POSITIVE MASS THEOREM FOR SOME ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We prove a positive mass theorem for some noncompact spin manifolds that are asymptotic to products of hyperbolic space with a compact manifold. As a conclusion we show the Yamabe inequality for some noncompact manifolds which are important to understand the behaviour of Yamabe invariants under surgeries.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. For compactly supported smooth functions $u\colon M \to \mathbb{R}$, $u \not\equiv 0$, we define

$$Q^*_M(u) := \frac{\int_M uL_g u \,\mathrm{dvol}_g}{\|u\|_p^2} \text{ and } Q^*(M, g) := \inf_u Q^*_M(u)$$

where $p = \frac{2m}{m-2}$ and $L_g = a_m\Delta_g + \mathrm{scal}_g$ is the conformal Laplacian, $a_m = 4\frac{m-1}{m-2}$. The quantity $Q^*(M, g)$ is a conformal invariant — the so-called Yamabe constant which was defined in order to solve the Yamabe problem. This problem consists of finding a metric with constant scalar curvature in a given conformal class on a given compact manifold. One of the main steps in the solution of the Yamabe problem is to prove the inequality

$$Q^*(M^m) < Q^*(\mathbb{S}^m) \tag{1}$$

for compact manifolds not conformally diffeomorphic to the standard sphere. For $m \geq 6$ and M not conformally flat, inequality (1) was proven by Aubin using a test function supported in an arbitrarily small ball around a point with non-vanishing Weyl curvature, see [9, p. 292]. This argument also holds for non-compact manifolds that are not conformally flat in the case $m \geq 6$. The remaining cases for compact M were solved by Schoen in [20] using the positive mass theorem [21, 22].

The present article is motivated by the question whether Schoen's method also provides inequality (1) for a certain class of non-compact manifolds, in particular we are interested in the manifolds $\mathbb{S}^n \times \mathbb{H}^{k+1}_c$, $c \in [0, 1]$, which play a central role in the surgery monotonicity formula for the Yamabe constant of compact manifolds in [3]. Here $\mathbb{H}^{k+1}_c$ denotes the simply connected complete Riemannian manifold of sectional curvature $-c^2$ and of dimension $k+1$ and g_c denotes its Riemannian metric, and $\mathbb{S}^n$ is the n-dimensional sphere with its standard metric σ^n. For $k = 0$ this means that $\mathbb{H}^1_c$ is the Euclidean space $\mathbb{R}^1$. The Yamabe constants of the spaces $\mathbb{S}^n \times \mathbb{H}^{k+1}_c$ were already intensively discussed in [5]. We now prove:

Theorem 1.1. *Let $m = n + k + 1$, $m \geq 3$, $k > 0$, and $c \in [0, 1)$. Then*

$$Q^*(\mathbb{S}^n \times \mathbb{H}^{k+1}_c, \sigma^n + g_c) < Q^*(\mathbb{S}^m, \sigma^m).$$

In the case $m \geq 6$ the corollary is an immediate consequence of Aubin's construction of a local test function as mentioned above. The case $n = 1$ follows from a known elementary estimate. The main objective of the article is to prove the theorem for the missing cases,

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i. e. for $m = 4, 5$ and $n \geq 2$. Their proof relies on a positive mass theorem (Theorem 1.2) presented below.
Note that if $k = 0$ or if $c = 1$, then $Q^*(\mathbb{S}^n \times \mathbb{H}_c^{k+1}, \sigma^n + g_c) = Q^*(\mathbb{S}^m, \sigma^m)$, as in this case $(\mathbb{S}^n \times \mathbb{H}_c^{k+1}, \sigma^n + g_c)$ is conformal to an open subset of $(\mathbb{S}^m, \sigma^m)$.
Theorem 1.1 is helpful as it implies, see [5, Section 10], that the infimum in the definition of $Q^*(\mathbb{S}^n \times \mathbb{H}_c^{k+1}, \sigma^n + g_c)$ and the infimum in the definition of its spinorial analogue are attained, and thus the Yamabe constants coincide with some related alternatively defined quantities, see again [5, Section 10].
Yamabe constants of related manifolds were already discussed in previous publications. The *cylindrical Yamabe constant* in [1] is in our notation $Q^*(N \times \mathbb{H}_1^{k+1})$ for $k = 0$ and N closed. Spaces of the form $N \times \mathbb{H}_c^{k+1}$, N closed, are also important in [2] where Yamabe constants on pseudomanifolds are studied. Assume that p is a point in a k-dimensional stratum of a pseudomanifold M and assume that the geometry in a neighborhood is modeled by $\mathbb{R}^k \times C(N)$, where $C(N)$ is the cone over N with the metric $dt^2 + t^2 g_N$. Note that $t^{-2}(g_0 + dt^2 + t^2 g_N) = e^{2u} g_0 + du^2 + g_N$, $u = -\log t$, is isometric to $N \times \mathbb{H}_1^{k+1}$ and thus $Q^*_{\text{loc}}(M, p) := \lim_{r \to 0} Q^*(B_r(p) \subset M)$ coincides with $Q^*(N \times \mathbb{H}_1^{k+1})$. In [2], the *local Yamabe invariant* was defined as the infimum of $Q^*_{\text{loc}}(M, p)$ over all singular points p of M. In particular if M is a pseudomanifold with simple edges whose links are round spheres, then the local Yamabe constant of M equals $\inf_c Q^*(\mathbb{S}^n \times \mathbb{H}_c^{k+1})$ where the infimum runs over all radii of the spheres representing the links.
Now let us turn to the positive mass theorem mentioned above which is the central ingredient in the proof of Theorem 1.1 in the non-trivial cases.

Theorem 1.2 (Positive mass theorem, special case of Theorem 5.1)**.** *Let $m = n + k + 1$, $m \in \{3, 4, 5\}$, $k \geq 0$, and $c \in [0, \infty)$. Let $(Z^m = N^n \times \mathbb{H}_c^{k+1}, g = g_N + g_c)$. Assume that N^n is a closed Riemannian spin manifold with constant scalar curvature and assume that*

$$\text{scal}_N > c^2 k \frac{n-1}{m-2}. \tag{2}$$

Then the mass of (Z, g) is nonnegative.
Moreover, if the mass of (Z, g) is zero, then $(Z, g) = \mathbb{S}_c^{m-1} \times \mathbb{R}$ or $(Z, g) = \mathbb{S}_c^{m-k-1} \times \mathbb{H}_c^{k+1}$, $k > 0$ and $c \in (0, \infty)$, where $\mathbb{S}_c^n$ is the rescaled standard sphere with sectional curvature c^2.

