A COMPARISON OF THE GEORGESCU AND VASY SPACES ASSOCIATED TO THE N-BODY PROBLEMS

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ABSTRACT. We show that the space introduced by Vasy in order to construct a pseudo-differential calculus adapted to the N-body problem can be obtained as the primitive ideal spectrum of one of the N-body algebras considered by Georgescu. In the process, we provide an alternative description of the iterated blow-up space of a manifold with corners with respect to a clean semilattice of adapted submanifolds (i.e. p-submanifolds). Since our constructions and proofs rely heavily on manifolds with corners and their submanifolds, we found it necessary to clarify the various notions of submanifolds of a manifold with corners.

CONTENTS

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INTRODUCTION

We show that the space introduced by Vasy in [21, 22] in relation to the N-body problem coincides with the primitive ideal spectrum of the algebras considered by Georgescu and others in [4, 6, 7, 18]. Here we use the variant introduced in [18]. We will briefly describe the construction of theses spaces here in the introduction, a more detailed account of these definitions will be provided in the main body of this article.

The space considered by Vasy was defined using blow-ups of manifolds with corners. Let throughout this paper M be a manifold with corners. A submanifold will be called *closed* if it is a closed subset of the larger manifold in the sense of a topological spaces. Recall that a *p-submanifold* $P \subset M$ is a closed submanifold of M that has a suitable tubular neighborhood in M (Definition 1.14), and hence one that has a tubular neighborhood $P \subset U_P \subset M$ in M.

For any p-submanifold $P \subset M$, recall that the *blow-up* $[M : P]$ *of* M *with respect to* P, is defined by replacing P with the set $N_+^M P$ of interior directions in the normal bundle $N^M P$ of P in M (see [8, 15], or Definiton 2.1). This construction has some special properties if $P = \emptyset$ or if P contains an open subset of M. For this reason, a manifold with one of these two properties will be called a *trivial submanifold* of M and sometimes will be excluded from our consideration.

Since Vasy's construction uses blow-ups, we begin this paper with their study. More precisely, we shall study and use the blow-up of a manifold with corners with respect to a *family* of p-submanifolds. If this family has clean intersections, the blow-up can be defined iteratively as in [1, 11, 15] and in other papers. Our method is based on an alternative definition of the blow-up with respect to a family of p-submanifolds. More precisely, let us consider a locally finite family $\mathcal{F}$ of *non-trivial* p-submanifolds of M and let $M_{\mathcal{F}} := M \smallsetminus \bigcup_{P\in\mathcal{F}} Y$ be the complement of all the submanifolds in $\mathcal{F}$. Then $M_{\mathcal{F}}$ is open and dense in M and is contained in each of the blow-up manifolds $[M : P]$, $P \in \mathcal{F}$. Then we define the *graph blow-up* $\{M : \mathcal{F}\}$ of M with respect to the family $\mathcal{F}$ as the closure

$$\{M : \mathcal{F}\} \;:=\; \overline{\delta(M_0)} \;\subset\; \prod_{P\in\mathcal{F}} [M : P]\,, \tag{1}$$

where δ is the diagonal embedding (see Definition 3.1 and the discussion following it for more details). For simplicity, in this paper, we shall consider the graph blow-up only with respect to a *locally finite* family of non-trivial p-submanifolds that are closed subsets of the ambient manifold.

In order to have a well-behaved graph blow-up, we shall impose some additional assumptions on $\mathcal{F}$. Recall that a family $\mathcal{S}$ of subsets of M is a *semilattice* (with respect to the inclusion) if, for all $P_1, P_2 \in \mathcal{S}$, we have $P_1 \cap P_2 \in \mathcal{S}$. Let then $\mathcal{S}$ be a semi-lattice of p-submanifolds of M and arrange its elements in an order $(\emptyset = P_0, P_1, P_2, \dots, P_n)$ that is assumed to be *compatible with the inclusion*, in the sense that

$$P_i \subset P_j \;\Rightarrow i \leq j\,. \tag{2}$$

We assume also that all non-empty members of $\mathcal{S}$ are non-trivial p-submanifold of M, that they are closed subsets of M, and that any finite subset of manifold in $\mathcal{S}$ intersect cleanly (in other words, we assume that $\mathcal{S}$ is a *clean semilattice*, see Definiton 4.3). Then we can successively blow-up M with respect to $(\emptyset = P_0, P_1, P_2, \dots, P_n)$ by first doing so with respect to P_1, then doing so with respect to the lift of P_2, and so on, to obtain in the end the *iterated blow-up* $[M : \mathcal{S}]$ [1, 11, 15]. One of our main results is to prove that, if $\mathcal{S}$ is a locally finite clean semilattice, then there exists a unique diffeomorphism

$$[M : \mathcal{S}] \simeq \{M : \mathcal{S}\} \tag{3}$$

that is the identity on the common open subset $M \smallsetminus \bigcup_{k=1}^{n} P_n$ (see Theorem 4.12). In particular, $[M : \mathcal{S}]$ is independent of the initially chosen order on $\mathcal{S}$, as long as it is compatible with the inclusion.

We apply these results to the study of the N-body problem in the following way. Let $\overline{X}$ denote be the spherical compactification of a finite-dimensional vector space X, with boundary at infinity $\mathbb{S}_X := \overline{X} \smallsetminus X$, with smooth structure defined by the central projection of Lemma 5.2. Let $\mathcal{F}$ be a finite semilattice of linear subspaces of X. To $\mathcal{F}$ we associate the semilattice $\mathcal{S} := \{\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F}\} \cup \{\emptyset\}$. In this application to the N-body problem, the role of M will be played by $\overline{X}$. Vasy has considered the space $[\overline{X} : \mathcal{S}]$ in order to define a pseudodifferential calculus adapted to the N-body problem, see [21, 22] and the references therein. Inspired by Georgescu [4, 6, 7, 18], let us consider the norm closed algebra

$$\mathcal{E}_{\mathcal{S}}(X) := \langle \mathcal{C}(\overline{X/Y}) \rangle \tag{4}$$

generated by all the spaces $\mathcal{C}(\overline{X/Y})$ in $L^\infty(X)$, with $Y \in \mathcal{F}$. Note that this algebra will be denoted $\mathcal{E}_{\mathcal{S}}(X)$ and not $\mathcal{E}_{\mathcal{F}}(X)$ in what follows, in order to keep with the notation of [18]. It was proved in [18] that the spectrum $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ of $\mathcal{E}_{\mathcal{S}}(X)$ is the closure of the image of X in the product $\prod_{Y \in \mathcal{F}} \overline{X/Y}$. In this paper, we show that this closure coincides with $\{\overline{X} : \mathcal{S}\}$. As a consequence, the identity $\mathrm{id}_X : X \to X$ extends to a homeomorphisms from the compactification $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ introduced by Georgescu to the compactification $[\overline{X} : \mathcal{S}]$ introduced by Vasy. As the two compactifications coincide, we will henceforth call both spaces "the Georgescu-Vasy space."

Here are the contents of the paper. Section 1 contains background material on manifolds with corners. In particular, we devote quite a little bit of effort to introduce and compare several classes of submanifolds of manifolds with corners. Section 2 recalls the definition of the blow-up of a manifold with corners with respect to a p-submanifold. In Section 3, we introduce the graph blow-up of a manifold with corners with respect to a *set* of p-submanifolds and prove some of its first properties. In Section 4 we prove that, for clean semilattices of p-submanifolds, the graph blow-up can also be obtained as an iterated blow-up, which is one of the main technical results of this paper. In the last section, we use the identification of the graph blow-up with the iterated blow-up to show that two spaces, introduced independently by Georgescu and Vasy, are in fact the same (canonically homeomorphic). Two appendices include some related topological results on proper maps and on summanifolds of manifolds with corners. The reader can thus see that this paper is heavily geometric, necessarily so since Vasy's construction is geometric.

1. Manifolds with corners and their submanifolds

We begin with some background material, mostly about manifolds with corners. This section contains few new results, but the presentation is new.

1.1. **Manifolds with corners.** We now introduce manifolds with corners and their smooth structure. We also set up some important notation to be used throughout the paper. The terminology used for manifold with corners is not uniform. Nevertheless, a good overview of the concept of a manifold with corners can be found in [8, 10, 11, 16, 20], to which we refer for the concepts not defined here and for further references. In this paper, we will mostly use the terminology introduced by Melrose and his coauthors, which predates most of the other ones.

1.1.1. *Smooth maps.* We have the following standard definition.

Definition 1.1. *Let $U \subset \mathbb{R}^n_k$ and $V \subset \mathbb{R}^m_l$ be two open subsets and $f = (f_1, \ldots, f_m) : U \to V$. We shall say that:*

(a) *f is* smooth *on U if there exists an open neighborhood W of U in $\mathbb{R}^n$ such that f extends to a smooth function $\tilde{f} : W \to \mathbb{R}^m$.*
(b) *f is a* diffeomorphism *between U and V if f is a bijection and both f and f^{-1} are smooth.*

1.1.2. *Notation.* For any finite dimensional real vector space Z, let $\mathbb{S}_Z$ denote the *set of vector directions* in Z, that is, the set of (non-constant) open half-lines $\mathbb{R}_+ v$, with $0 \neq v \in Z$ and $\mathbb{R}_+ := (0, \infty)$. We will also use the standard notation $\mathbb{S}^{n-1} := \mathbb{S}_{\mathbb{R}^n}$, for simplicity. In particular, if Z is a euclidean (real) vector space, we identify $\mathbb{S}_Z$ with the unit sphere in Z. Informally, a manifold with corners is a topological space locally modeled on the spaces

$$\mathbb{R}^n_k \, := \, [0, \infty)^k \times \mathbb{R}^{n-k}. \tag{5}$$

For $k, n \in \mathbb{N} = \{0, 1 \ldots\}$ with $k \leq n$, we let $\mathbb{S}^{n-1}_k \subset \mathbb{R}^n$ be

$$\mathbb{S}^{n-1}_k \, := \, \mathbb{S}^{n-1} \cap \mathbb{R}^n_k \, = \, \{\phi = (\phi_1, \ldots \phi_n) \mid \|\phi\| = 1 \text{ and } \phi_i \geq 0 \text{ for } 1 \leq i \leq k\}, \tag{6}$$

where $\|.\|$ is the euclidean norm.

Let us write 0_V for the neutral element of a vector space V, when we want to stress the space to which it belongs. We will often use maps between subsets of euclidean spaces, and, as a rule, we will try not to permute the coordinates, and, moreover, our embedding will be "first components embedings." More precisely, let $k' \leq k$ and $n' - k' \leq n - k$, we shall then use with priority the canonical "first components" embedding given by:

$$\begin{aligned} \mathbb{R}^{n'}_{k'} \, \simeq \, [0, \infty)^{k'} \times \{0_{\mathbb{R}^{k-k'}}\} \times \mathbb{R}^{n'-k'} \times \{0_{\mathbb{R}^{n-n'}}\} \, &\subseteq \, [0, \infty)^k \times \mathbb{R}^{n-k} \, = \, \mathbb{R}^n_k \\ (x', x'') \, \mapsto \, (x', 0_{\mathbb{R}^{k-k'}}, x'', 0_{\mathbb{R}^{n-n'}}) \end{aligned} \tag{7}$$

Other embeddings (involving permutations of the coordinates) between these sets will also be considered, and they will explained separately. For instance, we shall sometimes find it notationally convenient to use the *canonical permutation of coordinates* diffeomorphism

$$\begin{aligned} \mathrm{can} : \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \, &\simeq \, \mathbb{R}^{n+n'}_{k+k'} \\ (x', x'', y', y'') \, \mapsto \, (x', y', x'', y'') &\in [0, \infty)^{k+k'} \times \mathbb{R}^{n+n'-k-k'}, \end{aligned} \tag{8}$$

where $x' \in [0, \infty)^k$ and $y' \in [0, \infty)^{k'}$. (Compare with Equation (5).) However, if nothing else is mentioned, we consider the first components canonical embedding of Equation (7).

1.1.3. *Charts and atlases.* We shall use suitable charts to define the smooth structure on manifolds with corners. Let M be a Hausdorff space. We proceed as in the case of smooth manifolds (without corners).

Definition 1.2. *A* corner chart *on M (or simply,* "chart" *in what follows) is a couple (U, ϕ) with U an open subset of M and $\phi : U \to \Omega$ a homeomorphism onto an open subset Ω of $\mathbb{R}^n_k$. Let (U, ϕ) and (U', ϕ') be two corner charts with values in $\mathbb{R}^n_k$ and in $\mathbb{R}^{n'}_{k'}$, respectively. Let $V := U \cap U'$. We shall say that the corner charts (U, ϕ) and (U', ϕ') are* compatible *if $V = \emptyset$ or if*

$$\phi' \circ \phi^{-1} : \phi(V) \to \phi'(V)$$

is a diffeomorphism (see Definition 1.1) between the open subsets $\phi(V) \subset \mathbb{R}^n_k$ and $\phi'(V) \subset \mathbb{R}^n_{k'}$.

Given a point $m \in M$ and a corner chart (U, ϕ) with $m \in U$, we can always find a corner chart (U', ϕ'), $\phi' : U' \to \mathbb{R}^{n'}_{k'}$, compatible with (U, ϕ) such that $\phi'(m) = 0$ and k' is minimal.

Definition 1.3. *A* corner atlas $\mathcal{A} = \{(U_a, \phi_a), a \in A\}$ *on M is a family of compatible corner charts such that $M = \bigcup_{a \in A} U_a$. Two corner atlases are called* equivalent *if their union is again a corner atlas. A* manifold with corners *is defined to be a paracompact Hausdorff space M with an equivalence class of corner atlases (on M). In the following we will drop the word "corner" before the words "chart" and "atlas." In the context of a manifold with corners, the terms "atlas" and "chart" will always mean "corner atlas" and, respectively, "corner chart."*

A manifold with corners in the above sense is called a "t-manifold" in [15, Section 1.6]. If M is manifold with corners, then the union of all its atlases is again an atlas, the *maximal atlas* of M. An open subset of a manifold with corners is again, in an obvious way, a manifold with corners. Many concepts extend from the case of manifolds without corners to that of manifolds with corners.

Definition 1.4. *Let $f : M \to M'$ be a map between two manifolds with corners. We will say that f is* smooth *if, for any two charts (U, ϕ) of M and (U', ϕ') of M', the map $\phi' \circ f \circ \phi^{-1}$ is smooth on its domain of definition $\phi(f^{-1}(U'))$. If f is a bijection and both f and f^{-1} are smooth, we will say that f is a* diffeomorphism.

The following are some examples of manifolds with corners that will be used in this paper.

Example 1.5. Using the notation from Subsection 1.1.2, we have the following:

(i) Any open subset of $\mathbb{R}^n_k := [0, \infty)^k \times \mathbb{R}^{n-k}$ is a manifold with corners.
(ii) The sphere octant $\mathbb{S}^{n-1}_k := \mathbb{S}^{n-1} \cap \mathbb{R}^n_k$ of Equation (6) is a manifold with corners.
(iii) Any smooth manifold is a manifold with corners (even if it doesn't have a boundary or any true corners).

1.2. **The boundary and boundary faces.** We now fix some standard terminology to be used in what follows. In particular, we need the intrinsic definition of the boundary of a manifold with corners. The *depth* (in X) $\operatorname{depth}_X(p)$ of a point $p \in X$ is the number of non-negative coordinate functions vanishing at p in any local coordinate chart at p. It is the least k such that there exists a chart near U with values in $\mathbb{R}^n_k$. Let $(M)_k$ be the set of points of M of depth k. It is a smooth manifold (no corners). Its connected components are called the *open* boundary faces (or just the *open* faces) of codimension (or depth) k of M. A *boundary face* of depth k is the closure of an open boundary face of depth k. It is possible to construct a manifold with corners M that has a boundary face F such that F is not a manifold with corners for the induced smooth structure. More precisely, there are M and F such that $\{f|_F \mid f \in C^\infty(M)\}$ is not the set of smooth functions on F for some manifold-with-corners structure on F.

We will denote by $\mathcal{M}_k(M)$ the set of all *closed* boundary faces of codimension k. In particular, the *boundary* ∂M of M, defined as the set of all points of depth > 0, is given by

$$\partial M := \bigcup_{H \in \mathcal{M}_1(M)} H. \tag{9}$$

A boundary face of M of codimension one will be called a *hyperface* in what follows. Thus ∂M is the union of the hyperfaces of M. If H is a hyperface of M and $0 \leq x \in \mathcal{C}^\infty(M)$ is a function such that $H = x^{-1}(0)$ and $dx \neq 0$ on H, then x is called a *boundary defining function* of H. As above, there are examples of hypersurfaces, that do not have a boundary defining function. However, each boundary face F of codimension k can *locally* be represented as $F = \{x_1 = x_2 = \ldots = x_k = 0\}$, where x_j are boundary defining functions of the hyperfaces containing F. Here "locally" means that, given $p \in F$, there is an open neighborhood U of p in M such that the statement is true for M and F replaced with $M \cap U$ and, respectively, with $F \cap U$.

