap S^2$ and is equal to 0 in a neighborhood of

$(0,0,-1) = \{\phi = \pi\} \cap S^2$. Then we let

$$\tilde{\theta}_e(x) = \theta_e(x)\psi(\Theta_A(x))\psi(-\Theta_B(x))$$

where $\theta_e(x)$ is the θ coordinate of x in a cylindrical coordinate system (r, θ, z) in which the point A corresponds to the origin (i. e., $r = 0$ and $z = 0$) and the edge AB points in the positive direction of the z axis (i. e., B corresponds to $r = 0$ and $z > 0$). By choosing ψ to have support small enough in S^2 we may assume that the function $\tilde{\theta}_e$ is defined everywhere on $\mathbb{P} \smallsetminus e$. (This is why we need the cut-off function ψ.)

We then consider the function

$$\Phi : \mathbb{P} \to \mathbb{R}^N, \quad \Phi(x) = (x, \Theta_P(x), r_e(x), \tilde{\theta}_e(x)),$$

with $N = 3 + 3n_v + 2n_e$, n_v being the number of vertices of $\mathbb{P}$ and n_e being the number of edges of $\mathbb{P}$. Finally, we define $\Sigma(\mathbb{P})$ to be the closure of $\Phi(\mathbb{P})$ in $\mathbb{R}^N$. Then $\Sigma(\mathbb{P})$ is a manifold with corners that can be endowed with the structure of a Lie manifold with true boundary as follows. (Recall that a Lie manifold with boundary Σ is the closure $\overline{\Omega}$ of a regular open subset Ω in a Lie manifold M and the *true boundary* of Σ is the topological boundary $\partial_{\mathrm{top}}\Omega$.) The true boundary $\partial_{\mathrm{top}}\Sigma(\Omega)$ of $\Sigma(\Omega)$ is defined as the union of the closures of the faces D_j of $\mathbb{P}$ in $\Sigma(\mathbb{P})$. (Note that the closures of D_j in $\Sigma(\mathbb{P})$ are disjoint.) We can then take M to be the union of two copies of $\Sigma(\mathbb{P})$ with the true boundaries identified (i. e., the double of $\Sigma(\mathbb{P})$) and $\Omega = \Sigma(\mathbb{P}) \smallsetminus \partial_{\mathrm{top}}\Sigma(\mathbb{P})$. In particular, $\Omega_0 := \Omega \cap M_0$ identifies with $\mathbb{P}$.

To complete the definition of the Lie manifold with true boundary on $\Sigma(\mathbb{P})$, we now define the structural Lie algebra of vector fields $\mathcal{V}(\mathbb{P})$ of $\Sigma(\mathbb{P})$ by

$$\mathcal{V}(\mathbb{P}) := \{r_{\mathbb{P}}(\phi_1\partial_1 + \phi_2\partial_2 + \phi_3\partial_3),\ \phi_j \in \mathcal{C}^\infty(\Sigma(\mathbb{P}))\}. \tag{36}$$

(Here ∂_j are the standard unit vector fields. Also, the vector fields in $\mathcal{V}(\mathbb{P})$ are determined by their restrictions to $\mathbb{P}$.) This is consistent with the fact that $\partial_{\mathrm{top}}\Sigma(\mathbb{P})$, the true boundary of $\Sigma(\mathbb{P})$, is defined as the union of the boundary hyperfaces of $\Sigma(\mathbb{P})$ to which not all vector fields are tangent. This completes the definition of the structure of Lie manifold with boundary on $\Sigma(\mathbb{P})$.

The function $r_{\mathbb{P}}$ is easily seen to be an admissible weight on $\Sigma(\mathbb{P})$. It hence satisfies

$$r_{\mathbb{P}}(\partial_j r_{\mathbb{P}}) = r_{\mathbb{P}} \frac{\partial r_{\mathbb{P}}}{\partial x_j} \in \mathcal{C}^\infty(\Sigma(\mathbb{P})),$$

which is equivalent to the fact that $\mathcal{V}(\mathbb{P})$ is a Lie algebra. This is the analogue of Equation (11).

To check that $\Sigma(\mathbb{P})$ is a Lie manifold, let us notice first that $g = r_{\mathbb{P}}^{-2} g_E$ is a compatible metric on $\Sigma(\mathbb{P})$, where g_E is the Euclidean metric on $\mathbb{P}$. Then, let us denote by ν the outer unit normal to $\mathbb{P}$ (where it is defined), then $r_{\mathbb{P}}\partial_\nu$ is the restriction to $\partial_{\mathrm{top}}\Sigma(\Omega)$ of a vector field in $\mathcal{V}(\mathbb{P})$. Moreover $r_{\mathbb{P}}\partial_\nu$ is of length one and orthogonal to the true boundary in the compatible metric $g = r_{\mathbb{P}}^{-2} g_E$.

