The first eigenvalue of the Dirac operator in a conformal class

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Abstract

In this overview article, we study the first positive eigenvalue of the Dirac operator in a unit volume conformal class. In particular, we discuss the question whether the infimum is attained. In the first part, we explain the corresponding variational problem. In the following parts we discuss the relation to the spinorial mass endomorphism and an application to surfaces of constant mean curvature. The article also mentions some open problems and work in progress.

1 The associated variational problem

Let (M, g_0) be a compact n-dimensional Riemannian manifold equipped with a fixed spin structure that will not be mentioned explicitly in the notation, $\dim M = n \geq 2$. Let $[g_0]$ be the set of all metrics conformal to g_0 having volume 1. For any metric $g = f^2 g_0 \in [g_0]$ we obtain a spinor bundle $\Sigma^g M$ and a Dirac operator $D^g : \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$. We identify $\Sigma^g M$ with $\Sigma^{g_0} M$ such that [Hit74, Hij86, Hij01]

$$D^g \varphi = f^{-1} D^{g_0} \varphi,$$
$$|\varphi|_g = f^{(n-1)/2} |\varphi|_{g_0}.$$

In particular, with this identification the kernel of the Dirac operator is conformally invariant.

We study the first positive eigenvalue of the Dirac operator as a function on $[g_0]$, e.g. we are interested in the supremum and the infimum.

At first, one can show that the first positive eigenvalue of the Dirac operator $\lambda_1^+(g)$ is not bounded from above. Indeed in [ACHH] we construct a sequence of so-called Pinocchio metrics. Pinocchio metrics are a sequence of metrics (g_n) conformal to g_0 with an almost cylindrical part whose length tends to ∞ for $n \to \infty$ as indicated in Figure 1. Some analytic arguments involve that $\liminf \lambda_1^+(g_n) > 0$, but $\nu_n := \sqrt[n]{\mathrm{vol}(M, g_n)} \to \infty$. Hence, $\nu_n^{-2} g_n$ is a sequence of metrics in $[g_0]$, such that $\lambda_1^+(\nu_n^{-2} g_n) \to \infty$. As a result of these arguments we obtain

$$\sup_{g \in [g_0]} \lambda_1^+(g) = \infty.$$

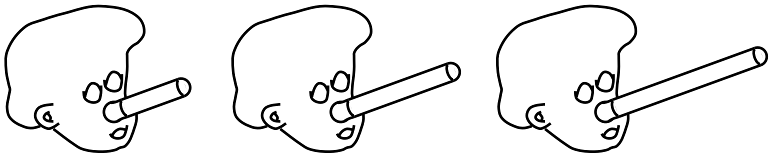

Figure 1: Pinocchio metrics with growing cylindrical part (= growing nose).

Now, let us turn our attention to the infimum of $\lambda_1^+(g)$. We define

$$\lambda_{\min}^+(M,[g_0]) := \inf_{g\in[g_0]} \lambda_1^+(g).$$

This invariant satisfies $\lambda_{\min}^+(M,[g_0]) \leq \lambda_{\min}^+(S^n) = (n/2)\omega_n^{1/n}$ where the sphere carries the standard Riemannian metric, and where ω_n denotes its volume [Hij86, Bär92, Amm03b, GH06].

The infimum $\lambda_{\min}^+(M,[g_0])$ is always positive, i.e. the first eigenvalue is uniformly bounded away from 0. This result is due to J. Lott if the Dirac operator is invertible [Lot86], for the general case see [Amm03b]. In order to prove this result it is helpful to reformulate the problem as a variational problem. Namely we define the conformally invariant functional

$$\mathcal{F}(\varphi) = \frac{\|D\varphi\|^2_{L^{2n/(n+1)}}}{\int \langle D\varphi, \varphi\rangle}.$$

One then shows that

$$\lambda_{\min}^+(M,[g_0]) := \inf \mathcal{F}(\varphi)$$

where the infimum runs over all spinors of regularity $H_1^{2n/(n+1)}$ with $\int\langle D\varphi,\varphi\rangle > 0$. Note that due to the boundedness of the Sobolev embedding $H_1^{2n/(n+1)} \hookrightarrow L^{2n/(n-1)}$ and due to the Hölder inequality the denominator $\int\langle D\varphi,\varphi\rangle$ is continuous on $H_1^{2n/(n+1)}$. The infimum of $\mathcal{F}$ is attained iff $\lambda_1^+(g)$ attains its infimum in a "generalized" metric. By a generalized metric we mean a $(2,0)$-tensor of the form $f^2 g_0$, where f is a real function that may have zeros (see [Amm03c, Amm03a] for details). We obtain the following result.