Remark 1.6. Note that in [15], by a "manifolds with corners" Melrose means a manifold with corners in our sense that has the further property that each of its hyperfaces (and hence each face) has a system of defining functions. Our definition is thus slightly more general. Furthermore, the submanifolds in [15] are sometimes required to be connected, which would be an inconvenient loss of generality for us. As the blow-up construction below is local, the results of [15] extend trivially to our framework.

It is also convenient to consider an alternative approach to the definition of manifolds with corners and of their smooth structure via embeddings, as in the next remark.

Remark 1.7. Every manifold with corners M is contained in a smooth manifold $\widetilde{M}$ of the same dimension [3, 8, 11, 16, 15, 20]. It is then convenient to define

$$TM \ := \ T\widetilde{M}|_M \,.$$

Up to a diffeomorphism, TM can be obtained by gluing the tangent spaces $T(\mathbb{R}^n_k) := \mathbb{R}^n_k \times \mathbb{R}^n$ using an atlas of M. We also let $T^+_x M$ be the set of tangent vectors of $T_x M$ that are inward-pointing or tangent to the boundary. It can be defined as the set of equivalence classes of curves starting at x and completely contained in M. We finally let $T^+M := \bigcup_{x \in M} T^+_x M$ with its projection map to M. Note that T^+M is not a fiber bundle, but a fiberwise conical closed subset of the tangent space. We note, however, that ∂M is intrinsically defined and sometimes it is *not* the *topological* boundary $\overline{M} \smallsetminus \mathring{M}$ of M, where the closure $\overline{M}$ and the interior $\mathring{M}$ are computed in $\widetilde{M}$. For instance, when $M := \{x \in \mathbb{R}^n \mid x_n \geq 0, \|x\| < 1\}$ and $\widetilde{M} = \mathbb{R}^n$, then $\partial M = \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| < 1\}$, whereas the topological boundary of M is $\partial M \cup \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| = 1\}$, a bigger set. In fact, we always have that ∂M is contained in the topological boundary of M in $\widetilde{M}$. Unlike ∂M, the topological boundary of M in $\widetilde{M}$ depends on $\widetilde{M}$.

1.3. **Submanifolds of manifolds with corners.** We now discuss several notions of submanifolds of a manifold with corners.

1.3.1. *Submanifolds and weak submanifolds.* We follow [15, Definition 1.7.3] to define submanifolds in manifolds with corners. Again, our definition differs slightly from Melrose's in that we do not require connectedness for the submanifold S or the ambient manifold $M \supset S$.

Definition 1.8. *A subset S of a manifold with corners M is a* submanifold *if, for every $p \in S$, there exists $k \in \{0, 1, \ldots, n\}$ and a (corner) chart $\phi : U \to \Omega \subset \mathbb{R}^n_k$, numbers $n' \leq n$ and $k' \leq n'$, and a matrix $G \in \mathrm{GL}(n, \mathbb{R})$ such that*

(1) $p \in U$
(2) $G \cdot \left(\mathbb{R}^{n'}_{k'} \times \{0\}\right) \subset \mathbb{R}^n_k$.

(3) *The chart ϕ maps $S\cap U$ bijectively to the intersection of this linear submanifold with Ω, in other words*

$$\phi(S\cap U) \;=\; G\cdot\Big(\mathbb{R}^{n'}_{k'}\times\{0\}\Big)\cap\Omega\,.$$

Let us comment on this definition.

Remark 1.9.

(1) In Definition 1.8, the symbol "$\cdot$" denotes the action of $\mathrm{GL}(n,\mathbb{R})$ on subsets of $\mathbb{R}^n$, action which is induced by the standard linear action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$.

(2) In Melrose's terminology, Property (2) of this definition is expressed by saying that $G\cdot\Big(\mathbb{R}^{n'}_{k'}\times\{0\}\Big)$ is a linear submanifolds of $\mathbb{R}^n_k$.

(3) If S is submanifold of M and $p\in S$, then the number n' and k' are uniquely determined. We say that n' is *the dimension* of S in p. This dimension is locally constant, but we allow it to depend on the connected component. A (connected) submanifold S of a manifold with corners M carries an induced structure of a manifold with corners. This structure can be characterised by the condition, that the embedding is a smooth map $i : S\to M$ and that the differential $d_pi : T_pS\to T_pM$ is injective for any $p\in S$. Alternatively this structure can be described by an atlas. For any ϕ and G as above, we the map $\tilde\phi(p)$ is defined for $p\in U\cap S$ by the relation $G^{-1}\phi(p) = (\tilde\phi(p),0)$. We set $\tilde U := U\cap S$ and $\tilde\Omega := \phi(\tilde U) = G^{-1}(\Omega)\cap\Big(\mathbb{R}^{n'}_{k'}\times\{0\}\Big)$. Then $\tilde\phi : \tilde U\to\tilde\Omega$ shall be taken as a chart for S around p, and if we do this construction for any $p\in S$, then we obtain an atlas for S. See [15, Lemma 1.7.1 and following] for further details.

(4) Note that in [2], a more restrictive notion of submanifold of a manifold with corners was used, see Remark 1.24 for more details.

The following example illustrates the definition of a submanifold of a manifold with corners and explains why we consider it in the first place.

Example 1.10 (Diagonal)**.** Let N be a manifold with corners. Then $M := N\times N$ is also a manifold with corners. Consider the diagonal $\Delta_N := \{(p,p)\in M \mid p\in N\}$. Then Δ_N is a submanifold of M.

In various applications it is helpful or even necessary to use variations of the concept of a "submanifold"; in our article we will require both a more general concept, that of a "weak submanifold," and a more restrictive one, namely, that of a "p-submanifold" of a manifold with corners, which will be discussed in the next section.

We explained above that a submanifold S of a manifold with corners M inherits an atlas from M. This property can be generalized to "weak submanifolds," which we will be define after the following example.

Example 1.11. The function $f(x,y) := (x+y^2,y)$ defines a injective immersion $\mathbb{R}^2_1\to\mathbb{R}^2_1$. It is a homeomorphism onto its image $S := f\big(\mathbb{R}^2_1\big)$. However, it can be easily seen that S is not a submanifold $\mathbb{R}^2_1$. On the other hand S is a submanifold of $\mathbb{R}^2$.

The situation of the last example is not convenient in applications as we would like the images of injective immersions with the homeomorphism property to defined some sort of submanifold. Our next aim is thus to generalize the notion of "submanifold" to allow for such examples.

Definition 1.12. *A subset S of a manifold with corners M is a* weak submanifold *if, for every $p \in S$, there exists $k \in \{0, 1, \dots, n\}$ and a chart $\phi : U \to \Omega \subset \mathbb{R}^n_k$, such that*

(1) $p \in U$ and

(2) $\phi(S \cap U)$ is a submanifold of $\mathbb{R}^n$.

The dimension of S at p *is by definition the dimension of $\phi(S \cap U)$ at $\phi(p)$.*

Equivalently, one can reformulate Definition 1.12 by saying that S is a weak submanifold if, and only if, M can be extended to a manifold $\tilde{M}$ without corners (or boundary), such that S is covered by charts $\phi : U \to \Omega$ of $\tilde{M}$ satisfying

$$\phi(S \cap U) \; = \; \left(\mathbb{R}^{n'}_{k'} \times \{0\}\right) \cap \Omega\,.$$

Note that if we replaced in *(2)* the ambient manifold $\mathbb{R}^n$ by $\mathbb{R}^n_k$, then S would be a weak submanifold if, and only if, S was a submanifold. However as written above, the class of weak submanifolds strictly contains the class of submanifolds. Indeed, the set S in Example 1.11 is a weak submanifold of $\mathbb{R}^2_1$, but not a submanifold.

For any chart $\phi : U \to \Omega$ of M the submanifold property provides an atlas on every $\phi(S \cap U)$ and thus an atlas on $S \cap U$. If $\phi' : U' \to \Omega'$ is another chart of M, then we similarly obtain an atlas on $S \cap U'$, and as $\phi' \circ \phi^{-1}|_{\phi(U \cap U')}$ is a diffeomorphism, the atlases on $S \cap U$ and $S \cap U'$ are compatible, i.e. their union is an atlas on $S \cap (U \cup U')$. By repeating this construction for all the domains of charts of an atlas of M, we obtain an atlas of S, the *induced atlas on* S. With this atlas, the set S is a manifold with corners.

By applying the implicit function theorem, one can prove that a subset $S \subset M$ of a manifold with corners is a weak submanifold, if, and only if, it is the image of an injective immersion $f : N \to M$, where N is a manifold with corners and where $f : N \to S$ is a homeomorphism. See Proposition B.1 in the appendix for a proof.

Furthermore, we have the following nice property which we will claim without proof: if P and Q are weak submanifolds of M and $P \subset Q$, then P is a weak submanifold of Q. Thus weak submanifolds also have nicer categorical properties than submanifolds. In the categorical language, the above property is expressed as follows: if we consider the category whose objects are manifolds with corners and the morphisms are inclusions as a weak submanifold, then this is a *full* subcategory of the category of sets with the inclusions as morphisms.

If we only consider submanifolds in the sense of Definition 1.8 as morphisms, then the above property does not hold as we have seen in Example 1.11, i.e. we obtain a subcategory that is not full.

1.3.2. *Submanifolds with tubular neighborhoods: p-submanifolds.* We now recall the definition of a p-submanifold of a manifold with corners M [1, 9, 15, 21]. In our paper, p-submanifolds are of central importance, as we blow-up manifolds with corners along closed p-submanifolds.

Definition 1.13. *Let I be a subset of $\{1, \dots, n\}$ and L be the subset of $\mathbb{R}^n_k$ defined by*

$$L_I \; := \; \{\, x = (x_1, \dots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \text{ if } i \in I \,\}\,. \tag{10}$$

The number $b := \#\big(I \cap \{1, \dots, k\}\big)$ will be called the boundary depth *of L_I; $c := \#I$ is the codimension of L_I and $d := n - c$ its dimension.*

After reordering the components, L_I is the first factor of $\mathbb{R}^n_k \cong \mathbb{R}^d_{k-b} \times \mathbb{R}^c_b$, in the sense that L_I is mapped to $\mathbb{R}^d_{k-b} \times \{0\}$. The boundary depth of L_I is the boundary depth of any interior point of L_I with respect to $\mathbb{R}^n_k$. The sets L_I are the local models for p-submanifolds [15, Definition 1.7.4].

Definition 1.14. *A subset P of a manifold with corners M is a* p-submanifold *if, for every $x \in P$, there exists a chart (U, ϕ) with $x \in U$ and $I \subset \{1, 2, \dots, n\}$ such that*

$$\phi(P \cap U) \;=\; L_I \cap \phi(U),$$

with L_I as defined in Equation (10)*. The number $n - \#I$ (respectively, $\#I$, respectively, $\#\big(I \cap \{1, \dots, l\}\big)$) will be called the* dimension *(respectively, the* codimension *of P in x, respectively, the* boundary depth *of P in x). We allow p-submanifolds Y of non-constant dimension. We define $\dim Y$ as the maximum of the dimensions of the connected components of Y and $\dim \emptyset = 0$.*

Obviously all p-submanifolds are submanifolds, and the definition of the dimension of P in x coincides with the dimension already defined above.

Remark 1.15. The numbers $n - \#I$ (respectively, $\#I$, respectively, $\#\big(I \cap \{1, \dots, l\}\big)$) introduced in Definition 1.14 are locally constant functions on P. For any interior point x in P and $\epsilon > 0$ small enough, these numbers are the dimension (respectively, the codimension, respectively, the boundary depth of x) of the intersection $B_\epsilon(x) \cap P$ in M. More generally: if P is a p-submanifold of M with boundary depth d on the component of $x \in P$, and if x is a (boundary) point of depth e in P, then x has depth $d + e$ in M. In particular, for a p-submanifold $P \subset M$, the difference of depths $\mathrm{depth}_M(x) - \mathrm{depth}_P(x)$ is constant on the connected components of P.

This definition of a p-submanifold comes from [15]. Note that "*p*" is used as an abbreviation for "product," reflecting the fact that, locally in coordinate charts, p-submanifolds are a factor of the product $\mathbb{R}^n_k \simeq \mathbb{R}^{n_1}_{k_1} \times \mathbb{R}^{n_2}_{k_2}$. A more general concept, that of an "interior binomial subvariety," was introduced and studied in [11].

Let $P \subset M$ be a p-submanifold. Then it is possible that $P \subset F$, for F a non-trivial face of M. If P is connected, then the boundary depth of P is the boundary depth of the smallest closed face F of M containing P.

We shall need the following lemma. Recall that a subset of a topological space is called *locally closed* if it is the intersection of a closed subset with an open subset.

Lemma 1.16. *Let $P \subset Q \subset M$ be manifolds with corners.*

(i) If P is a p-submanifold of M, then P is locally closed.
(ii) If both P and Q are p-submanifolds of M, then P is a p-submanifold of Q.
(iii) If P is a p-submanifold of Q and Q is a p-submanifold of M, then P is a p-submanifold of M.

Proof. Let us fix an atlas $\mathcal{A} := \{(U, \phi)\}$. The definition of a p-submanifold shows that it is a closed subset in every coordinate chart (U, ϕ). Hence it is locally closed. This proves (i).

In order to prove (ii) we consider functions $x^1, \dots, x^\ell$ defining a p-submanifold P of codimension ℓ in M locally in a neighborhood of $x \in P$. Choose $I \subset \{1, \dots, \ell\}$ such that $(dx^i|_p)_{i \in I}$ is a basis of $T^*_x Q$. Then in a possibly smaller neighborhood, the functions $(x^i)_{i \in I}$ define P as a p-submanifold of Q.

For (iii) we consider functions $x^1, \dots, x^k$ locally defining P as a p-submanifold of Q. We extend these functions to locally defined functions on M. Then we choose functions $x^{k+1}, \dots, x^l$ defining Q locally as a p-submanifold of M. Then $x^1, \dots, x^l$ locally define P as a p-submanifold of M. □

Example 1.17. The diagonal Δ_N in Example 1.10 is not a p-submanifold. If N is the 2-dimensional closed disc, then with argumets analogous to Remark 4.11, the diagonal is not a p-submanifold of $N \times N$. Alternatively, one could argue with [15], see Remark 1.23.

1.3.3. *The normal bundle of p-submanifolds.* The following standard concepts will be important in the definition of the blow-up of a manifold with corners by a p-submanifold.

Definition 1.18. *Let $P \subset M$ be a p-submanifold of the manifold with corners M. Then $N^M P := TM|_P/TP$ is called the* normal bundle of P in M. *The image $N^M_+ P$ of $T^+M|_P$ in $N^M P$ is called the* inward pointing normal fiber bundle of P in M. *In contrast to $T^+M|_P \to P$, which is not a fiber bundle over P, the projection map $N^M_+ P \to P$ defines a fiber bundle structure over P on $N^M_+ P$, called the* inward pointing normal bundle of P in M. *Finally, the set $\mathbb{S}(N^M_+ P)$ of unit vectors in $N^M_+ P$ is called the set of* inward pointing spherical normal bundle of P in M. *The inward pointing spherical normal bundle of P in M comes equipped with a fiber bundle projection*

$$\mathbb{S}(N^M_+ P) \ \to \ P\,.$$

We complete this section with a few remarks. We first notice the existence of suitable "tubular neighborhoods."

Remark 1.19. Let $P \subset M$ be a p-submanifold in the manifold with corners M. If M is compact, then P has a neighborhood $V_P \subset M$ such that V_P is diffeomorphic to the closed cone $N^M_+ P$ via a diffeomorphism that sends P to the zero section of $N^M_+ P \to M$ and induces the identity at the level of normal bundles. This was proved in [15, Proposition 2.10.1], under the additional assumption that P be closed. Moreover, the condition that M be compact is not necessary (since our p-manifolds are assumed to be locally closed). In this case, $N^M_+ P$ is a cone with corners in $N^M P$. Generalizing Example 1.5, we obtain that all of the sets $N^M P$, $N^M_+ P$, and $S^+(N^M P)$ introduced in the last definition are manifolds with corners. This is because the property of being a manifold with corners is a local property and the product of manifolds with corners is again a manifold with corners.

1.3.4. *The dimension of a non-connected p-submanifold.* We finish the discussion of p-submanifolds with a note on our terminology.