The definition of $\mathcal{V}(\mathbb{P})$ together with our definition of Sobolev spaces on Lie manifolds using vector fields shows that

$$\mathcal{K}_a^m(\mathbb{P}) = r_{\mathbb{P}}^{a-3/2} W^{m,2}(\Sigma(\mathbb{P})) = r_{\mathbb{P}}^{a-3/2} H^m(\Sigma(\mathbb{P})). \tag{37}$$

The induced Lie manifold structure on $\Sigma(\mathbb{P})$ consists of the vector fields on the faces D_j that vanish on the boundary of D_j. The Soblev spaces on the boundary are

$$\mathcal{K}_a^m(\partial\mathbb{P}) = r_{\mathbb{P}}^{a-1} W^{m,2}(\partial_{\mathrm{top}}\Sigma(\mathbb{P})) = r_{\mathbb{P}}^{a-1} H^m(\partial_{\mathrm{top}}\Sigma(\mathbb{P})). \tag{38}$$

The factors $-3/2$ and -1 in the powers of $r_{\mathbb{P}}$ appearing in the above two equations are due to the fact that the volume elements on $\mathbb{P}$ and $\Sigma(\mathbb{P})$ differ by these factors.

If P is an order m differential operator with smooth coefficients on $\mathbb{R}^3$ and $\mathbb{P} \subset \mathbb{R}^3$ is a polyhedral domain, then $r_{\mathbb{P}}^m P \in \mathrm{Diff}_{\mathcal{V}}^m(\Sigma(\mathbb{P}))$, by Equation (10). However, in general, $r_{\mathbb{P}}^m P$ will not define a smooth differential operator on $\overline{\mathbb{P}}$.

In particular, we have the following theorem, which is a direct analog of Proposition 4.8, if we replace "vertices" with "edges:"

Theorem 6.1. *Let $\mathbb{P} \subset \mathbb{R}^3$ be a polyhedral domain and P be a differential operator of order m with coefficients in $\mathcal{C}^\infty(\overline{\mathbb{P}})$. Then $P_\lambda := r_{\mathbb{P}}^\lambda P r_{\mathbb{P}}^{-\lambda}$ defines a continuous family of bounded maps $P_\lambda : \mathcal{K}_a^s(\mathbb{P}) \to \mathcal{K}_{a-m}^{s-m}(\mathbb{P})$, for any $s, a \in \mathbb{R}$. Let $\mathbb{P}'$ be $\overline{\mathbb{P}}$ with the* edges *removed. Then $\mathcal{C}_c^\infty(\mathbb{P}')$ is dense in $\mathcal{K}_a^m(\mathbb{P})$. Also, the restriction to the boundary extends to a continuous, surjective trace map $\mathcal{K}_a^s(\mathbb{P}) \to \mathcal{K}_{a-1/2}^{s-1/2}(\partial\mathbb{P})$. If $s = 1$, then the kernel of the trace map is the closure of $\mathcal{C}_c^\infty(\mathbb{P})$ in $\mathcal{K}_a^1(\mathbb{P})$.*

See [11] for applications of these results, especially of the above theorem.

Theorem 5.1 and the results of this section immediately lead to the proof of Theorem 0.1 formulated in the Introduction.

7. A non-standard boundary value problem

We present in this section a non-standard boundary value problem on a smooth manifold with boundary. Let $\mathcal{O}$ be a smooth manifold with boundary. We shall assume that $\mathcal{O}$ is connected and that the boundary is not empty.

Let $r : \overline{\mathcal{O}} \to [0, \infty)$ be a smooth function that close to the boundary is equal to the distance to the boundary and is > 0 on $\mathcal{O}$. Then we recall [14] that there exists a constant depending only on $\mathcal{O}$ such that

$$\int_{\mathcal{O}} r^{-2} |u(x)|^2 dx \leq C \int_{\mathcal{O}} |\nabla u(x)|^2 dx \tag{39}$$

for any $u \in H^1(\mathcal{O})$ that vanishes at the boundary. If we denote, as in Equation (2),

$$\mathcal{K}_a^m(\mathcal{O}; r) := \{u \in L^2_{loc}(\mathcal{O}),\ r^{|\alpha|-a}\partial^\alpha u \in L^2(\mathcal{O}),\quad |\alpha| \leq m\}, \quad m \in \mathbb{N} \cup \{0\},\, a \in \mathbb{R},$$

with norm $\|\cdot\|_{\mathcal{K}_a^m}$, the Equation (39) implies that $\|u\|_{\mathcal{K}_1^1} \leq C\|\nabla u\|_{L^2}$.

