MANIFOLDS WITH SMALL DIRAC EIGENVALUES ARE NILMANIFOLDS

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Abstract. Consider the class of n-dimensional Riemannian spin manifolds with bounded sectional curvatures and bounded diameter, and almost non-negative scalar curvature. Let $r = 1$ if $n = 2, 3$ and $r = 2^{[n/2]-1} + 1$ if $n \geq 4$. We show that if the square of the Dirac operator on such a manifold has r small eigenvalues, then the manifold is diffeomorphic to a nilmanifold and has trivial spin structure. Equivalently, if M is not a nilmanifold or if M is a nilmanifold with a non-trivial spin structure, then there exists a uniform lower bound on the r-th eigenvalue of the square of the Dirac operator. If a manifold with almost nonnegative scalar curvature has one small Dirac eigenvalue, and if the volume is not too small, then we show that the metric is close to a Ricci-flat metric on M with a parallel spinor. In dimension 4 this implies that M is either a torus or a $K3$-surface.

Keywords: Dirac operator, nilmanifold, small eigenvalues, almost positive scalar curvature
MSC: 53C27 (Primary), 58J50, 53C20, 53C21 (Secondary)

Contents

1. Introduction

The theorem of Bochner implies that if a connected compact Riemannian manifold has non-negative curvature operator, then $b^p(M) \leq \binom{n}{p}$. Furthermore if $b^p(M) = \binom{n}{p}$ for some p between 0 and n then M is isometric to a flat torus. In [35] it was shown that Riemannian manifolds with *almost*-nonnegative curvature operator, and $\binom{n}{p}$

Research at MSRI is supported in part by NSF grant DMS-9810361.

eigen-p-forms with small eigenvalue must be diffeomorphic to a nilmanifold. Given the Hodge-de Rham theorem, this could be viewed as a quantitative generalization of Bochner's theorem.

Here we discuss a similar result for the Dirac operator on Riemannian spin manifolds. Let $\lambda_i(\not{D}^2)$ denote the i-th eigenvalue of the square of the Dirac operator, and let $\lambda_i(\nabla^*\nabla)$ denote the i-th eigenvalue of the connection Laplacian on spinors. Here and throughout the article we assume that all eigenvalues are counted with multiplicity. All manifolds are connected. Let $r(n) = 2^{[\frac{n}{2}]-1} + 1$ for $n \geq 4$ and $r(n) = 1$ for $n \leq 3$. Our main result is:

Theorem 1.1. *Let (M^n, g, χ) be a compact Riemannian spin manifold with $|\sec| < K$, $\mathrm{diam} < D$. Then there is $\varepsilon = \varepsilon(n, K, D) > 0$, such that if $\lambda_r(\nabla^*\nabla) < \varepsilon$, then M is diffeomorphic to a nilmanifold. Furthermore, χ is the trivial spin structure on M.*

Using the Schrödinger-Lichnerowicz formula $\not{D}^2 = \nabla^*\nabla + \mathrm{scal}/4$ this implies:

Corollary 1.2. *Let (M^n, g, χ) be a compact Riemannian spin manifold with $|\sec| < K$, $\mathrm{diam} < D$. Then there is $\varepsilon = \varepsilon(n, K, D) > 0$, such that if $\mathrm{scal} > -\varepsilon$ and $\lambda_r(\not{D}^2) < \varepsilon$ then M is diffeomorphic to a nilmanifold. Furthermore, χ is the trivial spin structure on M.*

By reformulating we obtain.

Corollary 1.3. *If (M, χ) is not spin-diffeomorphic to a nilmanifold with a trivial spin structure, then among all metrics with bounded diameter and curvature, there is a uniform lower bound on the r-th $\nabla^*\nabla$-eigenvalue:*

$$\lambda_r(\nabla^*\nabla) \geq \varepsilon = \varepsilon(n, \max|\sec|, \mathrm{diam}) > 0.$$

In particular, any metric g on M with $\mathrm{scal} > -4\varepsilon(n, \max|\sec|, \mathrm{diam})$ has a non-trivial uniform lower bound on the r-th Dirac eigenvalue

$$\lambda_r(\not{D}^2) \geq \varepsilon(n, \max|\sec|, \mathrm{diam}) + \frac{\min \mathrm{scal}}{4}. \tag{1.4}$$

Recall that the Atiyah-Singer index theorem implies $\dim\ker\not{D} \geq |\hat{A}(M)|$. Therefore we obtain as a corollary: If M is an n-dimensional compact spin manifold with $|\hat{A}(M)| \geq r(n)$, then M does not carry a metric with $\mathrm{scal} > -4\varepsilon(n, \max|\sec|, \mathrm{diam})$ for the above $\varepsilon > 0$. This corollary already appears in Gallot's article [21] with $r(n)$ replaced by $2^{[n/2]}$.

We will give some examples that explain the special role of nilmanifolds and why we cannot replace r by a smaller number.

Examples.

(1) Any nilmanifold M^n carries a sequence of "left-invariant" metrics g_i with $\max|\sec_i| \to 0$, $\mathrm{diam}_i \to 0$ and $\mathrm{vol}_i \to 0$. If the spin structure on M is trivial, then for this sequence of metrics $\lambda_s(\not{D}_i^2) \to 0$ where $s = \mathrm{rank}(\Sigma_i M) = 2^{[n/2]} > r$.

(2) Let N be a K3-surface with a Calabi-Yau metric g^N and its unique spin structure. In particular, g^N is ricci-flat and $\ker\not{D} = \ker\nabla^*\nabla$ has complex dimension 2. For an integer $n \geq 4$, let M be the Riemannian product $N \times T^{n-4}$, where the

$n-4$ dimensional torus carries an arbitrary flat metric. We equip M with the product spin structure of the unique spin structure on N and the trivial spin structure on T^{n-4}. Then the spinor bundle ΣM on M is isomorphic (as a metric bundle with connection) to $\pi_1^*(\Sigma N) \otimes_{\mathbb{C}} \mathbb{C}^{\tilde{r}}$ where $\tilde{r} = 2^{[(n-4)/2]} = 2^{[n/2]-2}$ and $\pi_1 : N \times T^{n-4} \to N$ is the projection to the first component. If $\psi \in \Gamma(\Sigma N)$ is a parallel spinor on N and v is a constant section of $\mathbb{C}^{\tilde{r}}$, then $\pi_1^*(\psi) \otimes v$ is a parallel spinor on M. As N carries a 2-dimensional space of parallel spinors, the dimension of the space of parallel spinors on M is at least $2\tilde{r} = r-1$. And hence

$$\lambda_1(D\!\!\!/^2) = \ldots = \lambda_{r-1}(D\!\!\!/^2) = 0.$$

This example shows that we cannot replace r by $r-1$ in the above theorem.

The next example will show that the sectional curvature bounds in Theorem 1.1 are necessary. We need a lemma.