Remark 1.20. By taking I to be an empty subset, we obtain that the open subsets of M are p-submanifolds. The empty set $\emptyset$ also satisfies the conditions defining a p-submanifold. The empty set and the p-submanifolds containing open subsets of M will be called *trivial p-submanifolds of* M. We allow the different connected components of a p-submanifold to have *different* dimensions. Therefore, if Y is a p-submanifold of M, we shall denote by $\dim(Y)$ the *largest* dimension of a connected component of Y. Thus, the *non-trivial* p-submanifolds of M are the p-submanifolds that are non-empty and of lower dimension than M.

1.3.5. *Further classes of special submanifolds.* In the conclusion to this section, let us mention some further classes of submanifolds which put our article in the contex of the literature and which might be helpful to obtain possible extensions of our results. However, they are not needed to understand the statement or proof of our main results, thus may be skipped by the reader.

In some parts of our article, the notion of a submanifold in the sense of Definition 1.8 is too unspecific, and the notion of a p-submanifold to restrictive. In between these two classes there lies a class of submanifolds that we call "wib-submanifolds," for lack of a better name in the literature. Here "wib-submanifold" stands for a submanifold *without an interior boundary.*

Definition 1.21. *A submanifold $S \subset M$ is called a* wib-submanifold *or a* submanifold without interior boundary *if it can be defined locally in suitable charts as the kernel of a linear function. More precisely: $S \subset M$ is a* wib-submanifold *if, for every $x \in S$, there exists a (corner) chart $\phi : U \to \Omega \subset \mathbb{R}^n_k$, and a linear subspace L of $\mathbb{R}^n$, such that*

(1) $x \in U$ and
(2) $\phi(S \cap U) \ = \ L \cap \Omega\,.$

If $G \in \mathrm{GL}(n, \mathbb{R})$ is defined as above, then we necessarily have $L = G \cdot \Big(\mathbb{R}^{n'} \times \{0\}\Big)$. If $x \in S \cap U$, then $n' := \dim L$ is the dimension of S in x defined above.

Obviously all p-submanifolds are wib-manifolds, which can be easily seen by defining the L in the definition above as the linear extension of L_I in Definition 1.14.

Remark 1.22. In the above definition, we explicitly required S to be a submanifold. To justify this requirement, we will give an example of a closed subset $S \subset M$ which is not a submanifold, but fulfills all other requirements of the definition of a wib-submanifold. Indeed, let

$$K \ := \ \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 \leq x_3,\ x_2 \leq x_3\}\,,$$

which is a cone over a square. The map $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_1, x_3 - x_2)$ has the property $f^{-1}(\mathbb{R}^4_4) = K$. Then for $\phi = \mathrm{id}$, $x = 0$, and $L := f(\mathbb{R}^3)$ all requirements of the definition are satisfied, but $S := f(K)$ is not a submanifold of $\mathbb{R}^4_4$. It it were a submanifold, then its dimension would have to be 3, and then any boundary point of S is in at most 3 closed boundary hyperfaces. But $0 \in S$ is in 4 closed boundary hyperfaces of S.

Remark 1.23. Note that Melrose also introduces the notions d-submanifold [15, Def. 1.7.4] and b-submanifold [15, Def. 1.12.9], whose definitions will not be recalled here. They satisfy

$$S \text{ is a p-submanifold} \implies S \text{ is a d-submanifold} \quad \implies \quad S \text{ is a b-submanifold}$$

$$\implies \quad S \text{ is a submanifold} \implies S \text{ is a weak submanifold}\,.$$

However there are wib-manifolds that are not b-submanifolds, e.g. Melrose's example of the submanifold $\{x_3 = x_1 + x_2\} \in \mathbb{R}^3_3$. There are d-manifolds that are no wib-manifolds, e.g. $\mathbb{R}^1_1 = [0, \infty) \subset \mathbb{R}$ or any surface with boundary in $\mathbb{R}^3$. However all p-submanifolds introduced below are d-manifolds and wib-manifolds.

Melrose shows that the diagonal Δ_N is a b-submanifold of $N \times N$, but in general not a d-submanifold. It follows that Δ_N is not a p-submanifold.

Remark 1.24. Let us remark that the definition of a *tame* submanifold considered in [2, Sec. 2.3] is a submanifold in an essentially different sense. All notions of submanifolds discussed involve are properties that may hold or not for a subset N of a manifold with corners M. In contrast to this, tame submanifolds in [2, Sec. 2.3], are submanifolds of a *Lie manifold* (M, A), where M is a manifold with corners and A is a Lie algebroid on M with some compatibility conditions. The fact whether a subset N of M is a tame submanifold of (M, A) or not, depends also on the Lie algebroid A. In any case, a tame submanifold will have a tubular neighborhood in the strongest sense.

Similar remarks apply to the $A(\mathcal{G})$-tame submanifolds considered in [20].

2. The blow-up for manifolds with corners

We now introduce the blow-up of a manifold M with corners by a *closed* p-submanifold P with $\dim(P) < \dim(M)$ (see Remark 1.20 for the definition of the dimension of a p-submanifold). We also study some of the properties of the blow-up.

2.1. Definition of the blow-up and its smooth structure.

Recall (Remark 1.20) that if $P \subset M$ is a p-submanifold, then $\dim(P)$ denotes the largest dimension of the connected components of P (these connected components are not necessarily all of the same dimension).

2.1.1. *Definition of the blow-up as a set.* We now define the underlying set of the blow-up of a manifold with corners M with respect to a p-submanifold. If A and B are disjoint, we sometimes denote $A \sqcup B := A \cup B$ their union.

Definition 2.1. *Let M be a manifold with corners and P be a* closed *p-submanifold of M. Let $\mathbb{S}(N_+^M P)$ be the inward pointing spherical normal bundle of P in M (Definition 1.18). The* blow-up of M along P *(or* with respect to P*) is the following union of disjoint sets:*

$$[M : P] := (M \smallsetminus P) \sqcup \mathbb{S}(N_+^M P)\,.$$

In particular, $[M : \emptyset] = M$ and $[M : M] = \emptyset$. The blow-down map $\beta = \beta_{M,P} : [M : P] \to M$ is defined as the identity map on $M \smallsetminus P$ and as the fiber bundle projection $\mathbb{S}(N_+^M P) \to P$ on the complement.

The blow-up $[M : P]$ is therefore not defined if P is not closed, but we allow P to consist of the disjoint union of several closed, connected p-submanifolds of M of different dimensions.

Assume $P \subset M$ to be a trivial p-submanifold. Then, by definition, there will be $x \in P$ such that P is of codimension 0 in a neighborhood of x, and hence the fiber of $\mathbb{S}(N_+^M P)$ over x is $\emptyset$.

A general approach to smooth structures on the blow-up is contained in [11]. Here we recall an approach that suffices for our needs. We begin with the case of open subsets of a model space $\mathbb{R}^n_l$.

2.1.2. *The blow-up of the local models.* In the following, let I_j, $j = 1, 2, \dots n$, denote either $\mathbb{R}$ or $[0, \infty)$. We will write $N_1 \cong N_2$ if N_1 is a p-submanifold of $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$ and if there is a permutation σ of the components of $\mathbb{R}^n$ that induces a diffeomorphism from N_1 to the p-submanifold N_2 of $I_{\sigma(1)} \times I_{\sigma(2)} \times \cdots \times I_{\sigma(n)} \subset \mathbb{R}^n$. By contrast, when we write $N_1 \simeq N_2$, we will merely state that the indicated manifolds are diffeomorphic, without including further information on the diffeomorphism. In particular, $N_1 \cong N_2$ implies $N_1 \simeq N_2$.

To start with, the blow-up $[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$ of $\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \cong \mathbb{R}^{n+n'}_{l+l'}$ along its p-submanifold $\mathbb{R}^n_l \times \{0\} = \mathbb{R}^n_l \times \{0_{\mathbb{R}^{n'}}\}$ is, by Definition 2.1, the set

$$\begin{aligned}[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}] &:= \Big(\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \smallsetminus \mathbb{R}^n_l \times \{0\}\Big) \sqcup \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \\ &= \mathbb{R}^n_l \times \Big(\mathbb{S}^{n'-1}_{l'} \sqcup \big(\mathbb{R}^{n'}_{l'} \smallsetminus \{0\}\big)\Big)\,. \end{aligned} \tag{11}$$

Let us consider the map

$$(12)\qquad \begin{aligned} \kappa : \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty) \ &\to\ \mathbb{R}^n_l \times \Big(\mathbb{S}^{n'-1}_{l'} \sqcup (\mathbb{R}^{n'}_{l'} \smallsetminus \{0\})\Big)\,, \\ \kappa(x,\xi,r) \ &:=\ \begin{cases} (x,\xi) \in \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} & \text{if } r = 0 \\ (x, r\xi) \in \mathbb{R}^n_l \times (\mathbb{R}^{n'}_{l'} \smallsetminus \{0\}) & \text{if } r > 0\,. \end{cases} \end{aligned}$$

The map κ is immediately seen to be a bijection and we will use it to endow $[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$ with the structure of a manifold with corners induced from $\mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty)$. Under this diffeomorphism, the blow-down map becomes

$$(13)\qquad \beta : \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty) \to \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'}\,, \quad \beta(x,\xi,r) \ :=\ (x, r\xi)\,.$$

The blown-up space $[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$ is thus a space of "generalized spherical coordinates."

If $U \subset \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'}$ is an open subset, we endow

$$(14)\qquad [U : U \cap (\mathbb{R}^n_l \times \{0\})] \ =\ \beta^{-1}(U) \subset [\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$$

with the induced structure of a manifold with corners.

2.1.3. *The smooth structure of the blow-up.* The following lemmas will allow us to define a manifolds with corners structure on blow-ups.

Lemma 2.2. *Let $P_i \subset M_i$, $i = 1, 2$, be closed p-submanifolds and let $\phi : M_1 \to M_2$ be a diffeomorphism such that $\phi(P_1) = P_2$. Then there exists a unique map $\phi^\beta : [M_1 : P_1] \to [M_2 : P_2]$ that is bijective and makes the following diagram commute*

$$\begin{array}{ccc} [M_1 : P_1] & \xrightarrow{\ \phi^\beta\ } & [M_2 : P_2] \\ \downarrow{\scriptstyle \beta_{M_1,P_1}} & & \downarrow{\scriptstyle \beta_{M_2,P_2}} \\ M_1 & \xrightarrow{\ \phi\ } & M_2. \end{array}$$

This construction is functorial, in the sense that $(\phi \circ \psi)^\beta = \phi^\beta \circ \psi^\beta$. If M_i are open subsets of $\mathbb{R}^n_k$, then ϕ^β is a diffeomorphism.

Proof. The existence, uniqueness, and the functorial character of ϕ^β follows from the definition of the blow-up. The fact that ϕ^β is smooth if M_i are open subsets of the model space $\mathbb{R}^n_k$ is the content of Lemma 2.2 of [1]. □

Lemma 2.3. *Let $\mathcal{A} = \{(U_a, \phi_a) \mid a \in A\}$ be an* atlas *on a manifold with corners M, see Definition 1.3. Let $P \subset M$ be a closed p-submanifold and $\beta = \beta_{M,P} : [M : P] \to M$ be the blow-down map. We endow $[M : P]$ with the smallest topology that makes all the maps ϕ^β_a, $a \in A$, continuous (ϕ^β_a is defined on $\beta^{-1}(U_a)$). Then*

$$\beta^*(\mathcal{A}) \ :=\ \{(\beta^{-1}(U_a), \phi^\beta_a) \mid a \in A\}$$

is an atlas on $[M; P]$, where ϕ^β_a are the maps obtained from ϕ_a using Lemma 2.2. If we take another atlas $\mathcal{A}'$ of M that is compatible with $\mathcal{A}$, then $\beta^*(\mathcal{A})$ and $\beta^*(\mathcal{A}')$ will be compatible atlases on $[M : P]$.

Proof. This follows from Equation (14) and Lemma 2.2. □

Lemma 2.3 thus yields the desired smooth structure on $[M : P]$ that is moreover canonical (independent of any choices).

Definition 2.4. *Let M be a manifold with corners and $P \subset M$ be a closed p-submanifold. We endow $[M : P]$ with the smooth structure defined by the atlas $\beta^*(\mathcal{A})$ obtained from Lemma 2.3, for any atlas $\mathcal{A}$ on M.*

The smooth structure on $[M : P]$ is natural in the following strong sense.

Proposition 2.5. *With the notation of Lemma 2.2, we have that the map ϕ^β is a diffeomorphism (in general, not just in the case of open subsets of Euclidean spaces).*

Proof. If $\mathcal{A}$ is an atlas on M_2, then the pull-back of $\beta^*(\mathcal{A})$ to $[M_1 : P_1]$ is an atlas. □

The functoriality property of Lemma 2.2 then gives the following.

Corollary 2.6. *Let G be a discrete group acting smoothly on the manifold with corners M and let $P \subset M$ be a closed p-submanifold such that $g(P) = P$ for all $g \in G$. Then G acts smoothly on $[M : P]$.*

Proof. The action of every $g \in G$ on M defines a smooth action on $[M : P]$ by Proposition 2.5. It is a group action by the last part of Lemma 2.2 (the functoriality of the assignement $\phi \to \phi^\beta$. □

The blow-up $[M : P]$ of a manifold with corners is thus again a manifold with corners.

2.2. **Exploiting the local structure of the blow-up.** The local character of the definition of the smooth structure of the blow-up $[M : P]$ of the manifold with corners M along a p-submanifold P means that most of the proofs involving blow-ups can be conveniently treated by first treating the model case $P := \mathbb{R}^n_k = \mathbb{R}^n_k \times \{0\} \subset \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} = M$. To simplify notation, we shall often omit factors of the form $\{0\}$ when there is no danger of confusion. This is the case with the following results.

2.2.1. *The blow-down map is proper.* We shall need to prove that certain maps are closed. This will be conveniently done by proving that they are proper, since a proper map between manifolds with corners is closed. In particular, we will show that the blow-down map is proper.

Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. For instance, the map β of Equation (13) is immediately seen to be proper.

Corollary 2.7. *Let P be a closed p-submanifold of a manifold with corners M. The blow-down map $\beta_{M,P} : [M : P] \to M$ is proper.*

Proof. Using Lemma A.3 from the Appendix, we see that we can treat the problem in local coordinates. Then, in local coordinates, the blow-down map is given by Equation (13), which is a proper map, as we have already pointed out. □

2.2.2. *Blow-ups and products.* We have a simple, convenient behavior of the blow-up with respect to products.

Lemma 2.8. *Let M and M_1 be two manifolds with corners and P be a closed p-submanifold of M. Then $P \times M_1$ is a closed p-submanifold of $M \times M_1$ and the following diagram with smooth maps commutes:*

$$\begin{array}{ccc} [M \times M_1 : P \times M_1] & \xrightarrow{\simeq} & [M : P] \times M_1 \\ \downarrow{\scriptstyle \beta_{M \times M_1, P \times M_1}} & & \downarrow{\scriptstyle \beta_{M,P} \times \mathrm{id}} \\ M \times M_1 & \xrightarrow{\mathrm{id}} & M \times M_1 \, . \end{array} \tag{15}$$

Proof. Since the result is a local one and P is a p-submanifold of M, it is enough to treat the case

$$\begin{aligned} M &:= \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P &:= \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \subset M \\ M_1 &:= \mathbb{R}^l_{k_l}\,. \end{aligned}$$

In this local treatment, we will write $\cong$ to stress that a given diffeomorphism is given by a permutation of coordinates, more precisely in this proof, by the canonical permutation of coordinates diffeomorphism of Equation (8).

With this choice, we see that $P \times M_1$ is p-submanifold of $M \times M_1$. We have natural diffeomorphisms with the first one being obtained from the definition of the blow-up, Definition 2.1, and the last being induced by suitable permutations of coordinates)

$$\begin{aligned} [M \times M_1 : P \times M_1] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l}] \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l} \sqcup \Big((\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l}) \smallsetminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l}) \Big) \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \times \mathbb{R}^l_{k_l} \cong \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1} \end{aligned}$$

and

$$\begin{aligned} [M : P] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}] \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^p_{k_p} \sqcup \Big((\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p}) \smallsetminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}) \Big) \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \cong \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1}\,. \end{aligned}$$

The desired diffeomorphism $[M \times M_1 : P \times M_1] \xrightarrow{\simeq} [M : P] \times M_1$ is then induced by the above diffeomorphisms and the canonical permutation of coordinates diffeomorphism $\mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{R}^l_{k_l} \xrightarrow{\cong} \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1}$ of Equation (8). □

2.3. **Cleanly intersecting families and liftings.**

2.3.1. *Clean intersections.* We continue to exploit the local structure of the blow-up. Recall the following standard definition.

