A COMPARISONS OF THE GEORGESCU AND VASY SPACES ASSOCIATED TO THE N-BODY PROBLEMS AND APPLICATIONS

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ABSTRACT. We provide new insight into the analysis of N-body problems by studying a compactification M_N of $\mathbb{R}^{3N}$ that is compatible with the analytic properties of the N-body Hamiltonian H_N. We show that our compactification coincides with the compactification introduced by Vasy using blow-ups in order to study the scattering theory of N-body Hamiltonians and with a compactification introduced by Georgescu using C^*-algebras. Furthermore, we also provide a third description of the compactification as a submanifold of a product of elementary blowups. Our results allow many applications to the spectral theory of N-body problems and to some related approximation properties. For instance, results about the essential spectrum, the resolvents, and the scattering matrices (when they exist) of H_N may be related to the behavior at infinity on M_N of their distribution kernels, which can be efficiently studied by blow-up methods. The compactification M_N is compatible with the action of the permutation group which allows to implement bosonic and fermionic (anti-)symmetry relations. We also obtain a regularity result for the eigenfunctions of H_N.

CONTENTS

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1. Introduction

1.1. **A general introduction and motivation for our work.** The quantum behavior of an atomic system is often investigated via its associated Hamiltonian. A good modell for N non-relativistic particles interacting with each other by Coulomb type forces is given by the Hamiltonian [5, 15]

$$(H_N u)(x) \;:=\; \Big(-\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_{x_j} + \sum_{1\le j<k\le N} \frac{b_{jk}}{|x_j - x_k|}\Big) u(x)\,, \tag{1}$$

where $x_j \in \mathbb{R}^3$ describes the position of the j-th particle, $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N}$, the operator Δ_{x_j} is the Laplacian with respect to x_j, and $m_j \in \mathbb{R}_+$ and $b_{jk} \in \mathbb{R}$. As usual, by moving to the center of mass coordinates and effective operators, to an atom with $N-1$ electrons (so N particles) will correspond an operator H_{N-1}^{eff} acting on $\mathbb{R}^{3(N-1)}$, and we will adopt this simplification in order to illustrate certain results.

The way the mathematical properties of Hamiltonians are reflected in the properties of the physical system was explained in many works, including [18, 31, 43, 57]. In particular, the mathematical study of the operator H_N (and of its simplified version H_{N-1}^{eff}) is a very vast domain of study in quantum mechanics and in mathematics. We will not be able to make justice to all the people who have contributed to the field, but let us nevertheless mention some works that are among the closest to the methods of this paper, namely the monographs of Amrein, Boutet de Monvel, and Georgescu [5], Dereziński and Gérard [17] and Teschl [58], as well as the research papers [13, 15, 16, 22, 23, 25, 47]. More specific references even closer related to our work can be found below.

The mathematical study of H_N and H_{N-1}^{eff} is quite challenging, especially for $N > 2$. The simplest case is that of hydrogen type atoms, which corresponds to $N = 2$ and $m_1 \gg m_2$. Then

$$H_1^{\mathrm{eff}} u(x) \;:=\; \Big(-\frac{1}{2\mu}\Delta + \frac{b}{|x|}\Big) u(x)\,, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{2}$$

In order to understand the mathematical properties of this operator, one usually writes the Hamiltonian H_1^{eff} in spherical coordinates

$$(r, y) \in (0, \infty) \times \mathbb{S}^2\,, \quad r = |x|, \quad x = ry\,,$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, as usual. The use of spherical coordinates has led, for instance, to the determination of the spectrum of H_1^{eff} and to explicit formulas for its eigenfunctions [17, 19, 58, 62], which is the basis for the orbital model in (quantum) chemistry. For $N > 2$ explicit calculations seem to be impossible. Nevertheless, one can still try to find "more convenient coordinates" in which to perform our calculations than the usual, euclidean coordinates. In this vein, one of the main results of this paper is to provide convenient coordinates that generalize the polar coordinates and which are helpful to study both particle interactions at infinity and the regularity of eigenfunctions for $N > 2$ particles.

More precisely, our "convenient coordinates" patch together to yield a compact (smooth) manifold with corners M_N, whose interior $M_N \smallsetminus \partial M_N$ is $\mathbb{R}^{3N}$. Thus M_N is a *compactification* of $\mathbb{R}^{3N}$ in the usual mathematical sense. In turn, the construction of such a compactification M_N will yield convenient coordinate systems via its natural coordinate charts. In the earler literature, two compactifications of $\mathbb{R}^{3N}$ have played an important role in the study of the N-body problem. A first such compactification is Georgescu's compactification, which was obtained as the primitive ideal spectrum of a certain commutative C^*-algebra [23, 25, 26]. Another compactification was constructed by Vasy using blow-ups [59]. See also [37]. Recently, yet another compactification was introduced by Mougel, Nistor, and Prudhon in [50]. The construction in [50] will be recalled in the discussion surrounding Equation (12).

One of the *main results* of this paper is to show that both Georgescu's compactification and Vasy's compactification are naturally homeomorphic with the one introduced in [50] (see Equation (12)). This common compactification of $\mathbb{R}^{3N}$ will be the space M_N we were looking for and therefore we will call it eventually the "Georgescu-Vasy" space. The identification of Georgescu's and Vasy's compactifications as the space M_N will allow us to obtain more properties for M_N, since, as we will explain below, each of the constructions of M_N mentioned above has its own advantages. Of course, the motivation of these authors – as well as ours – is to use the the properties of the space M_N to obtain a better insight into the properties of H_N (or of related Hamiltonians). For instance, we immediately obtain a natural action of the symmetric group S_N on N-letters on Vasy's compactification, since there is a simple such action on Georgescu's compactification. See Section 7 for several applications to operators more general in certain regards than the operators H_N or H_{N-1}^{eff}.

For hydrogen type atoms (so $N = 2$ and hence $H = H_1^{\text{eff}}$), the relevant manifold with corners M_1 compatible with the convenient coordinate system on $\mathbb{R}^3$ that we have just discussed (*i. e.,* spherical coordinates) is just a manifold with boundary, namely the radial compactification

$$M_1 \ := \ \overline{\mathbb{R}^3} \ = \ \mathbb{R}^3 \cup \mathbb{S}_{\mathbb{R}^3} \, ,$$

where $\mathbb{S}_{\mathbb{R}^3}$ is the "sphere at infinity" of $\mathbb{R}^3$. (The smooth structure on M_1 is such that $\overline{\mathbb{R}^3} \smallsetminus \{0\}$ is diffeomorphic to $(0, \infty] \times \mathbb{S}^2$, see Section 6.) For $N > 2$ the construction of a good compactification M_N of $\mathbb{R}^{3N}$ is more involved as anybody familiar with the problem would expect [5, 17, 57] In particular, in order to realize the antisymmetry of the wave function (*i. e.,* the Pauli exclusion principle) for the fermions and the corresponding symmetry for the bosons, it is important to see that the action of the symmetric group S_N on $\mathbb{R}^{3N}$ by permutation extends to the compactification M_N, see Subsection 7.1.2. Finding a good compactification of $\mathbb{R}^{3N}$ that behaves well with respect to the action of S_N is a problem posed by Melrose and Singer in [48], which was solved in [37] using differential geometry and in [51] using C^*-algebras. We will use the example of hydrogen type atoms again in the next subsecton to explain our construction of the spaces M_N.

1.2. **Our setting and the construction of the compactification M_N.** Our results, in fact, will deal with operators that are somewhat different from the N-body Hamiltonians H_N or H_{N-1}^{eff} of Equation (1). The operators that we study are, in many regards, more general than the N-body Hamiltonians and, in any case, they retain most of the main features of the operators H_N and H_{N-1}^{eff} that are relevant to this paper. To describe the class of operators that we will study, let us thus first explain the following customary sequence of adjustements to be performed on H_N, following for instance [5] and the references therein. See also [15, 23].

The first adjustement to be performed is to "smooth out" the singularity in the potential. To see why this is reasonable, let us mention that, from the point of view of Partial Differential Equations, there are two main issues that distinguish H_N and H_{N-1}^{eff} from the customary differential operators studied in the introductory courses, namely:

- the behavior at infinity of the potential and
- the singularities in the potential.

Somewhat counterintuitively (from a pure mathematical point of view) is that the singularities of the potential are less important than the behavior at infinity, at least in what the spectral theory is concerned. In fact, it is known that many results concerning the essential spectrum of operators with potentials with Coulomb singularities can be obtained from the results for smooth potentials (see [26] and the references therein, for example; incidentally, in addition to the Hardy inequality, this involves norm closures and elementary C^*-algebra arguments). Thus, except in the very last subsection, in this paper, we will "smooth out" the singularities and hence look instead at a class of operators containing operators of the form

$$H'_N \;:=\; D + \sum_{1\le j\le N} v_j(x_j) + \sum_{1\le j<k\le N} v_{jk}(x_j - x_k)\,, \tag{3}$$

where D is a strongly elliptic differential operator with constant coefficients, and v_j and v_{jk} are *smooth functions* on $\mathbb{R}^3$ with uniform radial limits at infinity. (We suppress from the notation the function u on which H'_N acts.) The choice of functions with radial limits at infinity is motivated by the choice of "convenient coordinates" in the case of the hydrogen type atom and also because it leads to a less singular compactification of $\mathbb{R}^{3N}$. We note, however, that in the last subsection, the singularities of the potential will play a central role and are responsible for our regularity estimates

$$\rho^\alpha \partial^\alpha u \in L^2(\mathbb{R}^{3N})\,, \quad \rho(x) \;:=\; \min\{\operatorname{dist}(x, \textstyle\bigcup \mathcal{F}), 1\}\,, \tag{4}$$

for all multi-indices α, where u is an eigenfunction of H_N or H_N^{eff} (Theorem 7.4).

Our next adjustement will be to extend our setting from $\mathbb{R}^{3N}$ to an arbitrary real vector space X of finite dimension and to generalize the set of "collision planes" ($\{x_j - x_k = 0\}$ for H_N), we consider a suitable finite sets $\mathcal{F}$ of linear subspaces of X, as in [5, 15, 17, 25]. As in those works, which serve as a motivation for our approach, it will be convenient to assume that $\mathcal{F}$ is closed under intersection. Recall that a family $\mathcal{S}$ of subsets of M is a *semilattice* (with respect to the inclusion) if, for all $P_1, P_2 \in \mathcal{S}$, we have $P_1 \cap P_2 \in \mathcal{S}$. The set of subspaces $\mathcal{F}$ is "suitable" in the above sense, if it is a semilattice and if it contains the zero subspace, that is $\{0\} \in \mathcal{F}$. Our operator H'_N will then be replaced with a more general operator of the form

$$H \;:=\; D + \sum_{Y\in\mathcal{F}} v_Y\,, \tag{5}$$

where v_Y is a smooth function on X/Y with radial limits at infinity (more precisely $v_Y \in \mathcal{C}(\overline{X/Y})$, where $\overline{X/Y}$ is the radial compactification of X/Y). This completes our sequence of adjustements of H_N and provides us with the concepts needed to introduce our definition of compactification space X_{GV} (the Georgescu-Vasy space) as follows. (If $\mathcal{F}$ is the semilattice corresponding to the N-body problem or the effective N-body problem, we shall also use the notation M_N for its Georgescu-Vasy space.)

Let next $\mathcal{F}$ be a finite semilattice of linear subspaces of X with $\{0\} \in \mathcal{F}$ and

$$\delta : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y} \tag{6}$$

be the diagonal map obtained from all the projections $X \to X/Y$. We define the *Georgescu-Vasy space* X_{GV} as the closure

$$X_{GV} := \overline{\delta(X)} \tag{7}$$

of $\delta(X)$ in $\prod_{Y \in \mathcal{F}} \overline{X/Y}$. Since each $\overline{X/Y}$ is compact, X_{GV} is also compact.

Let us comment on the historic origins of the compactification X_{GV}. Georgescu's original definition used C^*-algebras. Also, initially, the one point compactifications $(X/Y)^+$ was used instead of $\overline{X/Y}$, which, by taking the closure, leads to the "small Georgescu space". This choice is more closely related to the form of the potential in the N-body problem, but leads to a to a space that is quite singular and hard to describe. In fact, our results give a determination of the small Georgescu space as a quotient of the (big) Georgescu-Vasy space X_{GV}. One of the advantages of Georgescu's approach is that we obtain right away an action of X on X_{GV}. Moreover, if our semilattice corresponds to the N-body problem (in which case, we recall, X_{GV} is also denoted M_N), we also obtain actions of the orthogonal group $O(3)$ and of the symmetric group S_N on M_N. At least for the latter group, this was non-trivial in Vasy's definition (explained next). Moreover, standard C^*-algebra techniques (see Subsection 7.3) combined with Georgescu's definition provide a determination of the essential spectrum of H_N, thus recovering the classical HVZ theorem describing the essential spectrum of N-body Hamiltonians [5, 17, 23, 57].

Vasy's definition of (what will also turn out to be) the space X_{GV} is using successive blow-ups of the spherical compactification $\overline{X}$ of X. Since it requires some more notation, we relegate it to Subsection 1.3. It is hence quite different in spirit from the one proposed by Georgescu. The advantage of this construction is that it equips right away X_{GV} with the structure of a smooth manifold with corners (and hence to a convenient coordinate system on X). It is also explicit enough to allow, in principle, the determination of the faces of X_{GV} and of the orbits of the action of X on X_{GV}. The smooth structure on X_{GV} provides powerful tools that are useful, for example, to prove uniform regularity results at infinity for the eigenfunctions of H_N and H^{eff}_{N-1} (Theorem 7.4).

By combining the three approaches (or definitions) of the space X_{GV} that we have described, we obtain further properties. For instance, the smooth structure on X_{GV} in connection with the smooth action of X then recovers right away a pseudo-differential calculus $\Psi_{NB}(\mathbb{R}^{3N})$ that is very closely related (maybe the same) to the one constructed by Vasy [60], see Subsection 7.2. In turn, our definition of this pseudodifferential calculus shows that it is spectrally invariant (*i. e.,*it is stable for holomorphic functional calculus) and thus leads to a description of the distribution kernels of the resolvents of H_N, Proposition 7.3. In the same spirit, by combining the results of this paper with those of [1], we can obtain a regularity result for the eigenfunctions of H_N, see Theorem 7.4 and [4] for full details.

1.3. **Vasy's blow-up and the graph blow-up.** Throughout this paper, M will be a manifold with corners (we recall the definition of manifolds with corners in the main body of the paper). A submanifold will be called *closed* if it is a closed subset of the larger manifold in the sense of a topological spaces. Recall that a *p-submanifold* $P \subset M$ is a closed submanifold of M that has a tubular neighborhood: $P \subset U_P \subset M$ that is locally of product form (see Definition 2.14 for details).

For any p-submanifold $P \subset M$, recall that the *blow-up* $[M : P]$ *of* M *with respect to* P is defined by replacing P with the set $\mathbb{S}(N_+^M P)$ of interior directions in the normal bundle $\mathbb{S}(N^M P)$ of P in M (see [32, 45, 55], or Definiton 3.1).

Since Vasy's construction uses blow-ups, we begin this paper with their study. More precisely, we shall study and use the blow-up of a manifold with corners with respect to a *family* of p-submanifolds. If this family has clean intersections, the blow-up can be defined iteratively as in [1, 38, 45] and in other papers. Our method is based on an alternative definition of the blow-up with respect to a family of p-submanifolds. More precisely, let us consider a locally finite family $\mathcal{F}$ of p-submanifolds of M and let $M_{\mathcal{F}} := M \smallsetminus \bigcup_{P\in\mathcal{F}} P$ be the complement of all the submanifolds in $\mathcal{F}$. Then $M_{\mathcal{F}}$ is open in M and is contained in each of the blow-up manifolds $[M : P]$, $P \in \mathcal{F}$. Then we define the *graph blow-up* $\{M : \mathcal{F}\}$ of M with respect to the family $\mathcal{F}$ as the closure

$$\{M : \mathcal{F}\} \ := \ \overline{\delta(M_{\mathcal{F}})} \ \subset \ \prod_{P\in\mathcal{F}} [M : P]\,, \tag{8}$$

where δ is the diagonal embedding (see Definition 4.1 and the discussion following it for more details). Note that $[M : \emptyset] = M$. For simplicity, in this paper, we shall consider the graph blow-up only with respect to a *locally finite* family of p-submanifolds that are closed subsets of the ambient manifold.

In order to have a well-behaved graph blow-up, we shall impose some additional assumptions on $\mathcal{F}$. Let then $\mathcal{S}$ be a semilattice of p-submanifolds of M and arrange its elements in an order $(P_0, P_1, P_2, \dots, P_n)$. We assume, as in [37] (see also [1]) that this order is *compatible with the inclusion*, in the sense that

$$P_i \subsetneq P_j \ \Rightarrow i \le j\,. \tag{9}$$

We also assume that any two manifolds in $\mathcal{S}$ intersect cleanly (in other words, we assume that $\mathcal{S}$ is a *clean semilattice*, see Definition 5.3). Under these assumptions, we will prove in Theorem 5.12 that $\{M : \mathcal{F}\}$ is a submanifold of the product space on the right hand side of (8) in some weak sense (Definition 2.13) and thus that it will inherit the structure of a manifold with corners. Note that if $\emptyset$ appears among the manifolds $(P_j)_{j=0}^N$ arranged in a compatible order, Equation (9), then $P_0 = \emptyset$, and since $[M : \emptyset] = M$, blowing up along $P_0 = \emptyset$ has no effect. So we shall assume that $P_0 = \emptyset$. Then we can successively blow-up M with respect to $(\emptyset = P_0, P_1, P_2, \dots, P_n)$ by first doing so with respect to P_1, then doing so with respect to the lift of P_2, and so on, to obtain in the end the *iterated blow-up* $[M : \mathcal{S}]$ [1, 38, 45]. One of our main results is to prove that, if $\mathcal{S} \ni \emptyset$ is a locally finite clean semilattice, then there exists a unique diffeomorphism

$$[M : \mathcal{S}] \ \simeq \ \{M : \mathcal{S}\} \tag{10}$$

that is the identity on the common open subset $M \smallsetminus \bigcup_{k=1}^n P_n$ (see Theorem 5.12). In particular, $[M : \mathcal{S}]$ is independent of the initially chosen order on $\mathcal{S}$, as long as it is compatible with the inclusion (so that the iterated blow-up is defined).

We apply these results to the study of the N-body problem in the following way. Let $\overline{X}$ denote be the spherical compactification of a finite-dimensional vector space X, with boundary at infinity the sphere $\mathbb{S}_X := \overline{X} \smallsetminus X$. Let $\mathcal{F}$ be a finite semilattice of linear subspaces of X containing the zero subspace. To $\mathcal{F}$ we associate the semilattice

$$\mathbb{S}_{\mathcal{F}} := \{\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F}\} \supset \{\emptyset\}\,, \tag{11}$$

In this application to the N-body problem, the role of M will be played by $\overline{X}$. Vasy has considered the space $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ in order to define a pseudodifferential calculus adapted to the N-body problem, see [60, 61] and the references therein. Inspired by Georgescu [25, 26, 50], let us consider the norm closed algebra (C^*-algebra)

$$\mathcal{E}_{\mathcal{F}}(X) := \langle \mathcal{C}(\overline{X/Y}) \rangle \tag{12}$$

generated by all the spaces $\mathcal{C}(\overline{X/Y})$ in $L^\infty(X)$, with $Y \in \mathcal{F}$. Here $\mathcal{C}(Z)$ denotes the space of continuous functions $Z \to \mathbb{C}$. It was proved in [50] that the spectrum $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X)$ of $\mathcal{E}_{\mathcal{F}}(X)$ is the closure of the image of X in the product $\prod_{Y \in \mathcal{F}} \overline{X/Y}$. In this paper, we show that this closure coincides with $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$. As a consequence, the identity $\mathrm{id}_X : X \to X$ extends to a homeomorphism from the compactification $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ introduced by Georgescu to the compactification $[\overline{X} : \mathcal{S}]$ introduced by Vasy. Since the two compactifications yield the same space, this justifies the name Georgescu-Vasy space for this space.

1.4. **Contents of the paper.** Section 2 contains background material on manifolds with corners. In particular, we devote quite a bit of effort to introduce and compare several classes of submanifolds of manifolds with corners. Section 3 recalls the definition of the blow-up of a manifold with corners with respect to a p-submanifold. In Section 4, we introduce the graph blow-up of a manifold with corners with respect to a *set* of p-submanifolds and prove some of its first properties. In Section 5 we prove that, for clean semilattices of p-submanifolds, the graph blow-up can also be obtained as an iterated blow-up, which is one of the main technical results of this paper. In Section 6, we use the identification of the graph blow-up with the iterated blow-up to show that two spaces, introduced independently by Georgescu and Vasy, coincide with the space X_{GV} introduced in [50], and hence that all three definitions of X_{GV} are, equivalent. In the last section, we provide four applications of the formalism that we have developped. The first one is in relation to the action of the symmetric group on S_N and Pauli exclusion principle. The second application is to make some connections between Vasy's pseudodifferential calculus and Georgescu's algebras. As a third application, we discuss the relation between our results and the HVZ theorem. Finally, the last application is a regularity result for bound states taken from [4]. Two appendices include some related topological results on proper maps and on submanifolds of manifolds with corners. The reader can thus see that this paper relies essentially on geometry, necessarily so since Vasy's construction is geometric.

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2. Manifolds with corners and their submanifolds

We begin with some background material, mostly about manifolds with corners. This section contains few new results, but the presentation is new.

2.1. **Manifolds with corners.** We now introduce manifolds with corners and their smooth structure. We also set up some important notation to be used throughout the paper. The terminology used for manifold with corners is not uniform. Nevertheless, a good overview of the concept of a manifold with corners can be found in [32, 36, 38, 46, 55], to which we refer for the concepts not defined here and for further references. In this paper, we will mostly use the terminology introduced by Melrose and his coauthors, which predates most of the other ones.

2.1.1. *Smooth maps.* We have the following standard definition.