Let $M = \overline{\mathcal{O}}$ with the structural Lie algebra of vector fields

$$\mathcal{V} = \mathcal{V}_0 := \{X, X = 0 \text{ at } \partial\mathcal{O}\} = r\Gamma(M; TM),$$

(see Example 1.7). Recall from Subsection 1.4 that $\mathrm{Diff}_{\mathcal{V}}^m(M)$ is the space of order m differential operators on M generated by multiplication with functions in $\mathcal{C}^\infty(M)$ and by differentiation with vector fields $X \in \mathcal{V}$. It follows that

$$r^m P \in \mathrm{Diff}_{\mathcal{V}}^m(M) \tag{40}$$

for any differential operator P of order m with smooth coefficients on M.

Lemma 7.1. *The pair $(M, \mathcal{V})$ is a Lie manifold with $M_0 = \mathcal{O}$ satisfying*

$$\mathcal{K}_a^m(\mathcal{O}; r) = r^{a-n/2} H^m(M). \tag{41}$$

If P is a differential operator with smooth coefficients on M, then $r^m P$ is a differential operator generated by $\mathcal{V}$, and hence $P_\lambda := r^\lambda P r^{-\lambda}$ gives rise to a continuous family of bounded maps $P_\lambda : \mathcal{K}^s_a(\mathcal{O}; r) \to \mathcal{K}^{s-m}_{a-m}(\mathcal{O}; r)$.

Because of the above lemma, it makes sense to define $\mathcal{K}^s_a(\mathcal{O}; r) = r^{a-n/2} H^s(M)$, for all $s, a \in \mathbb{R}$, with norm denoted $\| \cdot \|_{\mathcal{K}^s_a}$. The regularity result (Theorem 5.1) then gives

Lemma 7.2. *Let P be an order m elliptic differential operator with smooth coefficients defined in a neighborhood of $M = \overline{\mathcal{O}}$. Then, for any $s, t \in \mathbb{R}$, there exists $C = C_{st} > 0$ such that*

$$\|u\|_{\mathcal{K}^s_a} \leq C\big(\|Pu\|_{\mathcal{K}^{s-m}_{a-m}} + \|u\|_{\mathcal{K}^t_a}\big).$$

In particular, let $u \in \mathcal{K}^t_a(\mathcal{O}; r)$ be such that $Pu \in \mathcal{K}^{s-m}_{a-m}(\mathcal{O}; r)$, then $u \in \mathcal{K}^s_a(\mathcal{O}; r)$. The same result holds for elliptic systems.

Proof. We first notice that $r^m P \in \mathrm{Diff}^m_{\mathcal{V}}(M)$ is an *elliptic operator* in the usual sense (that is, its principal symbol $\sigma^{(m)}(r^m P)$ does not vanish outside the zero section of A^*). For this we use that $\sigma^{(m)}(r^m P) = r^m \sigma^{(m)}(P)$ and that A^* is defined such that multiplication by r^m defines an isomorphism $\mathcal{C}^\infty(T^*M) \to \mathcal{C}^\infty(A^*)$ that maps order m elliptic symbols to elliptic symbols. Then the proof is exactly the same as that of Theorem 5.1, except that we do not need strong ellipticity, because we do not have boundary conditions (and hence we have no condition of the form $u = 0$ on the boundary). □

An alternative proof of our lemma is obtained using pseudodifferential operators generated by $\mathcal{V}$ [3] and their L^p–continuity.

Theorem 7.3. *There exists $\eta > 0$ such that $\Delta : \mathcal{K}^{a+1}_{s+1}(\mathcal{O}; r) \to \mathcal{K}^{a-1}_{s-1}(\mathcal{O}; r)$ is an isomorphism for all $s \in \mathbb{R}$ and all $|a| < \eta$.*

Proof. The proof is similar to that of Theorem 2.1 in [10], so we will be brief. Consider

$$B : \mathcal{K}^1_1(\mathcal{O}; r) \times \mathcal{K}^1_1(\mathcal{O}; r) \to \mathbb{C}, \quad B(u, v) = \int_{\mathcal{O}} \nabla u \cdot \nabla \overline{v} dx.$$

Then $|B(u, v)| \leq \|u\|_{\mathcal{K}^1_1} \|v\|_{\mathcal{K}^1_1}$, so B is continuous.