Definition 2.9. *Let M be a manifold with corners and $X_1, X_2, \dots, X_k \subset M$ be p-submanifolds. We shall say that $X_1, X_2, \dots, X_k$* have a clean intersection *or that they* intersect cleanly *if*

(i) $Y := X_1 \cap X_2 \cap \dots \cap X_k$ is a p-submanifold of M (possibly empty),
(ii) for all $x \in Y$, $T_xY = T_xX_1 \cap T_xX_2 \cap \dots \cap T_xX_k$.

We consider the conditions (i) and (ii) of the Definition 2.9 to be automatically satisfied if $Y := X_1 \cap X_2 \cap \dots \cap X_k = \emptyset$. Similar conditions appear in Definition 2.7, [1]. They were used to define a *weakly transversal family* of connected submanifolds with corners. We shall need also the notion of a "cleanly intersecting family" (Definition 4.4), which roughly states that every subfamily intersects cleanly.

Lemma 2.10. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Then $P \cap Q$ is a p-submanifold of Q.*

Proof. According Definition 2.9 (i) $P \cap Q$ is a p-submanifold of M. Then Lemma 1.16 (ii) states that $P \cap Q$ is also a p-submanifold of Q. □

2.3.2. *Liftings of p-submanifolds to blowups.* We now consider the lifting of suitable p-submanifolds in M to $[M : P]$ as in [11, 15].

The local model for such lifts is given by the following lemma. (See Lemma 2.2 for the definition of j^β.)

Lemma 2.11. *If $l'' \geq l'$ and $n'' - l'' \geq n' - l'$, so that the canonical (first components) inclusion $j : \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \to \mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''}$ is defined, then there is a map j^β such that the diagram*

$$\begin{array}{ccc} [\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}] & \xrightarrow{j^\beta} & [\mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''} : \mathbb{R}^n_l \times \{0\}] \\ \beta_{\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'}, \mathbb{R}^n_l \times \{0\}} \downarrow & & \downarrow \beta_{\mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''}, \mathbb{R}^n_l \times \{0\}} \\ \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} & \xrightarrow{j} & \mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''}\,. \end{array} \tag{16}$$

commutes.

In fact the diagram (16) is obtained from

$$\begin{array}{ccc} [\mathbb{R}^{n'}_{l'} : \{0\}] & \xrightarrow{j^\beta_0} & [\mathbb{R}^{n''}_{l''} : \{0\}] \\ \beta_{\mathbb{R}^{n'}_{l'}, \{0\}} \downarrow & & \downarrow \beta_{\mathbb{R}^{n''}_{l''}, \{0\}} \\ \mathbb{R}^{n'}_{l'} & \xrightarrow{j_0} & \mathbb{R}^{n''}_{l''}\,. \end{array} \tag{17}$$

by taking for each space the product with $\mathbb{R}^n_l$ and extending the maps as a product with the identity map $\mathrm{id} : \mathbb{R}^n_l \to \mathbb{R}^n_l$, using the linear version of Lemma 2.8.

The lift j^β_0 is given by

$$[\mathbb{R}^{n'}_{l'} : \{0\}] \simeq \mathbb{S}^{n'-1}_{l'} \times [0, \infty) \xrightarrow{i \times \mathrm{id}} \mathbb{S}^{n''-1}_{l''} \times [0, \infty) \simeq [\mathbb{R}^{n''}_{l''} : \{0\}]\,, \tag{18}$$

where $i : \mathbb{S}^{n'-1}_{l'} \to \mathbb{S}^{n''-1}_{l''}$ is the restriction of j_0. In particular, j^β_0 and thus j^β are smooth.

Definition 2.12. *Let P be a p-submanifold of M and Q be a closed subset of M. The* lifting *$\beta^*_{M,P}(Q)$ of Q in $[M : P]$ is defined by*

$$\beta^*_{M,P}(Q) := \overline{\beta^{-1}_{M,P}(Q \smallsetminus P)} \quad \textit{(the closure is in } [M : P]\textit{)}\,.$$

A more general version of the lifting β^* was defined in [15, Chap 5, Section 7] (for $Q \subset P$, in which case, with the notation of Definition 2.12, $\beta^*(Q) := \beta^{-1}(Q)$), but that version will not be needed in this paper.

We have the following result on the blow-up of p-submanifolds, due, in part, to Melrose [15, Chapter 5, Section 7]. A proof in a slightly less general setting can be found also in Proposition 2.4 of [1]. For a p-submanifold $P \subset M$, recall the definition of $\mathbb{S}(N^M_+ P)$, the *inward pointing normal bundle of P in M* from Definition 1.18.

Proposition 2.13. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Then the inclusion $j : Q \to M$ lifts to a natural inclusion*

$$j^\beta : [Q : P \cap Q] := (Q \smallsetminus (P \cap Q)) \sqcup \mathbb{S}(N^Q_+(P \cap Q)) \to (M \smallsetminus P) \sqcup \mathbb{S}(N^M_+ P) =: [M : P]\,.$$

The map j^β is smooth for the natural p-submanifold structures. In particular, the inclusion $Q \smallsetminus P \subset [Q : P \cap Q]$ extends to a natural diffeomorphism

$$\beta^*_{M,Q}(Q) := \overline{\beta^{-1}_{M,Q}(Q \smallsetminus P)} \xrightarrow{\simeq} [Q : P \cap Q]\,.$$

Proof. The inclusion of Q into M restricts to a map $Q \smallsetminus (P \cap Q) \to M \smallsetminus P$. It also induces an inclusion $TQ \to TM$, extending the inclusion $T(P \cap Q) \to TP$. Since $T(P \cap Q) = TP \cap TQ$, we can pass to quotients to obtain an injective map

$$N^Q(P \cap Q) := TQ|_{P\cap Q}/T(P\cap Q) = TQ|_{P\cap Q}/(TP\cap TQ) \to TM|_P/TP =: N^M P$$

The injectivity of this map yields an inclusion $\mathbb{S}(N^Q_+ P) \to \mathbb{S}(N^M_+ P)$, which fits smoothly with a map $Q \smallsetminus (Q \cap P) \to M \smallsetminus P$, using the fact that P and Q intersect smoothly and the local description of the blow-up with half-spaces in [1]. The result then follows from the definition of the blow-up, Definition 2.1. □

3. The graph blow-up

We introduce also the blow-up with respect to more than one submanifold, called *graph blow-up*.

3.1. **Definition of the graph blow-up.** Let M be a manifold with corners and $\mathcal{F}$ be a locally finite set of p-submanifolds of M. Then $\bigcup \mathcal{F} := \bigcup_{Y\in\mathcal{F}} Y$ is nowhere dense in M and $M \smallsetminus \bigcup \mathcal{F}$ is a dense, open subset of $[M : Y]$, for each $Y \in \mathcal{F}$. Motivated by the results of [7, 18], we now introduce the following definition.

Definition 3.1. *Let $\mathcal{F}$ be a locally finite set of closed p-submanifolds of the manifold with corners M. Then the* graph blow-up $\{M : \mathcal{F}\}$ *of M along $\mathcal{F}$ is defined by*

$$\{M : \mathcal{F}\} := \overline{\{(x, x, \dots, x) \mid x \in M \smallsetminus \bigcup \mathcal{F}\}} \subset \prod_{Y\in\mathcal{F}} [M : Y] \,.$$

Let $\delta : M \smallsetminus \bigcup \mathcal{F} \to \prod_{Y\in\mathcal{F}}[M : Y]$ be the diagonal map of inclusions, $\delta(x) = (x, x, \dots, x)$. Thus the graph blow-up $\{M : \mathcal{F}\}$ is the closure of the image through δ of the complement $M \smallsetminus \bigcup \mathcal{F}$ in the product $\prod_{Y\in\mathcal{F}}[M : Y]$ of all the blown-up spaces $[M : Y]$, $Y \in \mathcal{F}$:

$$\{M : \mathcal{F}\} := \overline{\delta(M \smallsetminus \bigcup \mathcal{F})} \subset \prod_{Y\in\mathcal{F}} [M : Y] \,,$$

$$M \smallsetminus \bigcup \mathcal{F} \ni x \to \delta(x) := (x, x, \dots, x) \in \prod_{Y\in\mathcal{F}} [M : Y] \,.$$

Note that we have used here that $M \smallsetminus \bigcup \mathcal{F} \subset M \smallsetminus Y \subset [M : Y]$ for all $Y \in \mathcal{F}$. The graph blow-up will be compared in the next section to the iterated blow-up.

Definition 3.2. *If G is a Lie group acting smoothly on M and $\mathcal{F}$ is a locally finite set of closed p-submanifolds of M such that, for every $Y \in \mathcal{F}$ and $g \in G$, we have $g(Y) \in \mathcal{F}$, then we shall say that* $\mathcal{F}$ is a G-family of p-submanifolds of M.

Corollary 2.6 yields right away the following corollary

Corollary 3.3. *Let G be a discrete group and $\mathcal{F}$ be a G-family of p-submanifolds of M (see Definition 3.2). Then G acts continuously on $\{M : \mathcal{F}\}$.*

Proof. We have that each $g \in G$ acts on $M \smallsetminus \bigcup \mathcal{F}$ and on $\prod_{Y\in\mathcal{F}}[M : Y]$, with the action sending $[M : Y]$ to $[M : g(Y)]$, by Corollary 2.6, which also shows that this action is a smooth action of G on $\prod_{Y\in\mathcal{F}}[M : Y]$. The result follows since δ commutes with the action of G. □

Later on, we will show that $\{M : \mathcal{F}\}$ is a weak submanifold of a suitable manifold with corners, and thus that $\{M : \mathcal{F}\}$ inherits the structure of a manifold with corners. Then the proof above yields that G acts smoothly $\{M : \mathcal{F}\}$.

3.2. **Disjoint submanifolds.** We are allowing our p-submanifolds to have components of different dimensions. Blowing-up with respect to such a manifold amounts, as we will see, to blowing up successively with respect to each component.

We need first to discuss the gluing of open subsests. Let us assume that we have two manifolds with corners M_1 and M_2 and that $U_i \subset M_i$ are open subsets ($i = 1, 2$). Let us also assume that we are given a diffeomorphism $\phi : U_1 \to U_2$. Then we define

$$(19)\quad \begin{aligned} M_1 \cup_\phi M_2 \ &:= \ (M_1 \sqcup M_2)/\{x \equiv \phi(x) \mid x \in U_1\}\,, \\ M_1 \cup_{\mathrm{id}} M_2 \ &=: \ M_1 \cup_{U_1} M_2\,, \quad \text{if } U_1 = U_2 \text{ and } \phi \text{ is the identity map } \mathrm{id}\ . \end{aligned}$$

If ϕ is the identity, we shall call $M_1 \cup_{U_1} M_2$ the *union of* M_1 *and* M_2 *along* $U_1 = U_2$. Under favorable circumstances (but not always), $M_1 \cup_\phi M_2$ is also a manifold with corners.

We have the following simple lemma.

Lemma 3.4. *Let M be a manifold with corners (and hence Hausdorff) and $M_i \subset M$, $i = 1, 2$, be open subsets with $U := M_1 \cap M_2$ and $M_1 \cup M_2 = M$. Then there exists a unique structure of a manifold with corners on $M_1 \cup_U M_2$ that induces the given smooth structures on M_i, and hence we have a canonical diffeomorphism $M_1 \cup_U M_2 \simeq M$.*

Proof. Let $\mathcal{A}_i$ be an atlas for M_i. Then their union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas for M. It is also an atlas for any manifold with corners structure on $M_1 \cup_U M_2$ that induces the given one on each M_i. Hence the desired manifold with corners structure on $M_1 \cup_U M_2$ is given by the union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. □

This allows us to "commute" the procedures of taking blow-ups with respect to disjoint manifolds.

Lemma 3.5. *Let us assume that P and Q are closed p-submanifolds of M such that $P \cap Q = \emptyset$. Let $\beta_{M,Q} : [M : Q] \to M$ be the blow-down map. Then $\beta^*(P) := \beta_{M,Q}^{-1}(P) = P$ and the iterated blow-up $\big[[M : Q] : P\big]$ is defined and diffeomorphic to $([M : Q] \smallsetminus P) \sqcup_{M \smallsetminus (P \cup Q)} ([M : P] \smallsetminus Q)$, the union of $[M : Q] \smallsetminus P$ and $[M : P] \smallsetminus Q$ along $M \smallsetminus (P \cup Q)$, a common open subset. In particular,*

$$\big[[M : P] : Q\big] \ = \ \big[[M : Q] : P\big] \ = \ [M : P \cup Q]\,,$$

with the same smooth structure.

Proof. Since $Q \subset M \smallsetminus P$ and $\beta_{M,P}$ is the identity on $M \smallsetminus P$, we obtain

$$\beta_{M,P}^*(Q) \ := \ \overline{\beta_{M,P}^{-1}(Q \smallsetminus P)} \ = \ \overline{\beta_{M,P}^{-1}(Q)} \ = \ \overline{Q} \ = \ Q\,,$$

which is a p-submanifold of $M \smallsetminus P$ and hence also of $[M : P]$, since the property of being a p-submanifold is a local one. It also follows that $\mathbb{S}(N_+^M Q) = \mathbb{S}(N_+^{[M:P]} Q)$, since $\mathbb{S}(N_+^M Q)$ is also defined locally. We thus obtain

$$\begin{aligned} \big[[M : P] : Q\big] \ &:= \ ([M : P] \smallsetminus Q) \sqcup \mathbb{S}(N_+^{[M:P]} Q) \\ &= \ (M \smallsetminus (P \cup Q)) \sqcup \mathbb{S}(N_+^M P) \sqcup \mathbb{S}(N_+^M Q)\,, \end{aligned}$$

which is symmetric in Q and P, and hence $\big[[M : P] : Q\big] = \big[[M : Q] : P\big]$ as sets. The last part follows from Lemma 3.4 applied to the open subsets

$$\beta_{[M:P],Q}^{-1}([M : P] \smallsetminus Q) \ = \ [M : P] \smallsetminus Q \ \text{ and}$$
$$\beta_{[M:Q],P}^{-1}([M : Q] \smallsetminus P) \ = \ [M : Q] \smallsetminus P$$

of $\big[[M : P] : Q\big] = \big[[M : Q] : P\big]$. □

Definition 3.6. *Suppose $f_i : X \to Y_i$, $i = 1, \dots, N$, are continuous maps. We say $(f_1, \dots, f_N) : X \to \prod_{i=1}^N Y_i$, $x \mapsto (f_1(x), \dots, f_N(x))$ is proper in each component if each f_i is proper.*

We shall need the following lemma.

Lemma 3.7. *Let us assume that P and Q are closed, non-trivial,* disjoint *p-submanifolds of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : \big[[M : Q] : P\big] \to [M : P]$$

that restricts to the identity on $M \smallsetminus (P \cup Q)$. Moreover, the product map

$$\mathcal{B}_{M,Q,P} := (\zeta_{M,Q,P}, \beta_{[M:Q],P}) : \big[[M : Q] : P\big] \to [M : P] \times [M : Q]$$

is proper in each component. Its image is a weak submanifold in the sense of Definition 1.8 and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

Proof. Lemma 3.5 states that $\big[[M : Q] : P\big] = [M : P \cup Q] = \big[[M : P] : Q\big]$. This gives $\zeta_{M,Q,P} = \beta_{[M:P],Q}$. In particular, $\zeta_{M,Q,P}$ is proper, by Corollary 2.7. The map $\beta_{[M:Q],P}$ is proper by Corollary 2.7. As P and Q are disjoint, at each point, at least one component of $\mathcal{B}_{M,Q,P} = (\zeta_{M,Q,P}, \beta_{[M:Q],P})$ is a local diffeomorphism. Thus $\mathcal{B}_{M,Q,P}$ is an immersion. As it is injective and proper, it is a homeomorphism onto its image. Proposition B.1 implies that the image is thus a weak submanifold and that $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image. ☐

By iterating the above lemma, we obtain the following consequence.

Corollary 3.8. *Let $\mathcal{F} := \{P_1, P_2, \dots, P_k\}$ be a family of closed, non-trivial,* disjoint *p-submanifolds of a manifold with corners M. Then we have canonical diffeomorphisms inducing the identity on $M_0 := M \smallsetminus \bigcup_{j=1}^k P_j$ between the usual blow-ups and the graph blow-up (Definitions 2.1 and 3.1):*

$$[[\dots[[M : P_1] : P_2] : \dots : P_{k-1}] : P_k] \simeq [M : \bigcup_{j=1}^k P_j] \simeq \{M : \mathcal{F}\}\,.$$

Proof. This follows by induction from Lemmas 3.5 and 3.7 since P_j identifies naturally with a p-submanifold of $[[\dots[[M : P_1] : P_2] : \dots : P_{j-2}] : P_{j-1}]$. ☐

4. ITERATED BLOW-UPS

The graph blow-up $\{M : \mathcal{F}\}$ introduced in the previous subsection has the advantage that it is defined in great generality and is obviously independent of the order on the family of p-submanifolds $\mathcal{F}$, up to an isomorphism. However, it is not clear what is the structure of the graph blow-up. To this end, in this section, we shall consider an iterated blow-up, which is defined under much more restrictiv conditions, but will be, by construction, a manifold with corners. The main result will be that the iterated blow-up and the graph blow-up are diffeomorphic.