Theorem 1.1 ([Amm03a])**.** *Let $\alpha := 2/(n-1)$ if $n \geq 4$, and let $\alpha \in (0,1)$ if $n \in \{2,3\}$. Assume that*

$$\lambda_{\min}^+(M,[g_0]) < \lambda_{\min}^+(S^n) \tag{1.2}$$

holds.

(A) Then there is a spinor field $\varphi \in C^{2,\alpha}(\Sigma M) \cap C^\infty(\Sigma(M \setminus \varphi^{-1}(0)))$ on (M,g_0) minimizing $\mathcal{F}$ among all spinors with $\int\langle D\varphi,\varphi\rangle > 0$. In particular, after possibly adding an element of the kernel of D, the minimizer satisfies the Euler-Lagrange equation of $\mathcal{F}$:

$$D_{g_0}\varphi = \lambda_{\min}^+ |\varphi|^{2/(n-1)}\varphi, \qquad \|\varphi\|_{2n/(n-1)} = 1.$$

(B) *There is a generalized metric g conformal to g_0 with volume 1 such that*

$$\lambda_1^+(g) = \lambda_{\min}^+.$$

The generalized metric has the form $g = |\varphi|^{4/(n-1)} g_0$ where φ is a solution of the Euler-Lagrange equation in (A). The generalized metric is smooth outside the zero set of the conformal factor.

(C) *If $n = 2$, then the zero set of the conformal factor of g, denoted $\mathcal{S}_g$, is finite. The generalized metric is smooth everywhere including in the zero set $\mathcal{S}_g$. Furthermore,*

$$\#\mathcal{S}_g < \text{genus}(M).$$

In particular, if M is diffeomorphic to a 2-torus, then g is a metric in the ordinary sense.

Roughly speaking, the inequality (1.2) avoids concentration of minimizing sequences for our functional.

Inequality (1.2) is strongly related to the Yamabe problem. The Yamabe problem [Yam60] is the problem to find a metric of constant scalar curvature in the given conformal class $[g_0]$. It has been solved affirmatively in [Tru68], [Aub76], [Sch84], see also the well-written overview article [LP87]. The most difficult step in the solution of the Yamabe problem is to show that any manifold not conformal to the standard sphere satisfies

$$Y(M, [g_0]) < Y(S^n) \tag{1.3}$$

It is a direct consequence of Hijazi's inequality [Hij86] that inequality (1.2) implies inequality (1.3). Hence, proving (1.2) for a given conformal spin manifold $(M, [g_0], \chi)$ provides an alternative proof for the solvability of the Yamabe problem.

Obviously, one would like to determine all conformal manifolds $(M, [g_0])$ and all spin structures on M such that (1.2) holds. Unfortunately, this problem is widely open. It is not known whether (1.2) holds for all manifolds not conformal to S^n. However, some special cases are known. We will explain some of them in the following sections.

2 The spinorial mass endomorphism

We assume in this section that the Weyl curvature W of (M, g_0) vanishes in a neighborhood of a point $p \in M$. This is equivalent to saying that there is a metric $g \in [g_0]$ that is flat on a small ball $B_\varepsilon(p)$ centered in p.

Furthermore, we assume that the Dirac operator on M is invertible. The integral kernel of the inverse of the Dirac operator will be denoted as $G^D(x, y) \in \text{Hom}\,(\Sigma_y M, \Sigma_x M)$, which is a smooth $\Sigma M \boxtimes \Sigma M$-valued function on $M \times M \setminus \{(x, x) \,|\, x \in M\}$. In particular we obtain

$$\int_M \langle G^D(x, y)(\varphi_0), D\psi(x)\rangle \, dx = \langle \varphi_0, \psi(y)\rangle$$

for all smooth spinors $\psi \in \Gamma(\Sigma M)$ and $\varphi_0 \in \Sigma_y M$, $y \in B_\varepsilon(p)$.