Let us now discuss the proof, the relevance and the extensions of this theorem.
The condition 2 is needed to define the mass of (Z, g), the definition of the mass requires a distinguished Green function for the conformal Laplacian. For our purpose this will be achieved by demanding the invertibility of the conformal Laplacian as a densely defined unbounded operator on L^2-function, cf. Remark 2.5. This invertibility is equivalent to 2, see Example 5.3 for details.
The mass is then the zero order term in the asymptotics of the Green function in conformal normal coordinates with respect to a given base point, and this mass is well-defined if $m \in \{3, 4, 5\}$ of if (Z, g) is conformally flat in a neighborhood of the basepoint.
The proof for Theorem 1.2 actually applies to a larger class of manifolds. The basic idea of the proof follows Witten's proof the positive mass theorem for compact spin manifolds. In the proof we need some decay estimates for the Green functions which will be given in Section 4. We obtain a proof of Theorem 5.1 which is more general than Theorem 1.2. Theorem 5.1 allows manifolds of arbitrary dimension $m \geq 3$ that are asymptotic to manifolds Z as above assuming a certain symmetry. We refer to Section 5 for details, as we do not want to introduce the necessary notation here. We also think that Theorem 5.1 can even be generalized to a much larger class of non-compact manifolds, e.g. special cases of Lie manifolds, see e.g. [7]. This would however require other kinds of techniques and thus cannot be included in this article.
In the case $m = 3$, the manifolds $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{R}P^2 \times \mathbb{R}$ are the only manifolds satifying the assumptions of Theorem 1.2. It is already known that the mass of $\mathbb{S}^2 \times \mathbb{R}$ vanishes (as it is

conformal to $\mathbb{S}^3 \setminus \{p, -p\}$) and that the mass of $\mathbb{R}P^2 \times \mathbb{R}$ is positive (using methods from [22]). The important cases for Theorem 1.2 are thus $m = 4$ and $m = 5$.
The mass we are considering in this article is strongly related to the ADM mass introduced by Arnowitt-Deser-Misner [8, 11] of a manifold $(M \setminus \{p_0\}, \tilde{g})$ with an asymptotically Euclidean end. Here we define $\tilde{g} := \Gamma(., p_0)^{4/(m-2)} g$ (see also the proof of Theorem 3.1) where Γ is the Green function (see Theorem 2.2). Note that $(M \setminus \{p_0\}, \tilde{g})$ has two ends: the asymptotically hyperbolic one from (M, g) and an Euclidean one which is created by the above construction. Positivity of the ADM mass of $(M \setminus \{p_0\}, \tilde{g})$ is equivalent to positivity of the mass in our sense (this follows e.g. from [23, Prop. 4.1]).
To avoid missunderstandings we should point out that there are other notions of a mass of an asymptotically hyperbolic manifold, see e.g. [12, 24, 13], that measure how much the asymptotically hyperbolic end deviates from being hyperbolic in the strict sense. These quantities are not directly related to the mass used in our article.
Thus, this article does not aim to generalize previous positive mass theorems from asymptotically Euclidean ends to other types of ends. Our goal is to prove a positive mass theorem for manifolds which may have additional ends, in order to allow applications to the Yamabe problem on non-compact manifolds.
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2. Preliminaries

2.1. **Notations.** In the article a spin manifold always means a manifold admitting a spin structure together with a fixed choice of spin structure. The notion of spin structures can be defined for arbitrary oriented manifolds, but as soon as we have a Riemannian metric it yields a spin structure in the sense of Spin(n)-principal bundles.
For a Riemannian spin manifold (M, g) we will always write Σ_M for the spinor bundle. The Dirac operator on (M, g) is denoted by D^g.
Moreover, $\bar{B}_r(x)$ denotes the closed ball around $x \in M$ of radius r w.r.t. the metric g.
Furthermore, $\mathbb{S}^k$ always denotes the sphere with sectional curvature 1 and the corresponding metric is denoted by σ^k.
We often write $\mathbb{H}_c^{k+1}$ in polar coordinates, i.e. $\mathbb{H}_c^{k+1} \setminus \{p_0\} = \mathbb{S}^k \times (0, \infty)/\sim$ with the metric $g_c = \sinh_c(r)^2 \sigma^k + dr^2$ where $\sinh_c(r) := \frac{1}{c}\sinh(cr)$ for $c > 0$, $\sinh_0(r) := r$ and where σ^k denotes the standard metric on $\mathbb{S}^k$.

2.2. **Green functions and ADM mass.** In this section, we collect existence results for the Green function of the conformal Laplacian and the Dirac operator and give the definition of the mass which essentially goes back to Arnowitt, Deser and Misner.
In the following we will say that the Dirac operator resp. the conformal Laplacian is invertible if it is invertible in the sense of a densely defined unbounded operator on the space of L^2-spinors resp. L^2-functions.

Theorem 2.1. *Let (M^m, g) be a complete Riemannian spin manifold with positive injectivity radius. Assume that for an $r > 0$ that is smaller than the injectivity radius there are constants $C_1, C_2 > 0$ with $C_1 \leq \mathrm{vol}(B_r(x), g) \leq C_2$ for all $x \in M$. Let the Dirac operator D^g be invertible. Then, D^g posesses a unique Green function, i.e., there is a smooth section $G\colon M \times M \setminus \Delta \to \Sigma_M \boxtimes \Sigma_M^*$ that is locally integrable such that for any $x \in M$, $\psi_0 \in \Sigma_M|_x$, and $\varphi \in C_c^\infty(\Sigma_M)$*

$$\int_M \langle G(y, x)\psi_0, D^g \varphi(y)\rangle \, \mathrm{d}y = \langle \psi_0, \varphi(x)\rangle$$

holds and $G(., x) \in L^2(M \setminus B_r(x))$ for any $r > 0$. Here $\Delta := \{(x, x) \in M \times M\}$ is the diagonal in $M \times M$.

Proof. For a reference see [4, Prop. 3.6]. In the proof therein of the existence of a Green function we use the assumption of bounded geometry only for uniform volume bounds and the positive injectivity radius which we directly put in here as an assumption. □

Theorem 2.2. *Let (M^m, g) be a complete Riemannian manifold with positive injectivity radius $r_{\mathrm{inj}}(M)$. Assume that for some $r \in (0, r_{\mathrm{inj}}(M)/3)$ there are constants $C_1, C_2 > 0$ with $C_1 \le \mathrm{vol}(B_r(x), g) \le C_2$ for all $x \in M$.*

(a) *Let the conformal Laplacian L_g be invertible as an operator $H_2^2 \subset L^2 \to L^2$. Then L_g admits a unique Green function, i.e. a smooth function $\Gamma\colon M \times M \setminus \Delta \to \mathbb{R}$, such that*
   - $\Gamma(.,y) \in L^1_{\mathrm{loc}}$ *for all* $y \in M$,
   - $\Gamma(.,y)|_{M \setminus B_r(y)} \in L^2(M \setminus B_r(y))$ *for all* $y \in M$ *and all* $r > 0$,
   - *for all* $v \in C_c^\infty(M)$ *and all* $y \in M$

$$\int_M \Gamma(x,y) L_g v(x)\,\mathrm{d}x = v(y).$$

(b) *Moreover, assume additionally that (M, g) has bounded geometry, and that the spectrum of L_g is contained in $[\varepsilon, \infty)$ for an $\varepsilon > 0$. Then Γ is everywhere positive, and for every fixed $y \in M$ the function $x \mapsto \Gamma(x, y)$ and all its derivatives converge for $x \to \infty$ towards* 0.

(c) *Let $p_0 \in U$, and we now assume additionally to* (b) *one of the following conditions:*
   - $m \ge 6$ *and on an open subset* $U \ni p_0$ *the metric* g *is conformally flat,*
   - $m \in \{3, 4, 5\}$

   Then the Green function has the following expansion in conformal normal coordinates as $x \to p_0$

$$\Gamma(x, p_0) = \frac{1}{(m-2)\omega_{m-1} r^{m-2}} + m_{p_0} + o(1), \tag{3}$$

   where $r = \mathrm{dist}_g(x, p_0)$, ω_{m-1} is the volume of $\mathbb{S}^{m-1}$, and $m_{p_0} \in \mathbb{R}$.

Here $x \to \infty$ means by definition that x converges to ∞ in the Alexandrov compactification $M \cup \{\infty\}$.

Proof.

(a) The proof for the existence of the Green function is done analogously as for the Dirac operator in [4, Prop. 3.6].