4.1. **Definition of the iterated blow-up.** Recall the definition of the lifting $\beta^*(Q) = \beta^*_{M,P}(Q) := \overline{Q \smallsetminus P} \subset [M : P]$ (closure in $[M : P]$), Definition 2.12. We fix a manifold with corners M. We now introduce the *iterated version of the blow-up.*

Definition 4.1. *Let $(P_i)_{i=1}^k$, $P_i \subset M$, be a k-tuple of closed, non-trivial p-submanifolds of M and let $\beta_1 := \beta_{M,P_1} : [M : P_1] \to M$. (We do not assume any inclusion relations between the p-submanifolds P_i.) Whenever all the terms make sense, we define by induction on k the* iterated blow-up $[M : (P_i)_{i=1}^k]$ of M with respect to *or* along $(P_i)_{i=1}^k$ *by*

$$[M : (P_i)_{i=1}^k] := \begin{cases} [M : P_1] & \text{if } k = 1\,, \\ \big[[M : P_1] : \big(\beta_1^*(P_i)\big)_{i=2}^k\big] & \text{if } k > 1\,. \end{cases}$$

Note that we did not rule out the case $P_{i+1} = P_i$, for some i. In this case we can remove P_{i+1} from the family (P_i) without changing $[M : (P_i)]$. In particular, we can assume that all the manifolds P_i are distinct, without losing generality.

We stress that we do not assume any inclusions among the manifolds P_i, but, on the other hand, $[M : (P_i)_{i=1}^k]$ is not always defined (unlike the graph blow-up!), as we need additional conditions in order to guarantee that $\beta_{j-1}^*\beta_{j-2}^* \cdots \beta_1^*(P_j)$ is a closed p-submanifold for all j. We shall also write

$$[M : (P_i)_{i=1}^k] =: [M : P_1, P_2, \ldots, P_k]\,,$$

and hence, using the pull-back by the map β_1, we have

$$[M : P_1, P_2, \ldots, P_k] := \big[[M : P_1] : \beta_1^*(P_2), \ldots, \beta_1^*(P_k)\big]\,.$$

We generalize this relation in the following remark.

Remark 4.2. Let $\gamma_1 := \beta_1^*$ and $\gamma_j := \beta_j^* \circ \gamma_{j-1} = \beta_j^* \circ ... \circ \beta_1^*$, where

$$\beta_k := \beta_{[M,P_1,P_2,\ldots,P_{k-1}],P_k} : [M : P_1, P_2, \ldots, P_k] \to [M : P_1, P_2, \ldots, P_{k-1}]\,.$$

We then have

$$\begin{aligned} [M : P_1, P_2, \ldots, P_j] &= \big[[M : P_1] : \gamma_1(P_2), \ldots, \gamma_1(P_j)\big] \\ &= \Big[\big[[M : P_1] : \gamma_1(P_2)\big] : \gamma_2(P_3), \ldots, \gamma_2(P_j)\Big] \\ &= \ldots \\ &= [[\ldots[[[M : P_1] : \gamma_1(P_2)] : \gamma_2(P_3)] \ \ldots] : \gamma_{j-1}(P_j)]\,. \end{aligned}$$

Note that $[M : P_1, P_2, \ldots, P_j]$ is always defined if $j = 1$. Then the condition that the iterated blow-up $[M : P_1, P_2, \ldots, P_j]$ be defined can then be formulated by induction as follows:

(i) the iterated blow-up $[M : P_1, \ldots, P_{j-1}]$ is defined, and
(ii) the lift $\gamma_{j-1}(P_j)$ is defined and is a closed p-submanifold of $[M : P_1, \ldots, P_{j-1}]$.

4.2. **Clean semilattices.** We now investigate the iterated blow-up $[M : (P_i)_{i=1}^k]$ of a manifold with corners M with respect to a (suitably) *ordered* family of non-trivial p-submanifolds of M.

Definition 4.3. *Let $\mathcal{F}$ be a locally finite (unordered) set of p-submanifolds of M. We shall say that $\mathcal{F}$ is a* cleanly intersecting family *if any $X_1, X_2, \ldots, X_j \in \mathcal{F}$ have a clean intersection (Definition 2.9).*

We consider the iterated blow-up mostly with respect to semilattices. Recall that a *meet semilattice* (or, simply, *semilattice* in what follows) is a partially ordered set $\mathcal{L}$ such that, for every two $x, y \in \mathcal{L}$, there is a greatest common lower bound $x \cap y \in \mathcal{L}$ of x and y. We shall consider only semi-lattices of subsets of a given set where the order is given by $\subset$ and where $x \cap y$ is the usual intersection of sets. We can now introduce the semi-lattices we are interested in. We let $\mathcal{P}(M)$ denote the set of all subsets of M.

Definition 4.4. *A semilattice $\mathcal{S} \subset \mathcal{P}(M)$ of closed p-submanifolds of M will be called* clean *if $\mathcal{S}$ is a cleanly intersecting family of p-submanifolds of M.*

For the simplicity of the notation, we shall consider only semilattices $\mathcal{S} \subset \mathcal{P}(M)$ with $\emptyset \in \mathcal{S}$. This changes nothing in our results, but avoids us treating separately the cases $\emptyset \in \mathcal{S}$ and $\emptyset \notin \mathcal{S}$ in proofs. The concept of a clean semilattice introduced here is very closely related to that of a weakly transversal family considered in [1, Definition 2.7], except that in that paper, the authors considered only p-submanifolds that were *not* contained in the boundary.

Remark 4.5. Clean semilattices of closed p-submanifolds are useful for studying iterated blow-ups because, if P, Q are two p-submanifolds of a manifolds with corners M such that P and Q intersects cleanly, then the lifts of P and Q in $[M : P \cap Q]$ are disjoint p-submanifolds of $[M : P \cap Q]$. See also [1, Theorem 2.8].

The following result was essentially proved in [1]. The Lemma 5.11.2 of [15] treats also the lift of a family under the blow-up. The author proved that a the lift of a normal family remains a normal family if we do the blow-up by an element of the family.

Proposition 4.6. *Let $\mathcal{S} \ni \emptyset$ be a clean semilattice (of p-submanifolds) of M and let P be a minimal element of $\mathcal{S} \smallsetminus \{\emptyset\}$. Let $Q' := [Q : P \cap Q]$. Then*

$$\mathcal{S}' \ := \ \Big\{ Q' = [Q : P \cap Q] \,\Big|\, Q \in \mathcal{S} \Big\} .$$

is a clean semilattice of $[M : P]$ with $\emptyset = \emptyset' = P' \in \mathcal{S}'$.

Let $Q' := [Q : Q \cap P]$, so that $\mathcal{S}' = \{Q' \mid Q \in \mathcal{S}\}$. Recall that the minimality of P and the semilattice property of $\mathcal{S}$ imply that, for any $Q \in \mathcal{S}$, we have either $P \subset Q$ or $P \cap Q = \emptyset$. In the first case, we have $Q' := [Q : P \cap Q] = [Q : P]$ and in the second case we have $Q' := [Q : P \cap Q] = Q$. Thus

$$\mathcal{S}' \ := \ \{ [Q : P] \mid P \subset Q \in \mathcal{S} \} \cup \{ Q \mid Q \in \mathcal{S}, Q \cap P = \emptyset \} .$$

Let us also notice that $P' := [P : P \cap P] = \emptyset = [\emptyset : \emptyset \cap P] = \emptyset'$, whereas all the other manifolds Q' ($Q \in \mathcal{S} \smallsetminus \{\emptyset, P\}$) are different to each other and nonempty. Therefore, $|\mathcal{S}'| = |\mathcal{S}| - 1$ (i.e. $\mathcal{S}'$ has one element less than $\mathcal{S}$).

Proof. This result was proved in slightly less generality in [1, Theorem 2.8] (assuming that the p-manifolds are *not* contained in the boundary). The proof extends right away to the current setting. □

Let $\mathcal{S}$ be clean semilattice (of p-submanifolds) of M and let us arrange $\mathcal{S} \smallsetminus \{\emptyset\}$ in a sequence $(P_i)_{i=1}^k = (P_1, P_2, \ldots, P_k)$. Recall that $(P_i)_{i=1}^k$ is an ordering of $\mathcal{S} \smallsetminus \{\emptyset\}$ compatible with the inclusion if $i \le j$ whenever $P_i \subset P_j$, see Equation (2). An ordering with these properties was called an *admissible ordering* in [1, Definition 2.9].

Proposition 4.7. *Let $\mathcal{S}$ be clean semilattice (of closed p-submanifolds) and $(P_i)_{i=1}^k = (P_1, P_2, \ldots, P_k)$ be an ordering of $\mathcal{S} \smallsetminus \{\emptyset\}$ compatible with the inclusion ($P_i \subset P_j \Rightarrow i \le j$). Then $[M : (P_i)_{i=1}^k]$ is defined.*

Proof. We shall write $[M : \mathcal{S}] := [M : (P_i)_{i=1}^k]$. (This definition of $[M : \mathcal{S}]$ is implicitly assuming that a compatible order was chosen on $\mathcal{S}$. The notation is nevertheless justified since Theorem 4.12 will show that the result is independent of the order.) To prove that $[M : \mathcal{S}] := [M : (P_i)_{i=1}^k]$ is defined, we shall proceed by induction on the number of elements of $\mathcal{S}$. As $\emptyset \in \mathcal{S}$, let us assume, for the initial verification step, that $|\mathcal{S}| = 2$

and, more precisely, that $\mathcal{S} = \{\emptyset, P\}$, for some non-trivial p-submanifold P of M. Then $[M : (P_i)_{i=1}^1] := [M : P]$ is defined.

Let us assume that the result is true for semilattices $\mathcal{S}$ with j elements and prove it for lattices with $j+1$ elements. Then the semilattice $\mathcal{S}'$ obtained from $\mathcal{S}$ via Proposition 4.6 is a clean semilattice with j elements of $[M : P_1]$ by that same proposition. Therefore $\big[[M : P_1] : \mathcal{S}'\big]$ is defined by the induction hypothesis, and hence, using also Remark 4.2, we have that

$$[M : \mathcal{S}] \ := \ \big[[M : P_1] : \mathcal{S}'\big] \tag{20}$$

is also defined. □

Remark 4.8. Note that the normal sphere bundle of a submanifold of codimension 0 is the empty set, thus $[M : M] = \emptyset$. As a consequence, our definitions imply $[M : \mathcal{S}] = \emptyset$ in the case $M \in \mathcal{S}$. This is why all interesting examples satisfy $M \notin \mathcal{S}$.

4.3. **The pair blow-up lemma.** We now perform some essential calculations in local coordinates that will be needed for our main result. Recall from Equation (6) that

$$\mathbb{S}^n_k \ := \ \mathbb{S}^n \cap \mathbb{R}^{n+1}_k ,$$

where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^n$, as always. For $\psi \in \mathbb{S}^{n'+1}_{k'+1} := \mathbb{S}^{n'+1} \cap \mathbb{R}^{n'+2}_{k'+1}$, we shall write $\psi =: (\psi_1, \tilde{\psi})$, with $\psi_1 \in [0,1]$ and $\tilde{\psi} \in \mathbb{R}^{n'+1}_{k'}$, and we define the map

$$\begin{gathered} \Upsilon : \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \to \mathbb{S}^{n,n'}_{k,k'} \ := \ \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right) \\ (\phi, \psi) \mapsto (\psi_1 \phi, \tilde{\psi}) \, . \end{gathered} \tag{21}$$

We embed the sphere octant $\{0\} \times \mathbb{S}^{n'}_{k'} = \{0_{\mathbb{R}^n}\} \times \mathbb{S}^{n'}_{k'} \subset \mathbb{R}^{n+n'+1}$ into $\mathbb{R}^{n+n'+1}$ by mapping the sphere octant to the *last* components of $\mathbb{R}^{n+n'+1}$. Of course, we have an isomorphism

$$\mathbb{S}^{n,n'}_{k,k'} = \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right) \ \cong \ \mathbb{S}^{n+n'}_{k+k'} = \mathbb{S}^{n+n'} \cap \mathbb{R}^{n+n'+1}_{k+k'}$$

given by the canonical permutation of coordinates diffeomorphism of Equation (8).

We recall Proposition 5.8.1 of [15] and we give the proof to fix the notation.

Lemma 4.9. *Let again* $\mathbb{S}^{n,n'}_{k,k'} := \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right) \cong \mathbb{S}^{n+n'}_{k+k'}$ *and let the map* $\Upsilon : \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \to \mathbb{S}^{n,n'}_{k,k'}$ *be as in the last paragraph. If we define*

$$\Psi : \mathbb{S}^{n,n'}_{k,k'} \smallsetminus \left(\{0\} \times \mathbb{S}^{n'}_{k'}\right) \ \to \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} , \quad \Psi(\eta, \mu) \ = \ \Big(\frac{\eta}{|\eta|}, \, \big(|\eta|, \mu\big) \Big) ,$$

then $\Upsilon \circ \Psi$ *is the inclusion* $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus \left(\{0\} \times \mathbb{S}^{n'}_{k'}\right) \subset \mathbb{S}^{n,n'}_{k,k'}$ *and* Ψ *extends to a diffeomorphism*

$$\widetilde{\Psi} : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \ \xrightarrow{\sim} \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$$

such that $\beta_{\mathbb{S}^{n,n'}_{k,k'}, \{0\} \times \mathbb{S}^{n'}_{k'}} = \Upsilon \circ \widetilde{\Psi}$.

If we write by abuse of notation $\mathbb{S}^{n'}_{k'}$ for the image of $\{0\} \times \mathbb{S}^{n'}_{k'}$ in $\mathbb{S}^{n+n'}_{k+k'}$ under the permutation of coordinates described above, then we obtain a diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^{n+n'}_{k+k'} : \mathbb{S}^{n'}_{k'}] \ \xrightarrow{\sim} \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \, .$$

Proof. Let

$$\beta \ := \ \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\} \times \mathbb{R}^{n'+1}_{k'}} : [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}] \to \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \, .$$

denote the blow-down map. Also, recall that the lifting $\beta^*(\mathbb{S}^{n,n'}_{k,k'})$ is defined as the closure of $\beta^{-1}(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus \{0\} \times \mathbb{R}^{n'+1}_{k'})$ in $[\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}]$. Since

$$\mathbb{S}^{n,n'}_{k,k'} \cap \big(\{0\} \times \mathbb{R}^{n'+1}_{k'}\big) \ = \ \{0\} \times \mathbb{S}^{n'}_{k'} \, ,$$

Proposition 2.13 gives a diffeomorphism

$$\Phi : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \ \xrightarrow{\sim} \ \beta^*(\mathbb{S}^{n,n'}_{k,k'}),$$

uniquely determined by the condition that is the inclusion on $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus \{0\} \times \mathbb{S}^{n'}_{k'}$. (That is, the blow-up of $\mathbb{S}^{n,n'}_{k,k'}$ along $\{0\} \times \mathbb{S}^{n'}_{k'}$ is diffeomorphic to the lifting $\beta^*(\mathbb{S}^{n,n'}_{k,k'})$ of $\mathbb{S}^{n,n'}_{k,k'}$ to $[\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}]$ via the blow-down map $\beta \ := \ \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\} \times \mathbb{R}^{n'+1}_{k'}}$.)

