

WELL-POSEDNESS OF THE LAPLACIAN ON MANIFOLDS WITH BOUNDARY AND BOUNDED GEOMETRY

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ABSTRACT. Let M be a Riemannian manifold with a smooth boundary. The main question we address in this article is: “When is the Laplace-Beltrami operator $\Delta: H^{k+1}(M) \cap H_0^1(M) \rightarrow H^{k-1}(M)$, $k \in \mathbb{N}_0$, invertible?” We consider also the case of mixed boundary conditions. The study of this main question leads us to the class of manifolds with boundary and bounded geometry introduced by Schick (Math. Nach. 2001). We thus begin with some needed results on the geometry of manifolds with boundary and bounded geometry. Let $\partial_D M \subset \partial M$ be an open and closed subset of the boundary of M . We say that $(M, \partial_D M)$ has *finite width* if, by definition, M is a manifold with boundary and bounded geometry such that the distance $\text{dist}(x, \partial_D M)$ from a point $x \in M$ to $\partial_D M \subset \partial M$ is bounded uniformly in x (and hence, in particular, $\partial_D M$ intersects all connected components of M). For manifolds $(M, \partial_D M)$ with finite width, we prove a Poincaré inequality for functions vanishing on $\partial_D M$, thus generalizing an important result of Sakurai (Osaka J. Math, 2017). The Poincaré inequality then leads, as in the classical case to results on the spectrum of Δ with domain given by mixed boundary conditions, in particular, Δ is invertible for manifolds $(M, \partial_D M)$ with finite width. The bounded geometry assumption then allows us to prove the well-posedness of the Poisson problem with mixed boundary conditions in the higher Sobolev spaces $H^s(M)$, $s \geq 0$.

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Date: October 21, 2019.

B.A. has been partially supported by SFB 1085 Higher Invariants, Regensburg, funded by the DFG. V.N. has been partially supported by ANR-14-CE25-0012-01 (SINGSTAR).

Manuscripts available from <http://iecl.univ-lorraine.fr/Victor.Nistor/>

AMS Subject classification (2010): 58J32 (Primary), 35J57, 35R01, 35J70 (Secondary).

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1. INTRODUCTION

This is the first in a sequence of papers devoted to the spectral and regularity theory of differential operators on a suitable non-compact manifold with boundary M using analytic and geometric methods.

Let $\Delta := d^*d \geq 0$ be the (geometer's) Laplace operator acting on functions, also called the Laplace-Beltrami operator. Let k be a non-negative integer. The **Main Question** of this paper is:

“Is the operator $\Delta: H^{k+1}(M) \cap H_0^1(M) \rightarrow H^{k-1}(M)$ invertible?”

The range of k needs to be specified each time, and deciding the range for k is part of the problem, but in this paper we are mainly concerned with the cases $k = 0$ and $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Also part of the problem is, of course, to choose the “right definition” for the relevant function spaces. The Main Question of this paper turns out, in fact, to be mostly a geometric question, involving the underlying properties of our manifold M . It involves, in particular a geometric approach to the Poincaré inequality and to the regularity of the solution u of $\Delta u = f$. Most of our results on the Laplacian extend almost immediately to uniformly strongly elliptic operators, however, in order to keep the presentation simple, we concentrate in this paper on the Laplace-Beltrami operator. The general case will be discussed in [6]. Note also that our question is different (but related) to that of the presence of zero in the spectrum of the Laplacian [51].

For $k = 0$, it is easy to see that the invertibility of Δ in our Main Question is equivalent to the Poincaré inequality for L^2 -norms (see, for instance, Proposition 4.5). This answers completely our question for $k = 0$. Our main result (proved in the more general framework of mixed boundary conditions) is that our Main Question has an affirmative answer for all k *whenever M has finite width*. As a consequence, we obtain results on the spectral theory of the Laplacian with suitable mixed boundary conditions on manifolds with bounded geometry as well as on the regularity of the solutions of equations of the form $\Delta u = f$.