Definition 2.1. *Let $U \subset \mathbb{R}^n_k$ and $V \subset \mathbb{R}^m_l$ be two open subsets and $f = (f_1, \ldots, f_m) : U \to V$. We shall say that:*

(a) *f is* smooth *on U if there exists an open neighborhood W of U in $\mathbb{R}^n$ such that f extends to a smooth function $\tilde{f} : W \to \mathbb{R}^m$.*

(b) *f is a* diffeomorphism *between U and V if f is a bijection and both f and f^{-1} are smooth.*

2.1.2. *Notation.* For any finite dimensional real vector space Z, let $\mathbb{S}_Z$ denote the *set of vector directions* in Z, that is, the set of (non-constant) open half-lines $\mathbb{R}_+ v$, with $0 \neq v \in Z$ and $\mathbb{R}_+ := (0, \infty)$. We will also use the standard notation $\mathbb{S}^{n-1} := \mathbb{S}_{\mathbb{R}^n}$, for simplicity. In particular, if Z is a euclidean (real) vector space, we identify $\mathbb{S}_Z$ with the unit sphere in Z. Informally, a manifold with corners is a topological space locally modeled on the spaces

$$\mathbb{R}^n_k \ := \ [0,\infty)^k \times \mathbb{R}^{n-k}. \tag{13}$$

For $k, n \in \mathbb{N} = \{0, 1 \ldots\}$ with $k \leq n$, we let $\mathbb{S}^{n-1}_k \subset \mathbb{R}^n$ be

$$\mathbb{S}^{n-1}_k \ := \ \mathbb{S}^{n-1} \cap \mathbb{R}^n_k \ = \ \{\phi = (\phi_1, \ldots \phi_n) \mid \|\phi\| = 1 \text{ and } \phi_i \geq 0 \text{ for } 1 \leq i \leq k\}, \tag{14}$$

where $\|.\|$ is the euclidean norm.

Let us write 0_V for the neutral element of a vector space V, when we want to stress the space to which it belongs. We will often use maps between subsets of euclidean spaces, and, as a rule, we will try not to permute the coordinates, and, moreover, our embeddings will be “first components embedings.” More precisely, let $k' \leq k$ and $n' - k' \leq n - k$, we shall then use with priority the canonical “first components” embedding given by:

$$\begin{gathered} \mathbb{R}^{n'}_{k'} \ \simeq \ [0,\infty)^{k'} \times \{0_{\mathbb{R}^{k-k'}}\} \times \mathbb{R}^{n'-k'} \times \{0_{\mathbb{R}^{n-n'}}\} \ \subseteq \ [0,\infty)^k \times \mathbb{R}^{n-k} \ = \ \mathbb{R}^n_k \\ (x', x'') \ \mapsto \ (x', 0_{\mathbb{R}^{k-k'}}, x'', 0_{\mathbb{R}^{n-n'}}) \end{gathered} \tag{15}$$

Other embeddings (involving permutations of the coordinates) between these sets will also be considered, and they will explained separately. For instance, we shall sometimes find it notationally convenient to use the *canonical permutation of coordinates* diffeomorphism

$$\begin{gathered} \mathrm{can} : \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \ \simeq \ \mathbb{R}^{n+n'}_{k+k'} \\ (x', x'', y', y'') \ \mapsto \ (x', y', x'', y'') \in [0,\infty)^{k+k'} \times \mathbb{R}^{n+n'-k-k'}\,, \end{gathered} \tag{16}$$

where $x' \in [0,\infty)^k$ and $y' \in [0,\infty)^{k'}$. (Compare with Equation (13).) However, if nothing else is mentioned, we consider the first components canonical embedding of Equation (15).

2.1.3. *Charts and atlases.* We shall use suitable charts to define the smooth structure on manifolds with corners. Let M be a Hausdorff space. We proceed as in the case of smooth manifolds (without corners).

Definition 2.2. *A* corner chart *on M (or simply,* “chart” *in what follows) is a couple (U, ϕ) with U an open subset of M and $\phi : U \to \Omega$ a homeomorphism onto an open subset Ω of $\mathbb{R}^n_k$. Let (U, ϕ) and (U', ϕ') be two corner charts with values in $\mathbb{R}^n_k$ and in $\mathbb{R}^{n'}_{k'}$,* *respectively. Let $V := U \cap U'$. We shall say that the corner charts (U, ϕ) and (U', ϕ') are* compatible *if $V = \emptyset$ or if*

$$\phi' \circ \phi^{-1} : \phi(V) \to \phi'(V)$$

is a diffeomorphism (see Definition 2.1) between the open subsets $\phi(V) \subset \mathbb{R}^n_k$ and $\phi'(V) \subset \mathbb{R}^n_{k'}$.

Given a point $m \in M$ and a corner chart (U, ϕ) with $m \in U$, we can always find a corner chart (U', ϕ'), $\phi' : U' \to \mathbb{R}^{n'}_{k'}$, compatible with (U, ϕ) such that $\phi'(m) = 0$ and k' is minimal.

Definition 2.3. *A* corner atlas $\mathcal{A} = \{(U_a, \phi_a), a \in A\}$ *on M is a family of compatible corner charts such that $M = \bigcup_{a \in A} U_a$. Two corner atlases are called* equivalent *if their union is again a corner atlas. A* manifold with corners *is defined to be a paracompact Hausdorff space M with an equivalence class of corner atlases (on M). In the following we will drop the word "corner" before the words "chart" and "atlas." In the context of a manifold with corners, the terms "atlas" and "chart" will always mean "corner atlas" and, respectively, "corner chart."*

A manifold with corners in the above sense is called a "t-manifold" in [45, Section 1.6]. If M is manifold with corners, then the union of all its atlases is again an atlas, the *maximal atlas* of M. An open subset of a manifold with corners is again, in an obvious way, a manifold with corners. Many concepts extend from the case of manifolds without corners to that of manifolds with corners.

Definition 2.4. *Let $f : M \to M'$ be a map between two manifolds with corners. We will say that f is* smooth *if, for any two charts (U, ϕ) of M and (U', ϕ') of M', the map $\phi' \circ f \circ \phi^{-1}$ is smooth on its domain of definition $\phi(f^{-1}(U'))$. If f is a bijection and both f and f^{-1} are smooth, we will say that f is a* diffeomorphism.

The following are some examples of manifolds with corners that will be used in this paper.

Example 2.5. Using the notation from Subsection 2.1.2, we have the following:

(i) Any open subset of $\mathbb{R}^n_k := [0, \infty)^k \times \mathbb{R}^{n-k}$ is a manifold with corners.
(ii) The sphere orthant $\mathbb{S}^{n-1}_k := \mathbb{S}^{n-1} \cap \mathbb{R}^n_k$ of Equation (14) is a manifold with corners.
(iii) Any smooth manifold is a manifold with corners (even if it doesn't have a boundary or any true corners).

The following definition will be used to introduce p-submanifolds.

Definition 2.6. *Let I be a subset of $\{1, \ldots, n\}$ and L_I be the subset of $\mathbb{R}^n_k$ defined by*

$$L_I := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \text{ if } i \in I \} . \tag{17}$$

The number $b := \#\big(I \cap \{1, \ldots, k\}\big)$ will be called the boundary depth *of L_I; $c := \#I$ is the codimension of L_I and $d := n - c$ its dimension.*

2.2. **The boundary and boundary faces.** We now fix some standard terminology to be used in what follows, extending the local definitions of Definition 2.6. In particular, we need the intrinsic definition of the boundary of a manifold with corners. The *(boundary) depth* (in X) $\mathrm{depth}_X(p)$ of a point $p \in X$ is the number of non-negative coordinate functions vanishing at p in any local coordinate chart at p. It is the least k such that there exists a chart near U with values in $\mathbb{R}^n_k$. Let $(M)_k$ be the set of points of M of depth k. It is a smooth manifold (no corners). Its connected components are called the *open* boundary faces (or just the *open* faces) of codimension (or depth) k of M. A *boundary face* of depth k is the closure of an open boundary face of depth k. It is possible to construct a manifold with corners M that has a boundary face F such that F is not a manifold with corners for the induced smooth structure. More precisely, there are M and F such that $\{f|_F \mid f \in C^\infty(M)\}$ is not the set of smooth functions on F for some manifold-with-corners structure on F.

We will denote by $\mathcal{M}_k(M)$ the set of all *closed* boundary faces of codimension k. In particular, the *boundary* ∂M of M, defined as the set of all points of depth > 0, is given by

$$\partial M := \bigcup_{H \in \mathcal{M}_1(M)} H. \tag{18}$$

A boundary face of M of codimension one will be called a *hyperface* in what follows. Thus ∂M is the union of the hyperfaces of M. If H is a hyperface of M and $0 \leq x \in \mathcal{C}^\infty(M)$ is a function such that $H = x^{-1}(0)$ and $dx \neq 0$ on H, then x is called a *boundary defining function* of H. As above, there are examples of hypersurfaces, that do not have a boundary defining function. However, each boundary face F of codimension k can *locally* be represented as $F = \{x_1 = x_2 = \ldots = x_k = 0\}$, where x_j are boundary defining functions of the hyperfaces containing F. Here "locally" means that, given $p \in F$, there is an open neighborhood U of p in M such that the statement is true for M and F replaced with $M \cap U$ and, respectively, with $F \cap U$.

It is also convenient to consider an alternative approach to the definition of manifolds with corners and of their smooth structure via embeddings, as in the next remark.

Remark 2.7. Every manifold with corners M is contained in a smooth manifold $\widetilde{M}$ of the same dimension [2, 32, 38, 45, 46, 55]. It is then convenient to define

$$TM := T\widetilde{M}|_M .$$

Up to a diffeomorphism, TM can be obtained by gluing the tangent spaces $T(\mathbb{R}^n_k) := \mathbb{R}^n_k \times \mathbb{R}^n$ using an atlas of M. We also let $T^+_x M$ be the set of tangent vectors of $T_x M$ that are inward-pointing or tangent to the boundary. It can be defined as the set of equivalence classes of curves starting at x and completely contained in M. We finally let $T^+M := \bigcup_{x \in M} T^+_x M$ with its projection map to M. Note that T^+M is not a fiber bundle, but a fiberwise conical closed subset of the tangent space. We note, however, that ∂M is intrinsically defined and sometimes it is *not* the *topological* boundary $\overline{M} \smallsetminus \mathring{M}$ of M, where the closure $\overline{M}$ and the interior $\mathring{M}$ are computed in $\widetilde{M}$. For instance, when $M := \{x \in \mathbb{R}^n \mid x_n \geq 0, \|x\| < 1\}$ and $\widetilde{M} = \mathbb{R}^n$, then $\partial M = \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| < 1\}$, whereas the topological boundary of M is $\partial M \cup \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| = 1\}$, a bigger set. In fact, we always have that ∂M is contained in the topological boundary of M in $\widetilde{M}$. Unlike ∂M, the topological boundary of M in $\widetilde{M}$ depends on $\widetilde{M}$.

2.3. **Submanifolds of manifolds with corners.** We now discuss several notions of submanifolds of a manifold with corners.

2.3.1. *Submanifolds and weak submanifolds.* We start the discussion with the notation of submanifolds in manifolds with corners, following [45, Definition 1.7.3]. Again, our definition differs slightly from Melrose's in that we do not require connectedness for the submanifold S or the ambient manifold $M \supset S$.

Definition 2.8. *A subset S of a manifold with corners M is a* submanifold (in the sense of manifolds with corners) *if, for every $p \in S$, there exists $0 \leq k \leq n$ and a (corner) chart $\phi : U \to \Omega \subset \mathbb{R}^n_k := [0,\infty)^k \times \mathbb{R}^{n-k}$, numbers $n' \leq n$ and $k' \leq n'$, and a matrix $G \in \mathrm{GL}(n,\mathbb{R})$ such that*

(1) $p \in U$
(2) $G \cdot \Big(\mathbb{R}^{n'}_{k'} \times \{0\}\Big) \subset \mathbb{R}^n_k$.
(3) The chart ϕ maps $S \cap U$ bijectively to the intersection of this linear submanifold with Ω, in other words

$$\phi(S \cap U) \;=\; G \cdot \Big(\mathbb{R}^{n'}_{k'} \times \{0\}\Big) \cap \Omega\,.$$

Let us comment on this definition.

Remark 2.9.

(1) In Definition 2.8, the symbol "$\cdot$" denotes the action of $\mathrm{GL}(n,\mathbb{R})$ on subsets of $\mathbb{R}^n$, the action which is induced by the standard linear action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$.
(2) In Melrose's terminology, Property (2) of this definition is expressed by saying that $G \cdot \Big(\mathbb{R}^{n'}_{k'} \times \{0\}\Big)$ is a linear submanifold of $\mathbb{R}^n_k$.
(3) If S is submanifold of M and $p \in S$, then the number n' and k' are uniquely determined. We say that n' is *the dimension* of S in p. This dimension is locally constant, but we allow it to depend on the connected component. A (connected) submanifold S of a manifold with corners M carries an induced structure of a manifold with corners. This structure can be characterized by the condition, that the embedding is a smooth map $i : S \to M$ and that the differential $d_p i : T_pS \to T_pM$ is injective for any $p \in S$. Alternatively this structure can be described by an atlas. For any ϕ and G as above, the map $\tilde{\phi}(p)$ is defined for $p \in U \cap S$ by the relation $G^{-1}\phi(p) = (\tilde{\phi}(p), 0)$. We set $\tilde{U} := U \cap S$ and $\tilde{\Omega} := \phi(\tilde{U}) = G^{-1}(\Omega) \cap \Big(\mathbb{R}^{n'}_{k'} \times \{0\}\Big)$. Then $\tilde{\phi} : \tilde{U} \to \tilde{\Omega}$ shall be taken as a chart for S around p, and if we do this construction for any $p \in S$, then we obtain an atlas for S. See [45, Lemma 1.7.1 and following] for further details.
(4) Note that in [2] a more restrictive notion of submanifold of a manifold with corners was used. More precisely, a tame submanifold of M in the sense of [2] is a p-submanifold of M that is *not contained* in the boundary ∂M *together with an additional compatibilty condition, see* Remark 2.24 for details.

The following example illustrates the definition of a submanifold of a manifold with corners and explains why we consider it in the first place.

Example 2.10 (Diagonal)**.** Let N be a manifold with corners. Then $M := N \times N$ is also a manifold with corners. Consider the diagonal $\Delta_N := \{(p,p) \in M \mid p \in N\}$. Then Δ_N is a submanifold of M.

In various applications it is helpful or even necessary to use variations of the concept of a "submanifold"; in our article we will require both a more general concept, that of a "weak submanifold," and a more restrictive one, namely, that of a "p-submanifold" of a manifold with corners, which will be discussed in the next section.

We explained above that a submanifold S of a manifold with corners M inherits an atlas from M. This property can be generalized to "weak submanifolds," which we will be define after the following examples.

Example 2.11. The function $f : \mathbb{R}^2_1 := [0,\infty) \times \mathbb{R} \to \mathbb{R}^2_1$, $f(x,y) := (x+y^2, y)$, is an injective immersion. It is a homeomorphism onto its image $S := f\big(\mathbb{R}^2_1\big)$. However, it can be easily seen that S is not a submanifold (in the sense of manifolds with corners) of $\mathbb{R}^2_1$. On the other hand S is a submanifold of $\mathbb{R}^2$ in the sense of manifolds with corners.

Example 2.12. The function $f(x) := (x, x^2)$ defines a injective immersion $\mathbb{R}^1_1 \to \mathbb{R}^2_2$. It is a homeomorphism onto its image $S := f\big(\mathbb{R}^1_1\big)$. However, S is not a submanifold $\mathbb{R}^2_2$. On the other hand S is a submanifold of $\mathbb{R}^2_1$ and of $\mathbb{R}^2$ in the sense of manifolds with corners.

Images of injective immersion with the above homeomorphism property will play an important role in our article, but – unfortunately – not all of them are submanifolds in the sense of Definition 2.8 or in Melrose's sense. This motivates to introduce the more general notion of "weak submanifolds". The subsets S in the previous examples will turn out to be weak submanifolds.

Definition 2.13. *A subset S of a manifold with corners M is a* weak submanifold *if, for every $p \in S$, there exists $k \in \{0,1,\dots,n\}$ and a chart $\phi : U \to \Omega \subset \mathbb{R}^n_k$, such that*

(1) $p \in U$ and
(2) $\phi(S \cap U)$ is a submanifold of $\mathbb{R}^n$.

The dimension of S at p *is by definition the dimension of $\phi(S \cap U)$ at $\phi(p)$.*

Equivalently, one can reformulate Definition 2.13 by saying that S is a weak submanifold if, and only if, M can be extended to a manifold $\tilde{M}$ without corners (or boundary), such that S is covered by charts $\phi : U \to \Omega$ of $\tilde{M}$ satisfying

$$\phi(S \cap U) \;=\; \Big(\mathbb{R}^{n'}_{k'} \times \{0\}\Big) \cap \Omega\,.$$

Note that if we replaced in *(2)* the ambient manifold $\mathbb{R}^n$ by $\mathbb{R}^n_k$, then S would be a weak submanifold if, and only if, S was a submanifold. However as written above, the class of weak submanifolds strictly contains the class of submanifolds. Indeed, the set S in Example 2.11 is a weak submanifold of $\mathbb{R}^2_1$, but not a submanifold.

For any chart $\phi : U \to \Omega$ of M the submanifold property provides an atlas on every $\phi(S \cap U)$ and thus an atlas on $S \cap U$. If $\phi' : U' \to \Omega'$ is another chart of M, then we similarly obtain an atlas on $S \cap U'$, and as $\phi' \circ \phi^{-1}|_{\phi(U \cap U')}$ is a diffeomorphism, the atlases on $S \cap U$ and $S \cap U'$ are compatible, i.e. their union is an atlas on $S \cap (U \cup U')$. By repeating this construction for all the domains of charts of an atlas of M, we obtain an atlas of S, the *induced atlas on S*. With this atlas, the set S is a manifold with corners.

By applying the implicit function theorem, one can prove that a subset $S \subset M$ of a manifold with corners is a weak submanifold, if, and only if, it is the image of an injective immersion $f : N \to M$, where N is a manifold with corners and where $f : N \to S$ is a homeomorphism. See Proposition B.1 in the appendix for a proof.

Furthermore, we have the following nice property which we will claim without proof: if P and Q are weak submanifolds of M and $P \subset Q$, then P is a weak submanifold of Q. Thus weak submanifolds also have nicer categorical properties than submanifolds. In the categorical language, the above property is expressed as follows: if we consider the category whose objects are manifolds with corners and the morphisms are inclusions as a weak submanifold, then this is a *full* subcategory of the category of sets with the inclusions as morphisms.

If we only consider submanifolds in the sense of Definition 2.8 as morphisms, then the above property does not hold as we have seen in Example 2.11, i.e. we obtain a subcategory that is not full.

2.3.2. *Submanifolds with tubular neighborhoods: p-submanifolds.* We now recall the definition of a p-submanifold of a manifold with corners M [1, 37, 45, 60]. In our paper, p-submanifolds are of central importance, as we blow-up manifolds with corners along closed p-submanifolds.

Recall the subsets $L_I \subset \mathbb{R}^n_k$ of Definition 2.6. After reordering the components, L_I is the first factor of $\mathbb{R}^n_k \cong \mathbb{R}^d_{k-b} \times \mathbb{R}^c_b$, in the sense that L_I is mapped to $\mathbb{R}^d_{k-b} \times \{0\}$. The boundary depth of L_I is the boundary depth of any interior point of L_I with respect to $\mathbb{R}^n_k$. The sets L_I are the local models for p-submanifolds [45, Definition 1.7.4].

Definition 2.14. *A subset P of a manifold with corners M is a* p-submanifold *if, for every $x \in P$, there exists a chart (U, ϕ) with $x \in U$ and $I \subset \{1, 2, \dots, n\}$ such that*

$$\phi(P \cap U) \ = \ L_I \cap \phi(U),$$

with L_I as defined in Equation (17). *The number $n - \#I$ (respectively, $\#I$, respectively, $\#\big(I \cap \{1, \dots, l\}\big)$) will be called the* dimension *(respectively, the* codimension *of P in x, respectively, the* boundary depth *of P in x). We allow p-submanifolds Y of non-constant dimension. We define $\dim Y$ as the maximum of the dimensions of the connected components of Y and* $\dim \emptyset = 0$.

Obviously all p-submanifolds are submanifolds, and the definition of the dimension of P in x coincides with the dimension already defined above.

Remark 2.15. The numbers $n - \#I$ (respectively, $\#I$, respectively, $\#\big(I \cap \{1, \dots, l\}\big)$) introduced in Definition 2.14 are locally constant functions on P. For any interior point x in P and $\epsilon > 0$ small enough, these numbers are the dimension (respectively, the codimension, respectively, the boundary depth of x) of the intersection $B_\epsilon(x) \cap P$ in M. More generally: if P is a p-submanifold of M with boundary depth d on the component of $x \in P$, and if x is a (boundary) point of depth e in P, then x has depth $d + e$ in M. In particular, for a p-submanifold $P \subset M$, the difference of depths $\operatorname{depth}_M(x) - \operatorname{depth}_P(x)$ is constant on the connected components of P.

This definition of a p-submanifold comes from [45]. Note that "*p*" is used as an abbreviation for "product," reflecting the fact that, locally in coordinate charts, p-submanifolds are a factor of the product $\mathbb{R}^n_k \simeq \mathbb{R}^{n_1}_{k_1} \times \mathbb{R}^{n_2}_{k_2}$. A more general concept, that of an "interior binomial subvariety," was introduced and studied in [38].

Let $P \subset M$ be a p-submanifold. Then it is possible that $P \subset F$, for F a non-trivial face of M. If P is connected, then the boundary depth of P is the boundary depth of the smallest closed face F of M containing P.

We shall need the following lemma. Recall that a subset of a topological space is called *locally closed* if it is the intersection of a closed subset with an open subset.

Lemma 2.16. *Let $P \subset Q \subset M$ be manifolds with corners.*

(i) If P is a p-submanifold of M, then P is locally closed.
(ii) If both P and Q are p-submanifolds of M, then P is a p-submanifold of Q.
(iii) If P is a p-submanifold of Q and Q is a p-submanifold of M, then P is a p-submanifold of M.

Proof. Let us fix an atlas $\mathcal{A} := \{(U, \phi)\}$. The definition of a p-submanifold shows that it is a closed subset in every coordinate chart (U, ϕ). Hence it is locally closed. This proves (i).

In order to prove (ii) we consider functions $x^1, \dots, x^\ell$ defining a p-submanifold P of codimension ℓ in M locally in a neighborhood of $x \in P$. Choose $I \subset \{1, \dots, \ell\}$ such that $(dx^i|_p)_{i\in I}$ is a basis of $T^*_x Q$. Then in a possibly smaller neighborhood, the functions $(x^i)_{i\in I}$ define P as a p-submanifold of Q.