To identify more explicity the space $\beta^*(\mathbb{S}^{n,n'}_{k,k'})$, it is convenient to use the diffeomorphism $\kappa \ : \ \mathbb{S}^{n-1}_k \times [0, +\infty) \times \mathbb{R}^{n'+1}_{k'} \ \to \ [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \ : \ \{0\} \times \mathbb{R}^{n'+1}_{k'}]$ of Equation (12) with the order of its arguments reversed. To start with, the blow-down map $\beta := \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\} \times \mathbb{R}^{n'+1}_{k'}}$ is such that $\beta_1 := \beta \circ \kappa$ satisfies

$$\begin{gathered} \beta_1 := \beta \circ \kappa : \mathbb{S}^{n-1}_k \times [0, +\infty) \times \mathbb{R}^{n'+1}_{k'} \ \to \ \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \, , \\ \beta_1(z, r, x) \ = \ (rz, x). \end{gathered}$$

We have that $(z, r, x) \in \beta_1^{-1}(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'}))$ if, and only if $\|\beta_1(z, r, x)\| = 1$ and $r > 0$. Assume that $\|\beta_1(z, r, x)\| = 1$ and $r > 0$. Then $\|rz\|^2 + \|x\|^2 = 1$. Note that $z \in \mathbb{S}^{n-1}_k$, and hence $r^2 + \|x\|^2 = 1$. This leads to $(r, x) \in \mathbb{S}^{n'+1}_{k'+1} \subset \mathbb{R}^{n'+2}_{k'+1} = [0, \infty) \times \mathbb{R}^{n'+1}_{k'}$. We thus have

$$\beta_1^{-1}\big(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'})\big) \ = \ (\mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}) \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'}) \, .$$

The closure of this set is $\mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$, and hence we obtain a diffeomorphism $\Phi_1 := \kappa^{-1} \circ \Phi : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \ \xrightarrow{\sim} \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$. That $\Upsilon \circ \Psi$ is the inclusion follows from the defining formulas. The relation $\beta_{\mathbb{S}^{n,n'}_{k,k'}, \{0\} \times \mathbb{S}^{n'}_{k'}} = \Upsilon \circ \widetilde{\Psi}$ follows from the fact that they are both continuous and they coincide on the dense, open subset $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{S}^{n'}_{k'})$. This shows that $\Phi_1 = \Psi$ on $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{S}^{n'}_{k'})$ and hence $\tilde{\Psi} := \Phi_1$ is the desired extension. □

We now treat the basic case when the blow-up is defined, namely the simplest case when we blow up by two p-submanifolds P and Q with $Q \subset P$. The case when of two disjoint p-submanifolds was already treated in Lemma 3.7, so now we treat the remaining case, that is, that one submanifold is contained in the other.

Lemma 4.10. *Let us assume that Q is a p-submanifold of P and that P is a p-submanifold of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : [M : Q, P] \ := \ \big[[M : Q] : [P : Q]\big] \ \to \ [M : P]$$

that restricts to the identity on $M \smallsetminus P$. Moreover, the product map

$$\mathcal{B}_{M,Q,P} \ := \ (\zeta_{M,Q,P}, \beta_{[M:Q],[P:Q]}) : [M : Q, P] \to [M : P] \times [M : Q]$$

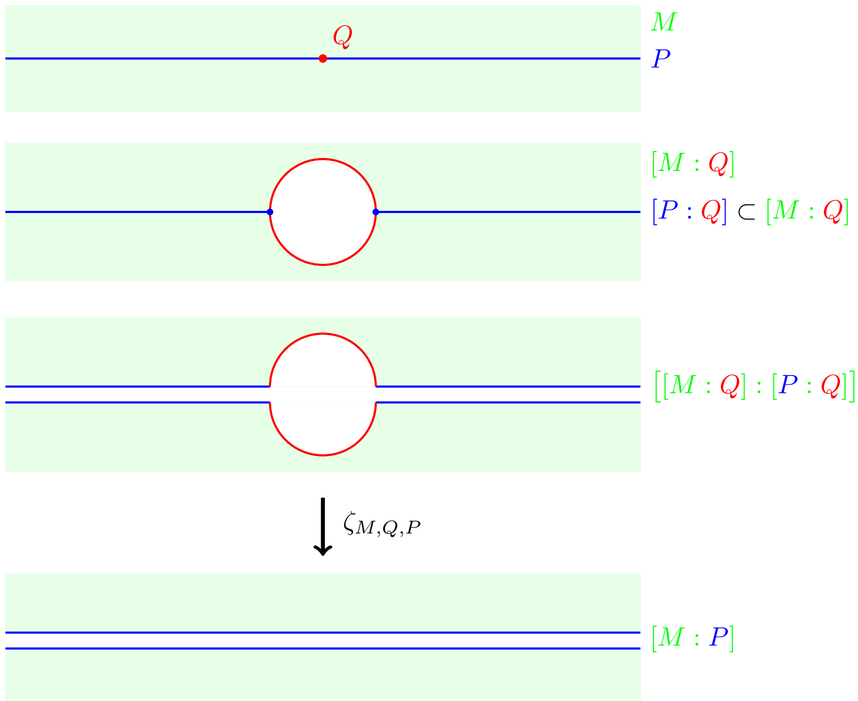

FIGURE 1. The blow-ups $[M : Q]$, $\big[[M : Q] : [P : Q]\big]$, and $[M : P]$

is proper in each component. The image of $\mathcal{B}_{M,Q,P}$ is a weak submanifold in the sense of Definition 1.12, and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

See Figure 4.3 for a local picture of these blow-ups in the example $M = \mathbb{R}^2$, $P = \mathbb{R} \times \{0\}$, $Q = \{0\}$.

Proof. The uniqueness of the map $\zeta_{M,Q,P}$ follows from the fact that it is the identity on the dense subset $M \smallsetminus (P \cup Q)$. The statement is local, so, in view of Lemma 2.8, we can assume that $Q = \{0\}$. That is, we can assume that

$$\begin{cases} M & := & \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P & := & \{0\} \times \mathbb{R}^p_{k_p} \\ Q & := & \{0\} \end{cases} \tag{22}$$

We have

$$\begin{aligned} [M : P] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0\} \times \mathbb{R}^p_{k_p}] \\ &= [\mathbb{R}^m_{k_m} : \{0\}] \times \mathbb{R}^p_{k_p} \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0, \infty) \times \mathbb{R}^p_{k_p} \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} . \end{aligned}$$

Its blow-down map is $\beta_{M,P}(x, t, y) = (tx, y)$.

On the other hand, we have (using the notation of Lemma 4.9):

$$[M : Q] = [\mathbb{R}^{m+p}_{k_m+k_p} : \{0\}] = \mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0, \infty).$$

Its blow-down map is $\beta_{M,Q}(x,t) = tx$. Lemma 2.13 gives that the lift of P to $[M : Q]$ is $P' := [P : Q] = \{0\} \times \mathbb{S}^{p-1}_{k_p} \times [0,\infty)$. Lemmas 2.8 and 4.9 (in this order) then give canonical diffeomorphisms

$$\begin{aligned}\big[[M:Q]:P'\big] &\simeq [\mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0,\infty) : \mathbb{S}^{p-1}_{k_p} \times [0,\infty)]\\ &= [\mathbb{S}^{m,p-1}_{k_m,\,k_p} : \mathbb{S}^{p-1}_{k_p}] \times [0,\infty)\\ &\simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^{p}_{k_p+1} \times [0,\infty)\,.\end{aligned}$$

The blow-down map $\beta_{[[M:Q]:P]} : \big[[M : Q] : P'\big] \to [M : Q]$ is given, up to canonical diffeomorphisms, by the map $\Upsilon \times \mathrm{id}$, where Υ is as defined in Equation (21). Hence $\Upsilon \times \mathrm{id}(\phi,\psi,t) = (\psi_1\phi,\tilde{\psi},t)$.

The desired map $\zeta_{M,Q,P}$ is then obtained from the blow-down map $\mathbb{S}^p_{k_p+1} \times [0,\infty) \to \mathbb{R}^{p+1}_{k_p+1} = [0,\infty) \times \mathbb{R}^p_{k_p}$, that is $\zeta_{M,Q,P}(x,y,t) = (x,ty)$. In particular, it is proper. It remains to check that this map is the identity on $M \smallsetminus P$. As we used for $x \in M \smallsetminus P$ the identifications $x \mathrel{\widehat{=}} \beta_{M,Q}(x) \mathrel{\widehat{=}} \beta_{M,P}(x) \mathrel{\widehat{=}} \beta_{[M:Q],[P:Q]}(x)$, it is enough to check

$$\beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ \beta_{[M:Q],[P:Q]} \tag{23}$$

on $M \smallsetminus P$. As this calculation is local, we can again assume (22) and the concrete presentations of $\beta_{M,Q}$, $\beta_{M,P}(x)$ and $\beta_{[M:Q],[P:Q]}$ described above, (23) turns into

$$\beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ (\Upsilon \times \mathrm{id}) \tag{24}$$

on $\mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0,\infty)$. Indeed for $x \in \mathbb{S}^{m-1}_{k_m}$, $y = (y_1,\tilde{y}) \in \mathbb{S}^p_{k_p+1} \subset \mathbb{R}^{p+1}_{k_p+1} = \mathbb{R}^1_1 \times \mathbb{R}^p_{k_p}$, $t \in [0,\infty) = \mathbb{R}^1_1$ we have

$$\beta_{M,P} \circ \zeta_{M,Q,P}(x,y,t) \;=\; \beta_{M,P}(x,ty) \;=\; \beta_{M,P}(x,ty) \;=\; (ty_1x,t\tilde{y})\,.$$

Together with

$$\beta_{M,Q} \circ (\Upsilon \times \mathrm{id})(x,y,t) \;=\; \beta_{M,Q}(y_1x,\tilde{y},t) \;=\; (ty_1x,t\tilde{y})\,,$$

this implies (24).

The map $\mathcal{B}$ is given in local coordinates by $\mathcal{B}(x,y,t) = \big(x,ty,(y_1x,\tilde{y}),t\big)$ with differentiable left inverse $(x,z,(w_1,w_2),t) \mapsto (x,(\|w_1\|,w_2),t)$. Hence by Corollary B.2 the image of $\mathcal{B}$ is a weak submanifold and $\mathcal{B}$ is a diffeomorphism onto its image. □

Remark 4.11. Note that, in general, the image of the map $\mathcal{B}_{M,Q,P}$ introduced in the proof above is not a p-submanifold of $[M : P] \times [M : Q]$. Indeed, let us consider the case when M is the closed unit disk in $\mathbb{R}^2$, and let p and q be two disjoint points in the interior of M. Let $P := \{p\}$ and $Q := \{q\}$. We claim that the image N of $\mathcal{B} = \mathcal{B}_{M,Q,P}$ is *not* a p-submanifold of $M_1 := [M : P] \times [M : Q]$. Suppose N were a p-submanifold of M_1. As N is connected, the function $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x)$ is constant on N, see Remark 1.15. However, the map $\mathcal{B}$ sends the interior points of $M \smallsetminus \{p,q\}$ to the interior of $M_1 = [M : P] \times [M : Q]$, thus $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x) = 0 - 0 = 0$ for $x = \mathcal{B}(y)$ with y in the interior of $M \smallsetminus \{p,q\}$. On the other hand $\mathcal{B}$ maps the boundary $\partial M = \partial(M \smallsetminus \{p,q\})$ to the corner $\partial M \times \partial M$ of $[M : P] \times [M : Q]$, which has boundary depth 2 in $M_1 = [M : P] \times [M : Q]$. Thus, if $x = \mathcal{B}(y)$, with $y \in \partial M$, we obtain $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x) = 2-1 = 1$. Therefore, the function $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x)$ is not constant on N, and hence N is not a p-submanifold of $M_1 = [M : P] \times [M : Q]$.

We conjecture that the image of $\mathcal{B}_{M,Q,P}$ is a b-submanifold and a wib-submanifold. We will discuss this in more detail in Remark 4.14.

Using the similar result for disjoint manifolds, Lemma 3.7, we obtain the following result. (Recall that our semilattices contain the empty set, but do not contain the ambient manifold M.)

Theorem 4.12. *Let $\mathcal{S} = (P_j)_{j=0,1,\dots,k}$ be a clean semilattice of closed p-submanifolds of M (so $\emptyset \in \mathcal{S}$). Then, for each $P \in \mathcal{S}$, there exists a unique smooth map $\phi_{\mathcal{S},P} : [M : \mathcal{S}] \to [M : P]$ that is the identity on $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$. These maps are such that the induced map*

$$\mathcal{B}_{\mathcal{S}} \,:=\, (\phi_{\mathcal{S},P_0}, \dots, \phi_{\mathcal{S},P_k}) : [M : \mathcal{S}] \to \prod_{j=0}^{k} [M : P_j]$$

is proper in each component. Furthermore the image of $\mathcal{B}_{\mathcal{S}}$ is a weak submanifold of $\prod_{j=0}^{k}[M : P_j]$ in the sense of Definition 1.12 and $\mathcal{B}_{\mathcal{S}}$ maps $[M : \mathcal{S}]$ diffeomorphically onto $\{M : \mathcal{S}\}$, i.e. we have a diffeomorphism

$$\mathcal{B}_{\mathcal{S}} : [M : \mathcal{S}] \xrightarrow{\sim} \{M : \mathcal{S}\} .$$

Proof. We shall proceed by induction on the number $k{+}1$ of elements of $\mathcal{S} = (P_j)_{j=0,1,\dots,k}$. We can assume that $P_i \neq \emptyset = P_0$ for all $i > 0$. (So k is the number of *non-empty* elements of $\mathcal{S}$.) The case $k = 0$ is trivial.
Case $k = 1$**:** If $\mathcal{S}$ has $1 + 1 = 2$ elements, we have $\mathcal{S} = (\emptyset, P)$ and $\mathcal{B}_{\mathcal{S}} = (\beta_{M,P}, \mathrm{id}_{[M:P]})$ so the claim is trivially satisfied, since the blow-down map is proper (Corollary 2.7).
Case $k = 2$**:** If $\mathcal{S}$ has $2 + 1 = 3$ elements, we have $\mathcal{S} = \{\emptyset, Q, P\}$ and we have $Q \subset P$ or $Q \cap P = \emptyset$.
1) In the first subcase, that is, if $Q \subset P$, the result was already proved in Lemma 4.10, with

$$\mathcal{B}_{\mathcal{S}} \,:=\, (\beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}, \beta_{[M:Q],[P:Q]}, \zeta_{M,Q,P}) ,$$

that is, we have, $\phi_{\mathcal{S},\emptyset} = \beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}$, $\phi_{\mathcal{S},Q} := \beta_{[M:Q],[P:Q]}$, $\phi_{\mathcal{S},P} := \zeta_{M,Q,P}$. In particular, the fact that $\mathcal{B}_{M,Q,P} = (\beta_{[M:Q],[P:Q]}, \zeta_{M,Q,P})$ is a diffeomorphism onto its image implies the same statement for $\mathcal{B}_{\mathcal{S}}$.
2) Similarly, in the second subcase, that is, if $Q \cap P = \emptyset$, the result was already proved in Lemma 3.7, with $\mathcal{B}_{\mathcal{S}} = (\beta_{M,P\cup Q}, \beta_{[M:Q],P}, \beta_{[M:P],Q})$, i.e. all the components of $\mathcal{B}_{\mathcal{S}}$ are given by blow-down maps. The diffeomorphism property for $\mathcal{B}_{\mathcal{S}}$ comes from the fact that its restriction to $[M \smallsetminus Q : P]$ and $[M \smallsetminus P : Q]$ has a component equal to the identity, so it is a local diffeomorphism onto its image, which is at the same time injective and proper, thus having a continuous inverse.
Case $k \geq 3$**:** Let us now proceed with the induction step from $k - 1$ to k, that is, let us assume that $\mathcal{S}$ has $k + 1$ elements $P_0 = \emptyset, P_1, \dots, P_k$. As always, the numbering of the sets P_j is chosen to be compatible with the inclusion (an admissible ordering), meaning that if $P_i \subset P_j$, then $i \leq j$. In particular, P_1 must be a minimal element of $\mathcal{S} \smallsetminus \emptyset$ with respect to the relation $\subset$. For $P := P_j \in \mathcal{S}$, $2 \leq j \leq k$, we thus have $P_1 \subset P$ or $P_1 \cap P = \emptyset$, by the minimality of P_1 in $(P_1, \dots, P_k)$ and by the fact that $\mathcal{S}$ is stable under intersections. Let $P' := [P : P \cap P_1]$. Thus we have $P' = [P : P_1]$, if $P_1 \subset P$, and $P' = P$, if $P_1 \cap P = \emptyset$. We shall use the notation of Proposition 4.6 with $P := P_1$, in particular, $Q' := [Q : Q \cap P_1]$. The semilattice $\mathcal{S}' = \big(P_j' := [P_j : P_j \cap P_1]\big)_{j=1,\dots k}$ of Proposition 4.6 is then clean. Note that $P_1' := [P_1 : P_1 \cap P_1] = \emptyset = \emptyset'$, and hence $\mathcal{S}'$ has k elements. By the induction hypothesis, the map $\mathcal{B}_{\mathcal{S}'}$ is a diffeomorphism onto its image. The same property is shared by the maps

$$\mathcal{B}_{M,P_1,P_j} : \big[[M : P_1] : [P_j : P_1]\big] \to [M : P_1] \times [M : P_j]$$

of the Lemmata 3.7 and 4.10. Let $\Phi := \mathrm{id} \times \prod_{j=2}^k \mathcal{B}_{M,P_1,P_j}$ and consider the composition

$$(25)\quad [M:\mathcal{S}] \;:=\; \big[[M:P_1]:\mathcal{S}'\big] \xrightarrow{\mathcal{B}_{\mathcal{S}'}} \prod_{j=1}^{k} \big[[M:P_1]:[P_j:P_1]\big]$$
$$\xrightarrow{\Phi} [M:P_1] \times \prod_{j=2}^{k} \Big([M:P_1]\times[M:P_j]\Big)\,.$$

The two maps of the composition are both injective immersions, and hence their composition is again an injective immersion. The desired map $\phi_{\mathcal{S},P_j}$ is the projection onto the P_j-component. The projection of the composite map onto any of the factors is the identity on $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$. Note that all components with factors of the form $[M : P_1]$ (which are repeated), yield the same projection, again because this projection is the identity map on $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$. By removing these repetitions, and by adding the iterated blow-down map $[M : \mathcal{S}] \to M$ we obtain the desired map $\mathcal{B}_{\mathcal{S}}$, which is consequently also an injective immersion. The map $\mathcal{B}_{\mathcal{S}}$, is proper in each component, and thus proper. It follows from Corollary A.2 that $\mathcal{B}_{\mathcal{S}}$ is a homeomorphism to its image $N := \mathcal{B}_{\mathcal{S}}([M : \mathcal{S}])$. With Proposition B.1 we see that N is a weak submanifold of $\prod_{j=0}^k [M : P_j]$, and that $\mathcal{B}_{\mathcal{S}}$ is a diffeomorphism onto N.