We work in normal coordinates for g based in p. As the metric g is flat on $B_\varepsilon(p)$, we can trivialize the spinor bundle on $B_\varepsilon(p)$ by parallel sections. Without explicitly stating it, we identify $\Sigma_y M$ and $\Sigma_x M$, $x, y \in B_\varepsilon(p)$ via parallel transport along a path from x to y in $B_\varepsilon(p)$. Similarly, $x - y$ can be identified with a vector in $T_pM \cong T_xM \cong T_yM$, and hence Clifford multiplication by $x - y$ defines an endomorphism in $\mathrm{End}(\Sigma_p M)$.

With these trivializations we then obtain on $B_\varepsilon(p)$ for $\varphi_0 \in \Sigma_y M$

$$G^D(x,y)\varphi_0 = -\frac{1}{\omega_{n-1}}\,\frac{x-y}{|x-y|^n} \cdot \varphi_0 + v(x,y)\varphi_0,$$

where $(x, y) \mapsto v(x, y)$ is a smooth $\mathrm{End}(\Sigma_p M)$-valued function on $B_\varepsilon(p) \times B_\varepsilon(p)$.

The section $x \mapsto \alpha_g(x) := v(x, x)$ of the bundle $\mathrm{End}(\Sigma M|_{B_\varepsilon(p)})$ is called the *mass endomorphism* of (M, g) on $B_\varepsilon(p)$. The index g indicates that it depends on the choice of $g \in [g_0]$. (One can show, that $\alpha_g(x)\mathrm{dvol}_g^{\frac{n-1}{n}}(x)$ *is* invariant under conformal changes g, hence $\alpha_g(x)\mathrm{dvol}_g^{\frac{n-1}{n}}(x)$ yields a well-defined smooth section defined on the interior of $\{x \in M | W_x = 0\}$.) The self-adjointness of the Dirac operator implies that $\alpha_g(x)$ is a self-adjoint endomorphism of $\Sigma_x M$, hence its eigenvalues are real.

Furthermore, in dimension $n \not\equiv 3 \mod 4$ there is a real linear endomorphism of ΣM anticommuting with the Dirac operator, and hence with $\alpha_g(x)$. Thus, in this case, the spectrum of $\alpha_g(x)$ is symmetric with respect to 0.

In dimension $n \equiv 3 \mod 4$, this is not the case, as we will see at the example of the real projective space $\mathbb{R}P^{4k+3}$.

Example 2.1. *(Real projective space.)* Suppose that (M, g_0) is the real projective space $\mathbb{R}P^{4k+3}$, $k \in \mathbb{N} \cup \{0\}$ with its standard metric with universal covering $\pi : S^{4k+3} \to \mathbb{R}P^{4k+3}$. The projective space $\mathbb{R}P^{4k+3}$ carries two spin structures. For one of the spin structures $n/2$ is in the spectrum of the Dirac operator, and the corresponding eigenspinors are so-called *Killing spinors* to the constant $-1/2$, i.e. spinors ψ satisfying

$$\nabla_X \psi = -\frac{1}{2}\, X \cdot \psi.$$

The value $-n/2$ is not in the spectrum. For the other spin structure, $-n/2$ is in the spectrum with eigenspinors being Killing spinors to the constant $1/2$, and $n/2$ is not in the spectrum. (See [Bär96, Section 4] for more informations about real projective spaces and other quotients of spheres.)

For each spin structure, we obtain a fiberwise isomorphism of vector bundles

$$\pi_* : \Sigma_p S^{4k+3} \to \Sigma_{\pi(p)} \mathbb{R}P^{4k+3}.$$

This allows us to calculate

$$G^D_{\mathbb{R}P^{4k+3}}(\pi x, \pi y) \circ \pi_* = \pi_* \circ G^D_{S^{4k+3}}(x, y) + \pi_* \circ G^D_{S^{4k+3}}(x, -y), \tag{2.2}$$

where $-y$ denotes the antipodal point of y. In order to calculate $\alpha_g(y)$ for some $g \in [g_0]$ we perform a stereographic projection. The first summand $\pi_* \circ G^D_{S^{4k+3}}(x,y) \circ (\pi_*)^{-1}$ goes over into the euclidean Green function