Lemma 1.5. *Let h be the standard metric on S^3. The ball of radius R around 0 in Euclidean space $\mathbb{R}^4$ will be denoted as $B_R^{\mathbb{R}^4}(0)$. For any $\varepsilon, R, \rho > 0$ there is $a > 0$ and a metric $g = dt^2 + \varphi^2(t)h$ on $S^3 \times (-(a+\rho), (a+\rho))$ such that*

(a) $g|_{(-(a+\rho),-a]\times S^3}$ *and* $g|_{[a,(a+\rho))\times S^3}$ *are isometric to* $B_{R+\rho}^{\mathbb{R}^4}(0) \setminus B_R^{\mathbb{R}^4}(0)$,
(b) $\mathrm{scal}_g \geq -\varepsilon$, *and*
(c) $\mathrm{diam}(M,g) \leq 6(R+\rho)$

For the proof one translates the desired properties into an ordinary differential inequality for φ. Details are available in [4].

Using the lemma we can construct an example showing that the curvature bound is necessary.

Example.

(3) Consider the flat torus T^4. Let $Z := Z(R_j, \varepsilon_j) := (-c, c) \times S^{n-1}$ carry a metric as in the above lemma for ε_j and R_j sufficiently small, that we will choose later, and for $\rho_j = R_j$. Let (M_j, g_j) be given by removing $2j$ small disks from T^4 and attaching j handles isometric to Z. The trivial spin structure on T^4 can be extended to a spin structure on M_j.[1] For a suitable choice of R_j and ε_j we obtain a family of Riemannian manifolds (M_j, g_j) with uniformly bounded diameter and $\liminf_{j\to\infty} \min \mathrm{scal}_j = 0$. They are pairwise non-diffeomorphic, and the sectional curvature is not uniformly bounded. Following the lines of [10] we use a cut-off function vanishing in the handles to construct a ($\mathrm{rank}\, \Sigma M_j = 4$)-dimensional space of test spinors. From this we see that $D\!\!\!/^2$ has at least 4 eigenvalues arbitrarily close to 0. This example shows that the sectional bound in Theorem 1.1 is necessary if the dimension of M is 4. We obtain similar examples for higher dimensions n by taking the product with a flat $(n-4)$-dimensional torus equipped with the trivial spin structure. See [4] for more details.

[1] This extension is not unique, but our construction works for any choice of spin structure.

Finally, we will give several examples in order to show that the bound (1.4) generalizes previously known bounds in dimension 2 and 3.

Examples.

(4) If M is diffeomorphic to the 2-dimensional sphere, then such a lower bound is already known. It is a result of Bär [7] that
$$\lambda_1(\not{D}^2) \geq \frac{4\pi}{\operatorname{area}(M)}.$$
If $K \geq -\delta^2$, $\delta > 0$ then $\operatorname{area}(M) \leq \frac{2\pi}{\delta^2}\,[\cosh(\delta \operatorname{diam} M) - 1]$. Hence,
$$\lambda_1(\not{D}^2) \geq \frac{2\delta^2}{\cosh(\delta \operatorname{diam} M) - 1} = \frac{4}{\operatorname{diam} M^2} - O(\delta^2 \operatorname{diam} M^2).$$

(5) Let (M, g) be diffeomorphic to the 2-dimensional torus T^2 equipped with a nontrivial spin structure and with $|\sec| < K$, $\operatorname{diam} < D$. It is not difficult to derive an explicit lower bound on $\lambda_1(\not{D}^2)$ from previously known estimates. To derive this bound, we use the uniformization theorem to find $u \in C^\infty(T^2)$ and a flat metric g_0 with $g = e^{2u} g_0$. Using the estimates in [2] together with some elementary calculations [4] one obtains
$$\max u - \min u \leq \tau(K|D),$$
where τ is an explicitly known, but long expression [2, Theorem 6.1]. Then one easily derives from [2, Corollary 2.3] that
$$\lambda_1(\not{D}^2) \geq \frac{\pi^2}{4D^2}\, e^{-4\tau(K|D)}.$$

(6) Surfaces of genus greater than 1 cannot have almost non-negative curvature in the above sense. Hence, (4) and (5) yield an explicit, but long formula for ε in dimension $n = 2$.

In higher dimension one expects ε to be an even more complicated expression. Thus, we want to restrict our attention to results that show only existence of a positive ε.

(7) Let (M, g, χ) be a compact spin 3-manifold with $\operatorname{scal} \geq 0$ and $\ker \not{D} \neq \{0\}$. Because of the Schrödinger-Lichnerowicz formula, any $\varphi \in \ker D \setminus \{0\}$ is a nontrivial parallel spinor, which implies that (M, g) is Ricci-flat, and hence flat. However, any flat compact 3-manifold admitting a nontrivial parallel spinor is diffeomorphic to a torus (see [36, Theorem 5.1]) and the spin structure is the trivial one.

We compare Corollary 1.2 which gives a uniform lower r-th eigenvalue bound to a theorem of J. Lott which provides a uniform upper bound on all eigenvalues.

Theorem (Lott [27, Theorem 4])**.** *Let $k \in \mathbb{Z}^+$. Then there is an $E_k = E(n, K, D, k)$ such that any compact Riemannian spin manifold (M, g, χ) with $|\sec| < K$,* diam $\leq$ *D satisfies either*

(a) $\lambda_k(\not{D}^2) \leq E_k$

(b) *M is the total space of an affine fiber bundle $M \to B$ with possible singularities, whose generic fiber is an infranilmanifold, and the spin structure along the generic fibers is* ***not trivial***.

Another result which will be proven in section 3 gives a different conclusion for manifolds with only one small Dirac eigenvalue, and additionally, a lower volume bound.

THEOREM 1.6. *Let (M, g) have $|\sec| < K$, $\mathrm{diam} < D$, $\mathrm{vol} > v$. Let $\lambda_1(\nabla^*\nabla)$ denote the first eigenvalue of the connection Laplacian $\nabla^*\nabla$ on the spinor bundle with respect to a spin structure χ. Then for all $\delta > 0$, there is an $\varepsilon = \varepsilon(n, v, K, D, \delta) > 0$ such that if $\lambda_1(\nabla^*\nabla) < \varepsilon$, then (M, g, χ) has $C^{1,\alpha}$-distance $\leq \delta$ to a Ricci-flat Einstein metric with a nontrivial parallel spinor.*

COROLLARY 1.7. *For $\delta > 0$, there is an $\varepsilon = \varepsilon(n, v, K, D, \delta) > 0$ such that the following holds: Let (M, g) have $|\sec| < K$, $\mathrm{diam} < D$, $\mathrm{vol} > v$ and $\mathrm{scal} > -\varepsilon$. Let $\lambda_1(\not{D}^2)$ denote the first eigenvalue of $\not{D}^2$ with respect to a spin structure χ. If $\lambda_1(\not{D}^2) < \varepsilon$, then (M, g, χ) has $C^{1,\alpha}$-distance $\leq \delta$ to a Ricci-flat Einstein metric with a nontrivial parallel spinor.*

A compact 4-dimensional manifolds M carrying a parallel spinor is either a flat torus or a $K3$-surface. Hence, any 4-manifold with one small Dirac eigenvalue is either diffeomorphic to a torus or a K3-surface, or is collapsed.

Example (1) shows that the volume bound in the above theorem and corollary is necessary.

If one applies the techniques that we will present in this paper to the Friedrich connection on the spinor bundle instead of the standard connection, one obtains analogs of Theorem 1.1, Corollary 1.2, Theorem 1.6 and Corollary 1.7. In particular, we obtain the following theorem. Here, once again, we define $r = 1$ if $n = 2, 3$ and $r = 2^{[n/2]-1} + 1$ if $n \geq 4$.