For (iii) we consider functions $x^1, \ldots, x^k$ locally defining P as a p-submanifold of Q. We extend these functions to locally defined functions on M. Then we choose functions $x^{k+1}, \ldots, x^l$ defining Q locally as a p-submanifold of M. Then $x^1, \ldots, x^l$ locally define P as a p-submanifold of M. □

Example 2.17. The diagonal Δ_N in Example 2.10 is not a p-submanifold. If N is the 2-dimensional closed disc, then with arguments analogous to Remark 5.11, the diagonal is not a p-submanifold of $N \times N$. Alternatively, one could argue with [45], see Remark 2.23.

2.3.3. *The normal bundle of p-submanifolds.* The following standard concepts will be important in the definition of the blow-up of a manifold with corners by a p-submanifold.

Definition 2.18. *Let $P \subset M$ be a p-submanifold of the manifold with corners M. Then $N^M P := TM|_P/TP$ is called the* normal bundle of P in M. *The image $N^M_+ P$ of $T^+M|_P$ in $N^M P$ is called the* inward pointing normal fiber bundle of P in M. *In contrast to $T^+M|_P \to P$, which is not a fiber bundle over P, the projection map $N^M_+ P \to P$ defines a fiber bundle structure over P on $N^M_+ P$, called the* inward pointing normal bundle of P in M. *Finally, the set $\mathbb{S}(N^M_+ P)$ of unit vectors in $N^M_+ P$ is called the set of* inward pointing spherical normal bundle of P in M. *The inward pointing spherical normal bundle of P in M comes equipped with a fiber bundle projection*

$$\mathbb{S}(N^M_+ P) \ \to \ P\,.$$

We complete this section with a few remarks. We first notice the existence of suitable "tubular neighborhoods."

Remark 2.19. Let $P \subset M$ be a p-submanifold in the manifold with corners M. If M is compact, then P has a neighborhood $V_P \subset M$ such that V_P is diffeomorphic to the closed cone $N^M_+ P$ via a diffeomorphism that sends P to the zero section of $N^M_+ P \to M$ and induces the identity at the level of normal bundles. This was proved in [45, Proposition 2.10.1], under the additional assumption that P be closed. Moreover, the condition that M be compact is not necessary (since our p-manifolds are assumed to be locally closed). In this case, $N^M_+ P$ is a cone with corners in $N^M P$. Generalizing Example 2.5, we obtain that all of the sets $N^M P$, $N^M_+ P$, and $S^+(N^M P)$ introduced in the last definition are manifolds with corners. This is because the property of being a manifold with corners is a local property and the product of manifolds with corners is again a manifold with corners.

2.3.4. *The dimension of a non-connected p-submanifold.* We finish the discussion of p-submanifolds with a note on our terminology.

Definition 2.20. *We allow the different connected components of a p-submanifold to have* different *dimensions. Therefore, if Y is a p-submanifold of M, we shall denote by* $\dim(Y)$ *the* largest *dimension of a connected component of Y.*

2.3.5. *Further classes of special submanifolds.* In the conclusion to this section, let us mention some further classes of submanifolds which put our article in the context of the literature and which might be helpful to obtain possible extensions of our results. However, they are not needed to understand the statement or proof of our main results, thus may be skipped by the reader.

In some parts of our article, the notion of a submanifold in the sense of Definition 2.8 is too unspecific, and the notion of a p-submanifold to restrictive. In between these two classes there lies a class of submanifolds that we call "wib-submanifolds," for lack of a better name in the literature. Here "wib-submanifold" stands for a submanifold *without an interior boundary.*

Definition 2.21. *A submanifold $S \subset M$ is called a* wib-submanifold *or a* submanifold without interior boundary *if it can be defined locally in suitable charts as the kernel of a linear function. More precisely: $S \subset M$ is a* wib-submanifold *if, for every $x \in S$, there exists a (corner) chart $\phi : U \to \Omega \subset \mathbb{R}^n_k$, and a linear subspace L of $\mathbb{R}^n$, such that*

(1) $x \in U$ and
(2) $\phi(S \cap U) \;=\; L \cap \Omega\,.$

If $G \in \mathrm{GL}(n, \mathbb{R})$ is defined as above, then we necessarily have $L = G \cdot \Big(\mathbb{R}^{n'} \times \{0\}\Big)$. If $x \in S \cap U$, then $n' := \dim L$ is the dimension of S in x defined above.

Obviously all p-submanifolds are wib-manifolds, which can be easily seen by defining the L in the definition above as the linear extension of L_I in Definition 2.14.

Remark 2.22. In the above definition, we explicitly required S to be a submanifold. To justify this requirement, we will give an example of a closed subset $S \subset M$ which is not a submanifold, but fulfills all other requirements of the definition of a wib-submanifold. Indeed, let

$$K \;:=\; \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 \leq x_3,\ x_2 \leq x_3\}\,,$$

which is a cone over a square. The map $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_1, x_3 - x_2)$ has the property $f^{-1}(\mathbb{R}^4_4) = K$. Then for $\phi = \mathrm{id}$, $x = 0$, and $L := f(\mathbb{R}^3)$ all requirements of the definition are satisfied, but $S := f(K)$ is not a submanifold of $\mathbb{R}^4_4$. It it were a submanifold, then its dimension would have to be 3, and then any boundary point of S is in at most 3 closed boundary hyperfaces. But $0 \in S$ is in 4 closed boundary hyperfaces of S.

Remark 2.23. Note that Melrose also introduces the notions d-submanifold [45, Def. 1.7.4] and b-submanifold [45, Def. 1.12.9], whose definitions will not be recalled here. They satisfy

$$S \text{ is a p-submanifold} \implies S \text{ is a d-submanifold} \implies S \text{ is a b-submanifold}$$

$$\implies S \text{ is a submanifold} \implies S \text{ is a weak submanifold}\,.$$

However there are wib-manifolds that are not b-submanifolds, e.g. Melrose's example of the submanifold $\{x_3 = x_1 + x_2\} \in \mathbb{R}^3_3$. There are d-manifolds that are no wib-manifolds, e.g. $\mathbb{R}^1_1 = [0, \infty) \subset \mathbb{R}$ or any surface with boundary in $\mathbb{R}^3$. However all p-submanifolds introduced below are d-manifolds and wib-manifolds.

Melrose shows that the diagonal Δ_N is a b-submanifold of $N \times N$, but in general not a d-submanifold. It follows that Δ_N is not a p-submanifold.

Remark 2.24. Let us remark that the definition of a *tame* submanifold considered in [2, Sec. 2.3] is a submanifold in an essentially different sense. All notions of submanifolds discussed involve are properties that may hold or not for a subset N of a manifold with corners M. In contrast to this, tame submanifolds in [2, Sec. 2.3], are submanifolds of a *Lie manifold* (M, A), where M is a manifold with corners and A is a Lie algebroid on M with some compatibility conditions. The fact whether a subset N of M is a tame submanifold of (M, A) or not, depends also on the Lie algebroid A. In any case, a tame submanifold will have a tubular neighborhood in the strongest sense.

Similar remarks apply to the $A(\mathcal{G})$-tame submanifolds considered in [55].

3. The blow-up for manifolds with corners

We now introduce the blow-up of a manifold M with corners by a *closed* p-submanifold. We also study some of the properties of the blow-up.

3.1. **Definition of the blow-up and its smooth structure.**

The blow-up $[M : P]$ of M by a closed p-submanifold P is obtained by replacing P with the inward spherical normal bundle $\mathbb{S}(N_+^M P)$ of P in M. More precisely, it is the disjoint union of $\mathbb{S}(N_+^M P)$ and $M \smallsetminus P$. The disjoint union of two subsets A and B will be usually denoted $A \sqcup B := A \cup B$.

3.1.1. *Definition of the blow-up as a set.* We now define the underlying set of the blow-up of a manifold with corners M with respect to a p-submanifold.

Definition 3.1. *Let M be a manifold with corners and P be a* closed *p-submanifold of M. Let $\mathbb{S}(N_+^M P)$ be the inward pointing spherical normal bundle of P in M (Definition 2.18). The* blow-up *of M* along *P (or* with respect to *P) is the following union of disjoint sets:*

$$[M : P] := (M \smallsetminus P) \sqcup \mathbb{S}(N_+^M P) .$$

In particular, $[M : \emptyset] = M$ and $[M : M] = \emptyset$. The blow-down map $\beta = \beta_{M,P} : [M : P] \to M$ is defined as the identity map on $M \smallsetminus P$ and as the fiber bundle projection $\mathbb{S}(N_+^M P) \to P$ on the complement.

The blow-up $[M : P]$ is therefore not defined if P is not closed, but we allow P to consist of the disjoint union of several closed, connected p-submanifolds of M of different dimensions.

If P has a connected component of codimension 0, or, equivalently, if P contains a connected component of M, then the above definitions imply that this connected component is not in the image of $\beta : [M : P] \to M$. In particular, β is not surjective in this case. As this is not what usually mathematicians have in mind when discussing blow-ups, we exclude this case for pedagogigal reasons. In other words, we will always assume $\dim(P) < \dim(M)$. Recall (Definition 2.20) that if $P \subset M$ is a p-submanifold, then $\dim(P)$ denotes the largest dimension of the connected components of P. Let us remark, however, that all statement would remain true for $\dim(P) = \dim(M)$ with minor modifications.

A general approach to smooth structures on the blow-up is contained in [38]. Here we recall an approach that suffices for our needs. We begin with the case of open subsets of a model space $\mathbb{R}^n_k$.

3.1.2. *The blow-up of the local models.* In the following, let I_j, $j = 1, 2, \dots n$, denote either $\mathbb{R}$ or $[0, \infty)$. We will write $N_1 \cong N_2$ if N_1 is a p-submanifold of $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$ and if there is a permutation σ of the components of $\mathbb{R}^n$ that induces a diffeomorphism from N_1 to the p-submanifold N_2 of $I_{\sigma(1)} \times I_{\sigma(2)} \times \cdots \times I_{\sigma(n)} \subset \mathbb{R}^n$. By contrast, when we write $N_1 \simeq N_2$, we will merely state that the indicated manifolds are diffeomorphic, without including further information on the diffeomorphism. In particular, $N_1 \cong N_2$ implies $N_1 \simeq N_2$. To start with, the blow-up $[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ of $\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \cong \mathbb{R}^{n+n'}_{k+k'}$ along its p-submanifold $\mathbb{R}^n_l \times \{0\} = \mathbb{R}^n_k \times \{0_{\mathbb{R}^{n'}}\}$ is, by Definition 3.1, the set

$$\begin{aligned} [\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}] &:= \Big(\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \smallsetminus \mathbb{R}^n_k \times \{0\}\Big) \sqcup \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \\ &= \mathbb{R}^n_k \times \Big(\mathbb{S}^{n'-1}_{k'} \sqcup \big(\mathbb{R}^{n'}_{k'} \smallsetminus \{0\}\big)\Big) . \end{aligned} \tag{19}$$

Let us consider the map

$$\kappa : \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \times [0,\infty) \ \to\ \mathbb{R}^n_k \times \left(\mathbb{S}^{n'-1}_{k'} \sqcup (\mathbb{R}^{n'}_{k'} \smallsetminus \{0\})\right) ,$$

(20)
$$\kappa(x,\xi,r) \ := \ \begin{cases} (x,\xi) \in \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} & \text{if } r = 0 \\ (x, r\xi) \in \mathbb{R}^n_k \times (\mathbb{R}^{n'}_{k'} \smallsetminus \{0\}) & \text{if } r > 0 \, . \end{cases}$$

The map κ is immediately seen to be a bijection and we will use it to endow $[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ with the structure of a manifold with corners induced from $\mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \times [0,\infty)$. Under this diffeomorphism, the blow-down map becomes

(21)
$$\beta : \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \times [0,\infty) \to \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \, , \quad \beta(x,\xi,r) \ := \ (x, r\xi) \, .$$

The blown-up space $[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ is thus a space of "generalized spherical coordinates."

If $U \subset \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}$ is an open subset, we endow

(22)
$$[U : U \cap (\mathbb{R}^n_k \times \{0\})] \ = \ \beta^{-1}(U) \subset [\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$$

with the induced structure of a manifold with corners.

3.1.3. *The smooth structure of the blow-up.* The following lemmas will allow us to define a manifolds with corners structure on blow-ups.

Lemma 3.2. *Let $P_i \subset M_i$, $i = 1, 2$, be closed p-submanifolds and let $\phi : M_1 \to M_2$ be a diffeomorphism such that $\phi(P_1) = P_2$. Then there exists a unique map $\phi^\beta : [M_1 : P_1] \to [M_2 : P_2]$ that is bijective and makes the following diagram commute*

$$\begin{array}{ccc} [M_1 : P_1] & \xrightarrow{\ \phi^\beta\ } & [M_2 : P_2] \\ \Big\downarrow{\scriptstyle \beta_{M_1,P_1}} & & \Big\downarrow{\scriptstyle \beta_{M_2,P_2}} \\ M_1 & \xrightarrow{\ \ \phi\ \ } & M_2 . \end{array}$$

This construction is functorial, in the sense that $(\phi \circ \psi)^\beta = \phi^\beta \circ \psi^\beta$. If M_i are open subsets of $\mathbb{R}^n_k$, then ϕ^β is a diffeomorphism.

Proof. The existence, uniqueness, and the functorial character of ϕ^β follows from the definition of the blow-up. The fact that ϕ^β is smooth if M_i are open subsets of the model space $\mathbb{R}^n_k$ is the content of Lemma 2.2 of [1]. □

Lemma 3.3. *Let $\mathcal{A} = \{(U_a, \phi_a) \mid a \in A\}$ be an* atlas *on a manifold with corners M, see Definition 2.3. Let $P \subset M$ be a closed p-submanifold and $\beta = \beta_{M,P} : [M : P] \to M$ be the blow-down map. We endow $[M : P]$ with the smallest topology that makes all the maps ϕ^β_a, $a \in A$, continuous (ϕ^β_a is defined on $\beta^{-1}(U_a)$). Then*

$$\beta^*(\mathcal{A}) \ := \ \{(\beta^{-1}(U_a), \phi^\beta_a) \mid a \in A\}$$

is an atlas on $[M : P]$, where ϕ^β_a are the maps obtained from ϕ_a using Lemma 3.2. If we take another atlas $\mathcal{A}'$ of M that is compatible with $\mathcal{A}$, then $\beta^*(\mathcal{A})$ and $\beta^*(\mathcal{A}')$ will be compatible atlases on $[M : P]$.

Proof. This follows from Equation (22) and Lemma 3.2. □

Lemma 3.3 thus yields the desired smooth structure on $[M : P]$ that is moreover canonical (independent of any choices).

Definition 3.4. *Let M be a manifold with corners and $P \subset M$ be a closed p-submanifold. We endow $[M : P]$ with the smooth structure defined by the atlas $\beta^*(\mathcal{A})$ obtained from Lemma 3.3, for any atlas $\mathcal{A}$ on M.*

The smooth structure on $[M : P]$ is natural in the following strong sense.

Proposition 3.5. *With the notation of Lemma 3.2, we have that the map ϕ^β is a diffeomorphism (in general, not just in the case of open subsets of Euclidean spaces).*

Proof. If $\mathcal{A}$ is an atlas on M_2, then the pull-back of $\beta^*(\mathcal{A})$ to $[M_1 : P_1]$ is an atlas. □

The functoriality property of Lemma 3.2 then gives the following.

Corollary 3.6. *Let G be a discrete group acting smoothly on the manifold with corners M and let $P \subset M$ be a closed p-submanifold such that $g(P) = P$ for all $g \in G$. Then G acts smoothly on $[M : P]$.*

Proof. The action of every $g \in G$ on M defines a smooth action on $[M : P]$ by Proposition 3.5. It is a group action by the last part of Lemma 3.2 (the functoriality of the assignment $\phi \to \phi^\beta$. □

The blow-up $[M : P]$ of a manifold with corners is thus again a manifold with corners.

3.2. **Exploiting the local structure of the blow-up.** The local character of the definition of the smooth structure of the blow-up $[M : P]$ of the manifold with corners M along a p-submanifold P means that most of the proofs involving blow-ups can be conveniently treated by first treating the model case $P := \mathbb{R}^n_k = \mathbb{R}^n_k \times \{0\} \subset \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} = M$. To simplify notation, we shall often omit factors of the form $\{0\}$ when there is no danger of confusion. This is the case with the following results.

3.2.1. *The blow-down map is proper.* We shall need to prove that certain maps are closed. This will be conveniently done by proving that they are proper, since a proper map between manifolds with corners is closed. In particular, we will show that the blow-down map is proper.

Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. For instance, the map β of Equation (21) is immediately seen to be proper.

Corollary 3.7. *Let P be a closed p-submanifold of a manifold with corners M. The blow-down map $\beta_{M,P} : [M : P] \to M$ is proper.*

Proof. Using Lemma A.3 from the Appendix, we see that we can treat the problem in local coordinates. Then, in local coordinates, the blow-down map is given by Equation (21), which is a proper map, as we have already pointed out. □

3.2.2. *Blow-ups and products.* We have a simple, convenient behavior of the blow-up with respect to products.

Lemma 3.8. *Let M and M_1 be two manifolds with corners and P be a closed p-submanifold of M. Then $P \times M_1$ is a closed p-submanifold of $M \times M_1$ and the following diagram with smooth maps commutes:*

$$
\begin{array}{ccc}
[M \times M_1 : P \times M_1] & \xrightarrow{\simeq} & [M : P] \times M_1 \\
\beta_{M \times M_1, P \times M_1} \big\downarrow & & \big\downarrow \beta_{M,P} \times \mathrm{id} \\
M \times M_1 & \xrightarrow{\mathrm{id}} & M \times M_1 \,.
\end{array}
\tag{23}
$$

Proof. Since the result is a local one and P is a p-submanifold of M, it is enough to treat the case

$$\begin{aligned} M &:= \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P &:= \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \subset M \\ M_1 &:= \mathbb{R}^l_{k_l}\,. \end{aligned}$$

In this local treatment, we will write $\cong$ to stress that a given diffeomorphism is given by a permutation of coordinates, more precisely in this proof, by the canonical permutation of coordinates diffeomorphism of Equation (16).

With this choice, we see that $P \times M_1$ is p-submanifold of $M \times M_1$. We have natural diffeomorphisms with the first one being obtained from the definition of the blow-up, Definition 3.1, and the last being induced by suitable permutations of coordinates

$$\begin{aligned} [M \times M_1 : P \times M_1] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l}] \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l} \sqcup \Big((\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l}) \smallsetminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l}) \Big) \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l} \cong \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1} \end{aligned}$$

and

$$\begin{aligned} [M : P] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}] \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^p_{k_p} \sqcup \Big((\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p}) \smallsetminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}) \Big) \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \cong \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1}\,. \end{aligned}$$

The desired diffeomorphism $[M \times M_1 : P \times M_1] \xrightarrow{\simeq} [M : P] \times M_1$ is then induced by the above diffeomorphisms and the canonical permutation of coordinates diffeomorphism $\mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{R}^l_{k_l} \xrightarrow{\cong} \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1}$ of Equation (16). □

3.3. **Cleanly intersecting families and liftings.** Clean intersections of manifolds are useful in studying the blow-up with respect to families of p-submanifolds.

3.3.1. *Clean intersections.* We continue to exploit the local structure of the blow-up. Recall the following standard definition.

Definition 3.9. *Let M be a manifold with corners and $X_1, X_2, \ldots, X_k \subset M$ be p-submanifolds. We shall say that $X_1, X_2, \ldots, X_k$* have a clean intersection *or that they* intersect cleanly *if*

(i) $Y := X_1 \cap X_2 \cap \ldots \cap X_k$ is a p-submanifold of M (possibly empty),
(ii) for all $x \in Y$, $T_xY = T_xX_1 \cap T_xX_2 \cap \ldots \cap T_xX_k$.

We consider the conditions (i) and (ii) of the Definition 3.9 to be automatically satisfied if $Y := X_1 \cap X_2 \cap \ldots \cap X_k = \emptyset$. Similar conditions appear in Definition 2.7, [1]. They were used to define a *weakly transversal family* of connected submanifolds with corners. We shall need also the notion of a "cleanly intersecting family" (Definition 5.4), which roughly states that every subfamily intersects cleanly.

Lemma 3.10. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Then $P \cap Q$ is a p-submanifold of Q (and also for P).*

Proof. According Definition 3.9 (i) $P \cap Q$ is a p-submanifold of M. Then Lemma 2.16 (ii) states that $P \cap Q$ is also a p-submanifold of Q. □

3.3.2. *Liftings of p-submanifolds to blowups.* We now consider the lifting of suitable p-submanifolds in M to $[M : P]$ as in [38, 45].

The local model for such lifts is given by the following lemma. (See Lemma 3.2 for the definition of j^β.)

Lemma 3.11. *If $k'' \geq k'$ and $n'' - k'' \geq n' - k'$, so that the canonical (first components) inclusion $j : \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \to \mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''}$ is defined, then there is a map j^β such that the diagram*

$$\begin{array}{ccc} [\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}] & \xrightarrow{j^\beta} & [\mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} : \mathbb{R}^n_k \times \{0\}] \\ \downarrow{\scriptstyle \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}, \mathbb{R}^n_k \times \{0\}}} & & \downarrow{\scriptstyle \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''}, \mathbb{R}^n_k \times \{0\}}} \\ \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} & \xrightarrow{j} & \mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''}\,. \end{array} \tag{24}$$

commutes.

In fact the diagram (24) is obtained from

$$\begin{array}{ccc} [\mathbb{R}^{n'}_{k'} : \{0\}] & \xrightarrow{j^\beta_0} & [\mathbb{R}^{n''}_{k''} : \{0\}] \\ \downarrow{\scriptstyle \beta_{\mathbb{R}^{n'}_{k'}, \{0\}}} & & \downarrow{\scriptstyle \beta_{\mathbb{R}^{n''}_{k''}, \{0\}}} \\ \mathbb{R}^{n'}_{k'} & \xrightarrow{j_0} & \mathbb{R}^{n''}_{k''}\,. \end{array} \tag{25}$$

by taking for each space the product with $\mathbb{R}^n_k$ and extending the maps as a product with the identity map $\mathrm{id} : \mathbb{R}^n_k \to \mathbb{R}^n_k$, using the linear version of Lemma 3.8.