(b) We fix an $r > 0$. Let $\Gamma_y : M \setminus B_r(y) \to \mathbb{R}$, $x \to \Gamma(x, y)$. In order to prove the decay estimates, we argue by contradiction. So we assume that there assume is an $\ell \in \mathbb{N}_0$ and a sequence $x_i \to \infty$ with $\liminf_{i\to\infty} |\nabla^\ell \Gamma_x(x_i)| > 0$. We can assume that the balls $B_r(x_i)$ are pairwise disjoint and that they are disjoint from $B_r(y)$. Then

$$\infty > \|\Gamma_y\|^2_{L^2(M\setminus B_r(y))} \ge \sum_i \|\Gamma_y\|^2_{L^2(B_r(x_i))},$$

thus $\|\Gamma_y\|_{L^2(B_r(x_i))} \to 0$ for $i \to \infty$. On these balls Γ_y satisfies $L_g\Gamma_y = 0$. Expressing this equation in normal coordinates centered in each x_i, we obtain a family of smooth functions u_i defined on the ball $\Omega := B_r(0)$ in $\mathbb{R}^m$ solving $L_i u_i = 0$ for some elliptic operators L_i whose coefficients satisfy the estimates in the assumptions of [15, Theorem 9.11] uniformly in i. The elliptic estimates in [15, Theorem 9.11] then yield

$$\|u_i\|_{H_2^2(B_{r/2}(0))} \le C_1 \|u_i\|_{L^2(B_r(0))}$$

for some constant C_1, and this is equivalent to the statement

$$\|\Gamma_y\|_{H_2^2(B_{r/2}(x_i))} \le C_2 \|\Gamma_y\|_{L^2(B_r(x_i))},$$

for some C_2, where H_k^2 is the Hilbert space of L^2-functions whose derivatives up to order k are again in L^2.

Similarly, we obtain, using [15, Theorem 8.10]

$$\|\Gamma_y\|_{H_k^2(B_{r/3}(x_i))} \le C_3 \|\Gamma_y\|_{H_1^2(B_{r/2}(x_i))} \le C_4 \|\Gamma_y\|_{L^2(B_r(x_i))} \to 0.$$

Together with the Sobolev embedding theorems this implies the stated decay estimates.

In order to obtain positivity, we fix $y \in M$ and define the function

$$u(x) := -\min\{\Gamma(x,y), 0\} = \begin{cases} 0 & \text{if } x = y, \\ 0 & \text{if } \Gamma(x,y) > 0 \\ -\Gamma(x,y) & \text{if } \Gamma(x,y) \le 0 \end{cases}$$

For $x = y$ we use the convention $\Gamma(y,y) = \infty > 0$. The function u is in H_1^2 and away from the zero set of Γ_y it is a smooth solution of $L_g u = 0$. For $\delta \ge 0$ we define $\Omega_\delta := \{x \in M \setminus B_r(y) \mid \Gamma_y(x) \le -\delta\}$. Using Sard's lemma, we can choose a sequence $\delta_i \to 0$, such that $-\delta_i < 0$ is a regular value of Γ_y. In particular, the boundary of $\Omega_i := \Omega_{\delta_i}$ is smooth. Let $\nu = -\mathrm{grad}\Gamma_y/|\mathrm{grad}\Gamma_y|$ be the inner normal of its boundary. We get

$$\begin{aligned}\int_M (a_m|du|^2 + \mathrm{scal}_g u^2)\,\mathrm{dvol}_g &= \int_{\Omega_0} (a_m|d\Gamma_y|^2 + \mathrm{scal}_g \Gamma_y^2)\,\mathrm{dvol}_g \\ &= \lim_{i\to\infty} \int_{\Omega_i} (a_m|d\Gamma_y|^2 + \mathrm{scal}_g \Gamma_y^2)\,\mathrm{dvol}_g \\ &= \lim_{i\to\infty} \left(\int_{\Omega_i} \Gamma_y \underbrace{(L_g\Gamma_y)}_{=0}\,\mathrm{dvol}_g - a_m \int_{\partial\Omega_i} \Gamma_y \underbrace{\partial_\nu \Gamma_y}_{-|\mathrm{grad}\Gamma_y|}\,\mathrm{ds}_g \right) \\ &= -a_m \lim_{i\to\infty} \delta_i \int_{\partial\Omega_i} |\mathrm{grad}\Gamma_y|\mathrm{ds}_g \\ &\le 0\end{aligned}$$

This implies the contradiction that the spectrum of L_g cannot be bounded from below by $\varepsilon > 0$.

(c) For the expansion of the Green function in normal coordinates see e.g. [17, Prop. 2.1].

□

Remark 2.3. In the case $(M^m = \mathbb{H}_c^{k+1} \times N^n, g = g_c + g_N)$, (N, g_N) closed, the invertibility of L_g already implies that the spectrum of L_g is contained in $[\varepsilon, \infty)$ for some $\varepsilon > 0$. To see this, we write $L_g = L_1 + L_2$ where $L_1 = a_m\Delta_{\mathbb{H}_c^{k+1}} - c^2k(k+1)$ and $L_2 = a_m\Delta_N + \mathrm{scal}_N$. This implies $\mathrm{Spec}(L_g) = \mathrm{Spec}(L_1) + \mathrm{Spec}(L_2)$. As the spectrum of the Laplacian on $\mathbb{H}_c^{k+1}$ is $[c^2k^2/4, \infty)$ we get $\mathrm{Spec}(L_1) = [a_m c^2 \frac{k^2}{4} - c^2k(k+1), \infty) = [-c^2k\frac{n-1}{m-2}, \infty)$. Now, if λ_2 is the smallest eigenvalue of L_2 (and thus $\lambda_2 := \inf \mathrm{Spec}(L_2) \ge \inf_N \mathrm{scal}_N$), then we obtain $\mathrm{Spec}(L_g) = [\lambda_2 - c^2k\frac{n-1}{m-2}, \infty)$. If $0 \notin \mathrm{Spec}(L_g)$, then we can choose $\varepsilon = \lambda_2 - c^2k\frac{n-1}{m-2} > 0$.

Remark 2.4. As a corollary of Theorem 2.2 we also obtain the symmetry of the Green function, i.e. $\Gamma(x,y) = \Gamma(y,x)$ for all $x \ne y$. We will not use this fact, but we give a short argument as a reference for future work.
Let $\Gamma_i(x) := \Gamma(x, y_i)$ for $i = 1, 2$ and let K_j be a compact exhaustion of M with smooth boundary. Then by Green's identity

$$\int_{K_j} (\Gamma_1 L_g \Gamma_2 - \Gamma_2 L_g \Gamma_1)\,\mathrm{dvol}_g = \int_{\partial K_j} (\Gamma_1 \partial_\nu \Gamma_2 - \Gamma_2 \partial_\nu \Gamma_1)\,\mathrm{ds}_g$$

For $j \to \infty$ the right hand side vanishes by Theorem 2.2 (b). Using the definition of Γ we get

$$0 = \lim_{j\to\infty} \int_{K_j} \Gamma_1(x)\delta_{y_2}(x) - \Gamma_2(x)\delta_{y_1}(x)\mathrm{dvol}_g = \Gamma(y_2, y_1) - \Gamma(y_1, y_2).$$

Remark 2.5. For the definition of a mass one needs an assumption on the asymptotics for a Green function. But in order to obtain a mass that – if defined – only depends on the underlying manifolds and a base point one needs a distinguished Green function.

This is achieved here by requiring L^2-invertibility which gives a unique Green function with L^2-decay.