On the other hand, by Equation (39), $B(u, u) \geq \theta \|u\|^2_{\mathcal{K}^1_1}$, for all u with compact support on $\mathcal{O}$ and for some $\theta > 0$ independent of u. Since $\mathcal{C}^\infty_{\mathrm{c}}(\mathcal{O})$ is dense in $\mathcal{K}^1_1(\mathcal{O}; r)$, by Theorem 4.2, the Lax-Milgram Lemma can be used to conclude that

$$\Delta : \mathcal{K}^1_1(\mathcal{O}; r) \to \mathcal{K}^{-1}_{-1}(\mathcal{O}; r) := \mathcal{K}^1_1(\mathcal{O}; r)^*$$

is an isomorphism. Since multiplication by $r^a : \mathcal{K}^1_1(\mathcal{O}; r) \to \mathcal{K}^1_{a+1}(\mathcal{O}; r)$ is an isomorphism and the family $r^a \Delta r^{-a}$ depends continuously on a by Lemma 7.1, we obtain that $\Delta : \mathcal{K}^1_{a+1}(\mathcal{O}; r) \to \mathcal{K}^{-1}_{a-1}(\mathcal{O}; r)$ is an isomorphism for $|a| < \eta$, for some $\eta > 0$ small enough.

Fix now a, $|a| < \eta$. We obtain that $\Delta : \mathcal{K}^{s+1}_{a+1}(\mathcal{O}; r) \to \mathcal{K}^{s-1}_{a-1}(\mathcal{O}; r)$ is a continuous, injective map, for all $s \geq 0$. The first part of the proof (for $a = 0$) together with the regularity result of Lemma 7.2 show that this map is also surjective. The Open Mapping Theorem therefore completes the proof for $s \geq 0$. For $s \leq 0$, the result follows by considering duals. □

It can be shown as in [10] that η is the least value for which $\Delta : \mathcal{K}^1_{\eta+1}(\mathcal{O}; r) \to \mathcal{K}^{-1}_{\eta-1}(\mathcal{O}; r)$ is not Fredholm. This, in principle, can be decided by using the Fredholm conditions in [43] that involve looking at the L^2 invertibility of the same differential operators when M is the half-space $\{x_{n+1} \geq 0\}$. See also [5] for some non-standard boundary value problems on exterior domains in weighted Sobolev spaces.

8. Pseudodifferential operators

We now recall the definition of pseudodifferential operators on M_0 generated by a Lie structure at infinity $(M, \mathcal{V})$ on M_0.

8.1. **Definition.** We fix in what follows a compatible Riemannian metric g on M_0 (that is, a metric coming by restriction from a metric on the bundle $A \to M$ extending TM_0), see Section 1. In order to simplify our discussion below, we shall use the metric g to trivialize all density bundles on M. Recall that M_0 with the induced metric is complete [4]. Also, recall that $A \to M$ is a vector bundle such that $\mathcal{V} = \Gamma(A)$.

Let $\exp_x : T_x M_0 \to M_0$ be the exponential map, which is everywhere defined because M_0 is complete. We let

$$\Phi : TM_0 \longrightarrow M_0 \times M_0, \quad \Phi(v) := (x, \exp_x(-v)),\ v \in T_x M_0, \tag{42}$$

If E is a real vector bundle with a metric, we shall denote by $(E)_r$ the set of all vectors v of E with $|v| < r$. Let $(M_0^2)_r := \{(x,y), x, y \in M_0,\, d(x,y) < r\}$. Then the map Φ of Equation (42) restricts to a diffeomorphism $\Phi : (TM_0)_r \to (M_0^2)_r$, for any $0 < r < r_{\mathrm{inj}}(M_0)$, where $r_{\mathrm{inj}}(M_0)$ is the injectivity radius of M_0, which was assumed to be positive. The inverse of Φ is of the form

$$(M_0^2)_r \ni (x,y) \longmapsto (x, \tau(x,y)) \in (TM_0)_r\,.$$

We shall denote by $S^m_{1,0}(E)$ the space of symbols of order m and type $(1,0)$ on E (in Hörmander's sense) and by $S^m_{cl}(E)$ the space of classical symbols of order m on E [21, 42, 57, 59]. See [3] for a review of these spaces of symbols in our framework.

Let $\chi \in \mathcal{C}^\infty(A^*)$ be a smooth function that is equal to 1 on $(A^*)_r$ and is equal to 0 outside $(A^*)_{2r}$, for some $r < r_{\mathrm{inj}}(M_0)/3$. Then, following [3], we define

$$q(a)u(x) = (2\pi)^{-n} \int_{T^*M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x,y)) a(x,\eta) u(y)\, d\eta\, dy\,.$$

This integral is an oscillatory integral with respect to the symplectic measure on T^*M_0 [22]. Alternatively, we consider the measures on M_0 and on $T^*_x M_0$ defined by some choice of a metric on A and we integrate first in the fibers $T^*_x M_0$ and then on M_0. The map $\sigma_{tot} : S^m_{1,0}(A^*) \to \Psi^m(M_0)/\Psi^{-\infty}(M_0)$,

$$\sigma_{tot}(a) := q(a) + \Psi^{-\infty}(M_0)$$

is independent of the choice of the function $\chi \in \mathcal{C}^\infty_c((A)_r)$ [3].