The lift j^β_0 is given by

$$[\mathbb{R}^{n'}_{k'} : \{0\}] \simeq \mathbb{S}^{n'-1}_{k'} \times [0, \infty) \xrightarrow{i \times \mathrm{id}} \mathbb{S}^{n''-1}_{k''} \times [0, \infty) \simeq [\mathbb{R}^{n''}_{k''} : \{0\}]\,, \tag{26}$$

where $i : \mathbb{S}^{n'-1}_{k'} \to \mathbb{S}^{n''-1}_{k''}$ is the restriction of j_0. In particular, j^β_0 and thus j^β are smooth.

Definition 3.12. *Let P be a p-submanifold of M and Q be a closed subset of M. The* lifting $\beta^*_{M,P}(Q)$ *of Q in $[M : P]$ is defined by*

$$\beta^*_{M,P}(Q) \;:=\; \overline{\beta^{-1}_{M,P}(Q \smallsetminus P)} \quad \textit{(the closure is in } [M : P]\textit{)}\,.$$

Note that the definition above differs rom Melrose's definition in [45, Chapter 5, Section 7] in the case $Q \subset P$. More precisely, if $Q \subset P$, our definition is such that $\beta^*_{M,P}(Q) = \emptyset$, whereas $\beta^*_{M,P}(Q) := \beta^{-1}_{M,P}(Q)$ in Melrose's definition. In view of our applications, our definition above will avoid the case $Q \subset P$ when defining $\beta^*_{M,P}(Q)$. Furthermore, our definition has the advantage of being local in the sense that, for any open subset $U \subset M$, we have

$$\beta^*_{U,P\cap U}(Q \cap U) = \beta^*_{M,P}(Q) \cap \beta^{-1}_{M,P}(U).$$

In Melrose's definition, locality may fail if Q is not connected.

We have the following result on the blow-up of p-submanifolds, due, in part, to Melrose [45, Chapter 5, Section 7]. A proof in a slightly less general setting can be found also in Proposition 2.4 of [1]. For a p-submanifold $P \subset M$, recall the definition of $\mathbb{S}(N^M_+ P)$, the *inward pointing normal bundle of P in M* from Definition 2.18.

Proposition 3.13. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Then the inclusion $j : Q \to M$ lifts to a natural inclusion*

$$j^\beta : [Q : P \cap Q] \;:=\; (Q \smallsetminus (P \cap Q)) \sqcup \mathbb{S}(N^Q_+(P \cap Q)) \to (M \smallsetminus P) \sqcup \mathbb{S}(N^M_+ P) \;=:\; [M : P]\,.$$

The map j^β is smooth for the natural p-submanifold structures. In particular, the inclusion $Q \smallsetminus P \subset [Q : P \cap Q]$ extends to a natural diffeomorphism

$$\beta^*_{M,P}(Q) \;:=\; \overline{\beta^{-1}_{M,P}(Q \smallsetminus P)} \xrightarrow{\simeq} [Q : P \cap Q]\,.$$

Proof. The inclusion of Q into M restricts to a map $Q \smallsetminus (P \cap Q) \to M \smallsetminus P$. It also induces an inclusion $TQ \to TM$, extending the inclusion $T(P \cap Q) \to TP$. Since $T(P \cap Q) = TP \cap TQ$, we can pass to quotients to obtain an injective map

$$N^Q(P{\cap}Q) \;:=\; TQ|_{P\cap Q}/T(P{\cap}Q) \;=\; TQ|_{P\cap Q}/(TP{\cap}TQ) \;\to\; TM|_P/TP \;=:\; N^M P$$

The injectivity of this map yields an inclusion $\mathbb{S}(N^Q_+ P) \to \mathbb{S}(N^M_+ P)$, which fits smoothly with a map $Q \smallsetminus (Q \cap P) \to M \smallsetminus P$, using the fact that P and Q intersect smoothly and the local description of the blow-up with half-spaces in [1]. The result then follows from the definition of the blow-up, Definition 3.1. ☐

4. The graph blow-up

We introduce also the blow-up with respect to more than one submanifold, called *graph blow-up*.

4.1. **Definition of the graph blow-up.** Let M be a manifold with corners and $\mathcal{F}$ be a locally finite set of p-submanifolds of M. We write $\bigcup \mathcal{F} := \bigcup_{Y \in \mathcal{F}} Y$. Then $M \smallsetminus \bigcup \mathcal{F}$ is an open subset of $[M : Y]$, for each $Y \in \mathcal{F}$. Motivated by the results of [26, 50], we now introduce the following definition.

Definition 4.1. *Let $\mathcal{F}$ be a locally finite set of closed p-submanifolds of the manifold with corners M. Then the* graph blow-up $\{M : \mathcal{F}\}$ *of M along $\mathcal{F}$ is defined by*

$$\{M : \mathcal{F}\} \;:=\; \overline{\{(x, x, \ldots, x) \mid x \in M \smallsetminus \bigcup \mathcal{F}\}} \;\subset\; \prod_{Y \in \mathcal{F}} [M : Y]\,.$$

Let $\delta : M \smallsetminus \bigcup \mathcal{F} \to \prod_{Y \in \mathcal{F}} [M : Y]$ be the diagonal map of inclusions, $\delta(x) = (x, x, \ldots, x)$. Thus the graph blow-up $\{M : \mathcal{F}\}$ is the closure of the image through δ of the complement $M \smallsetminus \bigcup \mathcal{F}$ in the product $\prod_{Y \in \mathcal{F}} [M : Y]$ of all the blown-up spaces $[M : Y]$, $Y \in \mathcal{F}$:

$$\{M : \mathcal{F}\} \;:=\; \overline{\delta(M \smallsetminus \bigcup \mathcal{F})} \;\subset\; \prod_{Y \in \mathcal{F}} [M : Y]\,,$$
$$M \smallsetminus \bigcup \mathcal{F} \ni x \to \delta(x) \;:=\; (x, x, \ldots, x) \in \prod_{Y \in \mathcal{F}} [M : Y]\,.$$

Note that we have used here that $M \smallsetminus \bigcup \mathcal{F} \subset M \smallsetminus Y \subset [M : Y]$ for all $Y \in \mathcal{F}$. The graph blow-up will be compared in the next section to the iterated blow-up.

Definition 4.2. *If G is a Lie group acting smoothly on M and $\mathcal{F}$ is a locally finite set of closed p-submanifolds of M such that, for every $Y \in \mathcal{F}$ and $g \in G$, we have $g(Y) \in \mathcal{F}$, then we shall say that $\mathcal{F}$* is a G-family of p-submanifolds *of M.*

Corollary 3.6 yields right away the following corollary

Corollary 4.3. *Let G be a discrete group and $\mathcal{F}$ be a G-family of p-submanifolds of M (see Definition 4.2). Then G acts continuously on $\{M : \mathcal{F}\}$.*

Proof. We have that each $g \in G$ acts on $M \smallsetminus \bigcup \mathcal{F}$ and on $\prod_{Y \in \mathcal{F}} [M : Y]$, with the action sending $[M : Y]$ to $[M : g(Y)]$, by Corollary 3.6, which also shows that this action is a smooth action of G on $\prod_{Y \in \mathcal{F}} [M : Y]$. The result follows since δ commutes with the action of G. $\square$

Later on, we will show that $\{M : \mathcal{F}\}$ is a weak submanifold of a suitable manifold with corners provided that $\mathcal{F}$ is a clean semilattice, see Subsection 5.2 for a definition. Thus it will follow that $\{M : \mathcal{F}\}$ inherits the structure of a manifold with corners. As soon as the differentiable structure is available on $\{M : \mathcal{F}\}$, the proof above immediately generalizes to yield the stronger result that G acts smoothly on $\{M : \mathcal{F}\}$.

4.2. **Disjoint submanifolds.** We are allowing our p-submanifolds to have components of different dimensions. Blowing-up with respect to such a manifold amounts, as we will see, to blowing up successively with respect to each component.

We need first to discuss the gluing of open subsets. Let us assume that we have two manifolds with corners M_1 and M_2 and that $U_i \subset M_i$ are open subsets ($i = 1, 2$). Let us also assume that we are given a diffeomorphism $\phi : U_1 \to U_2$. Then we define

$$\begin{gathered} M_1 \cup_\phi M_2 \ := \ (M_1 \sqcup M_2)/\{x \equiv \phi(x) \mid x \in U_1\}\,, \\ M_1 \cup_{\mathrm{id}} M_2 \ =: \ M_1 \cup_{U_1} M_2\,, \quad \text{if } \ U_1 = U_2 \ \text{ and } \ \phi \text{ is the identity map } \mathrm{id}\ . \end{gathered} \tag{27}$$

If ϕ is the identity, we shall call $M_1 \cup_{U_1} M_2$ the *union of* M_1 *and* M_2 *along* $U_1 = U_2$. Under favorable circumstances (but not always), $M_1 \cup_\phi M_2$ is also a manifold with corners.

We have the following simple lemma.

Lemma 4.4. *Let M be a manifold with corners (and hence Hausdorff) and $M_i \subset M$, $i = 1, 2$, be open subsets with $U := M_1 \cap M_2$ and $M_1 \cup M_2 = M$. Then there exists a unique structure of a manifold with corners on $M_1 \cup_U M_2$ that induces the given smooth structures on M_i, and hence we have a canonical diffeomorphism $M_1 \cup_U M_2 \simeq M$.*

Proof. Let $\mathcal{A}_i$ be an atlas for M_i. Then their union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas for M. It is also an atlas for any manifold with corners structure on $M_1 \cup_U M_2$ that induces the given one on each M_i. Hence the desired manifold with corners structure on $M_1 \cup_U M_2$ is given by the union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. $\square$

This allows us to "commute" the procedures of taking blow-ups with respect to disjoint manifolds. We thus have the following simple result, see for example [37, 45].

Lemma 4.5. *Let us assume that P and Q are closed p-submanifolds of M such that $P \cap Q = \emptyset$. Let $\beta_{M,Q} : [M : Q] \to M$ be the blow-down map. Then $\beta^*(P) := \beta_{M,Q}^{-1}(P) = P$ and the iterated blow-up $\big[[M : Q] : P\big]$ is defined and diffeomorphic to $([M : Q] \smallsetminus P) \sqcup_{M \smallsetminus (P \cup Q)} ([M : P] \smallsetminus Q)$, the union of $[M : Q] \smallsetminus P$ and $[M : P] \smallsetminus Q$ along $M \smallsetminus (P \cup Q)$, a common open subset. In particular,*

$$\big[[M : P] : Q\big] \ = \ \big[[M : Q] : P\big] \ = \ [M : P \cup Q]\,,$$

with the same smooth structure.

Recall that the blow-up is a local construction, hence, the lemma is trivially satisfied.

Definition 4.6. *Suppose $f_i : X \to Y_i$, $i = 1, \ldots, N$, are continuous maps. We say $(f_1, \ldots, f_N) : X \to \prod_{i=1}^N Y_i$, $x \mapsto (f_1(x), \ldots, f_N(x))$ is proper in each component if each f_i is proper.*

We shall need the following lemma.

Lemma 4.7. *Let us assume that P and Q are closed,* disjoint *p-submanifolds of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : \big[[M : Q] : P\big] \ \to \ [M : P]$$

that restricts to the identity on $M \smallsetminus (P \cup Q)$. Moreover, the product map

$$\mathcal{B}_{M,Q,P} \ := \ (\zeta_{M,Q,P}, \beta_{[M:Q],P}) : \big[[M : Q] : P\big] \to [M : P] \times [M : Q]$$

is proper in each component. Its image is a weak submanifold in the sense of Definition 2.8 and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

Again, our main focus lies on the case $\dim P < \dim M$ and $\dim Q < \dim M$. The statements, however remains (trivially) true if one of these dimensions is equal to $\dim M$, i.e. if a connected component of M is contained in P or Q. Then this component is removed both from $\big[[M : Q] : P\big]$ and from $[M : P]$ or $[M : Q]$.

Proof. Lemma 4.5 states that $\big[[M : Q] : P\big] = [M : P \cup Q] = \big[[M : P] : Q\big]$. This gives $\zeta_{M,Q,P} = \beta_{[M:P],Q}$. In particular, $\zeta_{M,Q,P}$ is proper, by Corollary 3.7. The map $\beta_{[M:Q],P}$ is proper by Corollary 3.7. As P and Q are disjoint, at each point, at least one component of $\mathcal{B}_{M,Q,P} = (\zeta_{M,Q,P}, \beta_{[M:Q],P})$ is a local diffeomorphism. Thus $\mathcal{B}_{M,Q,P}$ is an immersion. As it is injective and proper, it is a homeomorphism onto its image. Proposition B.1 implies that the image is thus a weak submanifold and that $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image. □

By iterating the above lemma, we obtain the following consequence.

Corollary 4.8. *Let $\mathcal{F} := (\emptyset, P_1, P_2, \dots, P_k)$ be a family of closed,* disjoint *p-submanifolds of a manifold with corners M. Then we have canonical diffeomorphisms inducing the identity on $M_0 := M \smallsetminus \bigcup_{j=1}^{k} P_j$ between the usual blow-ups and the graph blow-up (Definitions 3.1 and 4.1):*

$$[[\dots[[M : P_1] : P_2] : \ \dots \ : P_{k-1}] : P_k] \ \simeq \ [M : \bigcup_{j=1}^{k} P_j] \ \simeq \ \{M : \mathcal{F}\}\,.$$

Proof. This follows by induction from Lemmas 4.5 and 4.7 since P_j identifies naturally with a p-submanifold of $[[\dots[[M : P_1] : P_2] : \ \dots \ : P_{j-2}] : P_{j-1}]$. □

5. Iterated blow-ups

The graph blow-up $\{M : \mathcal{F}\}$ introduced in the previous subsection has the advantage that it is defined in great generality and is obviously independent of the order on the family of p-submanifolds $\mathcal{F}$, up to an isomorphism. However, it is not clear what is the structure of the graph blow-up. To this end, in this section, we shall consider an iterated blow-up, which is defined under much more restrictive conditions, but will be, by construction, a manifold with corners. The main result will be that the iterated blow-up and the graph blow-up are diffeomorphic.

5.1. **Definition of the iterated blow-up.** Recall the definition of the lifting $\beta^*(Q) = \beta^*_{M,P}(Q) := \overline{Q \smallsetminus P} \subset [M : P]$ (closure in $[M : P]$), Definition 3.12. We fix a manifold with corners M. We now introduce the *iterated version of the blow-up.*

Definition 5.1. *Let $(P_i)_{i=1}^k$, $P_i \subset M$, be a k-tuple of closed p-submanifolds of M and let $\beta_1 := \beta_{M,P_1} : [M : P_1] \to M$. Whenever all the terms make sense, we define by induction on k the* iterated blow-up $[M : (P_i)_{i=1}^k]$ of M with respect to *or* along $(P_i)_{i=1}^k$ *by*

$$[M : (P_i)_{i=1}^k] := \begin{cases} [M : P_1] & \text{if } k = 1\,, \\ \big[[M : P_1] : \big(\beta_1^*(P_i)\big)_{i=2}^k\big] & \text{if } k > 1\,. \end{cases}$$

Note that in this definition we did not assume any inclusion relations between the p-submanifolds P_i, although we will assume such relations later on.

Let us also notice that we did not rule out the case $P_j \subset P_i$, for some $j > i$. In this case we can remove P_j from the family (P_i) without changing $[M : (P_i)]$. In particular, we can assume that all the manifolds P_i are distinct, without losing generality.

We stress that we do not assume any inclusions among the manifolds P_i, but, on the other hand, $[M : (P_i)_{i=1}^k]$ is not always defined (unlike the graph blow-up!), as we need additional conditions in order to guarantee that $\beta_{j-1}^*\beta_{j-2}^* \cdots \beta_1^*(P_j)$ is a closed p-submanifold for all j. We shall also write

$$[M : (P_i)_{i=1}^k] =: [M : P_1, P_2, \ldots, P_k]\,,$$

and hence, using the pull-back by the map β_1, we have

$$[M : P_1, P_2, \ldots, P_k] := \big[[M : P_1] : \beta_1^*(P_2), \ldots, \beta_1^*(P_k)\big]\,.$$

We generalize this relation in the following remark.

Remark 5.2. Let $\gamma_1 := \beta_1^*$ and $\gamma_j := \beta_j^* \circ \gamma_{j-1} = \beta_j^* \circ ... \circ \beta_1^*$, where

$$\beta_k := \beta_{[M,P_1,P_2,\ldots,P_{k-1}],P_k} : [M : P_1, P_2, \ldots, P_k] \to [M : P_1, P_2, \ldots, P_{k-1}]\,.$$

We then have

$$\begin{aligned} [M : P_1, P_2, \ldots, P_j] &= \big[[M : P_1] : \gamma_1(P_2), \ldots, \gamma_1(P_j)\big] \\ &= \Big[\big[[M : P_1] : \gamma_1(P_2)\big] : \gamma_2(P_3), \ldots, \gamma_2(P_j)\Big] \\ &= \ \ldots \\ &= [[\ldots[[[M : P_1] : \gamma_1(P_2)] : \gamma_2(P_3)] \ \ldots] : \gamma_{j-1}(P_j)]\,. \end{aligned}$$

Note that $[M : P_1, P_2, \ldots, P_j]$ is always defined if $j = 1$. Then the condition that the iterated blow-up $[M : P_1, P_2, \ldots, P_j]$ be defined can then be formulated by induction as follows:

(i) the iterated blow-up $[M : P_1, \ldots, P_{j-1}]$ is defined, and
(ii) the lift $\gamma_{j-1}(P_j)$ is defined and is a closed p-submanifold of $[M : P_1, \ldots, P_{j-1}]$.

5.2. **Clean semilattices.** We now investigate the iterated blow-up $[M : (P_i)_{i=1}^k]$ of a manifold with corners M with respect to a (suitably) *ordered* family of p-submanifolds of M.

Definition 5.3. *Let $\mathcal{F}$ be a locally finite (unordered) set of p-submanifolds of M. We shall say that $\mathcal{F}$ is a* cleanly intersecting family *if any $X_1, X_2, \ldots, X_j \in \mathcal{F}$ have a clean intersection (Definition 3.9).*

We consider the iterated blow-up mostly with respect to semilattices. Recall that a *meet semilattice* (or, simply, *semilattice* in what follows) is a partially ordered set $\mathcal{L}$ such that, for every two $x, y \in \mathcal{L}$, there is a greatest common lower bound $x \cap y \in \mathcal{L}$ of x and y. We shall consider only semilattices of subsets of a given set where the order is given by $\subset$ and where $x \cap y$ is the usual intersection of sets. We can now introduce the semilattices we are interested in. We let $\mathcal{P}(M)$ denote the set of all subsets of M.

Definition 5.4. *A semilattice $\mathcal{S} \subset \mathcal{P}(M)$ of closed p-submanifolds of M will be called* clean *if $\mathcal{S}$ is a cleanly intersecting family of p-submanifolds of M.*

It is easier to deal with semilattices since a semilattice is clean if, and only if, any two members of the semilattice intersect cleanly. For the simplicity of the notation, we shall consider only semilattices $\mathcal{S} \subset \mathcal{P}(M)$ with $\emptyset \in \mathcal{S}$. This changes nothing in our results, but avoids us treating separately the cases $\emptyset \in \mathcal{S}$ and $\emptyset \notin \mathcal{S}$ in proofs. The concept of a clean semilattice introduced here is very closely related to that of a weakly transversal family considered in [1, Definition 2.7], except that in that paper, the authors considered only p-submanifolds that were *not* contained in the boundary. Similar concepts were also considered in [37, Theorem 3.2] and in [60]. The case considered in [60, Sec. 2] was that when all p-submanifolds with respect to which we blow-up *are contained in the boundary.* We want to recall this construction in the general situation, also in view of the applications in Section 7.

Remark 5.5. Clean semilattices of closed p-submanifolds are useful for studying iterated blow-ups because, if P, Q are two p-submanifolds of a manifolds with corners M such that P and Q intersects cleanly, then the lifts of P and Q in $[M : P \cap Q]$ are disjoint p-submanifolds of $[M : P \cap Q]$. See also [1, Theorem 2.8].

The following result was proved in special cases in [1] and in [60, Lemma 2.7] with similar proofs. In [45], Melrose proved that a the lift of a normal family remains a normal family if we do the blow-up by an element of the family. Lemma 5.11.2 of [45] also treats the lift of a family under the blow-up.

Proposition 5.6. *Let $\mathcal{S} \ni \emptyset$ be a clean semilattice (of p-submanifolds) of M and let P be a minimal element of $\mathcal{S} \smallsetminus \{\emptyset\}$. Let $Q' := [Q : P \cap Q]$. Then*

$$\mathcal{S}' \ := \ \Big\{ Q' = [Q : P \cap Q] \,\big|\, Q \in \mathcal{S} \Big\} .$$

is a clean semilattice of $[M : P]$ with $\emptyset = \emptyset' = P' \in \mathcal{S}'$.

Let $Q' := [Q : Q \cap P]$, so that $\mathcal{S}' = \{Q' \mid Q \in \mathcal{S}\}$. Recall that the minimality of P and the semilattice property of $\mathcal{S}$ imply that, for any $Q \in \mathcal{S}$, we have either $P \subset Q$ or $P \cap Q = \emptyset$. In the first case, we have $Q' := [Q : P \cap Q] = [Q : P]$ and in the second case we have $Q' := [Q : P \cap Q] = Q$. Thus

$$\mathcal{S}' \ := \ \{ [Q : P] \mid P \subset Q \in \mathcal{S} \} \cup \{ Q \mid Q \in \mathcal{S}, Q \cap P = \emptyset \} .$$

Let us also notice that $P' := [P : P \cap P] = \emptyset = [\emptyset : \emptyset \cap P] = \emptyset'$, whereas all the other manifolds Q' ($Q \in \mathcal{S} \smallsetminus \{\emptyset, P\}$) are different to each other and nonempty. Therefore, $|\mathcal{S}'| = |\mathcal{S}| - 1$ (i.e. $\mathcal{S}'$ has one element less than $\mathcal{S}$).

Proof. This result was proved in slightly less generality in [1, Theorem 2.8] (assuming that the p-manifolds are *not* contained in the boundary). The proof extends right away to the current setting. □

Let $\mathcal{S}$ be clean semilattice (of p-submanifolds) of M and let us arrange $\mathcal{S} \smallsetminus \{\emptyset\}$ in a size order $(P_i)_{i=1}^k = (P_1, P_2, \ldots, P_k)$. Recall that we have a size-order on $\mathcal{S} \smallsetminus \{\emptyset\}$ if $P_i \subsetneq P_j$ implies $i \le j$. It will be convenient to set $P_0 = \emptyset$. See also the related concept of an *admissible ordering* in [1, Definition 2.9]. We then have the following basic result, see [60, Lemma 2.8] and [37, Theorem 3.2]. We include a proof for the benefit of the reader.

