DETAILS TO EXAMPLES IN "MANIFOLDS WITH SMALL DIRAC EIGENVALUES ARE NILMANIFOLDS"

BERND AMMANN, CHAD SPROUSE

1. 3-dimensional nilmanifolds

We want to make the statements in Example (1) more explicit for the 3-dimensional nilmanifolds.

Let $H_3 := \{m(x,y,z)\,|\,x,y,z\in\mathbf{R}\}$ be the 3-dimensional Heisenberg group where

$$m(x,y,z) := \begin{pmatrix} 1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}.$$

Obviously the unit e is $m(0,0,0)$. Let $\Gamma := \{m(x,y,z)\,|\,x,y,z\in\mathbf{Z}\}$, and $M = \Gamma\backslash H_3$. For $\alpha>0$, there is a unique left invariant metric on H_3 which is given on T_eH_3 by $dx^2+dy^2+\alpha^2dz^2$. This defines a Riemannian metric on M, denoted by g_α. Then as $\alpha\to 0$ $|\sec_{g_\alpha}|\to 0$, $diam(M,g_\alpha)<C$ and $Vol(M,g,\alpha)=\alpha\to 0$. For the *trivial* spin structure on M one calculates $\lambda_1(D\!\!\!/^2)\to 0$ [1, Section 3]. This example shows that the lower volume bound in [2, Corollary 1.7] is necessary.

2. Details to Examples (3)

2.1. A metric on the cylinder.

Lemma 1. *Let h be the standard metric on S^3. The ball of radius R around 0 in euclidean space $\mathbb{R}^4$ will be denoted as $B_R^{\mathbb{R}^4}(0)$. For any $\epsilon,R,\rho>0$, there is $a>0$ and a metric $g=dt^2+\varphi^2(t)h$ on $S^3\times(-(a+\rho),(a+\rho))$ such that*

(1) $g|_{(-(a+\rho),-a]\times S^3}$ *and* $g|_{[a,(a+\rho))\times S^3}$ *are isometric to the* $B_{R+\rho}^{\mathbb{R}^4}(0)\setminus B_R^{\mathbb{R}^4}(0)$,
(2) $\mathrm{scal}_g\geq-\epsilon$, *and*
(3) $\mathrm{diam}(M,g)\leq 6(R+\rho)$

Proof. The scalar curvature of the metric
$g=dt^2+\varphi^2(t)h$ on $(a,b)\times S^3$ is

$$\mathrm{Scal}=6\left(-\frac{\varphi''}{\varphi}+\frac{1-(\varphi')^2}{\varphi^2}\right)$$

(see Petersen, Riemannian geometry, Section 3.2.3). Hence, the metric g is scalar flat iff φ is a solution of

$$-\frac{\varphi''}{\varphi}+\frac{1-(\varphi')^2}{\varphi^2}=0.$$

Obviously, this equation is equivalent to

$$\frac{1}{2}\frac{d^2}{dt^2}(\varphi^2)=1.$$

So, if we take the initial values $\varphi(0) = c > 0$ and $\varphi'(0) = 0$, then we obtain the unique solution

$$\varphi_c(t) = \sqrt{t^2 + c^2}.$$

Choose any $a_0 < R$, close to R. We want to modify φ_c such the metric $\tilde{g} = dt^2 + \tilde{\varphi}^2(t)h$ is euclidean on $[a_0, \infty) \times S^3$ and $(-\infty, -a_0] \times S^3$. For a small given $\delta > 0$ one can choose a small c such that $\kappa := 1 - \varphi_c'(a_0 - \delta) > 0$ is arbitrarily close to 0. We also can assume $\varphi_c(a_0) < R$. We define

$$\tilde{\varphi}(t) = \begin{cases} \varphi_c(t) & \text{for } |t| \leq a_0 - \delta \\ \frac{\kappa}{2\delta}(|t| - (a_0 - \delta))^2 + (1 - \kappa)(|t| - (a_0 - \delta)) + k_1 & \text{for } a_0 - \delta < |t| < a_0 \\ |t| + k_2 & \text{for } |t| \geq a_0 \end{cases}$$

where we choose k_1 and and k_2 such that the function $\tilde{\varphi}$ is continuous, i.e. $k_1 := \varphi(a_0 - \delta)$ and $k_2 := \delta - \frac{\kappa\delta}{2} + k_1 - a_0$. One directly sees that $\tilde{\varphi}$ is C^1. It is immediately clear from the construction that the metric $\tilde{g} = dt^2 + \tilde{\varphi}^2(t)h$ is euclidean on $[a_0, \infty) \times S^3$ and $(-\infty, -a_0] \times S^3$. It is also clear that if we choose δ and c sufficiently small, then $\tilde{\varphi}(a_0) = \delta - \frac{\kappa\delta}{2} + \varphi_c(a_0 - \delta) < R$, hence (1) holds for $a := a_0 + R - \tilde{\varphi}(a_0)$.