We now enlarge the class of order $-\infty$ operators that we consider. Any $X \in \mathcal{V} = \Gamma(A)$ generates a global flow $\Psi_X : \mathbb{R} \times M \to M$ because X is tangent to all boundary faces of M and M is compact. Evaluation at $t = 1$ yields a diffeomorphism

$$\psi_X := \Psi_X(1, \cdot) : M \to M. \tag{43}$$

We now define the pseudodifferential calculus on M_0 that we will consider following [3]. See [28, 29, 41, 44] for the connections between this calculus and groupoids.

Definition 8.1. Fix $0 < r < r_{\text{inj}}(M_0)$ and $\chi \in \mathcal{C}^\infty_{\text{c}}((A)_r)$ such that $\chi = 1$ in a neighborhood of $M \subseteq A$. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0)$ of *pseudodifferential operators generated by the Lie structure at infinity* $(M, \mathcal{V})$ is defined to be the linear space of operators $\mathcal{C}^\infty_{\text{c}}(M_0) \to \mathcal{C}^\infty_{\text{c}}(M_0)$ generated by $q(a)$, $a \in S^m_{1,0}(A^*)$, and $q(b)\psi_{X_1} \dots \psi_{X_k}$, $b \in S^{-\infty}(A^*)$ and $X_j \in \Gamma(A)$, $\forall j$.

Similarly, the space $\Psi^m_{cl,\mathcal{V}}(M_0)$ of *classical pseudodifferential operators generated by the Lie structure at infinity* $(M, \mathcal{V})$ is obtained by using classical symbols a in the construction above.

We have that $\Psi^{-\infty}_{cl,\mathcal{V}}(M_0) = \Psi^{-\infty}_{1,0,\mathcal{V}}(M_0) =: \Psi^{-\infty}_{\mathcal{V}}(M_0)$ (we dropped some subscripts).

8.2. **Properties.** We now review some properties of the operators in $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ from [3]. These properties will be used below. Let $\Psi^\infty_{1,0,\mathcal{V}}(M_0) = \bigcup_{m \in \mathbb{Z}} \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^\infty_{cl,\mathcal{V}}(M_0) = \bigcup_{m \in \mathbb{Z}} \Psi^m_{cl,\mathcal{V}}(M_0)$.

First of all, each operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ defines continuous maps $\mathcal{C}^\infty_{\text{c}}(M_0) \to \mathcal{C}^\infty(M_0)$, and $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, still denoted by P. An operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ has a distribution kernel k_P in the space $I^m(M_0 \times M_0, M_0)$ of distributions on $M_0 \times M_0$ that are conormal of order m to the diagonal, by [22]. If $P = q(a)$, then k_P has support in $(M_0 \times M_0)_r$. If we extend the exponential map $(TM_0)_r \to M_0 \times M_0$ to a map $A \to M$, then the distribution kernel of $P = q(a)$ is the restriction of a distribution, also denoted k_P in $I^m(A, M)$.

If $\mathcal{P}$ denotes the space of polynomial symbols on A^* and $\text{Diff}(M_0)$ denotes the algebra of differential operators on M_0, then

$$\Psi^\infty_{1,0,\mathcal{V}}(M_0) \cap \text{Diff}(M_0) = \text{Diff}^\infty_{\mathcal{V}}(M) = q(\mathcal{P}). \tag{44}$$

The spaces $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{1,0,\mathcal{V}}(M_0)$ are independent of the choice of the metric on A and the function χ used to define it, but depend, in general, on the Lie structure at infinity (M, A) on M_0. They are also closed under multiplication, which is a quite non-trivial fact.

Theorem 8.2. *The spaces $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ and $\Psi^\infty_{cl,\mathcal{V}}(M_0)$ are filtered algebras that are closed under adjoints.*

For $\Psi^m_{1,0,\mathcal{V}}(M_0)$, the meaning of the above theorem is that

$$\Psi^m_{1,0,\mathcal{V}}(M_0)\Psi^{m'}_{1,0,\mathcal{V}}(M_0) \subseteq \Psi^{m+m'}_{1,0,\mathcal{V}}(M_0) \text{ and } \big(\Psi^m_{1,0,\mathcal{V}}(M_0)\big)^* = \Psi^m_{1,0,\mathcal{V}}(M_0)$$

for all $m, m' \in \mathbb{C} \cup \{-\infty\}$.