Proposition 5.7. *Let $\mathcal{S}$ be clean semilattice of p-submanifolds of M and arrange $\mathcal{S} \smallsetminus \{\emptyset\}$ in a size-order $(P_i)_{i=1}^k = (P_1, P_2, \ldots, P_k)$. Then $[M : (P_i)_{i=1}^k]$ is defined.*

Proof. We shall write $[M : \mathcal{S}] := [M : (P_i)_{i=1}^k]$. (This definition of $[M : \mathcal{S}]$ is implicitly assuming that a compatible order was chosen on $\mathcal{S}$. The notation is nevertheless justified since Theorem 5.12 will show that the result is independent of the order.) To prove that $[M : \mathcal{S}] := [M : (P_i)_{i=1}^k]$ is defined, we shall proceed by induction on the number of elements of $\mathcal{S}$. As $\emptyset \in \mathcal{S}$, let us assume, for the initial verification step, that $|\mathcal{S}| = 2$ and, more precisely, that $\mathcal{S} = \{\emptyset, P\}$, for some p-submanifold P of M. Then $[M : (P_i)_{i=1}^1] := [M : P]$ is defined.

Let us assume that the result is true for semilattices $\mathcal{S}$ with j elements and prove it for lattices with $j+1$ elements. Then the semilattice $\mathcal{S}'$ obtained from $\mathcal{S}$ via Proposition 5.6 is a clean semilattice with j elements of $[M : P_1]$ by that same proposition. Therefore $\big[[M : P_1] : \mathcal{S}'\big]$ is defined by the induction hypothesis, and hence, using also Remark 5.2, we have that

$$[M : \mathcal{S}] \;:=\; \big[[M : P_1] : \mathcal{S}'\big] \tag{28}$$

is also defined. □

Remark 5.8. Note that the normal sphere bundle of a submanifold of codimension 0 is the empty set, thus $[M : M] = \emptyset$. As a consequence, our definitions imply $[M : \mathcal{S}] = \emptyset$ in the case $M \in \mathcal{S}$. This is why all interesting examples satisfy $M \notin \mathcal{S}$.

5.3. **The pair blow-up lemma.** We now perform some essential calculations in local coordinates that will be needed for our main result. Recall from Equation (14) that

$$\mathbb{S}_k^n \;:=\; \mathbb{S}^n \cap \mathbb{R}_k^{n+1},$$

where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, as always. For $\psi \in \mathbb{S}_{k'+1}^{n'+1} := \mathbb{S}^{n'+1} \cap \mathbb{R}_{k'+1}^{n'+2}$, we shall write $\psi =: (\psi_1, \tilde{\psi})$, with $\psi_1 \in [0,1]$ and $\tilde{\psi} \in \mathbb{R}_{k'}^{n'+1}$, and we define the map

$$\begin{aligned} \Upsilon : \mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1} \to \mathbb{S}_{k,k'}^{n,n'} \;&:=\; \mathbb{S}^{n+n'} \cap \big(\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1}\big) \\ (\phi, \psi) &\mapsto (\psi_1 \phi, \tilde{\psi})\,. \end{aligned} \tag{29}$$

We embed the sphere orthant $\{0\} \times \mathbb{S}_{k'}^{n'} = \{0_{\mathbb{R}^n}\} \times \mathbb{S}_{k'}^{n'} \subset \mathbb{R}^{n+n'+1}$ into $\mathbb{R}^{n+n'+1}$ by mapping the sphere orthant to the *last* components of $\mathbb{R}^{n+n'+1}$. Of course, we have an isomorphism

$$\mathbb{S}_{k,k'}^{n,n'} = \mathbb{S}^{n+n'} \cap \big(\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1}\big) \;\cong\; \mathbb{S}_{k+k'}^{n+n'} = \mathbb{S}^{n+n'} \cap \mathbb{R}_{k+k'}^{n+n'+1}$$

given by the canonical permutation of coordinates diffeomorphism of Equation (16).

We recall Proposition 5.8.1 of [45] and we give the proof to fix the notation.

Lemma 5.9. *Let again $\mathbb{S}_{k,k'}^{n,n'} \;:=\; \mathbb{S}^{n+n'} \cap \big(\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1}\big) \;\cong\; \mathbb{S}_{k+k'}^{n+n'}$ and let the map $\Upsilon : \mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1} \to \mathbb{S}_{k,k'}^{n,n'}$ be as in the last paragraph. If we define*

$$\Psi : \mathbb{S}_{k,k'}^{n,n'} \smallsetminus \big(\{0\} \times \mathbb{S}_{k'}^{n'}\big) \;\to\; \mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1}\,, \quad \Psi(\eta, \mu) \;=\; \Big(\frac{\eta}{|\eta|},\, \big(|\eta|, \mu\big)\Big)\,,$$

then $\Upsilon \circ \Psi$ is the inclusion $\mathbb{S}_{k,k'}^{n,n'} \smallsetminus \big(\{0\} \times \mathbb{S}_{k'}^{n'}\big) \subset \mathbb{S}_{k,k'}^{n,n'}$ and Ψ extends to a diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}_{k,k'}^{n,n'} : \{0\} \times \mathbb{S}_{k'}^{n'}] \;\xrightarrow{\sim}\; \mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1}$$

such that $\beta_{\mathbb{S}_{k,k'}^{n,n'},\, \{0\} \times \mathbb{S}_{k'}^{n'}} = \Upsilon \circ \widetilde{\Psi}$.

If we write by abuse of notation $\mathbb{S}^{n'}_{k'}$ for the image of $\{0\} \times \mathbb{S}^{n'}_{k'}$ in $\mathbb{S}^{n+n'}_{k+k'}$ under the permutation of coordinates described above, then we obtain a diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^{n+n'}_{k+k'} : \mathbb{S}^{n'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{n-1}_{k} \times \mathbb{S}^{n'+1}_{k'+1} .$$

Proof. Let

$$\beta := \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\}\times \mathbb{R}^{n'+1}_{k'}} : [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}] \to \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} .$$

denote the blow-down map. Also, recall that the lifting $\beta^*(\mathbb{S}^{n,n'}_{k,k'})$ is defined as the closure of $\beta^{-1}(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus \{0\} \times \mathbb{R}^{n'+1}_{k'})$ in $[\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}]$. Since

$$\mathbb{S}^{n,n'}_{k,k'} \cap \big(\{0\} \times \mathbb{R}^{n'+1}_{k'}\big) = \{0\} \times \mathbb{S}^{n'}_{k'} ,$$

Proposition 3.13 gives a diffeomorphism

$$\Phi : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \xrightarrow{\sim} \beta^*(\mathbb{S}^{n,n'}_{k,k'}),$$

uniquely determined by the condition that is the inclusion on $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus \{0\} \times \mathbb{S}^{n'}_{k'}$. (That is, the blow-up of $\mathbb{S}^{n,n'}_{k,k'}$ along $\{0\} \times \mathbb{S}^{n'}_{k'}$ is diffeomorphic to the lifting $\beta^*(\mathbb{S}^{n,n'}_{k,k'})$ of $\mathbb{S}^{n,n'}_{k,k'}$ to $[\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}]$ via the blow-down map $\beta := \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\}\times \mathbb{R}^{n'+1}_{k'}}$.)

To identify more explicitly the space $\beta^*(\mathbb{S}^{n,n'}_{k,k'})$, it is convenient to use the diffeomorphism $\kappa : \mathbb{S}^{n-1}_k \times [0, +\infty) \times \mathbb{R}^{n'+1}_{k'} \to [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}]$ of Equation (20) with the order of its arguments reversed. To start with, the blow-down map $\beta := \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\}\times \mathbb{R}^{n'+1}_{k'}}$ is such that $\beta_1 := \beta \circ \kappa$ satisfies

$$\beta_1 := \beta \circ \kappa : \mathbb{S}^{n-1}_k \times [0, +\infty) \times \mathbb{R}^{n'+1}_{k'} \to \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} ,$$
$$\beta_1(z, r, x) = (rz, x).$$

We have that $(z, r, x) \in \beta_1^{-1}(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'}))$ if, and only if $\|\beta_1(z, r, x)\| = 1$ and $r > 0$. Assume that $\|\beta_1(z, r, x)\| = 1$ and $r > 0$. Then $\|rz\|^2 + \|x\|^2 = 1$. Note that $z \in \mathbb{S}^{n-1}_k$, and hence $r^2 + \|x\|^2 = 1$. This leads to $(r, x) \in \mathbb{S}^{n'+1}_{k'+1} \subset \mathbb{R}^{n'+2}_{k'+1} = [0, \infty) \times \mathbb{R}^{n'+1}_{k'}$. We thus have

$$\beta_1^{-1}\big(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'})\big) = (\mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}) \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'}) .$$

The closure of this set is $\mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$, and hence we obtain a diffeomorphism $\Phi_1 := \kappa^{-1} \circ \Phi : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$. That $\Upsilon \circ \Psi$ is the inclusion follows from the defining formulas. The relation $\beta_{\mathbb{S}^{n,n'}_{k,k'}, \{0\}\times \mathbb{S}^{n'}_{k'}} = \Upsilon \circ \widetilde{\Psi}$ follows from the fact that they are both continuous and they coincide on the dense, open subset $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{S}^{n'}_{k'})$. This shows that $\Phi_1 = \Psi$ on $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{S}^{n'}_{k'})$ and hence $\tilde{\Psi} := \Phi_1$ is the desired extension. $\square$

We now treat the basic case when the blow-up is defined, namely the simplest case when we blow up by two p-submanifolds P and Q with $Q \subset P$. The case when of two disjoint p-submanifolds was already treated in Lemma 4.7, so now we treat the remaining case, that is, that one submanifold is contained in the other.

Lemma 5.10. *Let us assume that Q is a p-submanifold of P and that P is a p-submanifold of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : [M : Q, P] := \big[[M : Q] : [P : Q]\big] \to [M : P]$$

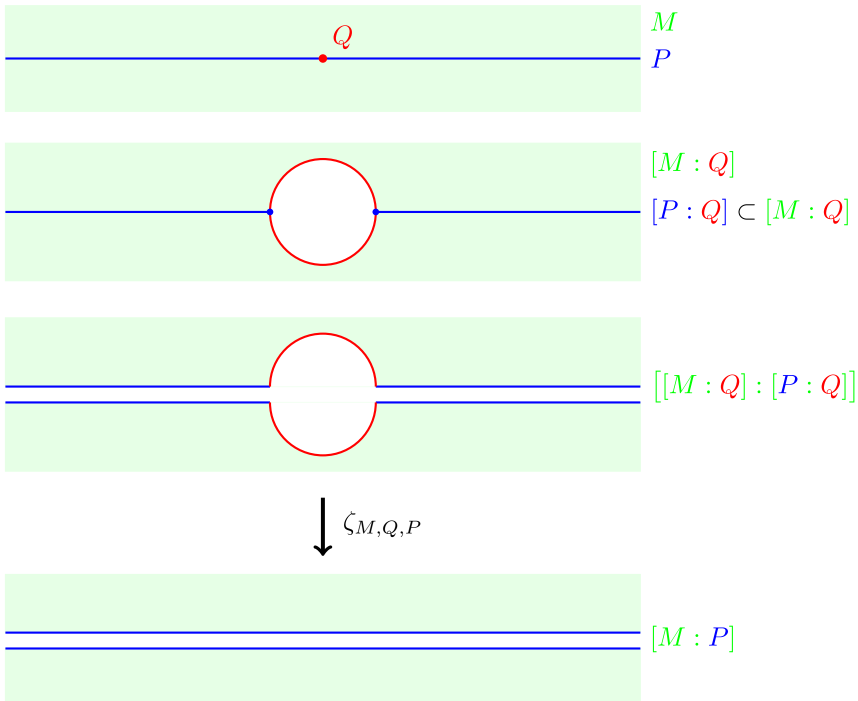

FIGURE 1. The blow-ups $[M : Q]$, $\big[[M : Q] : [P : Q]\big]$, and $[M : P]$

that restricts to the identity on $M \smallsetminus P$. Moreover, the product map

$$\mathcal{B}_{M,Q,P} := (\zeta_{M,Q,P}, \beta_{[M:Q],[P:Q]}) : [M : Q, P] \to [M : P] \times [M : Q]$$

is proper in each component. The image of $\mathcal{B}_{M,Q,P}$ is a weak submanifold in the sense of Definition 2.13, and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

See Figure 5.3 for a local picture of these blow-ups in the example $M = \mathbb{R}^2$, $P = \mathbb{R} \times \{0\}$, $Q = \{0\}$.

Proof. The uniqueness of the map $\zeta_{M,Q,P}$ follows from the fact that it is the identity on the dense subset $M \smallsetminus (P \cup Q)$. The statement is local, so, in view of Lemma 3.8, we can assume that $Q = \{0\}$. That is, we can assume that

$$\begin{cases} M & := & \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P & := & \{0\} \times \mathbb{R}^p_{k_p} \\ Q & := & \{0\} \end{cases} \tag{30}$$

We have

$$\begin{aligned} [M : P] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0\} \times \mathbb{R}^p_{k_p}] \\ &= [\mathbb{R}^m_{k_m} : \{0\}] \times \mathbb{R}^p_{k_p} \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0, \infty) \times \mathbb{R}^p_{k_p} \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} . \end{aligned}$$

Its blow-down map is $\beta_{M,P}(x,t,y) = (tx, y)$.

On the other hand, we have (using the notation of Lemma 5.9):

$$[M : Q] \;=\; [\mathbb{R}^{m+p}_{k_m+k_p} : \{0\}] \;=\; \mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0,\infty).$$

Its blow-down map is $\beta_{M,Q}(x,t) = tx$. Lemma 3.13 gives that the lift of P to $[M : Q]$ is $P' := [P : Q] = \{0\} \times \mathbb{S}^{p-1}_{k_p} \times [0,\infty)$. Lemmas 3.8 and 5.9 (in this order) then give canonical diffeomorphisms

$$\begin{aligned}\big[[M : Q] : P'\big] \;&\simeq\; [\mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0,\infty) : \mathbb{S}^{p-1}_{k_p} \times [0,\infty)]\\ &=\; [\mathbb{S}^{m,p-1}_{k_m,\,k_p} : \mathbb{S}^{p-1}_{k_p}] \times [0,\infty)\\ &\simeq\; \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^{p}_{k_p+1} \times [0,\infty)\,.\end{aligned}$$

The blow-down map $\beta_{[[M:Q]:P]} : \big[[M : Q] : P'\big] \to [M : Q]$ is given, up to canonical diffeomorphisms, by the map $\Upsilon \times \mathrm{id}$, where Υ is as defined in Equation (29). Hence $\Upsilon \times \mathrm{id}(\phi,\psi,t) = (\psi_1\phi, \tilde{\psi}, t)$.

The desired map $\zeta_{M,Q,P}$ is then obtained from the blow-down map $\mathbb{S}^{p}_{k_p+1} \times [0,\infty) \to \mathbb{R}^{p+1}_{k_p+1} = [0,\infty) \times \mathbb{R}^{p}_{k_p}$, that is $\zeta_{M,Q,P}(x,y,t) = (x,ty)$. In particular, it is proper. It remains to check that this map is the identity on $M \smallsetminus P$. As we used for $x \in M \smallsetminus P$ the identifications $x \mathrel{\widehat{=}} \beta_{M,Q}(x) \mathrel{\widehat{=}} \beta_{M,P}(x) \mathrel{\widehat{=}} \beta_{[M:Q],[P:Q]}(x)$, it is enough to check

$$\beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ \beta_{[M:Q],[P:Q]} \tag{31}$$

on $M \smallsetminus P$. As this calculation is local, we can again assume (30) and the concrete presentations of $\beta_{M,Q}$, $\beta_{M,P}(x)$ and $\beta_{[M:Q],[P:Q]}$ described above, (31) turns into

$$\beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ (\Upsilon \times \mathrm{id}) \tag{32}$$

on $\mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^{p}_{k_p+1} \times [0,\infty)$. Indeed for $x \in \mathbb{S}^{m-1}_{k_m}$, $y = (y_1,\tilde{y}) \in \mathbb{S}^{p}_{k_p+1} \subset \mathbb{R}^{p+1}_{k_p+1} = \mathbb{R}^1_1 \times \mathbb{R}^p_{k_p}$, $t \in [0,\infty) = \mathbb{R}^1_1$ we have

$$\beta_{M,P} \circ \zeta_{M,Q,P}(x,y,t) \;=\; \beta_{M,P}(x,ty) \;=\; \beta_{M,P}(x,ty) \;=\; (ty_1x, t\tilde{y})\,.$$

Together with

$$\beta_{M,Q} \circ (\Upsilon \times \mathrm{id})(x,y,t) \;=\; \beta_{M,Q}(y_1x,\tilde{y},t) \;=\; (ty_1x, t\tilde{y})\,,$$

this implies (32).

The map $\mathcal{B}$ is given in local coordinates by $\mathcal{B}(x,y,t) = \big(x, ty, (y_1x,\tilde{y}), t\big)$ with differentiable left inverse $(x,z,(w_1,w_2),t) \mapsto (x, (\|w_1\|, w_2), t)$. Hence by Corollary B.2 the image of $\mathcal{B}$ is a weak submanifold and $\mathcal{B}$ is a diffeomorphism onto its image. □

Remark 5.11. Note that, in general, the image of the map $\mathcal{B}_{M,Q,P}$ introduced in the proof above is not a p-submanifold of $[M : P] \times [M : Q]$. Indeed, let us consider the case when M is the closed unit disk in $\mathbb{R}^2$, and let p and q be two disjoint points in the interior of M. Let $P := \{p\}$ and $Q := \{q\}$. We claim that the image N of $\mathcal{B} = \mathcal{B}_{M,Q,P}$ is *not* a p-submanifold of $M_1 := [M : P] \times [M : Q]$. Suppose N were a p-submanifold of M_1. As N is connected, the function $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x)$ is constant on N, see Remark 2.15. However, the map $\mathcal{B}$ sends the interior points of $M \smallsetminus \{p,q\}$ to the interior of $M_1 = [M : P] \times [M : Q]$, thus $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x) = 0 - 0 = 0$ for $x = \mathcal{B}(y)$ with y in the interior of $M \smallsetminus \{p,q\}$. On the other hand $\mathcal{B}$ maps the boundary $\partial M = \partial(M \smallsetminus \{p,q\})$ to the corner $\partial M \times \partial M$ of $[M : P] \times [M : Q]$, which has boundary depth 2 in $M_1 = [M : P] \times [M : Q]$. Thus, if $x = \mathcal{B}(y)$, with $y \in \partial M$, we obtain

$\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x) = 2-1 = 1$. Therefore, the function $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x)$ is not constant on N, and hence N is not a p-submanifold of $M_1 = [M : P] \times [M : Q]$.

A careful investigation [39] shows that the image $\mathcal{B}_{M,Q,P}([M : Q, P]$ of $[M : Q, P]$ in $[M : P] \times [M : Q]$ may fail to be a submanifold of $[M : P] \times [M : Q]$ in Melrose's sense. This fact justifies our introduction of the notion of a "weak submanifold." In particular, a weak submanifold is neither a b-submanifold nor a wib-submanifold.

Using the similar result for disjoint manifolds, Lemma 4.7, we obtain the following result. (Recall that our semilattices contain the empty set, but do not contain the ambient manifold M.)

Theorem 5.12. *Let $\mathcal{S} = (P_j)_{j=0,1,\dots,k}$ be a clean semilattice of closed p-submanifolds of M (so $\emptyset \in \mathcal{S}$). Then, for each $P \in \mathcal{S}$, there exists a unique smooth map $\phi_{\mathcal{S},P} : [M : \mathcal{S}] \to [M : P]$ that is the identity on $M \smallsetminus \bigcup_{Q \in \mathcal{S}} Q$. These maps are such that the induced map*

$$\mathcal{B}_{\mathcal{S}} \ := \ (\phi_{\mathcal{S},P_0}, \dots, \phi_{\mathcal{S},P_k}) : [M : \mathcal{S}] \to \prod_{j=0}^{k} [M : P_j]$$

is proper in each component. Furthermore the image of $\mathcal{B}_{\mathcal{S}}$ is a weak submanifold of $\prod_{j=0}^{k} [M : P_j]$ in the sense of Definition 2.13 and $\mathcal{B}_{\mathcal{S}}$ maps $[M : \mathcal{S}]$ diffeomorphically onto $\{M : \mathcal{S}\}$, i.e. we have a diffeomorphism

$$\mathcal{B}_{\mathcal{S}} : [M : \mathcal{S}] \ \xrightarrow{\sim} \ \{M : \mathcal{S}\} \, .$$

Proof. We shall proceed by induction on the number $k+1$ of elements of $\mathcal{S} = (P_j)_{j=0,1,\dots,k}$. We can assume that $P_i \neq \emptyset = P_0$ for all $i > 0$. (So k is the number of *non-empty* elements of $\mathcal{S}$.) The case $k = 0$ is trivial.

Case $k = 1$: If $\mathcal{S}$ has $1 + 1 = 2$ elements, we have $\mathcal{S} = (\emptyset, P)$ and $\mathcal{B}_{\mathcal{S}} = (\beta_{M,P}, \mathrm{id}_{[M:P]})$ so the claim is trivially satisfied, since the blow-down map is proper (Corollary 3.7).

Case $k = 2$: If $\mathcal{S}$ has $2 + 1 = 3$ elements, we have $\mathcal{S} = \{\emptyset, Q, P\}$ and we have $Q \subset P$ or $Q \cap P = \emptyset$.

1) In the first subcase, that is, if $Q \subset P$, the result was already proved in Lemma 5.10, with

$$\mathcal{B}_{\mathcal{S}} \ := \ (\beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}, \beta_{[M:Q],[P:Q]}, \zeta_{M,Q,P}) \, ,$$

that is, we have, $\phi_{\mathcal{S},\emptyset} = \beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}$, $\phi_{\mathcal{S},Q} := \beta_{[M:Q],[P:Q]}$, $\phi_{\mathcal{S},P} := \zeta_{M,Q,P}$. In particular, the fact that $\mathcal{B}_{M,Q,P} = (\beta_{[M:Q],[P:Q]}, \zeta_{M,Q,P})$ is a diffeomorphism onto its image implies the same statement for $\mathcal{B}_{\mathcal{S}}$.