$$-\frac{1}{\omega_{n-1}} \frac{x-y}{|x-y|^n} \cdot .$$

Hence, $\alpha_g(y) = 0$ is equivalent to $G^D_{S^{4k+3}}(y,-y) = 0$, and using Moebius transformations it is clear that if $G^D_{S^{4k+3}}(y,-y)$ did vanish then the conformal change formula for $G^D_{S^{4k+3}}$ would imply that $G^D_{S^{4k+3}}$ vanishes everywhere. Hence $\alpha_g(y) \neq 0$. On the other hand, the isotropy group of the $\mathrm{Spin}(4k+4)$ action on S^{4k+3} in the point y is the subgroup $\mathrm{Spin}(4k+3)$. This isotropy subgroup acts on $\Sigma_y \mathbb{R}P^{4k+3}$. It is the unique irreducible representation of $\mathrm{Spin}(4k+3)$ and commutes with $\alpha_g(y)$. Hence, by Schur's lemma $\alpha_g(y) = \lambda \,\mathrm{Id}$ where $\lambda \in \mathbb{R} \setminus \{0\}$. On the other hand, it is easy to see that changing the spin structure changes the sign of the spinorial mass endomorphism.

Theorem 2.3 ([AHM03]). *Let (M, g_0) be a Riemannian spin manifold, flat in a neighborhood of $p \in M$, with invertible Dirac operator. If the spinorial mass endomorphism $\alpha(p)$ has a positive eigenvalue, then (1.2) holds.*

If $\alpha(p)$ has a negative eigenvalue, then the same result holds if we replace everywhere the first positive eigenvalue by the first negative one, namely

$$\inf_{g \in [g_0]} \left| \lambda_1^-(g) \right| < \frac{n}{2}\, \omega_n^{1/n}. \tag{2.4}$$

With this inequality, we can conclude in analogy to Theorem 1.1 that $|\lambda_1^-(g)|$ attains its infimum, and that one obtains a solution of the Euler-Lagrange equations.

Corollary 2.5. *Let (M, g_0) be a (locally) conformally flat Riemannian spin manifold of dimension n with invertible Dirac operator. Assume that the spinorial mass endomorphism does not vanish everywhere on M.*

(1) *If $n \not\equiv 3 \mod 4$, then (1.2)* **and** *(2.4) hold,*

(2) *If $n \equiv 3 \mod 4$, then (1.2)* **or** *(2.4) hold.*

Note that the "or" in the second case is not an "exclusive or".

One of the major problems around the spinorial mass endomorphism is that we need results that provide sufficient conditions for the non-vanishing of α.

We conjecture that for generic metrics the mass endormorphism should not vanish everywhere. A way to study this conjecture in dimensions 3 and 4 might be to study the perturbative methods explained in [Mai97].

In dimension $n \equiv 3 \mod 4$, a criterion for the non-vanishing of the spinorial mass endomorphism arises from the amazing connection of the spinorial mass endomorphism to the regularization of the trace of D^{-1}. Let us assume that (M, g_0) is conformally flat, i.e. that the Weyl curvature vanishes everywhere.

For any $y \in M$ let $g_y \in [g_0]$ be a flat metric that coincides with g_0 in y. In [Oki01] it is shown that the regularized trace $TR(D^{-1})$ satisfies

$$TR(D^{-1}) = \int \mathrm{tr}\ \alpha_{g_y}(y)\ dy$$

if $n \equiv 3 \mod 4$. Okikiolu's approach is tightly related to techniques developped in [KV95], but while Kontsevich and Vishik's techniques apply mainly to regularized traces of pseudo-differential operators of non-integer order, Okikiolu managed to control regularized traces of integer order by using a clever splitting of pseudo-differential operators into an "even" and an "odd" part. Okikiolu's work was motivated by calculating variation formulae for regularized traces.

Unfortunately, it seems even harder to calculate $TR(D^{-1})$ than to calculate the mass endomorphism itself. Hence, this criterion has not admitted strong applications until today for our problem.

3 Surfaces

For hypersurfaces in $\mathbb{R}^{n+1}$ there is another useful criterion for proving (1.2) which is particularly helpful for surfaces in $\mathbb{R}^3$. We will restrict to surfaces here, but similar techniques also apply in higher dimensions. The idea relies on the spinorial Weierstrass representation, [KS96], see also [Fri98], [Bär98] for more recent approaches, and see also [Amm03a, Section 9] for a history overview and further references.