It remains to argue that N coincides with

$$\{M:\mathcal{S}\} \overset{\text{(def)}}{=} \overline{\mathcal{B}_{\mathcal{S}}\Big(M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q\Big)}\,.$$

For any $x \in [M : \mathcal{S}]$ there is a sequence (x_i) in $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$ converging to x in $[M : \mathcal{S}]$. Thus

$$\mathcal{B}_{\mathcal{S}}\Big(M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q\Big) \ni \mathcal{B}_{\mathcal{S}}(x_i) \to \mathcal{B}_{\mathcal{S}}(x)\,,$$

thus $\mathcal{B}_{\mathcal{S}}(x) \in \{M : \mathcal{S}\}$. It follows that $N \subset \{M : \mathcal{S}\}$.

Conversely, for $y \in \{M : \mathcal{S}\}$ there is a sequence $y_i = \mathcal{B}_{\mathcal{S}}(x_i)$ in $\mathcal{B}_{\mathcal{S}}\Big(M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q\Big)$ converging to y in $\prod_{j=0}^k [M : P_j]$. Thus $\{y_i \mid i \in \mathbb{N}\} \cup \{y\}$ is compact, and by properness of $\mathcal{B}_{\mathcal{S}}$ the set

$$\big(\mathcal{B}_{\mathcal{S}}\big)^{-1}\left(\{y_i \mid i\in\mathbb{N}\}\cup\{y\}\right) = \{x_i \mid i\in\mathbb{N}\} \cup \big(\mathcal{B}_{\mathcal{S}}\big)^{-1}\left(\{y\}\right)$$

is compact as well. As a consequence a subsequence x_{i_k} has to converge to some $z \in [M : \mathcal{S}]$. We conclude that

$$N \ni \mathcal{B}_{\mathcal{S}}(z) = \lim_{k\to\infty} \mathcal{B}_{\mathcal{S}}(x_{i_k}) = \lim_{k\to\infty} y_{i_k} = y.$$

This yields $\{M : \mathcal{S}\} \subset N$. □

Again, the image of the map $\mathcal{B}_{\mathcal{S}}$ is, in general, not a p-submanifold, see Remark 4.11. We obtain the following corollary.

Corollary 4.13. *Let $\mathcal{S}$ be a clean semilattice of closed p-submanifolds of M. If G is a discrete group acting smoothly on M such that $g(\mathcal{S}) = \mathcal{S}$ for $g \in G$, then G acts smoothly on $[M : \mathcal{S}]$ and the action commutes with the above homeomorphism $\mathcal{B}_{\mathcal{S}}$.*

Proof. Let δ be the diagonal embedding $\delta(x) = (x, x, \dots, x)$ considered before. Theorem 4.12 gives that $\mathcal{B}_S = \delta$ on the dense open subset $M \smallsetminus \bigcup_{Q\in\mathcal{S}} Q$. Hence the image of the map $\mathcal{B}_{\mathcal{S}}$ is contained in the graph blow-up $\{M : \mathcal{S}\}$, by the definition of the later. We know that $\mathcal{B}_{\mathcal{S}}$ is continuous and proper, and hence with closed image. This gives that $\mathcal{B}_{\mathcal{S}}([M : \mathcal{S}]) = \{M : \mathcal{S}\}$. □

Remark 4.14. It is natural to ask whether we can replace "weak submanifold" by "wib-submanifold" or "b-submanifold" in Lemma 3.7, Corollary 3.8, Lemma 4.10, Theorem 4.12, and Equation (3). We conjecture that the submanifolds are both wib-submanifols and b-submanifolds. It is obvious by the discussion above, that they cannot be d-submanifold, and therefore they are not p-submanifolds. However, a proof – if it exists – of the wib- and b-submanifold properties would require considerable additional work. We do no want to carry this out here, as it is not needed for proving the main result of our article, namely to prove that Georgescu's compactification is homeomorphic to Vasy's compactification. Note that Georgescu's compactification, see Definition 5.8,does not come equipped naturally with a smoot structure, so it does not make sense to ask whether this homeomorphism is a diffeomorphism.

5. Applications to the N-body problem

5.1. **Spherical compactifications.** For any finite dimensional real vector space Z, recall that $\mathbb{S}_Z$ denotes the set of vector directions in Z, that is, the set of (non-constant) open half-lines $\mathbb{R}_+ v$, with $0 \neq v \in Z$ and $\mathbb{R}_+ := (0, \infty)$. The disjoint union

$$\overline{Z} \,:=\, Z \sqcup \mathbb{S}_Z \tag{26}$$

is then called the *radial compactification* of Z. For example, if $Z = \mathbb{R}$, then $\overline{Z} := [-\infty, \infty]$ with the usual topology. The action of the group $\mathrm{GL}(Z)$ of linear automorphisms of Z extends, by definition, to an action on $\overline{Z}$. Similarly, if $Y \subset Z$, then $\overline{Y} \subset \overline{Z}$. In particular, $\overline{Z}$ is the union of all *closed lines* $\overline{\mathbb{R}v}$, $0 \neq v \in Z$, with closure taken in $\overline{Z}$.

In the existing literature, $\overline{Z}$ carries a topology and a smooth structure, and our next goal is to recall their definitions. This will turn $\overline{Z}$ into a smooth manifold with boundary. For notational purposes it is convenient consider the case $Z = \mathbb{R}^n$ first. There is a bijection between the set of vector directions in $\mathbb{R}^{n+1}$ and its unit sphere $\mathbb{S}^n$. This allows us to regard $\mathbb{S}^n_1 := \{(x_1, x') \mid x_1 \geq 0\}$ as a subset of the set of vector directions in $\mathbb{R}^n$, where we used the usual notation of Equation (5). Now, let us regard $\mathrm{GL}(\mathbb{R}^n)$ as a subgroup of $\mathrm{GL}(\mathbb{R}^{n+1})$ with the action on the last n variables. This yields an action of $\mathrm{GL}(\mathbb{R}^n)$ on $\mathbb{S}^n_1$. We have the following standard lemma (see also [15, 21]), where we denote, as usual,

$$\langle x\rangle^2 \,:=\, 1 + \|x\|^2 \,=\, \|(1, x)\|^2 .$$

Definition 5.1. *We define a map* $\Theta_n : \overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup \mathbb{S}_{\mathbb{R}^n} \to \mathbb{S}^n_1$ *as follows. For* $x \in \mathbb{R}^n$ *we define* $\Theta_n(x) := \frac{1}{\langle x\rangle}(1, x) \in \mathbb{S}^n_1$. *For* $\mathbb{R}_+ v \in \mathbb{S}_{\mathbb{R}^n}$ *we define* $\Theta_n(\mathbb{R}_+ v) := \frac{1}{\|v\|}(0, v) \in \mathbb{S}^n_1$.

The map Θ_n is well defined as $\mathbb{R}_+ v = \mathbb{R}_+ w$ implies $v = \lambda w$ for some $\lambda \in \mathbb{R}_+$.

Note that the inverse of Θ_n is given by :

$$\begin{gathered} \Theta_n^{-1} : \mathbb{S}^n_1 \ni (y_0, y_1, \dots, y_n) \mapsto \frac{1}{y_0}(y_1, \dots, y_n) \in \overline{\mathbb{R}^n}\,, \\ \Theta_n^{-1}(0, v) = \mathbb{R}_+ v\,. \end{gathered} \tag{27}$$

As Θ_n and its inverse are $\mathrm{GL}(\mathbb{R}^n)$-invariant, we have obtained the following lemma:

Lemma 5.2. *The map Θ_n of Definition 5.1 is bijective and equivariant for the actions of $\mathrm{GL}(\mathbb{R}^n)$ on $\overline{\mathbb{R}^n}$ and on $\mathbb{S}^n_1$.*

We endow $\overline{\mathbb{R}^n}$ with the structure of a smooth manifold (with boundary) that makes Θ_n of Lemma 5.2 a diffeomorphism. This manifold structure on $\overline{\mathbb{R}^n}$ extends the standard manifold structure of $\mathbb{R}^n$. We extend now this construction to any n-dimensional real vector space Z in the usual way. First, choose a vector space isomorphism $Z \to \mathbb{R}^n$, which yields bijections

$$\overline{Z} \xrightarrow{\sim} \overline{\mathbb{R}^n} \xrightarrow{\sim} \mathbb{S}^n_1$$

which in turn can be used to define a smooth structure on the radial compactification $\overline{Z}$ of Z. The $\mathrm{GL}(\mathbb{R}^n)$-invariance of Lemma 5.2 above implies that this smooth structure does not depend on the isomorphism $Z \to \mathbb{R}^n$.

5.2. **Quotients and compactifications.** In what follows, the role played by M in the previous sections will be played by the spherical compactification $\overline{Z}$ of a vector space Z. It follows from the definition of the radial compactification and of its topology that if $Y \subset Z$ is a (linear) subspace, then $\overline{Y} \subset \overline{Z}$ is a p-submanifold and $\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y}$.

If Y is a proper linear subspace of X, then there is a well-defined map $\overline{X} \smallsetminus \overline{Y} \to \overline{X/Y}$, which is the projection $x \mapsto x + Y$ on X and, at the boundary, it is given by $\mathbb{R}_+ x \mapsto \mathbb{R}_+(xY)$. We would like to extend it to a map $\overline{X} \to \overline{X/Y}$, but this is not possible in a continuous way. However after blowing up along $\mathbb{S}_Y$ we may do this in a natural way. In fact, we have:

Proposition 5.3. *The canonical surjection $\pi_{X/Y} : X \to X/Y$ extends to a smooth map $\psi_Y : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ such that the induced map $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$ is a diffeomorphism onto its image, and this image is a weak submanifold. Let $G = \mathrm{GL}(X, Y) \subset \mathrm{GL}(X)$ be the group of linear isomorphisms $X \to X$ that map Y to itself. Then ψ_Y is G-equivariant.*

Again we conjecture that one can prove much better submanifold properties for this image, namely that it is both a wib-submanifold and a b-submanifold, but we omit this discussion here.

Proof. In view of Lemma 5.2, we can assume $X = \mathbb{R}^n$ and $Y = \{0\} \times \mathbb{R}^q$. We will write $\mathbb{S}^{q-1}$ and $\mathbb{R}^q$ instead of $\{0\} \times \mathbb{S}^{q-1}$ and $\{0\} \times \mathbb{R}^q$, for simplicity. Recall that Lemma 4.9 yields a diffeomorphism $\widetilde{\Psi} : [\mathbb{S}^{r,r'}_{k,k'} : \{0\} \times \mathbb{S}^{r'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{r-1}_{k} \times \mathbb{S}^{r'+1}_{k'+1}$. We shall use this result for $r = n - q + 1$, $r' = q - 1$, $k = 1$, and $k' = 0$. Since $\mathbb{S}^{q-1}_0 = \mathbb{S}^{q-1}$ and $\mathbb{S}^{n-q+1,q-1}_{1,0} = \mathbb{S}^n_1$ we obtain the diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^n_1 : \mathbb{S}^{q-1}] \xrightarrow{\sim} \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1 .$$

Let $p_1 : \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1 \to \mathbb{S}^{n-q}_1$ be the projection onto the first component.

By definition of the smooth structure on $\overline{X}$, the map $\Theta_n : \overline{X} \to \mathbb{S}^n_1 = \mathbb{S}^{n-q+1,q-1}_{1,0}$ of Lemma 5.2 is a diffeomorphism, and it maps diffeomorphically $\mathbb{S}_Y$ onto $\mathbb{S}^{q-1}$. Then by Lemma 2.2 we obtain a diffeomorphism $\Theta^\beta_n : [\overline{X} : \mathbb{S}_Y] \to [\mathbb{S}^n_1 : \mathbb{S}^{q-1}]$.

We define the composition

$$\psi_Y : [\overline{X} : \mathbb{S}_Y] \xrightarrow{\Theta^\beta_n} [\mathbb{S}^n_1 : \mathbb{S}^{q-1}] \xrightarrow{\widetilde{\Psi}} \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1 \xrightarrow{p_1} \mathbb{S}^{n-q}_1 \xleftarrow{\Theta_{n-q}} \overline{X/Y},$$

in other words

$$\psi_Y := \left(\Theta_{n-q}\right)^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta^\beta_n : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y},$$

and we claim that ψ_Y is the desired extension.

To prove the claim, recall that we defined $\widetilde{\Psi}$ in Lemma 4.9 as the unique continuous extension of the map

$$\Psi : \mathbb{S}_1^n \smallsetminus \mathbb{S}^{q-1} \;\to\; \mathbb{S}_1^{n-q} \times \mathbb{S}_1^q, \quad (\eta, \mu) \;\mapsto\; \Big(\frac{\eta}{|\eta|},\, \big(|\eta|, \mu\big) \Big),$$

where $v \in \mathbb{R}_1^{n-q+1}$ and $\mu \in \mathbb{R}^q$. We write $v \in X = Y^\perp \oplus Y$ as $v = (v_\perp, v_Y)$, i.e. $v_Y \in Y$ und $v_\perp \perp Y$ which means $v_\perp \in Y^\perp = \mathbb{R}^{n-q} \times \{0\}$. Then in the case $v_\perp \neq 0$ we have $\Theta_n(v) = \frac{1}{\langle v \rangle}(1, v) \in \mathbb{S}_1^n \smallsetminus \mathbb{S}^{q-1}$, and in this case we then calculate

$$\widetilde{\Psi} \circ \Theta_n(v) \;=\; \Psi\Big(\frac{1}{\langle v \rangle}(1, v) \Big) \;=\; \Big(\frac{1}{\langle v_{Y^\perp} \rangle}\,(1, v_{Y^\perp}),\; \frac{1}{\langle v \rangle}\,(\langle v_{Y^\perp} \rangle, v_Y) \Big)\,.$$

By continuity of the extension, this formula even holds for all $v \in [\mathbb{S}_1^n : \mathbb{S}^{q-1}]$.

By formula (27) we have $\big(\Theta_{n-q}\big)^{-1}(y_0, y_1, \ldots, y_{n-q}) = \frac{1}{y_0}(y_1, \ldots, y_{n-q})$, if $y_0 > 0$.

Finally, a straightforward calculation gives

$$\Theta_{n-q}^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta_n(v) = \Theta_{n-q}^{-1}\Big(\frac{1}{\langle v_{Y^\perp} \rangle}(1, v_{Y^\perp}) \Big) = v_{Y^\perp} = \pi_{X/Y}(v)\,.$$

So ψ_Y is indeed the desired extension of $\pi_{X/Y}$.

The restriction of the map $\beta_{\overline{X},\mathbb{S}_Y}$ to $\overline{X} \smallsetminus \mathbb{S}_Y$ is a diffeomorphism onto its image. The map $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$, when restricted to $\beta_{\overline{X},\mathbb{S}_Y}^{-1}(\mathbb{S}_Y) := \mathbb{S}N_+^{\overline{X}}\mathbb{S}_X \simeq \mathbb{S}_Y \times \overline{X/Y}$ becomes the inclusion map $\mathbb{S}_Y \times \overline{X/Y} \to \overline{X} \times \overline{X/Y}$.