2) Similarly, in the second subcase, that is, if $Q \cap P = \emptyset$, the result was already proved in Lemma 4.7, with $\mathcal{B}_{\mathcal{S}} = (\beta_{M,P\cup Q}, \beta_{[M:Q],P}, \beta_{[M:P],Q})$, i.e. all the components of $\mathcal{B}_{\mathcal{S}}$ are given by blow-down maps. The diffeomorphism property for $\mathcal{B}_{\mathcal{S}}$ comes from the fact that its restriction to $[M \smallsetminus Q : P]$ and $[M \smallsetminus P : Q]$ has a component equal to the identity, so it is a local diffeomorphism onto its image, which is at the same time injective and proper, thus having a continuous inverse.

Case $k \geq 3$: Let us now proceed with the induction step from $k - 1$ to k, that is, let us assume that $\mathcal{S}$ has $k + 1$ elements $P_0 = \emptyset, P_1, \dots, P_k$. As always, the numbering of the sets P_j is chosen to be compatible with the inclusion (an admissible ordering), meaning that if $P_i \subset P_j$, then $i \leq j$. In particular, P_1 must be a minimal element of $\mathcal{S} \smallsetminus \emptyset$ with respect to the relation $\subset$. For $P := P_j \in \mathcal{S}$, $2 \leq j \leq k$, we thus have $P_1 \subset P$ or $P_1 \cap P = \emptyset$, by the minimality of P_1 in $(P_1, \dots, P_k)$ and by the fact that $\mathcal{S}$ is stable under intersections. Let $P' := [P : P \cap P_1]$. Thus we have $P' = [P : P_1]$, if $P_1 \subset P$, and $P' = P$, if $P_1 \cap P = \emptyset$. We shall use the notation of Proposition 5.6 with $P := P_1$, in particular, $Q' := [Q : Q \cap P_1]$. The semilattice $\mathcal{S}' = \big(P_j' := [P_j : P_j \cap P_1]\big)_{j=1,\dots k}$ of Proposition 5.6 is then clean. Note that $P_1' := [P_1 : P_1 \cap P_1] = \emptyset = \emptyset'$, and hence $\mathcal{S}'$ has k elements. By the induction hypothesis, the map $\mathcal{B}_{\mathcal{S}'}$ is a diffeomorphism onto its image. The same property is shared by the maps

$$\mathcal{B}_{M,P_1,P_j} : \big[[M : P_1] : [P_j : P_1]\big] \to [M : P_1] \times [M : P_j]$$

of the Lemmata 4.7 and 5.10. Let $\Phi := \mathrm{id} \times \prod_{j=2}^{k} \mathcal{B}_{M,P_1,P_j}$ and consider the composition

$$(33) \quad [M : \mathcal{S}] := \big[[M : P_1] : \mathcal{S}'\big] \xrightarrow{\mathcal{B}_{\mathcal{S}'}} \prod_{j=1}^{k} \big[[M : P_1] : [P_j : P_1]\big] \xrightarrow{\Phi} [M : P_1] \times \prod_{j=2}^{k} \Big([M : P_1] \times [M : P_j]\Big)\,.$$

The two maps of the composition are both injective immersions, and hence their composition is again an injective immersion. The desired map $\phi_{\mathcal{S},P_j}$ is the projection onto the P_j-component. The projection of the composite map onto any of the factors is the identity on $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$. Note that all components with factors of the form $[M : P_1]$ (which are repeated), yield the same projection, again because this projection is the identity map on $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$. By removing these repetitions, and by adding the iterated blow-down map $[M : \mathcal{S}] \to M$ we obtain the desired map $\mathcal{B}_{\mathcal{S}}$, which is consequently also an injective immersion. The map $\mathcal{B}_{\mathcal{S}}$, is proper in each component, and thus proper. It follows from Corollary A.2 that $\mathcal{B}_{\mathcal{S}}$ is a homeomorphism to its image $N := \mathcal{B}_{\mathcal{S}}([M : \mathcal{S}])$. With Proposition B.1 we see that N is a weak submanifold of $\prod_{j=0}^{k}[M : P_j]$, and that $\mathcal{B}_{\mathcal{S}}$ is a diffeomorphism onto N.

It remains to argue that N coincides with

$$\{M : \mathcal{S}\} \overset{\text{(def)}}{=} \overline{\mathcal{B}_{\mathcal{S}}\Big(M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q\Big)}\,.$$

For any $x \in [M : \mathcal{S}]$ there is a sequence (x_i) in $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$ converging to x in $[M : \mathcal{S}]$. Thus

$$\mathcal{B}_{\mathcal{S}}\Big(M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q\Big) \ni \mathcal{B}_{\mathcal{S}}(x_i) \to \mathcal{B}_{\mathcal{S}}(x)\,,$$

thus $\mathcal{B}_{\mathcal{S}}(x) \in \{M : \mathcal{S}\}$. It follows that $N \subset \{M : \mathcal{S}\}$.

Conversely, for $y \in \{M : \mathcal{S}\}$ there is a sequence $y_i = \mathcal{B}_{\mathcal{S}}(x_i)$ in $\mathcal{B}_{\mathcal{S}}\Big(M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q\Big)$ converging to y in $\prod_{j=0}^{k}[M : P_j]$. Thus $\{y_i \mid i \in \mathbb{N}\} \cup \{y\}$ is compact, and by properness of $\mathcal{B}_{\mathcal{S}}$ the set

$$\big(\mathcal{B}_{\mathcal{S}}\big)^{-1}\left(\{y_i \mid i \in \mathbb{N}\} \cup \{y\}\right) = \{x_i \mid i \in \mathbb{N}\} \cup \big(\mathcal{B}_{\mathcal{S}}\big)^{-1}\left(\{y\}\right)$$

is compact as well. As a consequence a subsequence x_{i_k} has to converge to some $z \in [M : \mathcal{S}]$. We conclude that

$$N \ni \mathcal{B}_{\mathcal{S}}(z) = \lim_{k\to\infty} \mathcal{B}_{\mathcal{S}}(x_{i_k}) = \lim_{k\to\infty} y_{i_k} = y.$$

This yields $\{M : \mathcal{S}\} \subset N$. □

Again, the image of the map $\mathcal{B}_{\mathcal{S}}$ is, in general, not a p-submanifold, see Remark 5.11.

Remark 5.13. Until now we assumed at several place that $\emptyset$ is in $\mathcal{S}$ and that the family $\mathcal{S}$ is obtained by ordering the *set* $\mathcal{S}$. The latter condition implies $P_i \neq P_j$ for $i \neq j$. However, the iterative construction of $[M : \mathcal{S}]$ is not altered, if several p-submanifolds will appear repeatedly in $\mathcal{S}$ or if $\emptyset$ is not in $\mathcal{S}$, and the proof of Theorem 5.12 remains valid. Thus if $\mathcal{S}'$ is obtained from $\mathcal{S}$ by adding or removing $\emptyset$, possibly repeating some p-submanifolds or erasing repeated submanifolds, then Theorem 5.12 and its proof tell us that $\{M : \mathcal{S}\}$ is diffeomorphic to $\{M : \mathcal{S}\}$, although both spaces are weak submanifolds of other product spaces.

We obtain the following corollary of Theorem 5.12.

Corollary 5.14. *Let $\mathcal{S}$ be a clean semilattice of closed p-submanifolds of M. If G is a discrete group acting smoothly on M such that $g(\mathcal{S}) = \mathcal{S}$ for $g \in G$, then G acts smoothly on $[M : \mathcal{S}]$ and the action commutes with the above homeomorphism $\mathcal{B}_{\mathcal{S}}$.*

Proof. Let δ be the diagonal embedding $\delta(x) = (x, x, \ldots, x)$ considered before. Theorem 5.12 gives that $\mathcal{B}_S = \delta$ on the dense open subset $M \smallsetminus \bigcup_{Q \in \mathcal{S}} Q$. Hence the image of the map $\mathcal{B}_{\mathcal{S}}$ is contained in the graph blow-up $\{M : \mathcal{S}\}$, by the definition of the later. We know that $\mathcal{B}_{\mathcal{S}}$ is continuous and proper, and hence with closed image. This gives that $\mathcal{B}_{\mathcal{S}}([M : \mathcal{S}]) = \{M : \mathcal{S}\}$. □

Remark 5.15. Note that Georgescu's compactification, see Definition 6.6, does not come equipped naturally with a smooth structure, so it does not make sense to ask whether this homeomorphism is a diffeomorphism.

6. Identification of the Georgescu-Vasy space

In this section we apply the results of the previous sections to identify the spaces introduced by Georgescu and Vasy with the space $X_{GV} := \overline{\delta(X)}$ defined in the Introduction. In what follows, the role played by M in the previous sections will be played by the spherical compactification $\overline{Z}$ of a vector space Z, which we recall next.

6.1. Spherical compactifications.

For any finite dimensional real vector space Z, recall that $\mathbb{S}_Z$ denotes the set of vector directions in Z, that is, the set of (non-constant) open half-lines $\mathbb{R}_+ v$, with $0 \neq v \in Z$ and $\mathbb{R}_+ := (0, \infty)$. The disjoint union

$$\overline{Z} \ := \ Z \sqcup \mathbb{S}_Z \tag{34}$$

is then called the *radial compactification* of Z. For example, if $Z = \mathbb{R}$, then $\overline{Z} := [-\infty, \infty]$ with the usual topology. The action of the group $\mathrm{GL}(Z)$ of linear automorphisms of Z extends, by definition, to an action on $\overline{Z}$. Similarly, if $Y \subset Z$, then $\overline{Y} \subset \overline{Z}$. In particular, $\overline{Z}$ is the union of all *closed lines* $\overline{\mathbb{R}v}$, $0 \neq v \in Z$, with closure taken in $\overline{Z}$.

As is well known, $\overline{Z}$ carries a topology and a smooth structure, and our next goal is to recall their definitions that will turn $\overline{Z}$ into a smooth manifold with boundary. For notational purposes it is convenient consider the case $Z = \mathbb{R}^n$ first. We start by noticing that there is a bijection between the set of vector directions in $\mathbb{R}^{n+1}$ and its unit sphere $\mathbb{S}^n$. This allows us to regard $\mathbb{S}^n_1 := \{(x_1, x') \in [0, \infty) \times \mathbb{R}^n \mid x_1^2 + |x'|^2 = 1\}$ as the set of vector directions in $\mathbb{R}^{n+1}_1$, where we used the usual notation of Equation (13). Let

$$\langle x \rangle^2 \ := \ 1 + \|x\|^2 \ = \ \|(1, x)\|^2 \,,$$

as usual. We then have the following simple observation.

Remark 6.1. Let $\Theta_n : \overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup \mathbb{S}_{\mathbb{R}^n} \to \mathbb{S}^n_1$ be given by the formula:

$$\Theta_n(x) \ := \ \begin{cases} \frac{1}{\langle x \rangle}(1,x) & \text{if } x \in \mathbb{R}^n\,, \\ \frac{1}{\|v\|}(0,v) & \text{if } x = \mathbb{R}_+ v \in \mathbb{S}_{\mathbb{R}^n}\,. \end{cases}$$

First, the map Θ_n is well defined because $\mathbb{R}_+ v = \mathbb{R}_+ w$ implies $v = \lambda w$, for some $\lambda \in \mathbb{R}_+$. Second, Θ_n is $\mathrm{GL}(\mathbb{R}^n)$-invariant for the action defined in the last paragraph. Finally, it is bijective and its inverse is given by

$$\mathbb{S}^n_1 \ni (y_0, y_1, \ldots, y_n) \xmapsto{\Theta_n^{-1}} \begin{cases} \frac{1}{y_0}(y_1,\ldots,y_n) \in \mathbb{R}^n & \text{if } y_0 \neq 0 \\ \mathbb{R}_+(y_1,\ldots,y_n) \in \mathbb{S}_{\mathbb{R}^n} & \text{if } y_0 = 0\,. \end{cases} \tag{35}$$

We endow $\overline{\mathbb{R}^n}$ with the structure of a smooth manifold (with boundary) that makes Θ_n a diffeomorphism. This manifold structure on $\overline{\mathbb{R}^n}$ extends the standard manifold structure of $\mathbb{R}^n$. See also [45, 60].

We now extend the definition of the smooth structure on $\overline{\mathbb{R}^n}$ of Remark 6.1 to any n-dimensional real vector space Z in the usual way. First, choose a vector space isomorphism $Z \to \mathbb{R}^n$, which yields bijections

$$\overline{Z} \ \xrightarrow{\sim}\ \overline{\mathbb{R}^n} \ \xrightarrow{\sim}\ \mathbb{S}^n_1\,. \tag{36}$$

In turn, these bijections can be used to define a smooth structure on the radial compactification $\overline{Z}$ of Z. The $\mathrm{GL}(\mathbb{R}^n)$-invariance of Θ_n implies that the resulting smooth structure on $\overline{Z}$ does not depend on the isomorphism $Z \to \mathbb{R}^n$.

6.2. **Quotients and compactifications.** It follows from the definition of the radial compactification and of its topology that if $Y \subset Z$ is a (linear) subspace, then $\overline{Y} \subset \overline{Z}$ is a p-submanifold and $\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y}$.

If Y is a proper linear subspace of X, then the natural projection map $\pi_{X/Y} : X \to X/Y \to \overline{X/Y}$ extends to a well-defined map $\overline{X} \smallsetminus \mathbb{S}_Y \to \overline{X/Y}$, which, at the boundary, is given by $\mathbb{R}_+ x \mapsto \mathbb{R}_+(x+Y)$. This map does not extend to a continuous map on $\overline{X}$, but, as we will show next, it extends to the blow-up of $\overline{X}$ with respect to $\mathbb{S}_Y$.

Proposition 6.2. *The canonical surjection $\pi_{X/Y} : X \to X/Y$ extends to a smooth map $\psi_Y : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ such that the induced map*

$$\theta_Y \ := \ (\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$$

is a diffeomorphism onto its image, and this image is a weak submanifold of the product $\overline{X} \times \overline{X/Y}$. Let $G = \mathrm{GL}(X,Y) \subset \mathrm{GL}(X)$ be the group of automorphisms of X that map Y to itself. Then ψ_Y is G-equivariant.

Again, one can check, that the image θ_Y is not a submanifold in the sense of Definition 2.8, i.e. not a submanifold in Melrose's sense, it is only a submanifold in our weaker sense.

Proof. In view of the equivariance of Θ_n and of the bijections in (36), we can assume $X = \mathbb{R}^n$ and $Y = \{0\} \times \mathbb{R}^q$. We will write $\mathbb{S}^{q-1}$ and $\mathbb{R}^q$ instead of $\{0\} \times \mathbb{S}^{q-1}$ and $\{0\} \times \mathbb{R}^q$, for simplicity. Recall that Lemma 5.9 yields a diffeomorphism $\widetilde{\Psi} : [\mathbb{S}^{r,r'}_{k,k'} : \{0\} \times \mathbb{S}^{r'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{r-1}_k \times \mathbb{S}^{r'+1}_{k'+1}$. We shall use this result for $r = n-q+1$, $r' = q-1$, $k = 1$, and $k' = 0$. Since $\mathbb{S}^{q-1}_0 = \mathbb{S}^{q-1}$ and $\mathbb{S}^{n-q+1,q-1}_{1,0} = \mathbb{S}^n_1$, we obtain the diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^n_1 : \mathbb{S}^{q-1}] \ \xrightarrow{\sim}\ \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1\,.$$

Let $p_1 : \mathbb{S}_1^{n-q} \times \mathbb{S}_1^q \to \mathbb{S}_1^{n-q}$ be the projection onto the first component.

By definition of the smooth structure on $\overline{X}$, the map $\Theta_n : \overline{X} \to \mathbb{S}_1^n = \mathbb{S}_{1,0}^{n-q+1,q-1}$ of Remark 6.1 is a diffeomorphism, and it maps diffeomorphically $\mathbb{S}_Y$ onto $\mathbb{S}^{q-1}$. Then by Lemma 3.2 we obtain a diffeomorphism $\Theta_n^\beta : [\overline{X} : \mathbb{S}_Y] \to [\mathbb{S}_1^n : \mathbb{S}^{q-1}]$.

We define ψ_Y as the composition

$$[\overline{X} : \mathbb{S}_Y] \xrightarrow{\Theta_n^\beta} [\mathbb{S}_1^n : \mathbb{S}^{q-1}] \xrightarrow{\widetilde{\Psi}} \mathbb{S}_1^{n-q} \times \mathbb{S}_1^q \xrightarrow{p_1} \mathbb{S}_1^{n-q} \xleftarrow{\Theta_{n-q}} \overline{X/Y},$$

in other words

$$\psi_Y \ := \ \big(\Theta_{n-q}\big)^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta_n^\beta \ : \ [\overline{X} : \mathbb{S}_Y] \ \to \ \overline{X/Y},$$

and we claim that ψ_Y is the desired extension.

To prove the claim, recall that we defined $\widetilde{\Psi}$ in Lemma 5.9 as the unique continuous extension of the map

$$\Psi : \mathbb{S}_1^n \smallsetminus \mathbb{S}^{q-1} \ \to \ \mathbb{S}_1^{n-q} \times \mathbb{S}_1^q, \quad (\eta, \mu) \ \mapsto \ \Big(\frac{\eta}{|\eta|}, \, \big(|\eta|, \mu\big) \Big),$$

where $\eta \in \mathbb{R}_1^{n-q+1}$ and $\mu \in \mathbb{R}^q$. We write $v \in X = Y^\perp \oplus Y$ as $v = (v_\perp, v_Y)$, i.e. $v_Y \in Y$ and $v_\perp \perp Y$ which means $v_\perp \in Y^\perp = \mathbb{R}^{n-q} \times \{0\}$. Then in the case $v_\perp \neq 0$ we have $\Theta_n(v) = \frac{1}{\langle v \rangle}(1, v) \in \mathbb{S}_1^n \smallsetminus \mathbb{S}^{q-1}$, and in this case we then calculate

$$\widetilde{\Psi} \circ \Theta_n(v) \ = \ \Psi\Big(\frac{1}{\langle v \rangle}(1, v) \Big) \ = \ \Big(\frac{1}{\langle v_{Y^\perp} \rangle} \, (1, v_{Y^\perp}), \ \frac{1}{\langle v \rangle} \, (\langle v_{Y^\perp} \rangle, v_Y) \Big) \, .$$

By continuity of the extension, this formula even holds for all $v \in [\mathbb{S}_1^n : \mathbb{S}^{q-1}]$. By formula (35) we have $\big(\Theta_{n-q}\big)^{-1}(y_0, y_1, \ldots, y_{n-q}) = \frac{1}{y_0}(y_1, \ldots, y_{n-q})$, if $y_0 > 0$. This formula will be used in the following straightforward calculation:

$$\Theta_{n-q}^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta_n(v) = \Theta_{n-q}^{-1}\Big(\frac{1}{\langle v_{Y^\perp} \rangle}(1, v_{Y^\perp}) \Big) = v_{Y^\perp} = \pi_{X/Y}(v) \, .$$

So ψ_Y is indeed the desired extension of $\pi_{X/Y}$.

In the remaining part of the proof, we will use Proposition B.1 to show that $\theta_Y = (\beta_{\overline{X}, \mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$ is a diffeomorphism on its image, and that the image of this map is a weak submanifold of the product $\overline{X} \times \overline{X/Y}$. One of the conditions required by this proposition, is that θ_Y is an injective immersion, which we will check now.

The restriction of the map $\beta_{\overline{X}, \mathbb{S}_Y} : [\overline{X} : \mathbb{S}_Y] \to \overline{X}$ to $\overline{X} \smallsetminus \mathbb{S}_Y$ is a diffeomorphism onto its image, by the definition of the blow-up, and thus, $\theta_Y|_{\overline{X} \smallsetminus \mathbb{S}_Y}$ is an injective immersion as well. The complement of $\overline{X} \smallsetminus \mathbb{S}_Y$ in $[\overline{X} : \mathbb{S}_Y]$ is $\beta_{\overline{X}, \mathbb{S}_Y}^{-1}(\mathbb{S}_Y) := \mathbb{S}N_+^{\overline{X}} \mathbb{S}_Y \simeq \mathbb{S}_Y \times \overline{X/Y}$. On this set the map θ_Y becomes the inclusion map

$$\mathbb{S}_Y \times \overline{X/Y} \to \overline{X} \times \overline{X/Y} \, ,$$

and obviously this is smooth as well. As θ_Y maps $\overline{X} \smallsetminus \mathbb{S}_Y$ and $\mathbb{S}_Y \times \overline{X/Y}$ to disjoint sets, the injectivity of θ_Y follows.

It is not hard to check that the differential of the map θ_Y is also injective at the boundary points. Thus, θ_Y is an injective immersion.

Furthermore, θ_Y is defined on a compact set, and thus it is a homeomorphism onto its image. Using Proposition B.1, we see that its image, $\theta_Y([\overline{X} : \mathbb{S}_Y])$, is a weak submanifold of $\overline{X} \times \overline{X/Y}$, and the diffeomorphism property follows as well. This completes the proof. □

Remark 6.3. Let $\psi := p_1 \circ \widetilde{\Psi}$, using the notation of the proof of the last proposition. We thus have a commutative diagram

$$\begin{array}{ccc} [\overline{X} : \mathbb{S}_Y] & \xrightarrow{\psi_Y} & \overline{X/Y} \\ \Theta_n^\beta \big\downarrow & & \big\uparrow \Theta_{n-q}^{-1} \\ [\mathbb{S}_1^{n-1} : \mathbb{S}^{q-1}] & \xrightarrow{\psi} & \mathbb{S}_1^{n-q} \end{array} \tag{37}$$

The map ψ_Y was also considered in [26, 37].

6.3. **Georgescu's constructions using C^*-algebras.** One of the main motivations of our work was to prove that a compactification of $\mathbb{R}^{3N}$ introduced by Georgescu is naturally homeomorphic to a compactification introduced by Vasy in order to better understand and use their construction(s).

Georgescu's construction is that of a spectrum of a commutative C^*-algebra [23, 25, 26], whereas Vasy used blow-ups [60, 61]. Georgescu's construction provides a topological space, whereas Vasy's construction defines a smooth manifold with corners. Thus a homeomorphism of these spaces that extends the identity of $\mathbb{R}^{3N}$ is the best that we can hope for. In turn, however, this homeomorphism will then equip Georgescu's compactification with the structure of a smooth manifold with corners. To compare their approaches, we need to recall a few facts about commutative C^*-algebras. We refer to [20, 56] for basic facts about C^*-algebras. Let us mention, however, that a C^*-algebra is an algebra isomorphic to a closed, self-adjoint algebra of bounded operators on a Hilbert space.