Suppose that $(M, [g_0])$ is a compact Riemann surface and that the universal covering $\widetilde{M}$ admits a periodic branched conformal immersion $F : \widetilde{M} \to \mathbb{R}^3$. By "periodic", we mean that there exists a group homomorphism $h : \pi_1(M) \to \mathbb{R}^3$ such that

$$F(\gamma \cdot x) = F(x) + h(\gamma)$$

where $\gamma \cdot x$ denotes the image of x under the action of the Deck transformation $\gamma \in \pi_1(M)$.

If all branching points have even branching order, then the immersion F induces a spin structure on M. Let us explain this in more details: Let J be the complex structure on M and (X, JX) be a frame on M. Then outside the branching points

$$\left(\frac{dF(X)}{|dF(X)|}, \frac{dF(JX)}{|dF(JX)|}, \frac{dF(X)}{|dF(X)|} \wedge \frac{dF(JX)}{|dF(JX)|} \right)$$

is a frame in $\mathrm{SO}(3)$. Pulling back the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ defines a spin structure on $M \setminus \{\text{branching points}\}$. The spin structure does not extend over the branching points of odd order. However, if we assume that all branching points have even order, the spin structure extends to all of M. We obtain a spinor bundle. We can use the restriction of a parallel spinor of unit length as explained in [Bär98] in order to obtain a spinor φ on M satisfying

$$D\varphi = H|\varphi|^2\varphi \qquad |\varphi|^2 = |dF|.$$

We will call this spinor the *spinor induced by the immersion F*. Hence, if $\ker D = \{0\}$, then an easy Rayleigh quotient type argument implies that

$$(\lambda^+_{\min}(M, [g_0]))^2 \leq \int H^2,$$

where the integral has to be taken with respect to the volume measure induced from $\mathbb{R}^3$, and ranges over a fundamental domain of M.

The integral is the famous Willmore integral of F. It is not hard to prove that if F is the lift of a map $M \to \mathbb{R}^3$ (i.e. $h \equiv 0$), then $\int H^2 \geq 4\pi$. Hence, we do not get (1.2) in this case. However there are many Riemann surfaces together with branched conformal immersions $F : \widetilde{M} \to \mathbb{R}^3$ with non-trivial periodicity map h for which we obtain $\int H^2 < 4\pi$, and the resulting bound is exactly (1.2) for surfaces.

Alternatively, one can use perturbative methods [Mai97], or surgery methods [AH06a, AH06b] to prove (1.2) in some cases.

Now, let us assume that for a Riemann surface inequality (1.2) is satisfied. Then the functional attains its minimum, and the minimizer ψ satisfies the Euler-Lagrange equation

$$D\psi = \lambda^+_{\min}|\psi|^2\psi$$

with $\|\psi\|_{L^4} = 1$. Earlier in this section, we have explained how to get from a periodic conformal immersion to a spinor on (M, g) satisfying $D\psi = H|\psi|^2\psi$. This construction from a branched conformal immersion to a solution of $D\psi = H|\psi|^2\psi$ can be inversed. Namely by writing $\psi = (\psi_+, \psi_-)$ and by setting

$$\alpha := \mathrm{Re} \begin{pmatrix} \psi_+ \otimes \psi_+ + \overline{\psi_-} \otimes \overline{\psi_-} \\ i\psi_+ \otimes \psi_+ - i\,\overline{\psi_-} \otimes \overline{\psi_-} \\ 2i\psi_+ \otimes \overline{\psi_-} \end{pmatrix} \in \Gamma(T^*M \otimes_{\mathbb{R}} \mathbb{R}^3).$$

one obtains a closed $\mathbb{R}^3$-valued 1-form α, and this form is the differential of a periodic branched conformal map $F : \widetilde{M} \to \mathbb{R}^3$ with induced spinor ψ.

We obtain the following result.

PROPOSITION 3.1 ([Amm03b]). *Assume that the Riemann spin surface (M, g, σ) satisfies*

$$\lambda^+_{\min}(M, [g], \sigma) < 2\sqrt{\pi}. \tag{3.2}$$

Then there is a periodic branched conformal cmc immersion $F : \widetilde{M} \to \mathbb{R}^3$. The mean curvature is equal to $\lambda^+_{\min}(M, [g], \sigma)$ and the area of a fundamental domain is 1. The regular homotopy class of F is determined by the spin-structure σ. The indices of all branching points are even, and the sum of these indices is smaller than $2\mathrm{genus}(M)$. In particular, if M is a torus, then there are no branching points.