Let us formulate our results in more detail. Let (M, g) be a smooth m -dimensional Riemannian manifold with *smooth boundary and bounded geometry* (see Definition 2.5 and above). Our manifolds are assumed to be paracompact, but not necessarily second countable. See Remark 2.1. *It is important in applications not to assume M to be connected*. We denote the boundary of M by ∂M , as usual, and we assume that we are given a *disjoint union decomposition*

$$(1) \quad \partial M = \partial_D M \amalg \partial_N M,$$

where $\partial_D M$ or $\partial_N M$ are (possibly empty) open subsets of ∂M and \amalg denotes the disjoint union, as usual. We shall say that the boundary of M is *partitioned*. Of course, both $\partial_D M$ and $\partial_N M$ will also be closed.

Also, we shall say that the pair $(M, \partial_D M)$ has *finite width* if the distance from any point of M to $\partial_D M$ is *bounded uniformly* on M and $\partial_D M$ intersects all connected components of M (Definition 2.7). In particular, if $(M, \partial_D M)$ has finite width,

then $\partial_D M$ is not empty. The simplest example is $M = \mathbb{R}^{n-1} \times [0, T] \subset \mathbb{R}^n$ with $\partial_D M = \mathbb{R}^{n-1} \times \{0\}$ and $\partial_N M = \mathbb{R}^{n-1} \times \{1\}$.

Before stating our main result, Theorem 1.1, it is convenient to first state the Poincaré inequality that is used in its proof.

Theorem 1.1 (Poincaré inequality). *Let M be an m -dimensional smooth Riemannian manifold with smooth, partitioned boundary $\partial M = \partial_D M \amalg \partial_N M$. Assume that $(M, \partial_D M)$ has finite width. Then, for every $p \in [1, \infty]$, $(M, \partial_D M)$ satisfies the L^p -Poincaré inequality, that is, there exists $0 < c = c_{M,p} < \infty$ such that*

$$\|f\|_{L^p(M)} \leq c(\|f\|_{L^p(\partial_D M)} + \|df\|_{L^p(M)})$$

for all $f \in W_{\text{loc}}^{1,p}(M)$.

In a nice, very recent paper [62], Sakurai has proved this Poincaré inequality for $\partial_N M = \emptyset$, and $p = 1$. His proof (posted on the ArXiv preprint server shortly before we posted the first version this paper) can be extended to our case (once one takes care of a few delicate points; see Remarks 3.2 and 3.3). In particular, one can sharpen our Poincaré inequality result by relaxing the bounded-geometry condition by replacing it with a lower Ricci bound condition and a bound on the second fundamental form; see Remark 3.2. However, we do need the ‘full’ bounded geometry assumption for the higher regularity part of our results, so we found it convenient to consider the current slightly simplified setting.

For a manifold with boundary and bounded geometry, we set

$$(2) \quad H_D^1(M) := \{f \in H^1(M) \mid f = 0 \text{ on } \partial_D M\}$$

(note that this definition makes sense, due to the trace theorem for such manifolds, see Theorem 2.33) and

$$(3) \quad c_{M, \partial_D M} := \inf \{t \in \mathbb{R} \mid \|f\|_{L^2(M)} \leq t \|df\|_{L^2(M)}, (\forall) f \in H_D^1(M)\},$$

with the agreement that $c_{M, \partial_D M} = \infty$ if the set on the right hand side is empty. Thus $c_{M, \partial_D M} > 0$ is the best constant in the Poincaré inequality for $p = 2$ and vanishing boundary values (see Definition 4.1). Clearly $c_{M, \partial_D M} > 0$. The definition of $c_{M, \partial_D M}$ implies that $c_{M, \partial_D M} := \infty$ if and only if $(M, \partial_D M)$ does not satisfy the Poincaré inequality for $p = 2$.

Let ν be an outward unit vector at the boundary of our manifold with bounded geometry M and let ∂_ν be the directional derivative with respect to ν . Let d be the de Rham differential, d^* be its formal adjoint and $\Delta := d^*d$ be the scalar Laplace operator. Our main result is the following well-posedness result.

Theorem 1.2. *Let $(M, \partial_D M)$ be a manifold with boundary and bounded geometry and $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Then Δ with domain*

$$\mathcal{D}(\Delta) := H^2(M) \cap \{u = 0 \text{ on } \partial_D M \text{ and } \partial_\nu u = 0 \text{ on } \partial_N M\}$$

is self-adjoint on $L^2(M)$ and, for all $\lambda \in \mathbb{R}$ with $\lambda < \gamma := (1 + c_{M, \partial_D M}^2)^{-1}$, we have isomorphisms

$$\Delta - \lambda: H^{k+1}(M) \cap \mathcal{D}(\Delta) \rightarrow H^{k-1}(M), \quad k \geq 1.$$

If $(M, \partial_D M)$ has finite width, then $\gamma > 0$.