Definition 2.6. The constant m_{p_0} in (3) is called the mass at p_0.

$$\boxed{\begin{array}{l}\text{Fix } p_0 \in M \text{ and } \psi_0 \in \Sigma_M|_{p_0}. \text{ For the rest of the article we set} \\ \qquad\qquad u(.) := \Gamma(., p_0) \text{ and } \psi(.) := G(., p_0)\psi_0.\end{array}}$$

2.3. **Symmetries.** In this section we introduce the manifolds M_Z we merely want to consider in this paper. Moreover, we collect the implications of the symmetry assumptions on M_Z on the Green functions.

Definition 2.7 (Model spaces at infinities)**.** Let (N^n, g_N) be a closed Riemannian spin manifold of dimension $n \geq 0$, and let $f\colon [a, \infty) \to \mathbb{R}$ be a smooth positive function such that $f(r) \to \infty$ and $f^{-1}(r)f''(r) \to c^2$ as $r \to \infty$ for some $c \geq 0$. Set

$$(Z^m, g_Z) = (N \times \mathbb{S}^k \times (a, \infty), g_N + f(r)^2\sigma + dr^2)$$

with $m = n + k + 1$.

Remark 2.8.

(i) Similar to [4, Sect. 4.2] the action of $\mathrm{SO}(k+1)$ on $\mathbb{S}^k$ induces an isometric action on Z that lifts to a $\mathrm{Spin}(k+1)$-action on the spinor bundle.

(ii) The conditions $f(r) \to \infty$ and $f^{-1}(r)f''(r) \to c^2$ imply $f'(r)f^{-1}(r) \to c$ as $r \to \infty$. In order to see this, we rewrite this as a system of first order ordinary differential equations and use Theorem 4.1. Thus, by $\mathrm{scal}_g = \mathrm{scal}_N + f(r)^{-2}\mathrm{scal}_{\mathbb{S}^k} - n(n-1)f(r)^{-2}(\partial_r f(r))^2 - 2nf(r)^{-1}\partial_r^2 f$, cf. [14, Thm. 2.1], $\mathrm{scal}_g \to s := \mathrm{scal}_N - k(k+1)c^2$ as $r \to \infty$.

(iii) The unnormalized mean curvatures of $N \times \mathbb{S}^k$ and $\mathbb{S}^k$ in Z are both given by $H(r) = f'(r)f^{-1}(r)$ which due to (ii) is asymptotically constant.

(iv) The main examples we have in mind for the model spaces are $Z = \mathbb{H}^{k+1} \times N$ for $f(r) = \sinh(r)$, $a = 0$, $c = 1$ and $Z = \mathbb{R}^{k+1} \times N$ for $f(r) = r$, $a = 0$, $c = 0$.

Definition 2.9. From now on assume that (M_Z, g) is an m-dimensional connected Riemannian spin manifold with an isometric $\mathrm{SO}(k+1)$-action with a lift to a $\mathrm{Spin}(k+1)$-action on the spinor bundle. Moreover, M_Z shall be $\mathrm{Spin}(k+1)$-equivariantly spin isometric to (Z, g_Z) outside a compact subset.

Remark 2.10. Note that by $f^{-1}f'' \to c^2$ and $f \to \infty$, the manifold has positive injectivity radius and its curvature tensor is uniformly bounded. Thus, by comparison theorems we have the uniform volume bounds required in the results on the Green function in Subsection 2.3.

Lemma 2.11. *Let the conformal Laplacian on M_Z be positive, and let the Dirac operator on M_Z be invertible. Then, the Green function Γ of the conformal Laplacian on M_Z is* $\mathrm{SO}(k+1)$*-equivariant and, thus, $u|_Z$, cf. Definition 2.6, is only a function in r. Moreover, the Green function of the Dirac operator on M_Z^m is* $\mathrm{Spin}(k+1)$*-equivariant. Then, we have $(D^{\mathbb{S}^k})^2\psi|_Z = \frac{k^2}{4}\psi|_Z$ for ψ as in Definition 2.6 and where the $D^{\mathbb{S}^k}$ is the Dirac operator along the $\mathbb{S}^k$ component in Z.*

Proof. Since the Green function and the conformal Laplacian are both $\mathrm{SO}(k+1)$-invariant, it follows by the uniqueness of the Green function that $u|_Z$ is only a function of r.
The claim for ψ is proven completely analogously to [4, Lem. 5.3]. □

3. Positive mass theorem and decay of the Green functions

In [25] E. Witten gave a proof of the positive mass conjecture for spin manifolds. A simplified version of this proof was given by E. Humbert and the first author in [6]. In this section we examine conditions such that the result remains true for general non-compact spin manifolds. For that we need to assume decay conditions for the involved Green functions in order to show convergence of some integrals at infinity.

Theorem 3.1. *Let (M,g) be of bounded geometry. Let the spectrum of L_g be contained in $\mathbb{R}_{>0}$ and let the Dirac operator be invertible. Let $p_0 \in M$, u and ψ be as in Definition 2.6. Assume that if* $\dim \geq 6$ *the metric is conformally flat in a neighbourhood of p_0. Let K be a compact subset of M, and $R\colon M\setminus K \to (0,\infty)$ be a smooth proper function such that for all $r>0$ the set $S_r := \{x \in M\setminus K \mid R(x) = r\}$ is a smooth hypersurface in M and $B_r := K \cup \bigcup_{r'\leq r} S_{r'}$ is a compact subset of M with boundary S_r. Let ν be the outer normal of $S_r = \partial B_r$. Assume that there are constants $C_i > 0$ ($i = 1,2,3$), $r_0 > 0$, $\alpha_1 \geq \alpha_2 > 0$, $\beta > 0$ such that for all $x \in S_r$ with $r > r_0$ we have*

$$|u(x)| \geq C_1 e^{-\alpha_1 r} \quad |\partial_\nu u(x)| \leq C_2 e^{-\alpha_2 r}$$

and

$$|\psi(x)| \leq C_3 e^{-\beta r} \quad |\nabla_\nu \psi(x)| \leq C_3 e^{-\beta r}$$

and

$$\frac{m}{m-2}\alpha_1 - \alpha_2 < 2\beta. \tag{4}$$

Then, the mass at p_0 is nonnegative. If the mass is zero, then $M\setminus\{p_0\}$ carries a conformal metric that is flat and admits a basis of parallel spinors.

Note that we indeed require an estimate by below on $|u(x)|$ in contrast to the other estimates which are estimates from above.

Proof. We use the notations given in Subsection 2.2 and follow the idea of the proof of [6, Thm. 2.2]. Note that in the published version of [6] a small term was omitted, see the arXiv version. Moreover, note that by Theorem 2.2 u is a positive function. Set $\tilde{g} = u^{\frac{4}{m-2}} g$ and $\tilde{\psi} = u^{-\frac{m-1}{m-2}}\psi$. Then $(M\setminus\{p_0\}, \tilde{g})$ has zero scalar curvature and $\tilde{D}\tilde{\psi} = 0$ on $M\setminus\{p_0\}$. Thus, by the Schrödinger-Lichnerowicz formula $\tilde{\nabla}^*\tilde{\nabla}\tilde{\psi} = 0$. Moreover, $\partial_{\tilde{\nu}} = -u^{-\frac{2}{m-2}}\partial_\nu$ has length one w.r.t. $\tilde{g}$.
As in [6, Proof of Thm. 2.2], using $\tilde{\nabla}^*\tilde{\nabla}\tilde{\psi} = 0$ and partial integration we obtain

$$2\int_{M\setminus B_{p_0}(\varepsilon)} |\tilde{\nabla}\tilde{\psi}|^2 \mathrm{dvol}_{\tilde{g}} = \int_{S_{p_0}(\varepsilon)} \partial_{\tilde{\nu}} |\tilde{\psi}|^2 \mathrm{ds}_{\tilde{g}} + \lim_{r\to\infty} \int_{S_{p_0}(r)} \partial_{\tilde{\nu}} |\tilde{\psi}|^2 \mathrm{ds}_{\tilde{g}}.$$