The usual properties of the principal symbol remain true.

Proposition 8.3. *The principal symbol establishes isomorphisms*

$$\sigma^{(m)} : \Psi^m_{1,0,\mathcal{V}}(M_0)/\Psi^{m-1}_{1,0,\mathcal{V}}(M_0) \to S^m_{1,0}(A^*)/S^{m-1}_{1,0}(A^*) \tag{45}$$

and

$$\sigma^{(m)} : \Psi^m_{cl,\mathcal{V}}(M_0)/\Psi^{m-1}_{cl,\mathcal{V}}(M_0) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*). \tag{46}$$

Moreover, $\sigma^{(m)}(q(a)) = a + S^{m-1}_{1,0}(A^*)$ for any $a \in S^m_{1,0}(A^*)$ and $\sigma^{(m+m')}(PQ) = \sigma^{(m)}(P)\sigma^{(m')}(Q)$, for any $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $Q \in \Psi^{m'}_{1,0,\mathcal{V}}(M_0)$.

We shall need also the following result.

Proposition 8.4. *Let ρ be a defining function of some hyperface of M. Then $\rho^s \Psi^m_{1,0,\mathcal{V}}(M_0)\rho^{-s} = \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\rho^s \Psi^m_{cl,\mathcal{V}}(M_0)\rho^{-s} = \Psi^m_{cl,\mathcal{V}}(M_0)$ for any $s \in \mathbb{C}$.*

8.3. **Continuity on $W^{s,p}(M_0)$.** The preparations above will allow us to prove the continuity of the operators $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ between suitable Sobolev spaces. This is the main result of this section. Some of the ideas and constructions in the proof below have already been used in 5.1, which the reader may find convenient to review first. Let us recall from Equation (25) that an admissible weight ρ of M is a function of the form $\rho := \prod_H \rho_H^{a_H}$, where $a_H \in \mathbb{R}$ and ρ_H is a defining function of H.

Theorem 8.5. *Let ρ be an admissible weight of M and let $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $p \in (0,\infty)$. Then P maps $\rho^r W^{s,p}(M_0)$ continuously to $\rho^r W^{s-m,p}(M_0)$ for any $r, s \in \mathbb{R}$.*

Proof. We have that P maps $\rho^r W^{s,p}(M_0)$ continuously to $\rho^r W^{s-m,p}(M_0)$ if, and only if, $\rho^{-r} P \rho^r$ maps $W^{s,p}(M_0)$ continuously to $W^{s-m,p}(M_0)$. By Proposition 8.4 it is therefore enough to check our result for $r = 0$.

We shall first prove our result if the Schwartz kernel of P has support close enough to the diagonal. To this end, let us choose $\epsilon < r_{\mathrm{inj}}(M_0)/9$ and assume that the distribution kernel of P is supported in the set $(M_0^2)_\epsilon := \{(x,y), d(x,y) < \epsilon\} \subset M_0^2$. This is possible by choosing the function χ used to define the spaces $\Psi^m_{1,0,\mathcal{V}}(M_0)$ to have support in the set $(M_0^2)_\epsilon$. There will be no loss of generality then to assume that $P = q(a)$.

Then choose a smooth function $\eta : [0,\infty) \to [0,1]$, $\eta(t) = 1$ if $t \leq 6\epsilon$, $\eta(t) = 0$ if $t \geq 7\epsilon$. Let $\psi_x : B(x, 8\epsilon) \to B_{T_x M_0}(0, 8\epsilon)$ denote the normal system of coordinates induced by the exponential maps $\exp_x : T_x M_0 \to M_0$. Denote $\pi : A \to M$ be the natural (vector bundle) projection and

$$B := A \times_M A := \{(\xi_1, \xi_2) \in A \times A, \pi(\xi_1) = \pi(\xi_2)\}, \tag{47}$$

which defines a vector bundle $B \to M$. In the language of vector bundles, $B := A \oplus A$. For any $x \in M_0$, let η_x denote the function $\eta \circ \exp_x$, and consider the operator $\eta_x P \eta_x$ on $B(x, 13\epsilon)$. The diffeomorphism ψ_x then will map this operator to an operator P_x on $B_{T_x M_0}(0, 8\epsilon)$. Then P_x maps continuously $W^{s,p}(T_x M_0) \to W^{s-m,p}(T_x M_0)$, by the continuity of pseudodifferential operators on $\mathbb{R}^n$ [60, XIII, §5] or [56].