The purely Dirichlet version of this Theorem is contained in Theorem 4.13; for the general version, see Theorem 4.15. See Subsection 4.3 for further details. One obtains a version of the isomorphism in the Theorem also for $k = 0$. The example of

a smooth domain $\Omega \subset \mathbb{R}^n$ that coincides with the interior of a smooth cone outside a compact set shows that some additional assumptions on $(M, \partial_D M)$ are necessary in order for Δ to be invertible.

Our well-posedness result is of interest in itself, as a general result on analysis on non-compact manifolds, but also because it has applications to partial differential equations on singular and non-compact spaces, which are more and more often studied. Examples are provided by analysis on curved space-times in general relativity and conformal field theory, see [11, 12, 30, 31, 45, 41, 67] and the references therein. An important example of manifolds with bounded geometry is that of asymptotically hyperbolic manifolds, which play an increasingly important role in geometry and physics [5, 16, 25, 35]. It was shown in [52] that every Riemannian manifold is conformal to a manifold with bounded geometry. Moreover, manifolds of bounded geometry can be used to study boundary value problems on singular domains, see, for instance, [18, 19, 23, 24, 39, 47, 53, 54, 55] and the references therein.

The proof of Theorem 1.2 is based on a Poincaré inequality on M for functions vanishing on $\partial_D M$, under the assumption that the pair $(M, \partial_D M)$ has finite width, and on local regularity results (see Theorems 1.1 and 4.11). The higher regularity results is obtained using a description of Sobolev spaces on manifolds with bounded geometry using partitions of unity [34, 69] and is valid without the finite width assumption (we only need that M is with boundary and bounded geometry for our regularity result to be true). We also need the classical regularity of the Dirichlet and Neumann problems for strongly elliptic operators on smooth domains. The non-compactness of the boundary is dealt with using a continuity argument.

The paper is organized as follows. Section 2 is devoted to preliminary results on manifolds with boundary and bounded geometry. We begin with several equivalent definitions of manifolds with boundary and bounded geometry and their basic properties, some of which are new results; see in particular Theorem 2.10. The subsections 2.2 and 2.3 are devoted to some further properties of manifolds with boundary and bounded geometry that are useful for analysis. In particular, we introduce and study submanifolds with bounded geometry and we devise a method to construct manifolds with boundary and bounded geometry using a gluing procedure, see Corollary 2.24. In Subsection 2.4 we recall the definitions of basic coordinate charts on manifolds with boundary and bounded geometry and use them to define the Sobolev spaces, as for example in [34]. The Poincaré inequality for functions vanishing on $\partial_D M$ and its proof, together with some geometric preliminaries can be found in Section 3. The last section is devoted to applications to the analysis of the Laplace operator. We begin with the well-posedness in energy spaces (if $(M, \partial_D M)$ has finite width) and then we prove additional regularity results for solutions as well the fact that spectrum of Δ is contained in the closed half-line $\{\lambda \in \mathbb{R} \mid \lambda \geq \gamma\}$ (see Theorem 1.2). In particular, the proofs of Theorem 1.2, as well as that of its generalization to mixed boundary conditions, Theorem 4.15, can both be found in this section. Some of our results (such as regularity), extend almost without change to L^p -spaces, but doing that would require a rather big overhead of analytical technical background. On the other hand, certain results, such as the explicit determination of the spectrum using positivity, *do not extend* to the L^p setting. See [5, 46, 66] and the references therein.

2. MANIFOLDS WITH BOUNDARY AND BOUNDED GEOMETRY

In this section we introduce manifolds with boundary and bounded geometry and we discuss some of their basic properties. Our approach is to view both the manifold and its boundary as submanifolds of a larger manifold with bounded geometry, but without boundary. The boundary is treated as a submanifold of codimension one of the larger manifold, see Section 2.1. This leads to an alternative definition of manifolds with boundary and bounded geometry, which we prove to be equivalent to the standard one, see [64].

We also recall briefly in Section 2.4 the definition of the Sobolev spaces used in this paper.