Examples.

(a) Let (M, g) be a 2-dimensional torus. Via a conformal change we can achieve that g is flat, i.e. $M = \mathbb{R}^2/\Gamma$, equipped with the Euclidean metric. We assume that the lattice Γ is generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$, with $y > 0$. The spinor bundle of a flat manifold is flat as well, hence the holonomy is a map $\Gamma \to U(\Sigma_p M)$. Indeed, the image of this map is contained in $\{\pm \mathrm{Id}\}$. We obtain a homomorphism $\chi : \Gamma \to \{\pm \mathrm{Id}\}$. This homomorphism characterizes the spin structure σ in the sense that two spin structures on (M, g) are isomorphic iff the homomorphisms χ coincide, and to each such homomorphism there is a spin structure. The case $\chi \equiv +\mathrm{Id}$ corresponds to the so-called *trivial* spin structure σ_{tr}, the other cases correspond to *non-trivial* spin structures.

Let us concentrate in this summary to the case of a non-trivial spin structure $\sigma \neq \sigma_{\mathrm{tr}}$. After a possible rotation and rescaling, we can achieve that

$$\chi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{Id} \qquad \chi \begin{pmatrix} x \\ y \end{pmatrix} = -\mathrm{Id},$$

$$|x| \leq \frac{1}{2}, \quad y^2 + \left(|x| - \frac{1}{2}\right)^2 \geq \frac{1}{4}, \quad y > 0.$$

The Dirac operator is always invertible.

One easily sees $\lambda^+_{\mathrm{min}}(M, g, \sigma) \leq \frac{\pi}{\sqrt{y}}$. Hence, the proposition yields solutions for $y > \frac{4}{\pi}$. We conjecture that in some cases these minimizers are given (up to rotation and translation) by the parametrized cylinder

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R}^3 \\ \begin{pmatrix} a \\ b \end{pmatrix} &\mapsto \begin{pmatrix} \frac{\sqrt{y}}{2\pi} \cos \frac{2\pi b}{y} \\ \frac{\sqrt{y}}{2\pi} \sin \frac{2\pi b}{y} \\ \frac{a}{\sqrt{y}} \end{pmatrix} \end{aligned}$$

However, in some cases, e.g. if $x = 0$ and $4/\pi < y < 1$, the cylinders are represented by solutions of the Euler-Lagrange equations, but these solutions do no longer minimize the functional. We conjecture that in the case $x = 0$, $y < 1$ minimizers correspond to the unduloid immersions (see Figure 2). Recall that an unduloid is a surface of revolution of constant mean curvature.

(b) If M has genus 2, then as in the case of the torus, the dimension of the kernel is independent of the metric, however it depends on the spin structure. If σ is a spin structure such that (M, σ) is spin-cobordant 0, then the Dirac operator is invertible for any metric. Again, as in the torus case, one can find for any $\varepsilon > 0$ a conformal class $[g]$ on M with $\lambda^+_{\mathrm{min}}(M, [g], \sigma) < \varepsilon$ and one obtains periodic constant mean curvature surfaces.

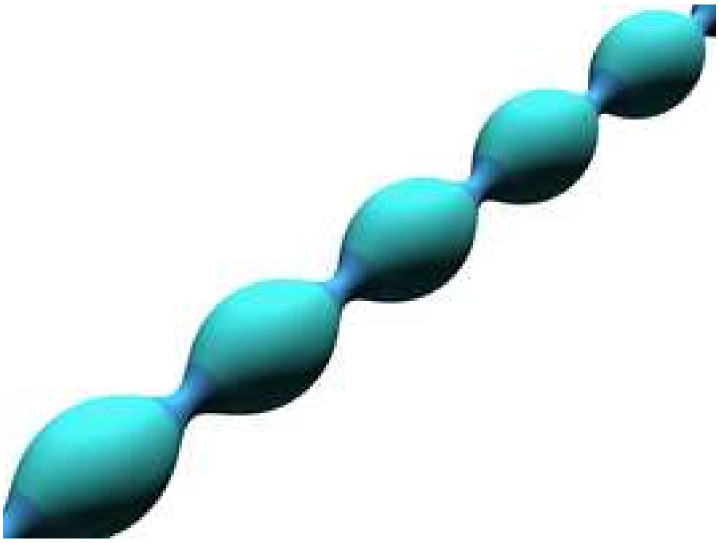

Figure 2: An unduloid in $\mathbb{R}^3$, visualized by Nick Schmitt.