THEOREM 1.8. *Let (M^n, g, χ) be a compact Riemannian manifold with $\mathrm{diam} < D$, $|\sec| < K$, and $\mathrm{scal} \geq n(n-1)\rho^2$ with a constant $\rho > 0$. Let $\not{D}$ be the Dirac operator on M. Then for any $\delta > 0$, there is $\varepsilon = \varepsilon(n, K, D, \rho, \delta) > 0$ such that if $\not{D}$ has $r = r(n)$ eigenvalues $\lambda_i \in [0, \frac{n\rho}{2} + \varepsilon)$, then M has $C^{1,\alpha}$-distance $\leq \delta$ to a manifold of constant curvature $\sec \equiv \rho^2$.*

However, motivated by Bär's classification of manifolds with real Killing spinors [8], we conjecture that the theorem still holds for a smaller number r.

In the following example, one sees that the bound on the sectional curvature is necessary.

Example.

(8) On any compact manifold M that admits a metric of positive scalar curvature, equipped with an arbitrary spin structure on M, C. Bär and M. Dahl [11] constructed a sequence of metrics g_i on M with scalar curvature $\geq n(n-1)$, but with $\lambda_{2^{[n/2]}}(\not{D}^2_{g_i}) \to n^2/4$.

The structure of our article is as follows. In Section 2 we will reformulate some previously known estimates on vector bundles. In Section 3 we will apply these estimates to prove Theorem 1.6. In the following sections Theorem 1.1 is proved. We will develop most of the tools in such a generality that we can easily replace the Dirac operator (acting on spinors) by other elliptic operators acting on sections of bundles with special holonomy. We begin this in Section 4 by defining the fixing dimension r of a faithful representation, which immediately gives the fixing dimension r of a vector bundle with special holonomy. In Section 5 we show that if there are r almost parallel sections on such a bundle, then the bundle is trivialized by almost parallel sections. Section 6 determines the fixing number of the spinor bundle, and we finally prove Theorem 1.1 in the last section.

Throughout the paper we adopt the convention that $\tau(x_1|x_2,\dots,x_m)$ represents a continuous function in $x_1,\dots,x_m$ such that $\tau\to 0$ as $x_1\to 0$ with $x_2,\dots,x_m$ fixed.

Acknowledgement. We want to thank C. Bär, A. Degeratu, M. Dahl, P. Petersen, and W. Tuschmann for several helpful discussions. The first named author wants to thank MSRI Berkeley, CA, USA for their hospitality. Research at MSRI is supported in part by NSF grant DMS-9810361.

2. General estimates on vector bundles

Let V be a complex vector bundle of rank k over M equipped with a connection ∇ and a metric $\langle\cdot,\cdot\rangle$. Recall that the second covariant derivative on sections of V is given by $\nabla^2_{X,Y}S=\nabla_X\nabla_Y S-\nabla_{\nabla_X Y}S$, and the curvature tensor on sections of V is given by $R^V(X,Y)S=\nabla^2_{X,Y}S-\nabla^2_{Y,X}S$.

Furthermore, we consider the connection Laplacian on V, which is given by

$$\nabla^*\nabla S=-\sum_{i=1}^n\nabla^2_{e_i,e_i}S,$$

where $\{e_i\}$ is an orthonormal set of vectors at any point $p\in M$. We say that S is an eigensection of V with eigenvalue λ if $\nabla^*\nabla S=\lambda S$. In this general situation, we recall the eigenvalue pinching theorems from ([33], [35]), which characterize eigensections with *small* eigenvalues.

Notation. We use the volume-normalized L^p-norm given by

$$\|u\|_p=\left(\frac{1}{\operatorname{vol}M}\int_M|u|^p\,\mathrm{dvol}\right)^{1/p}$$

and the volume-normalized L^2-scalar product

$$(u,v)=\frac{1}{\operatorname{vol}M}\int\langle u,v\rangle\,\mathrm{dvol}\,.$$

Theorem 2.1. *Suppose that V satisfies $|R^V|<K$, $|\nabla R^V|<K$, and M satisfies $|\sec|<K,\operatorname{diam}<D$. Suppose S is an eigensection of V with eigenvalue λ, normalized so that $\|S\|_2=1$. Then,*

$$\|S\|_\infty\le 1+\tau(\lambda|n,K,D)$$

$$\|\nabla S\|_\infty \leq \tau(\lambda|n, K, D)$$
$$\|\nabla\nabla S\|_2 \leq \tau(\lambda|n, K, D)$$

And furthermore,

Theorem 2.2. *Suppose that $S_1, \ldots, S_m$ are L^2-orthonormal eigensections of V, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m$. Then with K, D as above,*

$$\|\langle S_i, S_j\rangle - \delta_{ij}\|_\infty \leq \tau(\lambda_m|n, K, D).$$

We outline the proofs of these facts in Appendix A.

3. The first Dirac eigenvalue

In this section we characterize manifolds with $|\sec| < K$, diam $< D$, almost non-negative scalar curvature, and a single small Dirac eigenvalue. Our result is that such manifolds are either collapsed in the sense of Cheeger-Fukaya-Gromov or $C^{1,\alpha}$-close to an Einstein manifold with a parallel spinor. Note that in the case of the first eigenvalue on differential p-forms $\lambda_1^+(\Delta_p)$, such a manifold would always be collapsed. Or in other words, there is a lower bound on $\lambda_1^+(\Delta_p)$ in terms of a lower volume bound and the above curvature and diameter bounds ([17]). This is proved as follows. Suppose (M_i, g_i) is a sequence of manifolds as above with $\lambda_1^+(\Delta^p) \to 0$. Then the above conditions imply that there is a subsequence of M_i that converges to a limit manifold $\overline{M}$ in the $C^{1,\alpha}$ topology. But this is not possible since $\overline{M}$ would have a higher p-th Betti number than the limiting manifolds M_i.

On the other hand, since the number of harmonic spinors is not topologically invariant, this argument will not work in the spinor case. The reader interested in harmonic spinors may consult the classical reference [25] or several articles containing recent results [9],[10], [28], [3] about the dependence of $\dim\ker \not{D}$ on the metric.

We will now prove the following theorem from the introduction.

Theorem 1.6. *Let (M, g) have $|\sec| < K$, $\mathrm{diam} < D$, $\mathrm{vol} > v > 0$. Let $\lambda_1(\nabla^*\nabla)$ denote the first eigenvalue of the connection Laplacian $\nabla^*\nabla$ with respect to any spin structure. Then for any $\delta > 0$, there is $\varepsilon = \varepsilon(n, v, K, D, \delta) > 0$ such that if $\lambda_1(\nabla^*\nabla) < \varepsilon$, then (M, g) has $C^{1,\alpha}$-distance $\leq \delta$ to a Ricci-flat Einstein metric with a nontrivial parallel spinor.*

Proof. In the proof of the theorem we will use the following proposition about smoothing Riemannian metrics.