The metric is obviously scalar-flat outside of $[-a_0, -a_0 + \delta] \times S^3 \cup [a_0 - \delta, a_0] \times S^3$. In order to check (2) we calculate for $|t| \in [a_0 - \delta, a_0]$

$$\begin{aligned} -\frac{\tilde{\varphi}''}{\tilde{\varphi}} + \frac{1 - (\tilde{\varphi}')^2}{\tilde{\varphi}^2} &\geq -\frac{\tilde{\varphi}''}{\tilde{\varphi}} \\ &= -\frac{\kappa}{\delta\varphi_c(a_0 - \delta)} \\ &\geq -\frac{\kappa}{\delta(a_0 - \delta)} \end{aligned}$$

So as long as κ is small enough relative to δ and $a_0 - \delta$, then the minimum of the scalar curvature will be arbitrarily close to 0.

(3) is clear from the construction. $\square$

Remark. The obtained metric is only *piecewise* smooth and C^1 everywhere. However, it is easy to show that after a suitable smoothing, one obtains a smooth metric for which the lemma holds.

2.2. **Constructing the example.**

Theorem 2. *Let $n \geq 4$. We set $r := 2^{[n/2]}$ which is the rank of the spinor bundle in n dimensions. For given $\epsilon > 0$, $D > 0$ there is a sequence of compact n-dimensional Riemannian spin manifolds $(M_j, g_j, \chi_j)_{j \in \mathbb{N}}$ such that*

(1) $\operatorname{diam}(M_j, g_j) \leq D$
(2) $\operatorname{scal}_{g_j} \geq -\epsilon$
(3) *The smallest r eigenvalues of the square of the Dirac operator on (M_j, g_j, χ_j) are smaller than ϵ.*
(4) $b_1(M_j) \to \infty$ *for* $j \to \infty$.

Proof.

It is sufficient to prove the theorem for $n = 4$. As soon as we have a 4-dimensional family of manifolds (M_j, g_j, χ_j) with the above properties one obtains a higher dimensional family by taking the product with $(S^1)^{n-4}$ equipped with any flat metric and a trivial spin structure (i.e. the $\operatorname{Spin}(n-4)$-principal bundle admits a parallel section) also satisfies the above properties (with D replaced by a slightly larger number).

Let us assume $n = 4$. To obtain $(M_, g_j)$ we start with a flat torus $T^4 = (S^1)^4$ and remove $2k$ small disks and glue in k handles provided by the lemma above with small R, ρ and ϵ. Properties (1) and (2) are then obvious, and one calculates $b_1(M_k) = 4 + k$, which implies (4).

The spin structure is obtained by taking the trivial spin structure on T^4, which yields by restriction a spin structure on the torus with the disks removed. This spin structure extends to a spin structure on M_k. The extension is not unique, but our construction will be independent from the choice of this extension.

Let $\varphi_1, \ldots, \varphi_r$ be a basis of parallel section of ΣT^4.

Now we follow the lines of [Bär-Dahl, Surgery and the spectrum of the Dirac operator, First page of section 2.4]. We see that for any $\epsilon > 0$ there is a suitable cut-off function χ that vanishes on the disks to be removed such that any parallel spinor φ on T^4 satisfies

$$\frac{\int \langle D(\chi\varphi), D(\chi\varphi)\rangle}{\int \langle D\varphi, D\varphi\rangle} < \epsilon.$$

This spinor also lives naturally on M_k. While φ runs over all parallel spinors, we obtain an r-dimensional space of test spinors for which the above inequality holds. Hence, by the standard Rayleigh quotient argument we obtain (3). □

3. Details to Examples (5)

We assume that we have a torus. Let sys be a shortest non-contractible loop. Assume $K > -\delta^2$, $\delta > 0$, $|K| \le K_{\max}$, $diam \le D$. Then for any $p > 1$

$$\|K\|_{L^p} \le area^{1/p} K_{\max}$$

and

$$\|K\|_{L^1} = \int |K| = 2\int |K^-| \le \delta^2 area.$$

Then

$$area \le \frac{1}{\delta} \sinh(\delta diam M)\, sys.$$

As a consequence

$$\left(\int |K|\right) \frac{area}{sys^2} \le 2\left(\int |K^-|\right) \frac{area}{sys^2} \le 2\delta^2 \frac{area^2}{sys^2} \le 2\sinh^2(\delta diam M).$$

Similarly as in the sphere case:

$$area \le \frac{2\pi}{\delta^2}\left[\cosh(\delta diam M) - 1\right].$$

We can assume that $\delta^2 \le K_{\max}$. Hence,

$$\|K\|_{L^1} \le 2\pi\left[cosh(\sqrt{K_{\max}}D) - 1\right].$$

and

$$area \le \frac{2\pi}{K_{\max}}\left[cosh(\sqrt{K_{\max}}D) - 1\right]$$

Using the estimates in [ammann:02] this implies

$$\max u - \min u \le \mathcal{S}(K, D)$$

for a continuous function $\mathcal{S}$ with $\mathcal{S}(0,D)=0$. Obviously

$$diam \geq 1/2 \text{ spinsys}$$

References

[1] B. Ammann, C. Bär, *The Dirac Operator on Nilmanifolds and Collapsing Circle Bundles*, Ann. Global Anal. Geom. **16** no. 3 (1998), *221–253*
[2] B. Ammann, C. Sprouse, *Manifolds with small Dirac eigenvalues are nilmanifolds*,