(c) If the genus is larger than 2, then the kernel of the Dirac operator on a Riemannian spin manifold (M, σ) depends on the metric. For example if M is a surface of genus 3 equipped with the spin structure σ and the conformal structure g_0 associated to the periodic conformal immersion with vanishing mean curvature indicated in Figure 3.

This immersion induces a harmonic spinor on (M, g, σ). However, as (M, σ) is spin-cobordant 0, there is a perturbation $[g_t]$ of the conformal structure such that the Dirac operator on (M, g_t, σ) has a trivial kernel for small $t \neq 0$ [Mai97]. In this case

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \lambda^+_{\min}(M, [g_t], \sigma) = 0,$$

hence there exist solutions of $D\psi = c|\psi|^2\psi$. Such a solution is visualized in Figure 4.

Remark 3.3. *For $n \geq 4$, the* geometric *interpretation of the Euler-Lagrange equations is unknown, however we have the following unpublished partial result. Suppose that (M, g_0) is a* real-analytic *3-manifold, and φ a solution of the Euler-Lagrange equation. Then there is a (non-complete) ricci-flat 4-dimensional manifold (N, h) and an isometric embedding $(M \setminus \varphi^{-1}(0), |\varphi|^2 g_0)$ into (N, h) of constant mean curvature. We conjecture that (N, h) carries a parallel spinor. Note that 4-manifolds with a parallel spinor are Ricci-flat Kähler manifolds.*

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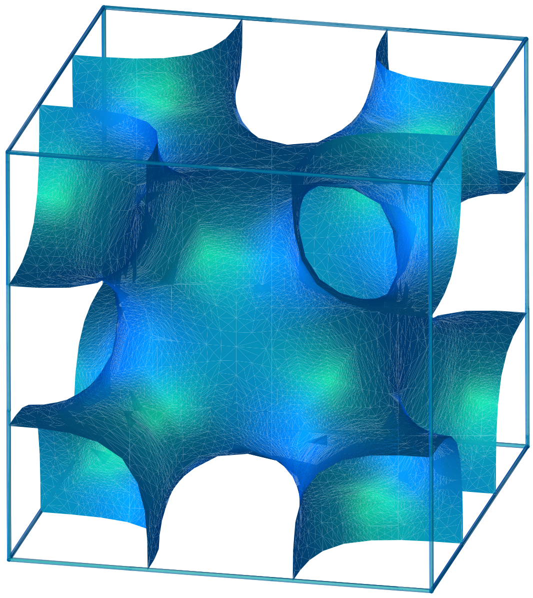

Figure 3: A periodic branched conformal minimal surface, visualized by K. Grosse-Brauckmann

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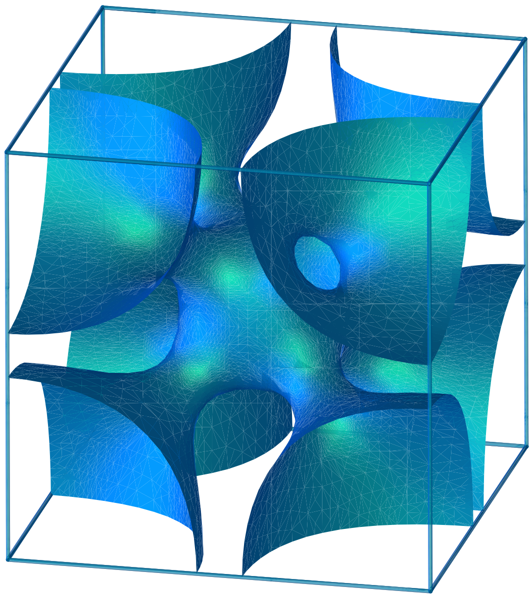

Figure 4: A periodic branched conformal cmc surface, visualized by K. Grosse-Brauckmann

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