Proposition 3.1. *For any $\delta = \delta(K, n) > 0$ there is $K_1 = K_1(n, K, \delta)$ such that on any n-dimensional complete Riemannian manifold (M, g) with $|\sec| \leq K$ there is a Riemannian metric $\tilde{g}$ on M such that*

$$e^{-\delta}\tilde{g}(X, X) \leq g(X, X) \leq e^{\delta}\tilde{g}(X, X) \qquad \forall X \in TM,$$
$$|\nabla^g - \nabla^{\tilde{g}}|_g \leq \delta.$$
$$|\sec_{\tilde{g}}|_{\tilde{g}} \leq K + \delta, \qquad |\nabla^{\tilde{g}} R_{\tilde{g}}|_{\tilde{g}} \leq K_1.$$

As far as we know, the proposition is due to Abresch [1], it also appears in [15, Theorem 1.12]. A modern proof using the Ricci flow is explained for example in the textbook [16].

Together with Proposition B.1 we see that one can assume in the proof of the theorem that $|\nabla R| < K_1(K,n,D)$. Let σ denote an eigenspinor to the eigenvalue λ_1 with $\|\sigma\|_2 = 1$. Then $\||\sigma|-1\|_\infty < \tau(\lambda_1|n,K,D)$ and $||\nabla\nabla\sigma||_2 < \tau(\lambda_1|n,K,D)$. If $e_1,\dots,e_n$ denotes a local orthonormal frame, then from the curvature formula for spinors $R(X,e_i)\sigma = \frac{1}{4}\sum_{j,k}\langle R(X,e_i)e_j,e_k\rangle\, e_i\cdot e_j\cdot\sigma$ one deduces (see e.g. [18])

$$\mathrm{Ric}(X)\cdot\sigma = -2\sum_{i=1}^n e^i\cdot R(X,e_i)\sigma,$$

and the fact that $R(\cdot,\cdot)\sigma$ is clearly bounded by $\nabla\nabla\sigma$. This implies $\|\,\mathrm{Ric}\,\|_2 < \tau(\lambda_1|n,K,D)$. The lower bound on the volume together with the upper bounds on $|\sec|$ and diam imply a lower bound on the injectivity radius of (M,g). Then from [32, Theorem 6.1] (see also [5], [22]), we have that for $\lambda_1 < \varepsilon(n,v,K,D)$, M is $C^{1,\alpha}$-close to a C^∞ Einstein manifold $\overline{M}$ with $\mathrm{Ric}\equiv 0$. As $\overline{M}$ is diffeomorphic to M if M and $\overline{M}$ are $C^{1,\alpha}$-close, we may assume in the following that M and $\overline{M}$ are equipped with the same spin structure.

It remains to show, that if ε has been chosen small enough, then $\overline{M}$ must carry a parallel spinor. Assume the opposite, then we have a sequence of manifolds M_i converging to $\overline{M}$ in the $C^{1,\alpha}$ topology, with $\lambda_1(\nabla^*\nabla, M_i)\to 0$. Proposition B.1 implies $\lambda_1(\nabla^*\nabla,\overline{M}) = 0$, or in other words $\overline{M}$ admits a parallel spinor. □

Remark. The above proof can be slightly simplified by using spinors on manifolds with a $C^{1,\alpha}$-metric. We avoided this for technical reasons.

4. The fixing dimension of a faithful representation

Here we discuss the fixing dimension for representations which we will need in Section 5. Let G be a compact Lie group, and let $\rho: G\to \mathrm{End}(V)$ be a faithful (i.e. injective) complex representation. For any subspace W of V let

$$\mathrm{Stab}^G(W) := \{g\in G\,|\,\rho(g)w = w\quad \forall w\in W\}$$

be the stabilizer. Note that faithfulness of ρ means that $\mathrm{Stab}^G(V) = \{1\}$.

Definition. The *fixing dimension* $\mathcal{F}(\rho)$ of ρ is defined to be the smallest number $r\in\{0,\dots,\dim V\}$ with the property that any r-dimensional subspace $W\subset V$ has a finite stabilizer $\mathrm{Stab}^G(W)$.

For the standard representation of $U(n)$ on $\mathbb{C}^n$, we have

$$\mathcal{F}\Big(U(n)\hookrightarrow \mathrm{GL}(\mathbb{C}^n)\Big) = n,$$

whereas

$$\mathcal{F}\Big(SU(n)\hookrightarrow \mathrm{GL}(\mathbb{C}^n)\Big) = n-1.$$

Proposition 4.1. *Let $\rho : G \to \mathrm{End}(V)$ be a faithful unitary representation with fixing dimension $\mathcal{F}(\rho)$. Then there is $N \in \mathbb{N}$ such that for all $\mathcal{F}(\rho)$-dimensional subrepresentations $W \subset V$, we have $\#\mathrm{Stab}^G(W) \le N$. We denote by $N(\rho)$ the smallest such N.*

Proof. Let $k = \mathcal{F}(\rho)$, and let K be the set of all orthonormal k-frames $F = (v_1, \dots, v_k) \in V^k$. The compact group G acts on the compact manifold K differentiably. The isotropy group of $F = (v_1, \dots, v_k)$

$$G_F := \{g \in G \mid gF = F\}$$

coincides with $\mathrm{Stab}^G(\mathrm{span}(v_1, \dots, v_k))$. Due to the definition of the fixing dimension, these isotropy groups are finite. In order to show that $\sup\{\#G_F \mid F \in K\} < \infty$ one can either suppose that there exists a sequence of $F(i) \in K$ with $\#G_{F(i)} \to \infty$ and easily prove that this yields a contradiction, or one can alternatively use the following theorem due to Mostow.

Theorem 4.2 ([30, Sec. 4],[31])**.** *Let G be a compact Lie group acting on a compact manifold K, then there is a finite set $\{H_1, \dots, H_j\}$ of subgroups of G such that any isotropy group of the action is conjugated to a H_i, $1 \le i \le j$.*

□

In section 6 we will determine the fixing dimension of the spinor representation.

5. Eigenvalue pinching on vector bundles with special holonomy

Let V be a vector bundle of rank $k > 0$ over M equipped with a connection ∇ and a metric $\langle \cdot, \cdot \rangle$. We fix $p \in M$. We assume that the holonomy group of the bundle is contained in a closed Lie group $H \subset U(V_p)$. For any $q \in M$ let H_q be the parallel transport of H to q.

Proposition 5.1. *We assume that M and V satisfy the conditions of Theorem 2.1. Assume that the rank-k bundle V has holonomy contained in H. Let $r = \mathcal{F}(H \subset U(T_pM))$ be the fixing dimension of the holonomy. Let $S_1, \dots, S_r$ be L^2-orthonormal sections of V such that*

$$\nabla^* \nabla S_i = \lambda_i S_i,$$

$1 \le i \le r$, $0 \le \lambda_i \le \varepsilon$. Then for small $\varepsilon > 0$, there is a finite cover $\pi : \widetilde{M} \to M$ and smooth sections $e_1, \dots, e_k$ of $\pi^*(V)$ with the following properties:

(1) $\mathcal{E} := (e_1, \dots, e_k)$ is a frame, i.e. $\mathcal{E}(q)$ is a basis of $\pi^*(V)_q$ for all $q \in \widetilde{M}$.

(2)

$$|S_i(\pi(q)) - e_i(q)| \le \tau(\varepsilon|n, K, D) \qquad \forall q \in \widetilde{M}, i = 1, \dots, r.$$

(3)

$$(\nabla \mathcal{E})_q = (\nabla e_1, \dots, \nabla e_k)_q \in T_q^* \widetilde{M} \otimes \mathrm{Lie}(H_{\pi(q)})$$

(4)

$$|\nabla e_i(q)| \le \tau(\varepsilon|n, K, D) \qquad \forall i = 1, \dots, k \quad \forall q \in \widetilde{M}.$$

Furthermore, if $N(\iota) = 1$ where $\iota : H \subset U(T_pM)) = 1$ is the inclusion, then we can choose $\widetilde{M} = M$.