If $\lim_{r\to\infty}\int_{S_{p_0}(r)} \partial_{\tilde{\nu}}|\tilde{\psi}|^2 \mathrm{ds}_{\tilde{g}} = 0$, then the non-negativity of the mass follows as in the compact case, see [6, Proof of Thm. 2.2 and Sect. 3]. Here we restrict to prove the vanishing of the limit from above: We have

$$\begin{aligned}\int_{S_{p_0}(r)} \partial_{\tilde{\nu}} |\tilde{\psi}|^2 \mathrm{ds}_{\tilde{g}} &= -\int_{S_{p_0}(r)} u^2 \partial_\nu |u^{-\frac{m-1}{m-2}}\psi|^2 \mathrm{ds}_g \\ &= -\int_{S_{p_0}(r)} u^{-\frac{2}{m-2}} \partial_\nu |\psi|^2 \mathrm{ds}_g + 2\frac{m-1}{m-2}\int_{S_{p_0}(r)} u^{-\frac{m}{m-2}} |\psi|^2 \partial_\nu u \mathrm{ds}_g.\end{aligned}$$

Using the decay estimates we obtain

$$\left| \int_{S_{p_0}(r)} u^{-\frac{2}{m-2}} \partial_\nu |\psi|^2 \mathrm{ds}_g \right| \leq C e^{(\frac{2}{m-2}\alpha_1 - 2\beta) r}$$

and

$$\left| \int_{S_{p_0}(r)} u^{-\frac{m}{m-2}} |\psi|^2 \partial_\nu u \mathrm{ds}_g \right| \leq C e^{(\frac{m}{m-2}\alpha_1 - \alpha_2 - 2\beta) r}.$$

Since $\frac{m}{m-2}\alpha_1 - \alpha_2 < 2\beta$ implies that $\frac{2}{m-2}\alpha_1 < 2\beta$, both integrals go to zero as $r\to\infty$. Thus, the mass at p_0 is nonnegative.
Assume now that the mass is zero, then as in the compact case $\tilde{\nabla}\tilde{\psi} = 0$, cp. [6, Thm. 2.2]. Moreover, since the choice of ψ_0 in the definition of ψ is arbitrary, we again obtain a basis of parallel spinors on $(M\setminus\{p_0\}, \tilde{g})$. Thus, $(M\setminus\{p_0\}, \tilde{g})$ is flat, and $(M\setminus\{p_0\}, g)$ is conformally flat. ∎

4. Green function estimates for M_Z.

We consider a spin manifold (M_Z^m, g) as introduced in Definition 2.9. The aim of this section is to provide estimates for the corresponding Green functions as required to apply Theorem 3.1.

4.1. **Perturbation of linear systems of ODE.** In order to estimate the Green functions we need the asymptotic behaviour for certain perturbed differential equations. We cite here a result that can be found in more general versions, e.g. in [16, I.2 Thm. (*)], [19, Satz 13].

Theorem 4.1. *Let $x(t)$ denote a function in $t \in [t_0, \infty)$ with values in $\mathbb{C}^d$. Let $J \in M_d(\mathbb{C})$, let $G\colon [t_0, \infty) \to M_d(\mathbb{C})$ be continuous where $M_d(\mathbb{C})$ denotes the sets of $d \times d$ complex matrices. Moreover let $G(t) \to 0$ as $t \to \infty$. Let $r_1, \dots, r_d$ be the real parts of the eigenvalues of J – listed with multiplicities. Then the differential equation*

$$x' = (J + G(t))x$$

has linearly independent solutions $x_1(t), \dots, x_d(t)$ with the following property: For all $\varepsilon > 0$ there is t_1 such that for all $t \geq t_1$

$$|x_i(t_1)| e^{(r_i - \varepsilon)(t - t_1)} \leq |x_i(t)| \leq |x_i(t_1)| e^{(r_i + \varepsilon)(t - t_1)}.$$

4.2. **Estimating the Green function of L_g.**

Lemma 4.2. *Let (M_Z^m, g) be as in Definition 2.9. Let the spectrum of conformal Laplacian L_g of (M_Z, g) be contained in $\mathbb{R}_{>0}$. We use the definitions and notations from Subsection 2.2. Then for every $\varepsilon > 0$ there is an $r_0 > 0$ such that for all $x = (x_N, x_{\mathbb{S}}, r) \in N \times \mathbb{S}^k \times [r_0, \infty)$ we get*

$$u(x) \geq C_0 e^{-\alpha_+ r} \text{ and } |\partial_r u(x)| \leq C_1 e^{-\alpha_- r},$$

where

$$\alpha_\pm = \frac{kc}{2} \pm \varepsilon + \mathrm{Re}\sqrt{\frac{k^2 c^2}{4} + \frac{\mathrm{scal}_\pm - c^2 k(k+1)}{a_m}},$$

$\mathrm{scal}_+ = \sup_N \mathrm{scal}_N$, $\mathrm{scal}_- = \inf_N \mathrm{scal}_N$, *and $C_0, C_1 > 0$.*

Proof. On $M \setminus \{p_0\}$ we have $L_g u = 0$ which reads

$$\frac{\partial^2}{\partial^2 r} u - \Delta^N u - \frac{1}{f(r)^2} \Delta^{\mathbb{S}^k} u + k \frac{f'(r)}{f(r)} \frac{\partial}{\partial r} u - \frac{\mathrm{scal}_g}{a_m} u = 0 \tag{5}$$

where $\mathrm{scal}_g = \mathrm{scal}_N + f(r)^{-2} \mathrm{scal}_{\mathbb{S}^k} - n(n-1) f(r)^{-2} (\partial_r f(r))^2 - 2n f(r)^{-1} \partial_r^2 f$, cf. [14, Thm. 2.1]. Note that by the assumptions on f and its derivatives we have $\mathrm{scal}_g \to \mathrm{scal}_N - k(k+1)c^2 := s$ as $r \to \infty$.

We decompose the space of smooth functions on $N \times \mathbb{S}^k$ into minimal subspaces which are generated by common eigenfunctions of the commuting operators Δ^N and $\Delta^{\mathbb{S}^k}$. If we extend those eigenfunctions constant in r-direction, we obtain a decomposition of the space of smooth functions on $N \times \mathbb{S}^k \times [a, \infty)$. In that sense we decompose $u = \sum_{i,l} u_{i,l}$ where $\Delta^N u_{i,l} = \mu_i u_{i,l}$ and $\Delta^{\mathbb{S}^k} u_{i,l} = \lambda_l u_{i,l}$ where $0 = \mu_0 < \mu_1 \leq \dots$ and $0 = \lambda_0 < \lambda_1 \leq \dots$. Then, (5) decomposes into the equations

$$\frac{\partial^2}{\partial^2 r} u_{i,l} + k \frac{f'(r)}{f(r)} \frac{\partial}{\partial r} u_{i,l} - \left(\mu_i + \lambda_l f(r)^{-2} + \frac{\mathrm{scal}_g}{a_m} \right) u_{i,l} = 0. \tag{6}$$

which can be written as

$$\frac{\partial}{\partial_r} \begin{pmatrix} u_{i,l} \\ \frac{\partial}{\partial_r} u_{i,l} \end{pmatrix} = \left(\underbrace{\begin{pmatrix} 0 & 1 \\ \mu_i + \frac{s}{a_m} & -kc \end{pmatrix}}_{=:J} + \underbrace{\begin{pmatrix} 0 & 0 \\ \frac{s - \mathrm{scal}_g}{a_m} + f(r)^{-2} \lambda_l & k\left(c - \frac{f'(r)}{f(r)}\right) \end{pmatrix}}_{=:R(r)} \right) \begin{pmatrix} u_{i,l} \\ \frac{\partial}{\partial_r} u_{i,l} \end{pmatrix}$$