The distribution kernel k_x of P_x is a distribution with compact support on

$$T_x M_0 \times T_x M_0 = A_x \times A_x = B_x$$

If $P = q(a) \in \Psi^m_{1,0,\mathcal{V}}(M_0)$, then the distributions k_x can be determined in terms of the distribution $k_P \in I^m(A, M)$ associated to P. This shows that the distributions k_x extend to a smooth family of distributions on the fibers of $B \to M$. From this, it follows that the family of operators $P_x : W^{s,p}(A_x) \to W^{s-m,p}(A_x)$, $x \in M_0$, extends to a family of operators defined for $x \in M$ (recall that $A_x = T_x M_0$ if $x \in M_0$). This extension is obtained by extending the distribution kernels. In particular, the resulting family P_x will depend smoothly on $x \in M$. Since M is compact, we obtain, in particular, that the norms of the operators P_x are uniformly bounded for $x \in M_0$.

By abuse of notation, we shall denote by $P_x : W^{s,p}(M_0) \to W^{s-m,p}(M_0)$ the induced family of pseudodifferential operators, and we note that it will still be a smooth family that is uniformly bounded in norm. Note that it is possible to extend P_x to an operator on M_0 because its distribution kernel has compact support.

Then choose the sequence of points $\{x_j\} \subset M_0$ and a partition of unity $\phi_j \in \mathcal{C}_c^\infty(M_0)$ as in Lemma 3.6. In particular, ϕ_j will have support in $B(x_j, 2\epsilon)$. Also, let $\psi_j : B(x_j, 4\epsilon) \to B_{\mathbb{R}^n}(0, 4\epsilon)$ denote the normal system of coordinates induced by the exponential maps $\exp_x : T_xM_0 \to M_0$ and some fixed isometries $T_xM_0 \simeq \mathbb{R}^n$. Then all derivatives of $\psi_j \circ \psi_k^{-1}$ are bounded on their domain of definition, with a bound that may depend on ϵ but does not depend on j and k [13, 54].

Let

$$\nu_{s,p}(u)^p := \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)}.$$

be one of the several equivalent norms defining the topology on $W^{s,p}(M_0)$ (see Proposition 3.10 and Equation (21). It is enough to prove that

$$\begin{aligned}(48)\quad \nu_{s,p}(Pu)^p := \sum_j \|(\phi_j Pu) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)}& \\ \leq C \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} &=: C\nu_{s,p}(u)^p,\end{aligned}$$

for some constant C independent of u.

We now prove this statement. Indeed, for the reasons explained below, we have the following inequalities.

$$\begin{aligned}\sum_j \|(\phi_j Pu) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} &\leq C \sum_{j,k} \|(\phi_j P\phi_k u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} \\ = C \sum_{j,k} \|(\phi_j P_{x_j}\phi_k u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} &\leq C \sum_{j,k} \|(\phi_j \phi_k u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} \\ &\leq C \sum_j \|(\phi_j u) \circ \psi_j^{-1}\|^p_{W^{s,p}(\mathbb{R}^n)} = C\nu_{s,p}(u)^p.\end{aligned}$$

Above, the first and last inequalities are due to the fact that the family ϕ_j is uniformly locally finite, that is, there exists a constant κ such that at any given point x, at most κ of the functions $\phi_j(x)$ are different from zero. The first equality is due to the support assumptions on ϕ_j, ϕ_k, and P_{x_j}. Finally, the second inequality is due to the fact that the operators P_{x_j} are continuous, with norms bounded by a constant independent of j, as explained above. We have therefore proved that $P = q(a) \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ defines a bounded operator $W^{s,p}(M_0) \to W^{s-m,p}(M_0)$, provided that the Schwartz kernel of P has support in a set of the $(M_0^2)_\epsilon$, for $\epsilon < r_{\mathrm{inj}}(M_0)/9$.

Assume now that $P \in \Psi^{-\infty}_{\mathcal{V}}(M_0)$. We shall check that P is bounded as a map $W^{2k,p}(M_0) \to W^{-2k,p}(M_0)$. For $k = 0$, this follows from the fact that the Schwartz kernel of P is given by a smooth function $k(x, y)$ such that $\int_{M_0} |k(x, y)| d\,\mathrm{vol}_g(x)$ and $\int_{M_0} |k(x, y)| d\,\mathrm{vol}_g(y)$ are uniformly bounded in x and y. For the other values of k, it is enough to prove that the bilinear form

$$W^{2k,p}(M_0) \times W^{2k,p}(M_0) \ni (u, v) \mapsto \langle Pu, v\rangle \in \mathbb{C}$$