Proof. According to Theorems 2.1 and 2.2 we have the estimates

$$\|S_i\|_\infty \leq 1 + \tau(\varepsilon|n, K, D) \tag{5.2}$$

$$\|\nabla S_i\|_\infty \leq \tau(\varepsilon|n, K, D) \tag{5.3}$$

and

$$\|\langle S_i, S_j\rangle - \delta_{ij}\|_\infty < \tau(\varepsilon|n, K, D). \tag{5.4}$$

We apply pointwise the Hilbert-Schmidt orthogonalization procedure to $S_1, \ldots, S_r$ and obtain new sections $\tilde{S}_1, \ldots, \tilde{S}_r$. All functions in this procedure and their first derivatives are controlled in terms of ε, n, K, and D. As a consequence these new sections also satisfy (5.2) and (5.3) and are pointwise orthonormal. We fix a point $p \in M$. The parallel transports of $(\tilde{S}_1(p), \ldots, \tilde{S}_r(p))$ define a principal bundle over M whose structure group is the holonomy group. By enlarging the structure group to H we obtain an H-principal bundle which we will denote by $P_H(M)$. The bundle $P_H(M)$ is a parallel subbundle of the frame bundle $P_{U(k)}(V)$.

We denote $G := \mathrm{Stab}^{U(k)}(\mathbb{R}^r) \cong U(k-r)$. Let $P_G(M)$ be the bundle of orthonormal bases of V such that the first r basis vectors coincide with $\tilde{S}_1, \ldots, \tilde{S}_r$ at each base point. Note that $P_G(M)$ is an $U(k-r)$ principal bundle. The bundles $P_G(M)$ and $P_H(M)$ have a common point $(\tilde{S}_1(p), \ldots, \tilde{S}_r(p))$. Because r is the fixing dimension of $H \subset U(T_pM)$ we get $\#G \cap H < \infty$.

Choose a bi-invariant metric on $U(k)$. This induces a metric on each fiber of $P_{U(k)}(V)$. For $x \in M$ let $\delta(x)$ be the distance of the fiber of $P_G(M)$ over x to the fiber of $P_H(M)$ over x with respect to this metric. Then, $\delta : M \to [0, \infty)$ is a function with $\delta(p) = 0$. Using (5.3) one sees that δ is a Lipschitz function with Lipschitz constant of the form $\tau(\varepsilon|n, K, D)$. We assume that ε is so small that 2δ is smaller than the injectivity radius of $U(k)$ and smaller than $\inf\{d(A, e) \mid A \in G \cap H, A \neq e\}$. Let $\widetilde{M}$ be the set of all points $p \in P_H(M)$ having minimal distance to $P_G(M)$, more exactly the points such that $d_{\pi(p)}(p, P_G(M)) = \delta(\pi(p))$, where $\pi(p) \in M$ is the basepoint of M and where $d_{\pi(p)}$ denotes the distance in the fiber of $P_{U(k)}$ over $\pi(p)$. Because of symmetry we have $\#G \cap H$ many points in $\widetilde{M}$ over each point in M. As δ is chosen as above, $\widetilde{M}$ is a smooth manifold and $\pi : \widetilde{M} \to M$ is a covering of M with $\#G \cap H$ many leaves. Any $q \in \widetilde{M} \subset P_{U(k)}V$ can be written as $q = (e_1(q), \ldots, e_k(q))$ with $e_j \in V_{\pi(q)}$, and e_j are clearly smooth sections of $\pi^*(V)$ satisfying (1).

Because of our construction the distance between e_i and $\tilde{S}_i$ is bounded by δ, and hence we obtain (2). As $\mathcal{E} := (e_1, \ldots, e_k)$ is a section of $\pi^*(P_H(M))$, we see that (3) holds.

For (4) we have to prove that for q in M and an arclength-parametrized curve c with $c(0) = q$,

$$|\nabla_{\dot{c}(0)} e_i(q)| \leq \tau(\varepsilon|n, K, D). \tag{5.5}$$

Let $\widehat{\mathcal{E}}$ be the parallel transport of $\mathcal{E}_q$ along c. Obviously, $\widehat{\mathcal{E}} \in P_H(M)$. Suppose that $A \in U(n)$ is the unique matrix such that $\mathcal{E}_q \cdot A$ is the closest point to $\mathcal{E}_q$ in $(P_G(M))_q$. Then $\widehat{\mathcal{E}} \cdot A$ is also a parallel frame of V along c, and by the construction of $P_G(M)$ and also (5.3) one sees that the distance from $\widehat{\mathcal{E}}(t) \cdot A$ to $(P_G(M))_{c(t)}$ is bounded by $|t|\tau(\varepsilon|n, K, D)$. On the other hand, $\mathcal{E}(c(t))$ is by definition a point such that it is — among all points in $(P_H(M))_{c(t)}$ — of minimal distance to $(P_G(M))_{c(t)}$. By applying the implicit function theorem one concludes that the distance between $\mathcal{E}(c(t))$ and $\widehat{\mathcal{E}}(t)$ is bounded by $|t|\tau(\varepsilon|n, K, D)$ for small t. This implies (5.5). □

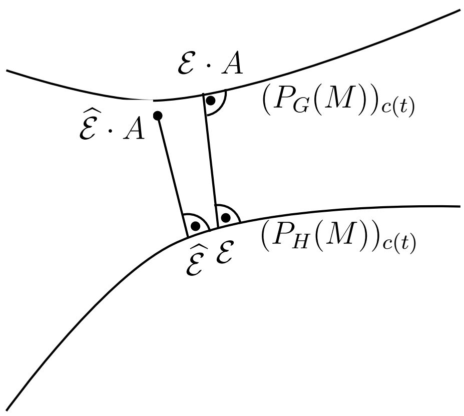

FIGURE 1. The shortest line between $(P_G(M))_{c(t)}$ and $(P_H(M))_{c(t)}$.

6. The fixing dimension for the spinor representation

Let V be an n-dimensional real vector space. We view $\mathrm{Spin}(V)$ as a subgroup of the group of invertible elements of the Clifford algebra of V (see e.g. [26], [19] or [24]). Let $e_1, \dots, e_n$ be an orthonormal basis of V. The complex spinor representation Σ of $\mathrm{Spin}(V)$ has dimension $2^{[n/2]}$.

PROPOSITION 6.1. *Let $g \in \mathrm{Spin}(V)$, $g \neq 1$. Then the multiplicity of the eigenvalue 1 of the endomorphism $g \in \mathrm{End}(\Sigma)$ is at most $2^{[n/2]-1}$.*

Proof. We set $A_j := e_{2j-1} \cdot e_{2j} \in Cl(V)$ for $j = 1, \dots, m$, $m := [n/2]$.