Since $u \in L^2$, each $u_{i,l} \in L^2$ as well. Multiplication of (6) with $u_{i,l}$ and partial integration shows that $\partial_r u_{i,l} \in L^2$.
Moreover, note that J is constant and that $R(r) \to 0$ as $r \to \infty$. The matrix J has the eigenvalues $\alpha_{i,\pm} = -\frac{kc}{2} \pm \sqrt{\frac{k^2c^2}{4} + \mu_i + \frac{s}{a_m}}$. Fix $\varepsilon > 0$. Thus, by Theorem 4.1 and since $u_{i,l}$ and $\partial_r u_{i,l}$ are L^2 we obtain an $r_0 > 0$ such that for all $r \geq r_0$

$$|u_{i,l}(r_0)|e^{(\mathrm{Re}\ \alpha_{i,-}-\varepsilon)(r-r_0)} \leq |u_{i,l}(r)| \leq |u_{i,l}(r_0)|e^{(\mathrm{Re}\ \alpha_{i,-}+\varepsilon)(r-r_0)}$$

and the analogous estimate for $\partial_r u_{i,l}$.
By Lemma 2.11 only $l = 0$ appears. Thus, only the different eigenvalues in N-direction need to be taken into account. Set $u_i := u_{i,0}$. Note that $\int_N u_i = 0$ for all $i > 0$. But since u is everywhere positive this implies that u_0 cannot be identically zero. Setting $\alpha_+ := \sup_N(-\mathrm{Re}\,\alpha_{0,-})$, this concludes the proof of the first estimate.
For the second estimate we first use that $\partial_r u$ is in L^2 on $\{r \geq r_0\} \subset Z$ and set $\alpha_i^- = \sup_N \mathrm{Re}\,\alpha_{i,-}$ to obtain

$$\begin{aligned}\infty > \int_{\{r\geq r_0\}} |\partial_r u|^2 \geq & C\sum_i \int_{r_0}^{\infty} |\partial_r u_i|^2 (f(r))^k \mathrm{d}r \\ \geq & C' \sum_i \int_{r_0}^{\infty} |\partial_r u_i(r_0)|^2 e^{\left(ck+2\alpha_i^- -2\varepsilon\right)(r-r_0)}\mathrm{d}r \\ \geq & C'' \sum_i |\partial_r u_i(r_0)|^2 \frac{1}{-2\alpha_i^- - ck + 2\varepsilon} > 0\end{aligned}$$

for positive constants C, C', C''.
Now we can estimate $\partial_r u$ by using the triangle inequality, Cauchy-Schwarz inequality und the estimates from above:

$$\begin{aligned}|\partial_r u| \leq & \sum_i |\partial_r u_i(r_0)| e^{(\alpha_i^- +\varepsilon)(r-r_0)} \\ \leq & \Bigg(\underbrace{\sum_i |\partial_r u_i(r_0)|^2 \frac{1}{-2\alpha_i^- - ck + 2\varepsilon}}_{<\infty}\Bigg)^{\frac{1}{2}} \left(\sum_i (-2\alpha_i^- - ck + 2\varepsilon) e^{(2\alpha_i^- + 2\varepsilon)(r-r_0)}\right)^{\frac{1}{2}} \\ \leq & Ce^{(\alpha_0^- +\varepsilon)r} \Bigg(\underbrace{\sum_i (-2\alpha_i^- + 2\varepsilon) e^{2(\alpha_i^- - \alpha_0^-)(r-r_0)}}_{=:A}\Bigg)^{\frac{1}{2}}.\end{aligned}$$

If we can show that A is bounded, then the upper bound is established for $-\alpha_- := \alpha_0^-$: Since $\alpha_i^- > \alpha_0^-$ for all $i > 0$, it is enough to bound $\sum_i(-2\alpha_i^- + 2\varepsilon)e^{2\alpha_i^-(r-r_0)}$.
Let $N(x) := |\{i \mid \mu_i \leq x\}|$. Then, $N(\mu_i - 1) \leq i \leq N(\mu_i)$. Together with Weyl's asymptotic law there is an $i_0 > 0$ and an $\delta > 0$ such that for all $i \geq i_0$, $x \geq \mu_{i_0}$ we have

$$2\sqrt{\mu_i} \geq -\alpha_i^- \geq \sqrt{\mu_i} \text{ and } |N(x)x^{-k/2} - c| \leq \delta$$

(where c depends on the geometry of $\mathbb{S}^k$). Thus, $(c-\delta)\mu_i^{k/2} \leq N(\mu_i) \leq i+1 \leq 2i$ and $(c+\delta)\mu_i^{k/2} \geq N(\mu_i) \geq i$ and we obtain for $r > r_1 > r_0$

$$\sum_{i\geq i_0} (-2\alpha_i^- + 2\varepsilon) e^{2\alpha_i^-(r-r_0)} \leq 2\sum_{i\geq i_0}(2\sqrt{\mu_i}+\varepsilon)e^{-2\sqrt{\mu_i}(r_1-r_0)} \leq c_0 \sum_{i\geq i_0}(i^{1/k}+c_1)e^{-c_2 i^{1/k}}$$

where c_0, c_1, c_2 are positive constants. The last sum converges since the integral $\int_R^\infty (x^{\frac{1}{k}} + c_1)e^{-c_2 x^{\frac{1}{k}}}\, dx$ is finite for $R > 0$. Thus, for $r \geq r_1$ the term A is bounded. $\square$

4.3. **Green function of the Dirac operator.** Similarly as in the last subsection we want to estimate the Green function of the Dirac operator on (M_Z, g). Again we use the notations and definitions of Subsection 2.2.

As in [4, Sect. 6] we decompose the space of spinors restricted to $N \times \mathbb{S}^k \times \{r_1\}$ into complex subspaces of minimal dimensions which are invariant under D^N, $D^{\mathbb{S}^k}$, and $\partial_r\cdot$. Such spaces have a basis of the form ψ, $\partial_r \cdot \psi$, $P\psi$, and $\partial_r \cdot P\psi$, where ψ satisfies $D^N\psi = \lambda\psi$, $(D^{\mathbb{S}^k})^2\psi = \rho^2\psi$, and $P := D^{\mathbb{S}^k}/\rho$ with $\lambda, \rho \in \mathbb{R}$. All these operations commute with parallel transport in r-direction, so by applying parallel transport in r-direction we obtain spinors ψ, $\partial_r \cdot \psi$, $P\psi$, and $\partial_r \cdot P\psi$ on $Z = N \times \mathbb{S}^k \times [a, \infty)$ with similar relations, and the space of all spinors of the form

$$\varphi = \varphi_1(r)\psi + \varphi_2(r)\partial_r \cdot \psi + \varphi_3(r)P\psi + \varphi_4(r)\partial_r \cdot P\psi$$

is preserved under the Dirac operator D on Z.