is continuous. Choose Q a parametrix of Δ^k and let $R = 1 - Q\Delta^k$ be as above. Let $R' = 1 - \Delta^k Q \in \Psi^{-\infty}_{\mathcal{V}}(M_0)$. Then

$$\langle Pu, v\rangle = \langle (QPQ)\Delta^k u, \Delta^k v\rangle + \langle (QPR)u, \Delta^k v\rangle + \langle (R'PQ)\Delta^k u, v\rangle + \langle (R'PR)u, v\rangle,$$

which is continuous since $QPQ, QPR, R'PQ$, and $R'PR$ are in $\Psi_{\mathcal{V}}^{-\infty}(M_0)$ and hence they are continuous on $L^p(M_0)$ and because $\Delta^k : W^{2k,p}(M_0) \to L^p(M_0)$ is continuous.

Since any $P \in \Psi_{1,0,\mathcal{V}}^m(M_0)$ can be written $P = P_1 + P_2$ with $P_2 \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$ and $P_1 = q(a) \in \Psi_{1,0,\mathcal{V}}^m(M_0)$ with support arbitrarily close to the diagonal in M_0, the result follows. □

We obtain the following standard description of Sobolev spaces.

Theorem 8.6. *Let $s \in \mathbb{R}_+$ and $p \in (1, \infty)$. We have that $u \in W^{s,p}(M_0)$ if, and only if, $u \in L^p(M_0)$ and $Pu \in L^p(M_0)$ for any $P \in \Psi_{1,0,\mathcal{V}}^s(M_0)$. The norm $u \mapsto \|u\|_{L^p(M_0)} + \|Pu\|_{L^p(M_0)}$ is equivalent to the original norm on $W^{s,p}(M_0)$ for any elliptic $P \in \Psi_{1,0,\mathcal{V}}^s(M_0)$.*

Similarly, the map $T : L^p(M_0) \oplus L^p(M_0) \ni (u, v) \mapsto u + Pv \in W^{-s,p}(M_0)$ is surjective and identifies $W^{-s,p}(M_0)$ with the quotient $(L^p(M_0) \oplus L^p(M_0))/\ker(T)$.

Proof. Clearly, if $u \in W^{s,p}(M_0)$, then $Pu, u \in L^p(M_0)$. Let us prove the converse. Assume $Pu, u \in L^p(M_0)$. Let $Q \in \Psi_{1,0,\mathcal{V}}^{-s}(M_0)$ be a parametrix of P and let $R, R' \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$ be defined by $R := 1 - QP$ and $R' = 1 - PQ$. Then $u = QPu + Ru$. Since both $Q, R : L^p(M_0) \to W^{s,p}(M_0)$ are defined and bounded, $u \in W^{s,p}(M_0)$ and $\|u\|_{W^{s,p}(M_0)} \leq C\big(\|u\|_{L^p(M_0)} + \|Pu\|_{L^p(M_0)}\big)$. This proves the first part.

To prove the second part, we observe that the mapping

$$W^{s,q}(M_0) \ni u \mapsto (u, Pu) \in L^q(M_0) \oplus L^q(M_0), \qquad q^{-1} + p^{-1} = 1,$$

is an isomorphism onto its image. The result then follows by duality using also the Hahn-Banach theorem. □

We conclude our paper with the sketch of a regularity results for solutions of elliptic equations. Recall the Sobolev spaces with weights $\rho^s W^{s,p}(\Omega_0)$ introduced in Equation (26).

Theorem 8.7. *Let $P \in \mathrm{Diff}_{\mathcal{V}}^m(M)$ be an order m elliptic operator on M_0 generated by $\mathcal{V}$. Let $u \in \rho^s W^{r,p}(M_0)$ be such that $Pu \in \rho^s W^{t,p}(M_0)$, $s, r, t \in \mathbb{R}$, $1 < p < \infty$. Then $u \in \rho^s W^{t+m,p}(M_0)$.*

Proof. Let $Q \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$ be a parametrix of P. Then $R = I - QP \in \Psi_{\mathcal{V}}^{-\infty}(M_0)$. This gives $u = Q(Pu) + Ru$. But $Q(Pu) \in \rho^s W^{t+m,p}(M_0)$, by Theorem 8.5, because $Pu \in \rho^s W^{t,p}(M_0)$. Similarly, $Ru \in \rho^s W^{t+m,p}(M_0)$. This completes the proof. □

Note that the above theorem was already proved in the case $t \in \mathbb{Z}$ and $m = 2$, using more elementary methods, as part of Theorem 5.1. The proof here is much shorter, however, it attests to the power of pseudodifferential operator algebra techniques.

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