Any $g \in \mathrm{Spin}(V)$ is contained in a maximal torus, i.e. there is an $h \in \mathrm{Spin}(V)$ and $t_j \in \mathbb{R}$ such that

$$g = h \cdot \exp(t_1 A_1) \cdot \exp(t_2 A_2) \cdot \dots \cdot \exp(t_m A_m) \cdot h^{-1}.$$

Let h' be the image of h under the map $\mathrm{Spin}(V) \to \mathrm{SO}(V)$. Then $h \cdot \exp(t_j A_j) \cdot h^{-1} = \exp(t_j h \cdot A_j \cdot h^{-1})$. We set

$$\widehat{A}_j := h \cdot A_j \cdot h^{-1} = h'(e_{2j-1}) \cdot h'(e_{2j})$$

and

$$g_j := \exp(t_j \widehat{A}_j) = \cos t_j + \sin t_j \widehat{A}_j.$$

The $\widehat{A}_j$ are pairwise commuting anti-self-adjoint endomorphisms. Hence they are simultaneously diagonalizable, with eigenvalues i and $-i$. Furthermore $h'(e_{2j})$ anti-commutes with $\widehat{A}_j$ and commutes with $\widehat{A}_k$, $j \neq k$. Hence, all simultaneous eigenspaces have the same dimension, which is 1.

We conclude that all g_j are simultaneously diagonalizable with eigenvalues $\exp(it_j)$ and $\exp(-it_j)$, having 1-dimensional simultaneous eigenspaces. Thus g has the eigenvalues

$$e^{i(\pm t_1 \pm t_2 \pm \cdots \pm t_m)}$$

where the signs vary independently, each sign combination providing an eigenspace of multiplicity 1. As a consequence, for any $g \neq 1$, the multiplicity of the eigenvalue 1 is at most 2^{m-1}. □

PROPOSITION 6.2. *The fixing dimension of the complex spinor representation of* $\mathrm{Spin}(n)$ *is*

$$r := \begin{cases} 1 & \text{if } n = 2, 3 \\ 2^{[\frac{n}{2}]-1} + 1 & \text{if } n \geq 4 \end{cases}$$

Furthermore, for any r-dimensional subspace W of Σ we have $\#\mathrm{Stab}^G(W) = 1$, i.e. the N in Proposition 4.1 equals 1.

Proof. If $n = 2$, the spinor representation is

$$S^1 \to \mathrm{SU}(2), \quad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

There are no invariant subspaces, hence $r = 1$ and $N = 1$.

If $n = 3$, then the spinor representation is the identity $\mathrm{Spin}(3) = \mathrm{SU}(2) \to \mathrm{SU}(2)$. Let W be a 1-dimensional subspace of $\mathbb{C}^2$. Any $h \in \mathrm{Stab}^{\mathrm{SU}(2)}(W)$ can be diagonalized with eigenvalues λ and λ^{-1}. However, as W is fixed by h, we obtain $\lambda = 1$, and hence $h = 1$. We have thus shown that $r \leq 1$. Obviously $r \geq 1$. We also see $N = 1$.

If $n \geq 4$, then example (2) in the introduction shows that $r > 2^{[\frac{n}{2}]-1}$. The previous proposition implies $r \leq 2^{[\frac{n}{2}]-1} + 1$ and $N = 1$. □

7. Proof of Theorem 1.1

We briefly recall some definitions from spin geometry. Details can be found for example in [26].

Definition. Let (M, g) be a Riemannian manifold. Let $P_{\mathrm{SO}}(M)$ be the frame bundle over M. A *spin structure* is a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}}(M)$ together with a $\Theta : \mathrm{Spin}(n) \to SO(n)$-equivariant fiber map $\chi : P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$ over the identity $M \to M$.

Example. Let G be an n-dimensional Lie group, and Γ a lattice in G. The frame bundle of G is trivialized by left invariant frames, i.e. $\mathrm{SO}(G) = G \times \mathrm{SO}(n)$. Hence, there is a spin structure on G given by $\mathrm{Spin}(G) = G \times \mathrm{Spin}(n)$ where χ is the identity in the first component and the standard map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ in the second component. The frame bundle of $\Gamma\backslash G$ is $(\Gamma\backslash G) \times \mathrm{SO}(n)$. One possible spin structure on $\Gamma\backslash G$ is $(\Gamma\backslash G) \times \mathrm{Spin}(n)$ together with the equivariant map $\mathrm{id} \times \Theta$. This spin structure is called the *trivial spin structure*.

Definition. Let $\rho : \mathrm{Spin}(n) \to U(\Sigma)$ be the complex spinor representation. The *spinor bundle* is defined as the associated vector bundle

$$\Sigma M := P_{\mathrm{Spin}}(M) \times_\rho \Sigma.$$

As ρ is a $2^{[n/2]}$-dimensional complex representation, the complex vector bundle Σ has rank $k := 2^{[n/2]}$. The holonomy is contained in $\mathrm{Spin}(n)$, the inclusion $\mathrm{Spin}(n) \hookrightarrow \mathrm{U}(k)$ given by the spinor representation.

As before we denote the fixing dimension of the spinor representation by

$$r = r(n) = \begin{cases} 1 & \text{if } n = 2, 3 \\ 2^{\left[\frac{n}{2}\right]-1} + 1 & \text{if } n \geq 4. \end{cases}$$

THEOREM 1.1. *Let (M^n, g, χ) be a compact Riemannian spin manifold with $|\sec| < K$, $\operatorname{diam} < D$. Then there is $\varepsilon = \varepsilon(n, K, D) > 0$, such that if $\lambda_r(\nabla^*\nabla) < \varepsilon$, then M is diffeomorphic to a nilmanifold. Furthermore, χ is the trivial spin structure on M.*

Proof. We will use Abresch's smoothing theorem (Proposition 3.1). A brief survey of such results is contained in Section 5 of [20].) This theorem states that for any $\delta > 0$ there is a constant $K_1(K, n, \delta)$ such that any metric g on a compact manifold M^n with $|\sec_g| < K$ can be approximated by another metric $\tilde{g}$ on M with
(1) $e^{-\delta} g \leq \tilde{g} \leq e^\delta g$,
(2) $|\nabla^g - \nabla^{\tilde{g}}| \leq \delta$,
(3) $|\sec_{\tilde{g}}| \leq K + \delta$, and
(4) $|\nabla R_{\tilde{g}}| \leq K_1$.
As the eigenvalues of the connection Laplacian $\nabla^*\nabla$ on the spinor bundle are uniformly continuous under C^1-perturbations of the metric (Proposition B.1), this shows that it is sufficient to prove the theorem under the additional assumption that $|\nabla R|$ is bounded. We will formulate the remaining step as a lemma. □

LEMMA 7.1. *Let (M^n, g, χ) be a compact Riemannian spin manifold with $|\sec| < K$, $|\nabla R| < K$, $\operatorname{diam} < D$. Then there is $\varepsilon = \varepsilon(n, K, D) > 0$, such that if $\lambda_r(\nabla^*\nabla) < \varepsilon$, then M is diffeomorphic to a nilmanifold. Furthermore, χ is the trivial spin structure on M.*

Proof of the lemma. We apply Proposition 5.1 for $V = \Sigma M$, $H = \mathrm{Spin}(n)$. We obtain a frame $\mathcal{E}$ of ΣM with $|\nabla \mathcal{E}| = \tau(\lambda | n, K, D)$. The spin structure $\chi : P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$ maps $\mathcal{E}$ to $\chi(\mathcal{E})$ with

$$|\nabla \chi(\mathcal{E})| \leq \tau(\varepsilon | n, K, D),$$

i.e. an almost parallel frame of TM. Now, using [23] we see that M is C^0-close and diffeomorphic to a nilmanifold $\Gamma\backslash N$ with Γ a cocompact lattice in the nilpotent Lie group N. Let $\mathcal{F}'$ be a frame on M which is sufficiently close to $\chi(\mathcal{E})$ and whose pullback to N is left-invariant. It can be lifted, i.e. there is a frame $\mathcal{E}'$ with $\chi(\mathcal{E}') = \mathcal{F}'$, hence the spin structure is trivial. □

A. Some analytical tools

Here we outline the results from [33],[34] which we need. The main analytic tool is the following lemma which follows from Moser iteration. Note that Lemma 3.1 in [33] is incorrect. A correct version is as follows [34]. Similar bounds were also obtained in [6] and [12], where a version of Lemma A.4 was derived which does not depend on $|\operatorname{div} R^V|$.