Then the operators discussed above restricted to such a minimal subspace are represented by matrices and the equation $D\varphi = 0$ on Z reads as, cp. [4, Proof of Prop. 6.2],

$$\Phi'_{\lambda,\rho}(r) = \Bigg(\underbrace{A - \frac{kc}{2}\mathrm{Id}}_{=:J} + \underbrace{\frac{k}{2}\Big(c - \frac{f'(r)}{f(r)}\Big)\mathrm{Id} + \frac{\rho}{f(r)}B}_{:=G(r)}\Bigg)\Phi_{\lambda,\rho}(r).$$

where Id is the identity matrix and

$$A := \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & -\lambda & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

The eigenvalues of J are $\beta_{\lambda,\pm} = -\frac{kc}{2} \pm \lambda$ – both of them have multiplicity two. Thus, by Theorem 4.1 and since $\Phi_{\lambda,\rho}$ is L^2, for each $\varepsilon > 0$ there is $r_0 > 0$ such that

$$|\Phi_{\lambda,\rho}(r_0)|e^{(\mathrm{Re}\beta_{\lambda,-}-\varepsilon)(r-r_0)} \leq |\Phi_{\lambda,\rho}(r)| \leq |\Phi_{\lambda,\rho}(r_0)|e^{(\mathrm{Re}\beta_{\lambda,-}+\varepsilon)(r-r_0)}.$$

By Lemma 2.11 only $\rho^2 = \frac{k^2}{4}$ occurs, i.e., $\rho = \pm\frac{k}{2}$. Thus, we can proceed completely analogously as for the estimate of $\partial_r u$ in the proof of Lemma 4.2 in order to obtain the upper decay of ψ and $\partial_r\psi$. Then, in total we obtain the following

Lemma 4.3. *Let (M_Z^m, g) be as in Definition 2.9. Let the spectrum of conformal Laplacian L_g of (M_Z, g) be contained in $\mathbb{R}_{>0}$. Let the Dirac operator D^g of (M_Z, g) be invertible, and let λ_N^2, $\lambda_N \geq 0$, be the lowest Dirac eigenvalue for the square of the Dirac operator $(D^N)^2$ on (N, g_N). Then for $\varepsilon > 0$, there are constants C_0, C_1 and r_0, such that for all $r \geq r_0$ and $x = (x_N, x_{\mathbb{S}}, r) \in Z$ we have the estimates (using Definition 2.6)*

$$|\psi(x)| \leq C_0 e^{-\beta r} \text{ and } |\nabla_r \psi(x)| \leq C_1 e^{-\beta r},$$

where $\beta = \frac{kc}{2} + \lambda_N - \varepsilon$.

5. Proof of Theorem 1.2

With the preparations on the Green function decay on (M_Z, g) in the last two subsections we are now ready to apply Theorem 3.1.

Theorem 5.1 (Positive mass theorem for asymptotically hyperbolic manifolds)**.** *Let (M_Z^m, g) be a Riemannian spin manifold as described above, for a precise definition see Definitions 2.7 and 2.9. If $m \geq 6$, assume additionally that there is a point $p \in M$ such that (M_Z, g) is conformally flat on a neighbourhood of p. We assume that the spectrum of the conformal Laplacian on (M_Z, g) is contained in $[\delta, \infty)$ for a $\delta > 0$ and that the Dirac operator on*

(M_Z, g) is invertible as densely defined unbounded operator on L^2-spinors. Furthermore, we assume

$$\frac{kc(3-m)}{m-2} + \frac{m}{m-2}\mathrm{Re}\sqrt{b + \frac{\sup_N \mathrm{scal}_N}{a_m}} - \mathrm{Re}\sqrt{b + \frac{\inf_N \mathrm{scal}_N}{a_m}} < 2\lambda_N \qquad (7)$$

where λ_N^2, $\lambda_N \geq 0$, is the smallest eigenvalue of the square of the Dirac operator on N, and

$$b := -\frac{kc^2(n-1)}{4(m-1)}.$$

Moreover, let $a_m b + \inf_N \mathrm{scal}_N > 0$. Then the mass of (M_Z, g) is nonnegative.
Moreover, if the mass of (M_Z, g) is zero, then (M_Z, g) is conformally equivalent to $\mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$.

Proof. We are in the case $n = m - k - 1 > 1$, since $a_m b + \inf_N \mathrm{scal}_N = 0$ for $n = 1$. By Lemmata 4.2 and 4.3 we can apply Theorem 3.1 for

$$\begin{aligned}
\alpha_1 =& \frac{kc}{2} + \varepsilon + \mathrm{Re}\sqrt{\frac{k^2c^2}{4} + \frac{\sup_N \mathrm{scal}_N - c^2k(k+1)}{a_m}} \\
\alpha_2 =& \frac{kc}{2} - \varepsilon + \mathrm{Re}\sqrt{\frac{k^2c^2}{4} + \frac{\inf_N \mathrm{scal}_N - c^2k(k+1)}{a_m}} \\
\beta =& \frac{kc}{2} + \lambda_N - \varepsilon.
\end{aligned}$$

Since ε can be chosen arbitrarily small, (7) is exactly the condition assumed in (4). Thus, we obtain that the mass at p_0 is nonnegative.
Assume now that the mass is zero. Then, by Theorem 3.1 $M \setminus \{p_0\}$ is conformally flat. We thus can apply the following lemma which was shown in Section B of [18].

Lemma 5.2. *Let $(M^m = M_1^{m_1} \times M_2^{m_2}, g_1 + g_2)$ be a product manifold. Let $m_1, m_2 \geq 1$ and $m = m_1 + m_2 \geq 3$.*
Then, M is conformally flat if and only if M_1 and M_2 both have constant sectional curvature and in case that $m_1, m_2 > 1$ these sectional curvature have the same magnitude and opposite sign.

It follows from this lemma that one of the following cases occurs (where G is always finite):

	N	Z	
1.	$\mathbb{R}^n/G$	$(\mathbb{R}^n/G \times \mathbb{S}^k \times (a,\infty), g_E + c^2r^2\sigma^k + \mathrm{d}r^2)$	$c \in [0,1]$, $n > 1$
2.	$\mathbb{S}_c^n/G$	$(\mathbb{S}_c^n/G \times \mathbb{S}^k \times (a,\infty), \sigma_c^n + \sinh_c(r)^2\sigma^k + \mathrm{d}r^2)$	$n > 1$, $c > 0$
3.	$\mathbb{S}_c^{m-1}/G$	$\mathbb{S}_c^{m-1}/G \times (a,\infty)$	$c > 0$
4.	$\mathbb{H}_c^{m-1}/G$	$\mathbb{H}_c^{m-1}/G \times (a,\infty)$	$c > 0$

The first and fourth case are excluded since in that case the conformal Laplacian on M_Z is not invertible.
We now consider the second case: $Z = \mathbb{S}_c^n/G \times (\mathbb{H}_c^{k+1} \setminus \bar{B}_a(0))$ for G finite, $n > 1$ and $c > 0$. Let $(\hat{M} = \mathbb{S}_c^n \times (\mathbb{H}_c^{k+1} \setminus \bar{B}_a(0)), \hat{g})$ be the G-cover of M. The corresponding projection is denoted by $\pi\colon \hat{M} \to M$. Then, $\hat{M}$ still fulfills the assumptions in Theorem 2.2 and, thus, $L_{\hat{g}}$ possesses a Green function $\hat{\Gamma}$. Since $\lambda_N = \lambda_{\mathbb{S}^n}$, [10, p. 62], $\hat{M}$ also fullfills the assumptions of this theorem. Thus, from above we know that the mass of $\hat{M}$ at $\hat{p}_0$ with $\pi(\hat{p}_0) = p_0$ is nonnegative. On the other hand together with [22, Prop. 4.3] we obtain for nontrivial G that $0 = m_{p_0} > m_{\hat{p}_0} \geq 0$ which gives a contradiction. Hence, G is the trivial group. Analogously, we obtain in the third case that G is trivial. Summarizing we know that $Z = \mathbb{S}_c^n \times (\mathbb{H}_c^{k+1} \setminus \bar{B}_a(0))$ with $c > 0$, $n > 1$ and $k \geq 0$ or $Z = \mathbb{S}_c^{m-1} \times [a, \infty)$ for $c > 0$. In all those cases (M_Z, g) is conformally compactifiable to a manifold $(\widetilde{M}, \tilde{g})$ by considering