2.1. Definition of manifolds with boundary and bounded geometry. *Unless explicitly stated otherwise, we agree that throughout this paper, M will be a smooth Riemannian manifold of dimension m with smooth boundary ∂M , metric g and volume form dvol_g . We are interested in non-compact manifolds M . The boundary will be a disjoint union $\partial M = \partial_D M \amalg \partial_N M$, with $\partial_D M$ and $\partial_N M$ both open (and closed) subsets of ∂M , as in Equation (1). (That is, the boundary of M is partitioned.) Typically, M will be either assumed or proved to be with (boundary and) bounded geometry.*

Remark 2.1 (Definition of a manifold). In textbooks, one finds two alternative definitions of a smooth manifold. Some textbooks define a manifold as a locally Euclidean, second countable Hausdorff space together with a smooth structure, see e.g. [48, Chapter 1]. Other textbooks, however, replace second countability by the weaker requirement that the manifold be a paracompact topological space. The second choice implies that every connected component is second countable, thus both definitions coincide if the number of connected components is countable. Thus, a manifold in the second sense is a manifold in the first sense if, and only if, the set of connected components is countable. As an example, consider \mathbb{R} with the discrete topology. It is not a manifold in the first sense, but it is a 0-dimensional manifold in the second sense. For most statements in differential geometry, it does not matter which definition is chosen. However, in order to allow uncountable index sets I in Theorem 3.1, we will only assume that our manifolds are paracompact (thus we will *not require* our manifolds to be second countable). This is needed in some applications.

For any $x \in M$, we define

$$\mathcal{D}_x := \{v \in T_x M \mid \text{there exists a geodesic } \gamma_v: [0, 1] \rightarrow M \text{ with } \gamma'_v(0) = v\}.$$

This set is open and star-shaped in $T_x M$ and

$$(4) \quad \mathcal{D} := \bigcup_{x \in M} \mathcal{D}_x$$

is an open subset of TM . The map $\exp^M: \mathcal{D} \rightarrow M$, $\mathcal{D}_x \ni v \mapsto \exp^M(v) = \exp_x^M(v) := \gamma_v(1)$ is called the *exponential map*.

If $x, y \in M$, then $\text{dist}(x, y)$ denotes the distance between x and y , computed as the infimum of the set of lengths of the paths in M connecting x to y . If x and y are not in the same connected component, we set $\text{dist}(x, y) = \infty$. The map $\text{dist}: M \times M \rightarrow [0, \infty]$ satisfies the axioms of an extended metric on M . This means

that it satisfies the usual axioms of a metric, except that it takes values in $[0, \infty]$ instead of $[0, \infty)$. If $A \subset M$ is a subset, then

$$(5) \quad U_r(A) := \{x \in M \mid \exists y \in A, \text{dist}(x, y) < r\}$$

will denote the r -neighborhood of A , that is, the set of points of M at distance $< r$ to A . Thus, if E is a Euclidean space, then

$$(6) \quad B_r^E(0) := U_r(\{0\}) \subset E$$

is simply the ball of radius r centered at 0.

We shall let $\nabla = \nabla^Y$ denote the Levi-Civita connection on a Riemannian manifold Y and R^Y denote its curvature. Now let Y be a submanifold of M with its induced metric. Recall the maximal domain $\mathcal{D} \subset TM$ of the exponential map from Equation (4). Let $\mathcal{D}^\perp := \mathcal{D} \cap \nu_Y$ be the intersection of \mathcal{D} with the normal bundle ν_Y of Y in M and

$$(7) \quad \exp^\perp := \exp|_{\mathcal{D}^\perp} : \mathcal{D}^\perp \rightarrow M,$$

be the *normal exponential map*. By choosing a local unit normal vector field ν to Y to locally identify \mathcal{D}^\perp with a subset of $Y \times \mathbb{R}$, we obtain $\exp^\perp(x, t) := \exp_x^M(t\nu_x)$. As \mathcal{D} is open in TM , we know that \mathcal{D}^\perp is a neighborhood of $Y \times \{0\}$ in $Y \times \mathbb{R}$.

Definition 2.2. We say that a Riemannian manifold (M, g) has *totally bounded curvature*, if its curvature R^M satisfies

$$\|\nabla^k R^M\|_{L^\infty} < \infty \quad \text{for all } k \geq 0.$$

where ∇ is the Levi-Civita connection on (M, g) , as before.