One can check that the differential of $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y)$ is injective also in the boundary points. Thus, it is a injective immersion. It is defined a compact set, and thus a homeomorphism onto its image. Using Proposition B.1 we see that it is a weak submanifold. □

Remark 5.4. Let $\psi := p_1 \circ \widetilde{\Psi}$, using the notation of the proof of the last proposition. We thus have a commutative diagram

$$\begin{array}{ccc} [\overline{X} : \mathbb{S}_Y] & \xrightarrow{\;\psi_Y\;} & \overline{X/Y} \\ {\scriptstyle \Theta_n^\beta}\big\downarrow & & \big\uparrow{\scriptstyle \Theta_{n-q}^{-1}} \\ [\mathbb{S}_1^{n-1} : \mathbb{S}^{q-1}] & \xrightarrow{\;\psi\;} & \mathbb{S}_1^{n-q} \end{array} \tag{28}$$

Remark 5.5. The extension ψ_Y was also considered in [7, 9]. It satisfies the following property. Let Y be a linear subspace of X. If $x_n \in X$ converges to $\overline{x} \in \overline{X}$ and $\overline{x} \notin \mathbb{S}_Y$, then $x_n + Y$ converges in $\overline{X/Y}$ to $\psi_Y(\overline{x})$.

5.3. **Induced maps on C^*-algebras.** The main motivation of our constructions was to prove that certain spaces introduced by Georgescu and Vasy are naturally homeomorphic. Georgescu's construction is that of a spectrum of a commutative C^*-algebra [4, 5, 7], whereas Vasy used blow-ups [21, 22]. Georgescu's construction provides a topological space, whereas Vasy's construction defines a smooth manifold with corners. Thus a homeomorphism is the best we can obtain, and this homeomorphism then equips Georgescu's conpactification with the structure of a smooth manifold with corners. To compare their approaches, we need to recall a few facts about commutative C^*-algebras.

Definition 5.6. *A* C^*-algebra A *is an algebra over* $\mathbb{C}$ *with a norm* $\|.\|$ *and with a map* $* : A \to A$ *such that* A *is a Banach algebra and for every* $\lambda, \mu \in \mathbb{C}$ *and* $a, b \in A$*, we have*

(i) $(a^*)^* = a$*,*

(ii) $(ab)^* = b^* a^*$,

(iii) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$,

(iv) $\|aa^*\| = \|a\|^2$.

The C^*-algebra is commutative *if $ab = ba$ for all $a, b \in A$.*

The next definition provides, up to isomorphism, all examples of commutative and unital C^*-algebras.

Definition 5.7. *For a compact and Hausdorff topological space Z let $\mathcal{C}(Z)$ be the algebra of complex-valued continuous function on Z. We endow $\mathcal{C}(Z)$ with the involution $f^* = \overline{f}$ (the complex conjugation) and with the norm $\|f\|_\infty = \sup_{z \in Z} |f(z)|$. With this structure, $\mathcal{C}(Z)$ is a commutative, unital C^*-algebra.*

Give two vector spaces X and $Y \subset X$, the composition

$$X \xrightarrow{\pi_{X/Y}} X/Y \xrightarrow{\text{incl}} \overline{X/Y}$$

induces by pullback an injective map $\mathcal{C}(\overline{X/Y}) \xrightarrow{\pi^*_{X/Y}} \mathcal{C}(X)$.

Let X be a vector space and $\mathcal{F}$ be a finite semilattice of linear subspaces of X, $X \notin \mathcal{F}$, $\{0\} \in \mathcal{F}$. We shall be interested in the induced semilattice $\mathcal{S}$ on the boundary:

$$\mathcal{S} := \{ \mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F} \} . \tag{29}$$

Then $\emptyset \in \mathcal{S}$, as it corresponds to the subspace $\{0\} \subset X$ that was assumed to be in $\mathcal{F}$. As in the Introducton, Equation (4), let $\mathcal{E}_\mathcal{S}(X)$ be the norm closed algebra generated by the pullbacks of the spaces $\mathcal{C}(\overline{X/Y})$ where Y runs over $\mathcal{F}$. Then $\mathcal{E}_\mathcal{S}(X)$ is a C^*-algebra by construction.

If $\mathcal{A}$ is a commutative C^*-algebra, we define its spectrum $\operatorname{Spec}(\mathcal{A})$ as the set of primitive ideals of $\mathcal{A}$. Recall that an ideal of A is *primitive* if it is the kernel of an irreducible representation of A and if A is commutative the set of primitives ideals and the set of maximal ideals coincides. A *character* of a C^*-algebra is a non-zero $*$-morphism $A \to \mathbb{C}$. If A is a commutative C^*-algebra, then there is a one-to-one correspondence between the characters $\chi : A \to \mathbb{C}$ and the maximal ideals of A, given by $\chi \mapsto \ker(\chi)$. The w^* topology on the space of characters defines a locally compact topology on $\operatorname{Spec}(\mathcal{A})$ and then yields an isomorphism $\mathcal{A} \to \mathcal{C}_0(\operatorname{Spec}(\mathcal{A}))$. Similarly it yields for a locally compact Hausdorff space M a homeomorphism $M \to \operatorname{Spec}(\mathcal{C}_0(M))$, sending $x \in M$ to the maximal ideal $\{f : M \to \mathbb{C} \mid f(x) = 0\}$, or equivalently, to the character e_x, where $e_x(f) := f(x)$.

Definition 5.8. *Let $\mathcal{F}$ be a finite semilattice of linear subspaces of the finite dimensional vector space X as above. Then the spectrum* $\operatorname{Spec}(\mathcal{E}_\mathcal{S}(X))$ *of the algebra introduced in Equation* (4) *is called* Georgescu's compactification of X with respect to $\mathcal{F}$ *or* with respect to $\mathcal{S}$.

In [18], two of the authors of this paper, together with N. Prudhon, have proved the following result.

Proposition 5.9. *The spectrum* $\operatorname{Spec}(\mathcal{E}_\mathcal{S}(X))$ *of* $\mathcal{E}_\mathcal{S}(X)$ *is homeomorphic to the closure of the image of X in the product $\prod_{Y \in \mathcal{F}} \overline{X/Y}$ under the "diagonal" map $\delta : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$, $\delta(x) := (\pi_Y(x))_{Y \in \mathcal{F}}$. More precisely, the homeomorphism $\Phi : \overline{\delta(X)} \to$* $\operatorname{Spec}(\mathcal{E}_\mathcal{S}(X))$ *is given as follows. Let $z = (z_Y)_{Y \in \mathcal{F}}$ be in the closure of $\delta(X)$. Then the homeomorphism Φ sends z to the C^*-algebra generated by*

$$\bigcup_{Y \in \mathcal{F}} \{\pi^*_{X/Y} f \mid f \in \mathcal{C}(\overline{X/Y}),\ f(z_Y) = 0\}.$$

In particular, this C^*-algebra is a maximal ideal in $\mathcal{E}_{\mathcal{S}}(X)$.

5.4. **Identification of the Georgescu and Vasy spaces.** If we apply Theorem 4.12 to the setting of the current subsection, then the Y-component of the map $\mathcal{B}_{\mathcal{S}}$ in Theorem 4.12 is the map $\phi_{\mathcal{S},\mathbb{S}_Y} : [\overline{X} : \mathcal{S}] \to [\overline{X} : \mathbb{S}_Y]$. We then compose this map with the map $\psi_Y : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ defined in Proposition 5.3 to obtain the following result.

Corollary 5.10. *The product map*

$$\Xi_{\mathcal{S}} : [\overline{X} : \mathcal{S}] \;\to\; \prod_{Y \in \mathcal{F}} \overline{X/Y} \tag{30}$$

of the composite maps $\psi_Y \circ \phi_{\mathcal{S},\mathbb{S}_Y} : [\overline{X} : \mathcal{S}] \to [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ is a diffeomorphism onto its image. (For $Y = 0$, the map $\psi_Y \circ \phi_{\mathcal{S},\mathbb{S}_Y} = \psi_Y \circ \phi_{\mathcal{S},\emptyset}$ is simply the blow-down map $[\overline{X} : \mathcal{S}] \to \overline{X}$.) Let G be a discrete group of linear autormorphisms of X that map elements of $\mathcal{F}$ to elements of $\mathcal{F}$ (thus $g(\mathcal{S}) = \mathcal{S}$ for all $g \in G$). Then G acts on $[\overline{X} : \mathcal{S}]$ and the map $\Xi_{\mathcal{S}}$ is G-equivariant.

Proof. If we replace in the definition of $\Xi_{\mathcal{S}}$ the maps ψ_Y with the maps $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) \circ \phi_{\mathcal{S},\mathbb{S}_Y}$ of Proposition 5.3, then the resulting map $\Xi'_{\mathcal{S}} = \big((\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) \circ \phi_{\mathcal{S},\mathbb{S}_Y}\big)_{Y \in \mathcal{F}}$ is a diffeomorphism by Proposition 5.3 together with Theorem 4.12. The map $\Xi'_{\mathcal{S}}$ differs from $\Xi_{\mathcal{S}}$ by simply repeating several times the factors $\beta : [\overline{X} : \mathcal{S}] \to \overline{X}$. The corollary is obtained by keeping only one of these repeated factors, which still insures that the resulting map is a diffeomorphism onto its image. The action of G and the fact that $\Xi_{\mathcal{S}}$ is G-equivariant follow from the definition of $\Xi_{\mathcal{S}}$ and from Theorem 4.12. □

Combining Corollary 5.10 with Proposition 5.9, we obtain the following result.

Theorem 5.11. *There exists a unique homeomorphism*

$$\mathrm{Spec}(\mathcal{E}_{\mathcal{S}}(X)) \;\simeq\; [\overline{X} : \mathcal{S}]$$

that is the identity on X.

Proof. Let $\delta : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$ be the diagonal map. Proposition 5.9 states that we have a homeomorphism $\mathrm{Spec}(\mathcal{E}_{\mathcal{S}}(X)) \to \overline{\delta(X)}$. Corollary 5.10 states that the map $\Xi_{\mathcal{S}}$ defined on $[\overline{X} : \mathcal{S}]$ is a diffeomorphism onto its image. Since $[\overline{X} : \mathcal{S}]$ is compact, the image is closed. It moreover contains $\delta(X)$ as a dense open subset. Therefore $[\overline{X} : \mathcal{S}]$ is also homeomorphic to $\overline{\delta(X)}$. □

We obtain the following description for the spaces introduced in [4, 5, 6].

Remark 5.12. In [4, 5, 6], Georgescu and his collaborators have considered the norm closed subalgebra of functions $\mathfrak{A}_{\mathcal{S}}$ of $L^\infty(X)$ generated by all the algebras $\mathcal{C}_0(X/Y)$. This corresponds to potentials that have zero limit at infinity on X/Y. The spectrum of this algebra (after adjoining a unit) identifies with the closure of the image of X in $\prod_{Y \in \mathcal{S}} (X/Y)^+$, where Z^+ denotes the one point compactification of a locally compact space Z. Since $\mathfrak{A}_{\mathcal{S}} \subset \mathcal{E}_{\mathcal{S}}(X)$, we obtain that $\mathrm{Spec}(\mathfrak{A}_{\mathcal{S}})$ is a quotient of $\mathrm{Spec}(\mathcal{E}_{\mathcal{S}}(X))$, and hence also a quotient of $[\overline{X} : \mathcal{S}]$, by Theorem 5.11. Generally, the topology on $\widehat{\mathfrak{A}_{\mathcal{S}}}$ is rather complicated and singular, see also [13, 14] for concrete examples when $\dim(X) = 2$.

Example 5.13. In the concrete case of the N-body problem, we take $X := \mathbb{R}^{3N}$ and consider the subspaces

$$Y_j \;:=\; \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_j = 0\} \quad \text{and}$$
$$Y_{ij} \;:=\; \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_j = x_j\}\,, \;\; i \neq j\,.$$

Thus each $x_i \in \mathbb{R}^3$. We let $\mathcal{F}$ be the semilattice generated by the subspaces Y_i and Y_{ij}, $i, j \in \{1, 2, \dots, N\}$. Let $\mathcal{S}$ be the finite semilattice of p-submanifolds of $\overline{X}$ as in Equation (29). Then our results imply that $[\overline{X} : \mathcal{S}]$ will be endowed with the following natural smooth actions:

- of $X := \mathbb{R}^{3N}$ by translation,
- of S_N, the symmetric group on N variables, by permutation, and
- of $O(3)$ acting diagonally on the components of $X := \mathbb{R}^{3N}$.

These actions are easy to obtain at the level of spectra of C^*-algebras or for the graph-family blow-up, but more difficult to obtain geometrically using iterated blow-ups. In particular, this explains the answer to the question of Melrose and Singer [17] provided in [19].

APPENDIX A. PROPER MAPS

Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$.

Lemma A.1 (Generalizes [12, Prop 4.32])**.** *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. If f is proper, then f is closed.*

In [12, Prop 4.32] the lemma is stated with the additional requirement that X is locally compact. However in the proof the locally compactness of X is not needed. Furthermore we will apply the lemma only if X is locally compact. Thus we omit the proof.

Corollary A.2. *Let $f : X \to Y$ be a continuous injective map between two Hausdorff spaces with Y locally compact. If f is proper, then f is a homeomorphism onto its image.*

Proof. The map $f : X \to f(X)$ is bijective continuous and closed and thus a homeomorphism. □

We shall say that f is *locally proper* if, for every $y \in Y$, there exists an open neighborhood V_y of y in Y such that the map $f^{-1}(V_y) \to V_y$ induced by f is proper.

Lemma A.3. *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. Then f is proper if, and only if, it is locally proper.*

Proof. Clearly, every proper map is locally proper, by definition. Let us assume that f is locally proper and let $K \subset Y$ be a compact subset. For any $y \in K$ we choose the open neighborhood V_y as above (in the definition of a locally proper map). As Y is locally compact, there is an open neighborhood W_y of y in V_y such that its closure $\overline{W}_y$ in Y is a compact subset of V_y. The local properness of f together with the choice of V_y implies that $f^{-1}(\overline{W}_y \cap K)$ is compact. By the compactness of K we can coose $y_1, \dots, y_N$ such that K is covered by $\left(W_{y_j}\right)_{1 \le j \le N}$. Then $K = \bigcup_{j=1}^N \left(\overline{W}_{y_j} \cap K\right)$. Then

$$f^{-1}(K) \quad = \quad \bigcup_{j=1}^{N} f^{-1}(\overline{W}_{y_j} \cap K)$$

is also compact. This completes the proof. □

APPENDIX B. SUBMANIFOLD CRITERIA

Proposition B.1. *Let N and M be manifolds with corners. Let $f : N \to M$ be a smooth map which is an immersion and a homeomorphism onto its image. Then $f(N)$ is a weak submanifold of M in the sense of Definition 1.12, and f is a diffeomorphism from N to $f(N)$.*

Proof. Note that the homeomorphism property implies that we have a local statement, i.e. we can restrict to small neighborhoods in N and M to prove it. Without loss of generality we can assume by passing to a chart of M that we have $M = \mathbb{R}^n_\ell$, $n = \dim M$. We will show that $f(N)$ is a submanifold of $\mathbb{R}^n$. It is sufficient to do this on a neighborhood of $p \in N$ with $f(p) = 0$. By choosing an appropriate chart for N, we can also assume that $p = 0$ and that N is an open neighborhood of 0 in $\mathbb{R}^{n'}_{\ell'}$, where $n' := \dim N = \dim d_0 f(T_0 N)$.

We choose vectors $v_1, \dots, v_{n-n'}$ such that they form a basis of a complement of $d_0 f(T_0 N)$ in $T_0 M \cong \mathbb{R}^n$. We extend f to a smooth map

$$F : N \to \mathbb{R}^{n-n'} \to M, \quad F(q, t_1, \dots, t_{n-n'}) \mapsto f(q) + \sum_{j=1}^{n-n'} t_j v_j \,.$$

Obviously $d_{(p,0)}F$ is an invertible linear map, and F can be extended to a smooth map $\tilde{F} : B \to \mathbb{R}^n$, defined on an open ball B around 0 in $\mathbb{R}^n$. Thus $\tilde{F}$ is a diffeomorphism onto its image on a small neighborhood V of $(p, 0)$. By taking $\phi := \big(\tilde{F}|V\big)^{-1}$ as a chart for $\mathbb{R}^n$, the conditions in Definition 1.8 are satisfied for $k := 0$, $k' := \ell'$, $G = 1 \in \mathrm{GL}(n, \mathbb{R})$.

Thus $f(N)$ is a submanifold of $\mathbb{R}^n$ and thus a weak submanifold of M.

In the chart ϕ constructed this way, the map f is a linear injective map and thus a diffeomorphism onto its image. □

Corollary B.2. *Let N and M be manifolds with corners. Let $f : N \to M$ be a smooth map. If there is a smooth map $F : M \to N$ with $F \circ f = \mathrm{id}_N$, then $f(N)$ is a submanifold of M in the sense of Definition 1.12.*

Proof. The relation $\mathrm{id}_{T_x N} = d_{f(x)} F \circ d_x f$ implies that $d_x f : T_x N \to T_{f(x)} M$ is injective. As $F|_{f(N)}$ is continuous, f is a homeomorphism onto its image. □

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