Lemma A.1. *Let* (M, g) *satisfy* $\operatorname{Ric} \geq -k^2$, $\operatorname{diam} < D$. *Then for a function* u *on* M *satisfying* $\Delta u \leq \alpha u + \beta$, $\alpha, \beta \geq 0$ *we have* $\|u\|_\infty \leq \tau(\|u\|_2 | \alpha, \beta, n, k, D)$. *If* $\beta = 0$ *then in fact* $\|u\|_\infty \leq (1 + \tau(\alpha | n, k, D))\|u\|_2$.

Here the diameter and Ricci curvature bounds give a bound on the Sobolev constant used in Moser iteration by a result of Gallot, and the lower Ricci curvature bound is implied by the bounds on sectional curvature which we have assumed. Then, a standard argument yields the following.

Lemma A.2. *Let* V *be a vector bundle over* M. *Suppose* M *has* $\operatorname{Ric} \geq -k^2$, $\operatorname{diam} < D$. *Then for any section of* V *satisfying* $\langle \nabla^*\nabla S, S\rangle \leq \lambda |S|^2$, $\|S\|_2 = 1$, *we have* $\|S\|_\infty \leq 1 + \tau(\lambda | n, k, D)$.

Proof.

$$\begin{aligned} \Delta|S|^2 &= 2\langle \nabla^*\nabla S, S\rangle - 2|\nabla S|^2 \\ &\leq 2\langle \nabla^*\nabla S, S\rangle - 2\Big|\nabla|S|\Big|^2 \\ &\leq 2\lambda|S|^2 - 2\Big|\nabla|S|\Big|^2 \end{aligned}$$

Now we can use that we also have

$$\Delta|S|^2 = 2|S|\,\Delta|S| - 2\Big|\nabla|S|\Big|^2$$

and solving for $\Delta|S|$ we get $\Delta|S| \leq \lambda|S|$, and hence A.1 gives the desired result. □

To bound $|\nabla S|$ we apply the following Bochner formula

Lemma A.3. *Let* $R^V : TM \otimes TM \otimes V \to V$ *denote the curvature of the vector bundle* V. *Let* $\operatorname{div}^1 R^V$ *be minus the metric contraction of* ∇R^V *in the first two slots. Then for any section* S *of* V

$$\nabla(\nabla^*\nabla)S = (\nabla^*\nabla)\nabla S - (\operatorname{div}^1 R^V)S + \nabla_{\operatorname{Ric}(.)}S + 2c_{12}(\operatorname{id} \otimes R^V)(\nabla S)$$

Proof. First note that, as the metric is parallel, metric contraction is parallel. The metric contraction of the i-th slot with the j-th slot is denoted by c_{ij}. Let $\tau : TM \otimes TM \to TM \otimes TM, \quad X \otimes Y \mapsto Y \otimes X$. Note that $R^V S = \nabla\nabla S - (\tau \otimes \mathrm{id})\nabla\nabla S$. We calculate

$$\begin{aligned} \nabla(\nabla^*\nabla)S &= -\nabla c_{12}(\nabla\nabla S) \\ &= -c_{23}(\nabla\nabla\nabla S) \\ &= -c_{23}(\tau \otimes \mathrm{id} \otimes \mathrm{id})(\nabla\nabla\nabla S + R^{T^*M\otimes V}\nabla S) \\ &= -c_{23}R^{T^*M\otimes V}\nabla S - c_{13}\nabla\nabla\nabla S \end{aligned}$$

The first summand gives

$$-c_{23}R^{T^*M\otimes V}\nabla S = c_{23}\nabla_{R(\cdot,\cdot)\cdot}S - c_{13}(\mathrm{id} \otimes R^V)(\nabla S) = \nabla_{\mathrm{Ric}(\cdot)}S + c_{12}(\mathrm{id} \otimes R^V)(\nabla S).$$

For the second term,

$$\begin{aligned} -c_{13}\nabla\nabla\nabla S &= -c_{13}\nabla\left((\tau \otimes \mathrm{id})\nabla\nabla S + R^V S\right) \\ &= -c_{13}(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\nabla\nabla\nabla S) + c_{12}\nabla R^V S \\ &= (\nabla^*\nabla)\nabla S - \mathrm{div}^1 R^V S + c_{12}(\mathrm{id} \otimes R^V)(\nabla S), \end{aligned}$$

where we have used the definition $\mathrm{div}^1 R^V S = -c_{12}(\nabla R^V)S$. □

As a consequence of the Lemmas A.1 and A.3 we have

Lemma A.4. *Suppose that M, V have $\mathrm{Ric} \geq -k^2$, $|c_{12}(\mathrm{id} \otimes R^V)|, |\mathrm{div}^1 R^V| < K$, and diam $< D$. Let S be an eigensection of V with $\nabla^*\nabla S = \lambda S$, and $\|S\|_2 = 1$. Then $\|\nabla S\|_\infty \leq \tau(\lambda|n, k, K, D)$.*

Note that $|\mathrm{Ric}|$ and $|c_{12}(\mathrm{id} \otimes R^V)|$ are bounded by $|\sec|$ and $|R^V|$, and $|\mathrm{div}^1 R^V|$ is bounded by $|\nabla R^V|$.

Proof. First of all

$$\begin{aligned} \Delta|\nabla S|^2 &= 2\langle \nabla^*\nabla(\nabla S), \nabla S\rangle - 2|\nabla\nabla S|^2 \\ &\leq 2\langle \nabla^*\nabla(\nabla S), \nabla S\rangle - 2\Big|\nabla|\nabla S|\Big|^2 \end{aligned}$$

Again we can use

$$\Delta|\nabla S|^2 = 2|\nabla S|\,\Delta|\nabla S| - 2\Big|\nabla|\nabla S|\Big|^2 \tag{A.5}$$

Which gives

$$\begin{aligned} |\nabla S|\Delta|\nabla S| &\leq \langle \nabla^*\nabla(\nabla S), \nabla S\rangle \\ &= \langle \nabla(\nabla^*\nabla)S - \nabla_{\mathrm{Ric}(.)}S + \mathrm{div}^1(R^V S) - 2c_{12}(\mathrm{id} \otimes R^V)(\nabla S), \nabla S\rangle \\ &\leq (\lambda + k^2 + 2|c_{12}(\mathrm{id} \otimes R^V)|)|\nabla S|^2 + |\mathrm{div}^1 R^V||S||\nabla S|. \end{aligned}$$

Hence

$$\begin{aligned} \Delta|\nabla S| &\leq (\lambda + k^2 + 2|c_{12}(\mathrm{id} \otimes R^V)|)|\nabla S| + |\mathrm{div}^1 R^V||S| \\ &\leq (\lambda + k^2 + 2|c_{12}(\mathrm{id} \otimes R^V)|)|\nabla S| + |\mathrm{div}^1 R^V|(1 + \tau(\lambda|n, k, D)). \end{aligned}$$

Then finally we can use Lemma A.1, along with the fact that $\|\nabla S\|_2 = \lambda\|S\|_2$. □

We then note that Lemma A.3 gives us a bound on $\nabla^*\nabla(\nabla S)$ from which we can conclude from Lemma A.4 that $\int_M \langle \nabla^*\nabla(\nabla S), \nabla S\rangle dV = \|\nabla\nabla S_i\|_2$ is small. Hence, Theorem 2.1 is proven.