Definition 6.4. *A* C^*-algebra *A is an algebra over $\mathbb{C}$ with a norm $\|.\|$ and with a map $* : A \to A$ such that A is a Banach algebra and for every $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$, we have*

(i) $(a^*)^* = a$,
(ii) $(ab)^* = b^* a^*$,
(iii) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$,
(iv) $\|aa^*\| = \|a\|^2$.

The C^*-algebra is commutative *if $ab = ba$ for all $a, b \in A$.*

The next example provides, up to isomorphism, all examples of commutative and unital C^*-algebras.

Example 6.5. For a compact and Hausdorff topological space Z let $\mathcal{C}(Z)$ be the algebra of complex-valued continuous function on Z. We endow $\mathcal{C}(Z)$ with the involution $f^* = \overline{f}$ (the complex conjugation) and with the norm $\|f\|_\infty = \sup_{z \in Z} |f(z)|$. With this structure, $\mathcal{C}(Z)$ is a commutative, unital C^*-algebra.

Give two vector spaces X and $Y \subset X$, the composition

$$X \xrightarrow{\pi_{X/Y}} X/Y \xrightarrow{\text{incl}} \overline{X/Y}$$

induces by pullback an injective map $\mathcal{C}(\overline{X/Y}) \xrightarrow{\pi_{X/Y}^*} \mathcal{C}_b(X)$, where $\mathcal{C}_b(X)$ is the C^*-algebra of continuous and bounded complex-valued functions on X, again equipped with the supremum norm. Let also $\mathcal{F}$ be a finite semilattice of linear subspaces of X, $X \notin \mathcal{F}$, $\{0\} \in \mathcal{F}$.

As in the Introduction, Equation (12), let $\mathcal{E}_\mathcal{F}(X)$ be the norm closed sub-algebra of $\mathcal{C}_b(X)$ generated by the pullbacks of the spaces $\mathcal{C}(\overline{X/Y})$, where Y runs over $\mathcal{F}$. Then $\mathcal{E}_\mathcal{F}(X)$ is stable for complex conjugation, and hence a C^*-algebra.

Let $\mathcal{A}$ be a C^*-algebra, we recall that an ideal of $\mathcal{A}$ is *primitive* if it is the kernel of an irreducible representation of $\mathcal{A}$. If $\mathcal{A}$ is a commutative C^*-algebra, we define its spectrum $\mathrm{Spec}(\mathcal{A})$ as the set of primitive ideals of $\mathcal{A}$. In this case, the set of primitives ideals and the set of maximal ideals coincide. A *character* of a C^*-algebra is a non-zero $*$-morphism $\mathcal{A} \to \mathbb{C}$. In the commutative case, there is a one-to-one correspondence between the set of characters $\chi : \mathcal{A} \to \mathbb{C}$ of A and the set of maximal ideals of A. This correspondence is given by $\chi \mapsto \ker(\chi)$. The w^* topology on the space of characters defines a locally compact topology on $\mathrm{Spec}(\mathcal{A})$ and then yields an isomorphism $\mathcal{A} \to \mathcal{C}_0(\mathrm{Spec}(\mathcal{A}))$. Similarly it yields for a locally compact Hausdorff space M a homeomorphism $M \to \mathrm{Spec}(\mathcal{C}_0(M))$, sending $x \in M$ to the maximal ideal $\{f : M \to \mathbb{C} \mid f(x) = 0\}$, or equivalently, to the character e_x, where $e_x(f) := f(x)$. If $\mathcal{A}$ is a commutative C^*-algebra, its spectrum $\mathrm{Spec}(\mathcal{A})$ is canonically in bijection with the set of non-zero algebra morphisms $\mathcal{A} \to \mathbb{C}$.

Definition 6.6. *Let $\mathcal{F}$ be a finite semilattice of linear subspaces of the finite dimensional vector space X as above. Then the spectrum* $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ *of the algebra introduced in Equation* (12) *is called* Georgescu's compactification of X with respect to $\mathcal{F}$.

In [50, Theorem 4.4], two of the authors of this paper, together with N. Prudhon, have proved the following result.

Proposition 6.7. *The spectrum* $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ *of* $\mathcal{E}_{\mathcal{F}}(X)$ *is homeomorphic to the closure of the image of X in the product* $\prod_{Y\in\mathcal{F}} \overline{X/Y}$ *under the "diagonal" map* $\delta : X \to \prod_{Y\in\mathcal{F}} \overline{X/Y}$, $\delta(x) := (\pi_Y(x))_{Y\in\mathcal{F}}$. *More precisely, the homeomorphism* $\Phi : \overline{\delta(X)} \to \mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ *is given as follows. Let* $z = (z_Y)_{Y\in\mathcal{F}}$ *be in the closure of* $\delta(X)$. *Then the homeomorphism* Φ *sends* z *to the character* χ_z *defined by* $\chi_z(f_Y) = f_Y(z_Y)$ *whenever* $f_Y \in \mathcal{C}(\overline{X/Y})$.

This result (Proposition 6.7) thus identifies the spectrum of $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ (Georgescu's space) with the space $\overline{\delta(X)}$ introduced in [50].

6.4. **Identification of the Georgescu and Vasy spaces.** Let us consider the semilattices

$$\mathbb{S}_{\mathcal{F}} := \{\, \mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F} \,\} \text{ and } \overline{\mathcal{F}} := \{\, \overline{Y} \mid Y \in \mathcal{F} \,\}. \tag{38}$$

Then $\emptyset \in \mathbb{S}_{\mathcal{F}}$, as it corresponds to the subspace $\{0\} \subset X$ that was assumed to be in $\mathcal{F}$.

We now apply Theorem 5.12 to the semilattice $\mathbb{S}_{\mathcal{F}}$ of p-submanifolds of M, then the pairs $(\mathcal{S}, P_j)$ of that Theorem need to be replaced with pairs of the form $(\mathbb{S}_{\mathcal{F}}, \mathbb{S}_Y)$. Then the component corresponding to $\mathbb{S}_Y$ of the map $\mathcal{B}_{\mathcal{S}}$ in Theorem 5.12 is the map $\phi_{\mathbb{S}_{\mathcal{F}},\mathbb{S}_Y} : [\overline{X} : \mathbb{S}_{\mathcal{F}}] \to [\overline{X} : \mathbb{S}_Y]$. We then compose this map with the map $\psi_Y : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ defined in the proof of Proposition 6.2 to obtain the following result.

Proposition 6.8. *The product map*

$$\Xi_{\mathcal{F}} : [\overline{X} : \mathbb{S}_{\mathcal{F}}] \to \prod_{Y\in\mathcal{F}} \overline{X/Y} \tag{39}$$

of the composite maps $\psi_Y \circ \phi_{\mathbb{S}_{\mathcal{F}},\mathbb{S}_Y} : [\overline{X} : \mathbb{S}_{\mathcal{F}}] \to [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ *is a diffeomorphism onto its image, which is a weak submanifold of the product. (For* $Y = 0$, *the map* $\psi_Y \circ \phi_{\mathbb{S}_{\mathcal{F}},\mathbb{S}_Y} = \psi_Y \circ \phi_{\mathbb{S}_{\mathcal{F}},\emptyset}$ *is simply the blow-down map* $[\overline{X} : \mathbb{S}_{\mathcal{F}}] \to \overline{X}$.*) Let G be a discrete group of linear automorphisms of X that map elements of $\mathcal{F}$ to elements of $\mathcal{F}$ (thus* $g(\mathbb{S}_{\mathcal{F}}) = \mathbb{S}_{\mathcal{F}}$ *for all* $g \in G$*). Then G acts on* $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ *and the map* $\Xi_{\mathbb{S}_{\mathcal{F}}}$ *is G-equivariant.*

Proof. If we replace in the definition of $\Xi_{\mathcal{S}}$ the maps ψ_Y with the maps $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) \circ \phi_{\mathcal{S},\mathbb{S}_Y}$ of Proposition 6.2, then the resulting map $\Xi'_{\mathcal{S}} = \big((\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) \circ \phi_{\mathcal{S},\mathbb{S}_Y}\big)_{Y\in\mathcal{F}}$ is a diffeomorphism by Proposition 6.2 together with Theorem 5.12. The map $\Xi'_{\mathcal{S}}$ differs from $\Xi_{\mathcal{S}}$ by simply repeating several times the factors $\beta : [\overline{X} : \mathcal{S}] \to \overline{X}$. The corollary is obtained by keeping only one of these repeated factors, which still insures that the resulting map is a diffeomorphism onto its image. The action of G and the fact that $\Xi_{\mathcal{S}}$ is G-equivariant follow from the definition of $\Xi_{\mathcal{S}}$ and from Theorem 5.12. □

This result (Proposition 6.8) thus identifies the blown-up space $[X : \mathbb{S}_{\mathcal{F}}]$ (Vasy's space) with the space $\overline{\delta(X)}$ introduced in [50]. Combining Propositions 6.7 and 6.8, we obtain the following result.

Theorem 6.9. *Let $\mathcal{F}$ be a finite semilattice of subspaces of X containing $\{0\}$ and $\mathbb{S}_{\mathcal{F}} := \{\mathbb{S}_Y \mid Y \in \mathcal{F}\}$ be as in Equation* (38). *There exists a unique homeomorphism*

$$\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X)) \simeq [\overline{X} : \mathbb{S}_{\mathcal{F}}]$$

that is the identity on X.

Proof. Let $\delta : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$ be the diagonal map. Proposition 6.7 states that we have a homeomorphism $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X)) \to \overline{\delta(X)}$. Proposition 6.8 states that the map $\Xi_{\mathbb{S}_{\mathcal{F}}}$ defined on $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ is a diffeomorphism onto its image. Since $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ is compact, the image is closed. It moreover contains $\delta(X)$ as a dense open subset. Therefore $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ is also homeomorphic to $\overline{\delta(X)}$. □

To conclude, the above result shows that the following spaces: $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ (Georgescu's space), $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ (Vasy's space), and $\overline{\delta(X)}$ (introduced in [50]) are all homeomorphic. Any of these spaces will be denoted X_{GV} and called the *Georgescu-Vasy space*.

We obtain as a corollary the following description for the space introduced in [25, 26] (the "small Georgescu space").

Remark 6.10. In [25, 26], Georgescu and his collaborators have considered the norm closed sub-algebra of functions $\mathfrak{A}_{\mathcal{F}}$ of $L^\infty(X)$ generated by all the algebras $\mathcal{C}_0(X/Y)$ with $Y \in \mathcal{F}$. This corresponds to potentials that have zero limit at infinity on X/Y. The spectrum of this algebra (after adjoining a unit) identifies with the closure of the image of the diagonal map of X to $\prod_{Y \in \mathcal{S}} (X/Y)^+$, where Z^+ denotes the one point compactification of a locally compact space Z. Since $\mathfrak{A}_{\mathcal{F}} \subset \mathcal{E}_{\mathcal{F}}(X)$, we obtain that $\mathrm{Spec}(\mathfrak{A}_{\mathcal{F}})$ is a quotient of $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$, and hence also a quotient of

$$X_{GV} := [\overline{X} : \mathbb{S}_{\mathcal{F}}],$$

by Theorem 6.9. Generally, the topology on $\mathrm{Spec}\,(\mathfrak{A}_{\mathcal{F}})$ is rather complicated and singular, see also [44, Section 5] for concrete examples when $\dim(X) = 2$.

7. Applications to the N-body problem

Our hope is that by identifying the spaces appearing in Georgescu's and Vasy's constructions (Theorem 6.9), we can then combine the results and the techniques in their papers (see Georgescu's works [5, 13, 23, 25, 26] and Vasy's papers [60, 61] as well as in [17, 37, 38, 53, 54], and in the references therein) to obtain new results. In this spirit, in this section, we briefly discuss some applications of our results. We provide a brief, but complete account of these applications based on a complete set of references.

7.1. **The N-body semilattice and Pauli exclusion principle.** The setting considered in the previous sections of a semilattice $\mathcal{F}$ of linear subspaces of a vector space X is inspired from the N-body problem. In this subsection, we explain the concrete form of our setting in the case of the N-body problem and notice that it is compatible with symmetry and antisymmetry assumptions as, for instance, the Pauli exclusion principle. More precisely, the space M_N (the Georgescu-Vasy space associated to the semilattice of the effective N-body problem) carries a natural, concrete action of the symmetric group S_N (the permutation group on N letters). This subsection, while relevant on its own, also prepares for the applications in the following sections.

7.1.1. *The semilattice of the N-body problem.* Here is what the choices of X and $\mathcal{F}$ are for the N-body problem.

Example 7.1. In the concrete case of the effective N-body problem, we take $X := \mathbb{R}^{3N}$ and consider the subspaces

$$\begin{aligned} Y_j \ &:= \ \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_j = 0\} \quad \text{and} \\ Y_{ij} \ &:= \ \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_j = x_j\}\,, \ \ i \neq j\,. \end{aligned}$$

Thus each $x_i \in \mathbb{R}^3$. We let $\mathcal{F} := \mathcal{F}_N$, be the semilattice generated by the subspaces Y_i and Y_{ij}, $i, j \in \{1, 2, \ldots, N\}$ [5, 17].

In particular, our results give the following.

Remark 7.2. Let the vector space $X := \mathbb{R}^{3N}$ and the semilattice $\mathcal{F} := \mathcal{F}_N$ be as in the last example, Exemple 7.1. Let $\mathbb{S}_{\mathcal{F}_N} := \{\mathbb{S}_Y \mid Y \in \mathcal{F}_N\}$ be the finite semilattice of p-submanifolds of $\overline{X}$ as in Equation (38). Then our results, especially Theorem 6.9 and Proposition 6.8, imply that $M_N := [\overline{X} : \mathbb{S}_{\mathcal{F}_N}]$ (the Georgescu-Vasy space X_{GV} associated to the semilattice $\mathcal{F}_N$) will be endowed with natural, smooth actions of the following groups:

- of S_N, the symmetric group on N variables, by permutation;
- of $O(3)$ acting diagonally on the components of $X := \mathbb{R}^{3N}$; and
- of X, extending the action by translation on itself. (This is valid for all semilattices $\mathcal{F}$ of linear subspaces of X, not just for $\mathcal{F}_N$, yielding a smooth action of X on X_{GV}).

These actions are easy to obtain at the level of spectra of C^*-algebras or for the graph-family blow-up, but more difficult to obtain geometrically using iterated blow-ups. In particular in [48], the main problem is to to construct a compactification such these actions are smooth (and also enjoy some additional proprieties). Answers to this problem are provided in [37, 51]. The smoothness of these actions is based on Theorem 6.9, since the groups act smoothly on each $\overline{X/Y}$, $Y \in \mathcal{F}$.

See also [7, 8, 11, 24, 27, 33, 34] for physically relevant results that can point out to further extensions of our work, including to Quantum Field Theory on a curved space-time.

7.1.2. *Symmetry, antisymmetry and the Pauli exclusion principle.* As already remarked above the action of the symmetric group S_N on $M_N := [\overline{X} : \mathbb{S}_{\mathcal{F}_N}]$ is important for applications. Recall that in the motivational part of the Introduction we allowed mixed systems of particles. Some of them might be bosons, and the wave function will be symmetric under permuation of two variables corresponding to bosons of the same kind. Other particles will be fermions, e.g. electrons, then the wave function is antisymmetric under odd permutations of two variables corresponding to fermions of the same kind. This is commonly known as the *Pauli exclusion principle*. In total, we consider the subgroup $G \subset S_N$ of permutations of particles of the same kind, and we obtain a map $\chi : G \to \{-1, +1\}$ such that only functions with

$$f : \mathbb{R}^{3N} \to \mathbb{C}, \quad f(gx) = \chi(g) \cdot f(x)$$

are allowed as wave functions for physical reason, where G acts on $\mathbb{R}^{3N}$ by permutation of components. It is thus helpful that we have proven, see Corollary 5.14, that this G-actions extends to the compactification M_N.

In fact the situation gets slightly more complicated as some particles may have spin, or – in mathematical terms – they are vector valued. As an example which hopefully already indicates the general case, let us explain the case of N electrons. As electrons have spin $1/2$, we should enlarge the target of the wave function and discuss functions

$$\Psi : \mathbb{R}^{3N} \to (\mathbb{C}^2)^{\otimes N}\,.$$

Here the tensor product is the tensor product over $\mathbb{C}$ and the k-th factor models the spin of the k-th electron. Let S_N act on the target $(\mathbb{C}^2)^{\otimes N}$ by permutation of the factors, i.e. $g(v_1 \otimes \cdots \otimes v_N) = (v_{g(1)} \otimes \cdots \otimes v_{g(N)})$ and on $\mathbb{R}^{3N}$ be exchanging the component vectors, i.e. mapping $g(x_1, \ldots, x_N) = (x_{g(1)}, \ldots, x_{g(N)})$. The Pauli exclusion principle states that the physically allowed wave functions are described by functions satisfying

$$\Psi(g(x)) = \mathrm{sgn}(g) g(\Psi(x))\,.$$

Thus, the action of S_N, given by Corollary 5.14, on M_N given by Example 7.1 and Remark 7.2 is essential for our applications to the N-body problem in the next subsections.

7.2. **Vasy's pseudodifferential calculus and Georgescu's algebra.** We will now return to the more general setting of a general semilattice $\mathcal{F}$ of linear subspaces of a finite dimensional vector space X, using the notation of (11).

The action of X by translation on

$$X_{GV} \;:=\; [\overline{X} : \mathbb{S}_{\mathcal{F}}] \tag{40}$$

can be used to define Georgescu's algebra and (possibly) Vasy's pseudodifferential calculus, along the lines of Georgescu's method [23, 25] (see also [3]). Let us outline this construction and derive some consequences.

Let $\mathscr{S}(X)$ denote the Schwartz space of rapidly decreasing functions. Any $f \in \mathscr{S}(X)$ gives rise to a convolution operator $f(T) : L^2(X) \to L^2(X)$, $h \mapsto f * h$. In the notation $T : X \to \mathcal{L}(L^2(X))$, $q \mapsto T_q$ stands for the translation operator $T_q f(x) = f(x+q)$ as, for instance, in [25, 26]. In fact, much more general functions f can be allowed here, e.g. function whose Fourier transform is a classical symbol of pseudodifferential operators. Similarly, a function $g \in \mathcal{C}(X_{GV})$ gives rise to a multiplication operator M_g on $L^2(X)$. By results of Georgescu [25] (using also Theorem 6.9), the Georgescu's algebra $\mathcal{C}(X_{GV}) \rtimes X$ is the norm closure of the algebra generated by operators of the form $M_g f(T)$ acting on $L^2(X)$. So, if $\mathcal{L}(\mathcal{H})$ denotes the algebra of bounded operators on a Hilbert space $\mathcal{H}$, then $\mathcal{C}(X_{GV}) \rtimes X \subset \mathcal{L}(L^2(X))$. The resulting sub-algebra $\mathcal{C}(X_{GV}) \rtimes X$, called crossed product, is norm closed and closed under adjoints, hence is a C^*-algebra. See [25, 23] for the details on the link between the crossed product of such a crossed-product of commutative C^*-algebra by $\mathbb{R}^n$ and operators of the form $M_g f(T)$.

By requiring f to be such that its Fourier transform is a classical symbol of order m on X, we have that $P := M_g f(T)$ becomes a pseudodifferential operator. Classically then, its distribution kernel $k_K \in \mathcal{D}'$ is a classical conormal distribution in $I^m(X \times X; X)$, with X diagonally embedded in $X \times X$. (See [29] for the definition of (classical) conormal distributions.) The map $(x_1, x_2) \to x_1 - x_2$ extends then to a smooth map of pairs $(X \times X, X) \to (X_{GV} \times X; X_{GV})$, with the embedding $X_{GV} \simeq X_{GV} \times \{0\} \subset X_{GV} \times X$. This embedding sends k_P to $g \otimes f$, and hence k_P can be identified with a classical conormal distribution in $I^m(X_{GV} \times X; X_{GV})$ (this is a particular case of the construction in [3]). Let $I^m_c(X_{GV} \times X; X_{GV})$ be the set of such distributions with compact support. Then it follows that $I^\infty_c(X_{GV} \times X; X_{GV}) := \bigcup_{m \in \mathbb{Z}} I^m_c(X_{GV} \times X; X_{GV})$ is a filtered algebra which acts by convolution on functions $X_{GV} \to \mathbb{C}$ as an algebra of pseudodifferential operators [3]. (Recall that we are considering only classical conormal distributions and the index "c" comes from "compact support." Vasy's N-body calculus $\Psi^\infty_N(X)$ is certainly bigger and better than $I^\infty_c(X_{GV} \times X; X_{GV})$ in the sense that it contains the resolvents of its L^2-invertible operators. Let

$$I^{-\infty}_c(X_{GV} \times X; X_{GV}) \subset \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV}) \subset I^{-\infty}(X_{GV} \times X; X_{GV})$$

be the projective tensor product. We have strong reasons to believe that Vasy's N-body calculus is

$$\Psi^\infty_N(X) \;=\; \mathfrak{A}^\infty \;:=\; I^\infty(X_{GV} \times X; X_{GV}) + \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV}) \tag{41}$$

We need to include $\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV}) := \mathscr{S}(X; \mathcal{C}^\infty(X_{GV}))$ on the right hand side to accomodate operators of the form $M_g f(T)$ with $f \in \mathscr{S}(X)$ *with non-compact support,* since $M_g f(T) \in I^m_c(X_{GV} \times X; X_{GV})$ if, and only if, f is compactly supported (recall that $\hat{f}$ is a classical symbol of order m).

Proposition 7.3. *Let*

$$\mathfrak{A}^n \;:=\; I^n(X_{GV} \times X; X_{GV}) + \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV})\,.$$

The space $\mathfrak{A}^\infty := \bigcup_{n \in \mathbb{Z}} \mathfrak{A}^n$ is a filtered algebra that is closed under holomorphic functional calculus. Let D be a strongly elliptic differential operator of order $m > 0$, with constant coefficients and $v_Y \in \mathcal{C}^\infty(\overline{X/Y})$. Then $H'_N := D + \sum_{Y \in \mathcal{S}} v_Y \in \mathfrak{A}^m$. Consequently, for all $\lambda \notin \operatorname{Spec}(H'_N)$, we have

$$(H'_N - \lambda)^{-1} \in \mathfrak{A}^{-m} := I^{-m}_c(X_{GV} \times X; X_{GV}) + \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV})\,.$$

If $X := \mathbb{R}^{3N}$, the action of the symmetric group S_N on X induces an order-preserving automorphism of the algebra $\mathfrak{A}^\infty$.