$\tilde{g} = h^2 g$ where $h(r) = \cosh^{-1}(r)$ for $r \geq a$, cp. [3, Prop. 3.1], and where $\widetilde{M} \setminus M_Z$ is a totally geodesic $\mathbb{S}^k$. Note that on a neighbourhood U of the $\mathbb{S}^k$ the metric $\tilde{g}$ is the standard metric on $\mathbb{S}^m$. Set $s = \mathrm{dist}_{\tilde{g}}(., \mathbb{S}^k)$. Then, $\cosh^{-1}(r) = \sin s$ for $r \geq a$.
Let $\tilde{u} = h^{-\frac{m-2}{2}} u$. Then, $L_{\tilde{g}} \tilde{u} = \delta_{p_0}$ on $\widetilde{M} \setminus \mathbb{S}^k$. Since $u \in L^2(Z, g)$, $\int_Z \frac{1}{\sin^2 s} \tilde{u}^2 \,\mathrm{dvol}_{\tilde{g}} = \int_Z u^2 \,\mathrm{dvol}_g < \infty$ and, hence, $\int_Z \frac{1}{s^2} \tilde{u}^2 \,\mathrm{dvol}_{\tilde{g}} < \infty$. Thus, analogously as in [5, Lem. 7.3] we can remove the singularity. Thus, $\tilde{u}$ is the Green function of $(\widetilde{M}, \tilde{g})$ around p_0 and the mass of $(\widetilde{M}, \tilde{g})$ is zero. It follows that $(\widetilde{M} \setminus \{p_0\}, \tilde{u}^{\frac{4}{m-2}} \tilde{g})$ is flat, complete and asymptotically Euclidean. Thus, it has to be $\mathbb{R}^m$. Thus, M is conformally equivalent to $\mathbb{S}^m$, i.e., there is a diffeomorphism $\varphi\colon M \setminus \to \mathbb{S}^m$ with $\varphi^* \sigma^m = f^2 g$ for some $f \in C^\infty(M, \mathbb{R}_{>0})$. By Liouville's theorem $\varphi|_U \colon U \subset \mathbb{S}^m \to \varphi(U) \subset \mathbb{S}^m$ is the restriction of a conformal transformation $\psi\colon \mathbb{S}^m \to \mathbb{S}^m$. Then, the diffeomorphism $\psi^{-1} \circ \varphi\colon M \to \mathbb{S}^m$ is the identity on U. Thus, on U we have $\sigma^m = f^2 \sigma^m$ and, hence, $f|_U = 1$. Summarizing we obtain that M_Z is conformally equivalent to $\mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$. ☐

Example 5.3. Let $(M_Z^m = N^n \times \mathbb{H}_c^{k+1}, g = g_N + g_c)$ where $(\mathbb{H}_c^{k+1} = \mathbb{S}^k \times \mathbb{R}, g_c = \sinh_c(r)^2 \sigma^k + \mathrm{d}\, r^2)$ where $\sinh_c(r) := \frac{1}{c} \sinh(cr)$ for $c > 0$ and $\sinh_0(r) := r$. Moreover, we assume that scal_N is constant, and that N is closed.
Then, $L_g = a_m \Delta_N + a_m \Delta_{\mathbb{H}_c^{k+1}} + \mathrm{scal}_N - c^2 k(k+1)$. Since the scalar curvature of M_Z is constant, we get that the spectrum of L_g is given by $[d, \infty)$ with $d := \frac{a_m c^2 k^2}{4} + \mathrm{scal}_N - c^2 k(k+1) = \mathrm{scal}_N + a_m b$. Thus, L_g has positive spectrum if and only if $d > 0$ which is equivalent to (2). In particular, if $n > 1$, then $\mathrm{scal}_N > 0$, and if $n \in \{0, 1\}$ this is not possible. Let now $n > 1$. Since the whole real line is the spectrum of the Dirac operator $\mathbb{H}_c^{k+1}$, the Dirac operator on M_Z is invertible if and only if the Dirac operator on N is. This is automatically fulfilled since $\mathrm{scal}_N > 0$. Hence, if

$$\frac{kc(3-m)}{m-2} + \frac{2}{m-2} \sqrt{b + \frac{\mathrm{scal}_N}{a_m}} < 2\lambda_N \quad \text{with } b = -\frac{kc^2(n-1)}{4(m-1)}$$

is fulfilled, then Theorem 5.1 applies to $M_Z = N \times \mathbb{H}_c^{k+1}$. Since $m \geq 3$ and $b \leq 0$, we always have

$$\frac{kc(3-m)}{m-2} + \frac{2}{m-2} \sqrt{b + \frac{\mathrm{scal}_N}{a_m}} \leq \frac{2}{m-2} \sqrt{\frac{\mathrm{scal}_N}{a_m}} \leq \frac{2}{\sqrt{(m-2)(m-1)}} \lambda_N < 2\lambda_N$$

where the second inequality uses the Schrödinger-Lichnerowicz formula for N.
In particular, for $M = \mathbb{H}_c^{k+1} \times \mathbb{S}^n$, $n > 1$, and $m \geq 3$, all the assumptions of Theorem 5.1 are fulfilled.

Proof of Theorem 1.2. Theorem 1.2 now follows directly from Theorem 5.1 and Example 5.3. ☐

6. Application to the Yamabe invariant

Lemma 6.1. *Let (M_Z^m, g) be as in Subsection 2.2 and we use the notations therein. Assume that the Dirac operator on (N, g_N) is invertible. Let $m = 3, 4$ or 5. Let L_g be an invertible operator and assume that (7) holds. Then, $Q^*(M_Z, g) < Q^*(\mathbb{S}^m, \sigma^m)$ unless M_Z is conformally equivalent to $\mathbb{S}^n \times \mathbb{H}^{k+1}$.*

Proof. By Theorem 1.2 the mass is positive. In order to obtain $Q^*(M_Z) < Q^*(\mathbb{S}^m)$ we then use the test function φ that was constructed by Schoen out of Γ, cp. [20, p. 482]. Since $\Gamma \in H_1^2$, the calculation remains completely the same. ☐

Note that the conformal Laplacian of $\mathbb{S}^n \times \mathbb{H}^{k+1}$ is invertible only if $n > 1$.
We now recall and prove Theorem 1.1 from the introduction.

Theorem 1.1. Let $m = n + k + 1$, $m \geq 3$ $k > 0$, and $c \in [0, 1)$. Then

$$Q^*(\mathbb{S}^n \times \mathbb{H}_c^{k+1}, \sigma^n + g_c) < Q^*(\mathbb{S}^m, \sigma^m).$$

Proof. For $m \geq 6$, $c \in [0,1)$ and $n > 1$ the manifold $\mathbb{S}^n \times \mathbb{H}_c^{k+1}$ is not conformally flat. Thus, Aubin's construction mentioned in the introduction yields the result in this case. For $n > 1$ and $m = 3, 4, 5$ this follows from Theorem 6.1 together with Example 5.3. For the remaining case $n = 1$ the claim follows from $Q^*(\mathbb{S}^1 \times \mathbb{H}_c^{m-1}, \sigma^n + g_c) = c^{\frac{2}{m}} Q^*(\mathbb{S}^m, \sigma^m)$, cp. [5, Rem. 9.9]. □

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