Finally we include a proof of Theorem 2.2.

Theorem 2.2. *Suppose that $S_1, \dots, S_m$ are L^2-orthonormal eigensections of V, with eigenvalues $\lambda_1 \le \dots \le \lambda_m$. Then with for k, K, D as above*

$$\|\langle S_i, S_j\rangle - \delta_{ij}\|_\infty \le \tau(\lambda_m | n, K, k, D).$$

Proof. For any unit length vector X we calculate for

$$\begin{aligned} |\nabla_X \langle S_i, S_j\rangle| &= |\langle \nabla_X S_i, S_j\rangle + \langle S_i, \nabla_X S_j\rangle| \\ &\le \|\nabla S_i\|_\infty \|S_j\|_\infty + \|S_i\|_\infty \|\nabla S_j\|_\infty \\ &\le \tau(\max(\lambda_i, \lambda_j) | n, k, K, D) \end{aligned}$$

where we used Lemmata A.2 and A.4 in the last inequality. Together with $\int_M \langle S_i, S_j\rangle = \delta_{ij}$ the statement easily follows. □

B. Connection Laplacians under perturbations of the metric

We assume here that M is a compact spin manifold with a fixed (topological) spin structure. The topological spin structure defines for any metric g on M a (metric) spin structure $P_{\mathrm{Spin}}(M, g) \to P_{\mathrm{SO}}(M, g)$.

Proposition B.1. *Let ∇^g be the Levi-Civita-connection on the spinor bundle with respect to the metric g. Let $\lambda_1(g) \le \lambda_2(g) \dots$ be the eigenvalues of the connection Laplacian $\nabla^{g*}\nabla^g$. For two metrics g and $\tilde g$ let $\delta = \delta(g, \tilde g)$ be the smallest number such that*

$$e^{-\delta}\tilde g(X, X) \le g(X, X) \le e^\delta \tilde g(X, X) \qquad \forall X \in TM,$$
$$|\nabla^g - \nabla^{\tilde g}|_g \le \delta.$$

Then

$$e^{-\frac{1001}{1000}\delta}\lambda_k(\tilde g) - \tau(\delta) \le \lambda_k(g) \le e^{\frac{1001}{1000}\delta}\lambda_k(\tilde g) + \tau(\delta),$$

where $\tau(\delta) \to 0$ for $\delta \to 0$.

Proof. Let A be the unique positive selfadjoint endomorphism of TM such that

$$\tilde g(AX, AY) = g(X, Y).$$

As a consequence

$$e^{-\delta/2}\,\mathrm{Id} \le A \le e^{\delta/2}\,\mathrm{Id}$$

in the sense of symmetric operators and

$$|\widetilde\nabla(A^2)|_{\tilde g} = |\widetilde\nabla g|_{\tilde g} = |(\widetilde\nabla - \nabla)g|_{\tilde g} \le \tau_1(\delta).$$

Hence also $|\widetilde\nabla A|_{\tilde g} \le \tau_2(\delta)$. Let $e_1, \dots, e_n$ be a local orthonormal frame for g. Then $Ae_1, \dots, Ae_n$ is an orthonormal frame for $\tilde g$. The connection-1-forms ω and $\tilde\omega$ are defined as

$$\omega(X)_j^k := g(\nabla_X e_j, e_k) \qquad \widetilde\omega(X)_j^k := \tilde g(\widetilde\nabla_X Ae_j, Ae_k).$$

We calculate

$$\begin{aligned}|\omega(X)_j^k - \widetilde{\omega}(X)_j^k| &\leq |g((\nabla_X - \widetilde{\nabla}_X)e_j, e_k)| + |g(A^{-1}(\widetilde{\nabla}_X A)e_j, e_k)| \\ &\leq \left(|\nabla - \widetilde{\nabla}|_g + e^{\delta/2}|\widetilde{\nabla} A|_{\tilde{g}}\right)|X| \leq \tau_3(\delta)\,|X|\end{aligned}$$

The map A induces an $\mathrm{SO}(n)$ equivariant fiber map $P_{\mathrm{SO}}(M,g) \to P_{\mathrm{SO}}(M,\widetilde{g})$ which lifts to the spin structure (if the spin structures coincide as topological spin structures). The corresponding vector bundle map of the associated bundles is an isomorphism of vector bundle $A_\Sigma : \Sigma(M,g) \to \Sigma(M,\widetilde{g})$ which preserves length fiberwise and such that $A_\Sigma(V\cdot\varphi) = A(V)\cdot A_\Sigma(\varphi)$ for all $V \in T_pM$, $\varphi \in \Sigma_p M$.

$$A_\Sigma(\nabla_X\varphi) - \widetilde{\nabla}_X(A_\Sigma\varphi) = \frac{1}{4}\sum_{j,k=1}^n (\omega(X)_j^k - \widetilde{\omega}(X)_j^k) A_\Sigma(e_j\cdot e_k\cdot\varphi).$$

Hence $|A_\Sigma(\nabla_X\varphi) - \widetilde{\nabla}_X(A_\Sigma\varphi)| \leq \frac{n^2\tau_3(\delta)}{4}$.

$$|d(\det A)|_{\widetilde{g}} \leq \tau(\delta).$$

We set $\widetilde{\psi} := (\det A)A_\Sigma\psi$. The map $L^2(\Sigma(M,g)) \to L^2(\Sigma(M,\tilde{g}))$, $\psi \mapsto \widetilde{\psi}$ is an isometry. Then

$$\langle\nabla^*\nabla\psi,\psi\rangle^{1/2} \leq e^{\delta/2}\langle\widetilde{\nabla}^*\widetilde{\nabla}\widetilde{\psi},\widetilde{\psi}\rangle^{1/2} + \tau(\delta).$$

From this we deduce

$$(1-\tau(\delta))\,e^{-\delta}\lambda_k(\tilde{g}) - \tau(\delta) \leq \lambda_k(g) \leq (1+\tau(\delta))\,e^{\delta}\lambda_k(\tilde{g}) + \tau(\delta).$$

As a consequence

$$e^{-\frac{1001}{1000}\delta}\lambda_k(\tilde{g}) - \tau(\delta) \leq \lambda_k(g) \leq e^{\frac{1001}{1000}\delta}\lambda_k(\tilde{g}) + \tau(\delta).$$

□

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