Proof. (Sketch) There are two main things to prove here: first, that the convolution product makes $\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV}) := \mathscr{S}(X; \mathcal{C}^\infty(X_{GV}))$ an algebra and, second, that it is stable for holomorphic functional calculus (equivalently in this case, that the algebra with adjoint unit contatins the resolvents of its L^2-invertible elements). The first question is answered by noticing that the action of X on $\mathcal{C}^\infty(\overline{X/Y})$ is with polynomial growth (this is quite unusual for the action of X on a manifold!) and hence it is again with polynomial growth on X_{GV} in view of our Theorem 6.9. The second question is answered by using the results of [40] as follows. We consider three families of operators on $L^2(X_{GV} \times X)$, possible unbounded (so not defined everywhere). Let A_1 be the set of differential operators on X_{GV}, let A_2 the set of multiplication operators with polynomials on X and, finally, let A_3 be the set of constant coefficients differential operators on X. Then

$$\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV}) \;:=\; \{\, f \in \mathcal{C}(X_{GV}) \rtimes X \mid [[f, P_1], P_2]P_3 \text{ is bounded} \,\}\,.$$

The results of [40], especially Theorems 2 and 3, then give that $\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{GV})$ is spectrally invariant (*i. e.,* stable under holomorphic functional calculus). ☐

We ignore if one can replace H'_N with $H_N \notin \mathfrak{A}^\infty$ in the resolvent estimate of the last proposition, since the later allows Coulomb singularities in the potential. This brings us to one of the main reasons for considering Georgescu's algebras $\mathcal{E}_{\mathcal{F}}(X) \rtimes X$ instead of a pseudodifferential calculus (and one of the reasons why we may need to take norm closures) is that Georgescu's algebra does not suffer from this deficiency, and thus one has

$$(H_N - \lambda)^{-1} \in \mathcal{E}_{\mathcal{F}}(X) \rtimes X \, . \tag{42}$$

[5, 25, 23] (this is a consequence of Hardy's inequality and is explained also in [26]). Of course, the above proposition provides a much more precise result, when applicable, but is also much more difficult to prove than the relation of Equation (42). Let us notice, moreover, that $\mathcal{E}_{\mathcal{F}}(X) \rtimes X$ is the norm closure of $\mathfrak{A}^{-1}$ in $\mathcal{L}(L^2(X))$, the algebra of bounded operator on $L^2(X)$.

7.3. **The HVZ theorem.** The algebras considered in the previous subsection were introduced, in part, in order to obtain conceptual proofs and extensions of the classical HVZ theorem, named after Hunziker, van Winter, and Zhislin, describing the essential spectrum of N-body Hamiltonians H [17, 23, 18, 54, 58]. Is well-known that the operators H considered here are selfadjoint. One sees, that λ *is not* in the essential spectrum of H if, and only if, $H - \lambda$ is Fredholm. Our next application is of a conceptual nature on how to relate the HVZ theorem with other classical Fredholm results in PDE theory. There exist many refinements of the HVZ theorem, in the simple context of an atom with N-electrons with refer to [58, Section 11, Theorem 11.2]; for a more general version see [57, Theorem XIII.12]. The HVZ theorem determines the essential spectrum of the Hamiltonian H_N in terms of other, simpler Hamiltonians H_{N_α}, where α ranges over a certain index set. The operators H_{N_α} are usually called "limit operators," and can indeed be obtained as strong limits of translations of H_N. Very powerful generalizations of the HVZ theorem were obtained by Georgescu using C^*-algebras [5, 23] and others [26] – more on this below.

Nowadays, there are many results telling us when (pseudo)differential operators on non-compact or singular spaces are Fredholm, and typically they are also in terms of certain "generalized limit operators", the terminologies "indicial operator" or "normal symbol" are also used by Melrose and Schulze independently. We refer to [41] for a overview and comparison between these two approach. Results of this type go back at least to Kondratiev's 1967 celebrated paper [35]. Some of the strongest current results are based on groupoids, see [9, 10, 52] and the references therein. We also refer [12, 54] for the case that the groupoid is obtained from the action of a group on a space, as it is our case in this paper. The results are in terms of orbits, their isotropies, and the induced operators. In fact, each of these induced operators, referred to as "a generalized limit operator" above, acts on the product of the corresponding orbit with the corresponding isotropy group and is invariant with respect to that group.

A natural question is to reconcile the classical results on H_N using limit operators with the classical PDEs results based on "generalized limit operators". This is almost done by [59], except that it is not clear whether the resolvents of H_N belong to Vasy's pseudodifferential calculus. (We do know, however, that the resolvents of H_N belong to the norm closure of the pseudodifferential calculus introduced in the previous subsection, as discussed in the previous subsection. Note that here we are using our Theorem 6.9.)

The Fredholm results just mentioned, do apply, however, also to the norm closure of the corresponding pseudodifferential calculi, and hence, in principle, the HVZ theorem could then be obtained from the structure of the orbits of the action of X on $X_{GV} := [\overline{X} : \mathcal{S}]$ and their isotropies (both the orbits and the isotropies are linear subspaces of X) and the explicit form of the generalized limit operators. However, in order not to increase too much the length of this paper, we leave this for a future publication. Nevertheless, it is interesting to point out that that this approach has the potential to provide Fredholm conditions for the restrictions of H_N and its variants to the isotypical components of the action of S_N or some subgroup of S_N. Results in this direction (for operators on compact manifolds) were recently obtained in [6] and the references therein. See also [12, 14, 26, 49, 50, 52] for related results.

7.4. **A regularity result for bound states.** An application of our results to regularity for bound states for Schrödinger operators with inverse square potentials is contained in our recent preprint [4]. Here we just quickly explain the result. Let $X_{\mathcal{F}} := [X_{GV} : \overline{\mathcal{F}}] = [\overline{X} : \mathcal{S} \cup \overline{\mathcal{F}}]$. Let, for each $Y \in \mathcal{F}$, $a_Y \in \mathcal{C}^\infty(X_{\mathcal{F}})$ and d_Y denote the distance to Y in some fixed euclidean metric on X. A function of the form

$$V(x) \ := \ \sum_{Y \in \mathcal{F}} a_Y(x) d_Y(x)^{-2}$$

will be called an *inverse square potential (associated to the semilattice $\mathcal{F}$).* Let

$$\rho(x) \ := \ \min \big\{ \operatorname{dist}(x, \bigcup \mathcal{F}), 1 \big\} \,. \tag{43}$$

The following result (which combines techniques of this paper with those in [1]) was proved in [4].

Theorem 7.4. *Let V be an inverse square potential associated to the semilattice $\mathcal{F}$ of linear subspaces of the euclidean space X, $\rho(x) := \min\{\operatorname{dist}(x, \cup\mathcal{F}), 1\}$, and assume $u \in L^2(X)$ is an eigenfunction of $\mathcal{H} := -\Delta + V$, then, for all multi-indices α, we have*

$$\rho^{|\alpha|} \partial^\alpha u \in L^2(X) \,.$$

The regularity of bound states and, in general, the geometry of the Georgescu–Vasy space $X_{GV} := [X : \mathbb{S}_{\mathcal{F}}]$ may be useful for approximation pourposes. In fact, another, related motivation of our work is the approximation of the isolated eigenfunctions of N-body Hamiltonians using the Finite Element Method. The role of the Georgescu–Vasy space X_{GV} here is to provide a good underlying support for the construction of the approximation spaces. This is very tentative yet, but see [21, 28, 30, 63] for some results in this direction, including more references.

APPENDIX A. PROPER MAPS

Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$.

Lemma A.1 (Generalizes [42, Prop 4.32])**.** *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. If f is proper, then f is closed.*

In [42, Prop 4.32] the lemma is stated with the additional requirement that X is locally compact. However in the proof the locally compactness of X is not needed. Furthermore we will apply the lemma only when X is locally compact. Thus we omit the proof.

Corollary A.2. *Let $f : X \to Y$ be a continuous injective map between two Hausdorff spaces with Y locally compact. If f is proper, then f is a homeomorphism onto its image.*

Proof. The map $f : X \to f(X)$ is bijective continuous and closed and thus a homeomorphism. □

We shall say that f is *locally proper* if, for every $y \in Y$, there exists an open neighborhood V_y of y in Y such that the map $f^{-1}(V_y) \to V_y$ induced by f is proper.

Lemma A.3. *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. Then f is proper if, and only if, it is locally proper.*

Proof. Clearly, every proper map is locally proper, by definition. Let us assume that f is locally proper and let $K \subset Y$ be a compact subset. For any $y \in K$ we choose the open neighborhood V_y as above (in the definition of a locally proper map). As Y is locally compact, there is an open neighborhood W_y of y in V_y such that its closure $\overline{W}_y$ in Y is a compact subset of V_y. The local properness of f together with the choice of V_y implies that $f^{-1}(\overline{W}_y \cap K)$ is compact. By the compactness of K we can choose $y_1, \dots, y_N$ such that K is covered by $\left(W_{y_j}\right)_{1 \le j \le N}$. Then $K = \bigcup_{j=1}^N \left(\overline{W}_{y_j} \cap K\right)$. Then

$$f^{-1}(K) = \bigcup_{j=1}^{N} f^{-1}(\overline{W}_{y_j} \cap K)$$

is also compact. This completes the proof. □

APPENDIX B. SUBMANIFOLD CRITERIA

Proposition B.1. *Let N and M be manifolds with corners. Let $f : N \to M$ be a smooth map which is an immersion and a homeomorphism onto its image. Then $f(N)$ is a weak submanifold of M in the sense of Definition 2.13, and f is a diffeomorphism from N to $f(N)$.*

Proof. Note that the homeomorphism property implies that we have a local statement, i.e. we can restrict to small neighborhoods in N and M to prove it. Without loss of generality we can assume by passing to a chart of M that we have $M = \mathbb{R}^n_\ell$, $n = \dim M$. We will show that $f(N)$ is a submanifold of $\mathbb{R}^n$. It is sufficient to do this on a neighborhood of $p \in N$ with $f(p) = 0$. By choosing an appropriate chart for N, we can also assume that $p = 0$ and that N is an open neighborhood of 0 in $\mathbb{R}^{n'}_{\ell'}$, where $n' := \dim N = \dim d_0 f(T_0 N)$.

We choose vectors $v_1, \dots, v_{n-n'}$ such that they form a basis of a complement of $d_0 f(T_0 N)$ in $T_0 M \cong \mathbb{R}^n$. We extend f to a smooth map

$$F : N \to \mathbb{R}^{n-n'} \to M, \quad F(q, t_1, \dots, t_{n-n'}) \mapsto f(q) + \sum_{j=1}^{n-n'} t_j v_j \,.$$

Obviously $d_{(p,0)} F$ is an invertible linear map, and F can be extended to a smooth map $\tilde{F} : B \to \mathbb{R}^n$, defined on an open ball B around 0 in $\mathbb{R}^n$. Thus $\tilde{F}$ is a diffeomorphism onto its image on a small neighborhood V of $(p, 0)$. By taking $\phi := \left(\tilde{F}|V\right)^{-1}$ as a chart for $\mathbb{R}^n$, the conditions in Definition 2.8 are satisfied for $k := 0$, $k' := \ell'$, $G = 1 \in \mathrm{GL}(n, \mathbb{R})$.

Thus $f(N)$ is a submanifold of $\mathbb{R}^n$ and thus a weak submanifold of M.

In the chart ϕ constructed this way, the map f is a linear injective map and thus a diffeomorphism onto its image. □

Corollary B.2. *Let N and M be manifolds with corners. Let $f : N \to M$ be a smooth map. If there is a smooth map $F : M \to N$ with $F \circ f = \mathrm{id}_N$, then $f(N)$ is a submanifold of M in the sense of Definition 2.13.*

Proof. The relation $\mathrm{id}_{T_x N} = d_{f(x)} F \circ d_x f$ implies that $d_x f : T_x N \to T_{f(x)} M$ is injective. As $F|_{f(N)}$ is continuous, f is a homeomorphism onto its image. □

REFERENCES

[1] B. Ammann, C. Carvalho, and V. Nistor. Regularity for eigenfunctions of Schrödinger operators. *Lett. Math. Phys.*, 101(1):49–84, 2012.

[2] B. Ammann, A. D. Ionescu, and V. Nistor. Sobolev spaces on Lie manifolds and regularity for polyhedral domains. *Doc. Math.*, 11:161–206, 2006.

[3] B. Ammann, R. Lauter, and V. Nistor. Pseudodifferential operators on manifolds with a Lie structure at infinity. *Ann. of Math. (2)*, 165(3):717–747, 2007.

[4] B. Ammann, J. Mougel, and V. Nistor. A regularity result for the bound states of N-body Schrödinger operators: Blow-ups and Lie manifolds. 2020. Arxiv preprint http://arxiv.org/abs/2012.13902.

[5] W. Amrein, A. Boutet de Monvel, and V. Georgescu. *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*. Modern Birkhäuser Classics. Birkhäuser/Springer, Basel, 1996. 2013, reprint of the 1996 edition.

[6] A. Baldare, R. Côme, and V. Nistor. Fredholm conditions for operators invariant with respect to compact lie group actions. ArXiv preprint 2012.03944, December, 2020.

[7] C. Bär. Green-hyperbolic operators on globally hyperbolic spacetimes. *Comm. Math. Phys.*, 333(3):1585–1615, 2015.

[8] C. Bär and N. Ginoux. Classical and quantum fields on Lorentzian manifolds. In *Global differential geometry*, volume 17 of *Springer Proc. Math.*, pages 359–400. Springer, Heidelberg, 2012.

[9] C. Carvalho, R. Côme, and Y. Qiao. Gluing action groupoids: Fredholm conditions and layer potentials. *Rev. Roumaine Math. Pures Appl.*, 64(2-3):113–156, 2019.

[10] C. Carvalho, V. Nistor, and Y. Qiao. Fredholm conditions on non-compact manifolds: theory and examples. In *Operator theory, operator algebras, and matrix theory*, volume 267 of *Oper. Theory Adv. Appl.*, pages 79–122. Birkhäuser/Springer, Cham, 2018.

[11] P. Chruściel and M. Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific J. Math.*, 212(2):231–264, 2003.

[12] R. Côme. The Fredholm Property for Groupoids is a Local Property. *Results Math.*, 74(4):Paper No. 160, 2019.

[13] M. Damak and V. Georgescu. Self-adjoint operators affiliated to C^*-algebras. *Rev. Math. Phys.*, 16(2):257–280, 2004.

[14] C. Debord, J.-M. Lescure, and F. Rochon. Pseudodifferential operators on manifolds with fibred corners. *Ann. Inst. Fourier (Grenoble)*, 65(4):1799–1880, 2015.

[15] J. Dereziński. Asymptotic completeness of long-range N-body quantum systems. *Ann. of Math. (2)*, 138(2):427–476, 1993.

[16] J. Dereziński, J. Faupin, Q. Nguyen, and S. Richard. On radial Schrödinger operators with a Coulomb potential: general boundary conditions. *Adv. Oper. Theory*, 5(3):1132–1192, 2020.

[17] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N-particle systems*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[18] J. Dereziński and C. Gérard. *Mathematics of quantization and quantum fields*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2013.

[19] J. Dereziński and M. Wrochna. Exactly solvable Schrödinger operators. *Ann. Henri Poincaré*, 12(2):397–418, 2011.

[20] J. Dixmier. *C^*-algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[21] H.-J. Flad, G. Harutyunyan, R. Schneider, and B.-W. Schulze. Explicit Green operators for quantum mechanical Hamiltonians. I. The hydrogen atom. *Manuscripta Math.*, 135(3-4):497–519, 2011.

[22] W. Fulton and R. MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[23] V. Georgescu. On the essential spectrum of elliptic differential operators. *J. Math. Anal. Appl.*, 468(2):839–864, 2018.

[24] V. Georgescu, C. Gérard, and D. Häfner. Resolvent and propagation estimates for Klein-Gordon equations with non-positive energy. *J. Spectr. Theory*, 5(1):113–192, 2015.

[25] V. Georgescu and A. Iftimovici. Localizations at infinity and essential spectrum of quantum Hamiltonians. I. General theory. *Rev. Math. Phys.*, 18(4):417–483, 2006.

[26] V. Georgescu and V. Nistor. On the essential spectrum of N-body Hamiltonians with asymptotically homogeneous interactions. *J. Operator Theory*, 77(2):333–376, 2017.

[27] C. Gérard. An introduction to quantum field theory on curved spacetimes. In *Asymptotic analysis in general relativity*, volume 443 of *London Math. Soc. Lecture Note Ser.*, pages 171–218. Cambridge Univ. Press, Cambridge, 2018.

[28] M. Griebel and J. Hamaekers. Sparse grids for the Schrödinger equation. *M2AN Math. Model. Numer. Anal.*, 41(2):215–247, 2007.

[29] L. Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[30] E. Hunsicker, Hengguang Li, V. Nistor, and V. Uski. Analysis of Schrödinger operators with inverse square potentials II: FEM and approximation of eigenfunctions in the periodic case. *Numer. Methods Partial Differential Equations*, 30(4):1130–1151, 2014.

[31] W. Hunziker and I. M. Sigal. The quantum N-body problem. *J. Math. Phys.*, 41(6):3448–3510, 2000.

[32] D. Joyce. A generalization of manifolds with corners. *Adv. Math.*, 299:760–862, 2016.

[33] W. Junker and E. Schrohe. Adiabatic vacuum states on general spacetime manifolds: definition, construction, and physical properties. *Ann. Henri Poincaré*, 3(6):1113–1181, 2002.

[34] S. Klainerman, I. Rodnianski, and J. Szeftel. The bounded L^2 curvature conjecture. *Invent. Math.*, 202(1):91–216, 2015.

[35] V. A. Kondrat′ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Transl. Moscow Math. Soc.*, 16:227–313, 1967.

[36] C. Kottke. Blow-up in Manifolds with Generalized Corners. *Int. Math. Res. Not. IMRN*, (8):2375–2415, 2018.

[37] C. Kottke. Functorial compactification of linear spaces. *Proc. Amer. Math. Soc.*, 147(9):4067–4081, 2019.

[38] C. Kottke and R. Melrose. Generalized blow-up of corners and fiber products. *Trans. Amer. Math. Soc.*, 367(1):651–705, 2015.

[39] A. König. Master thesis. University of Regensburg.

[40] R. Lauter, B. Monthubert, and V. Nistor. Spectral invariance for certain algebras of pseudodifferential operators. *J. Inst. Math. Jussieu*, 4(3):405–442, 2005.

[41] R. Lauter and J. Seiler. Pseudodifferential analysis on manifolds with boundary—a comparison of b-calculus and cone algebra. In *Approaches to singular analysis (Berlin, 1999)*, volume 125 of *Oper. Theory Adv. Appl.*, pages 131–166. Birkhäuser, Basel, 2001.

[42] J. Lee. *Introduction to topological manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, first edition, 2000.

[43] E. Lieb and R. Seiringer. *The stability of matter in quantum mechanics*. Cambridge University Press, Cambridge, 2010.

[44] A. Mageira. Some examples of graded C^*-algebras. *Math. Phys. Anal. Geom.*, 11(3-4):381–398, 2008.

[45] R. Melrose. Differential analysis on manifolds with coners. Book in preparation. Manuscript available at math.mit.edu/∼rbm/book.html.

[46] R. Melrose. Calculus of conormal distributions on manifolds with corners. *Internat. Math. Res. Notices*, (3):51–61, 1992.

[47] R. Melrose. Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. In *Spectral and scattering theory (Sanda, 1992)*, volume 161 of *Lecture Notes in Pure and Appl. Math.*, pages 85–130. Dekker, New York, 1994.

[48] R. Melrose and M. Singer. Scattering configuration spaces. arxiv: 0808.2022.

[49] J. Mougel. Essential spectrum, quasi-orbits and compactifications: application to the heisenberg group. *Rev. Roumaine Math. Pures Appl.*, 2019.

[50] J. Mougel, V. Nistor, and N. Prudhon. A refined HVZ-theorem for asymptotically homogeneous interactions and finitely many collision planes. *Rev. Roumaine Math. Pures Appl.*, 62(1):287–308, 2017.

[51] J. Mougel and N. Prudhon. Exhaustive families of representations of C^*-algebras associated with N-body Hamiltonians with asymptotically homogeneous interactions. *C. R. Math. Acad. Sci. Paris*, 357(2):200–204, 2019.

[52] M. Măntoiu and V. Nistor. Spectral theory in a twisted groupoid setting: spectral decompositions, localization and Fredholmness. *Münster J. Math.*, 13(1):145–196, 2020.

[53] M. Măntoiu, R. Purice, and S. Richard. Twisted crossed products and magnetic pseudodifferential operators. In *Advances in operator algebras and mathematical physics*, volume 5 of *Theta Ser. Adv. Math.*, pages 137–172. Theta, Bucharest, 2005.

[54] M. Măntoiu. Essential spectrum and Fredholm properties for operators on locally compact groups. *J. Operator Theory*, 77(2):481–501, 2017.

[55] V. Nistor. Desingularization of Lie groupoids and pseudodifferential operators on singular spaces. *Comm. Anal. Geom.*, 27(1):161–209, 2019.

[56] G. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press Inc., London, 1979.
[57] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press, New York, 1978.
[58] G. Teschl. *Mathematical methods in quantum mechanics*, volume 157 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2014. With applications to Schrödinger operators.
[59] A. Vasy. Asymptotic behavior of generalized eigenfunctions in N-body scattering. *J. Funct. Anal.*, 148(1):170–184, 1997.
[60] A. Vasy. Propagation of singularities in many-body scattering. *Ann. Sci. École Norm. Sup. (4)*, 34(3):313–402, 2001.
[61] A. Vasy. Geometry and analysis in many-body scattering. In *Inside out: inverse problems and applications*, volume 47 of *Math. Sci. Res. Inst. Publ.*, pages 333–379. Cambridge Univ. Press, Cambridge, 2003.
[62] H. Weyl. *The theory of groups and quantum mechanics*. Dover Publications, Inc., New York, 1950. Translated from the second (revised) German edition by H. P. Robertson, Reprint of the 1931 English translation.
[63] H. Yserentant. *Regularity and approximability of electronic wave functions*, volume 2000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.

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