Recall the definition of the ball B_r^E of Equation (6). The injectivity radius of M at p , respectively, the *injectivity radius* of M are then

$$r_{\text{inj}}(p) := \sup\{r \mid \exp_p : B_r^{T_p M}(0) \rightarrow M \text{ is a diffeomorphism onto its image}\}$$

$$r_{\text{inj}}(M) := \inf_{p \in M} r_{\text{inj}}(p) \in [0, \infty].$$

Recall also following classical definition in the case $\partial M = \emptyset$.

Definition 2.3. A Riemannian manifold without boundary (M, g) is said to be of *bounded geometry* if M has *totally bounded curvature* and $r_{\text{inj}}(M) > 0$.

This definition cannot be carried over in a straightforward way to manifolds with boundary, as manifolds with non-empty boundary always have $r_{\text{inj}}(M) = 0$. We use instead the following alternative approach.

Let N^n be a submanifold of a Riemannian manifold M^m . Let $\nu_N \subset TM|_N$ be the normal bundle of N in M . Informally, the second fundamental form of N in M is defined as the map

$$(8) \quad \mathbb{I} : TN \times TN \rightarrow \nu_N, \quad \mathbb{I}(X, Y) := \nabla_X^M Y - \nabla_X^N Y.$$

Let us specialize this definition to the case when N is a hypersurface in M (a submanifold with $\dim N = \dim M - 1$) assumed to carry a globally defined normal vector field ν of unit length, called a *unit normal field*. Then one can identify the normal bundle of N in M with $N \times \mathbb{R}$ using ν , and hence, the second fundamental form of N is simply a smooth family of symmetric bilinear maps $\mathbb{I}_p : T_p N \times T_p N \rightarrow \mathbb{R}$, $p \in N$. In particular, we see that \mathbb{I} defines a smooth tensor. See [26, Chapter 6] for details.

Definition 2.4. Let (M^m, g) be a Riemannian manifold of bounded geometry with a hypersurface $N^{m-1} \subset M$ with a unit normal field ν to N . We say that N is a *bounded geometry* hypersurface if the following conditions are fulfilled:

- (i) N is a closed subset of M ;
- (ii) $(N, g|_N)$ is a manifold of bounded geometry;
- (iii) The second fundamental form \mathbb{I} of N in M and all its covariant derivatives along N are bounded. In other words:

$$\|(\nabla^N)^k \mathbb{I}\|_{L^\infty} \leq C_k \text{ for all } k \in \mathbb{N}_0;$$

- (iv) There is a number $\delta > 0$ such that $\exp^\perp: N \times (-\delta, \delta) \rightarrow M$ is injective.

See Definition 2.22 for the definition of arbitrary codimension submanifolds with bounded geometry. We prove in Section 2.3 that Axiom (ii) of Definition 2.4 is redundant. In other words it already follows from the other axioms. We keep Axiom (ii) in our list in order to make the comparison with the definitions in [64] and [34] more apparent.

Definition 2.5. A Riemannian manifold (M, g) with (smooth) boundary has *bounded geometry* if there is a Riemannian manifold $(\widehat{M}, \widehat{g})$ with bounded geometry satisfying

- (i) $\dim \widehat{M} = \dim M$
- (ii) M is contained in \widehat{M} , in the sense that there is an isometric embedding $(M, g) \hookrightarrow (\widehat{M}, \widehat{g})$
- (iii) ∂M is a bounded geometry hypersurface in \widehat{M} .

As unit normal vector field for ∂M we choose the outer unit normal field. Similar definitions were considered by [3, 4, 17, 64]. We will show in Section 2.3 that our definition coincides with [64, Definition 2.2]. In this section we also discuss further conditions, which are equivalent to the conditions in Definition 2.5.

Note that if M is a manifold with boundary and bounded geometry, then each connected component of M is a complete metric space. To simplify the notation, we say that a Riemannian manifold is *complete*, if any of its connected components is a complete metric space. The classical Hopf-Rinow theorem [26, Chapter 7] then tells us that a Riemannian manifold is complete if, and only if, all geodesics can be extended indefinitely.

Example 2.6. An important example of a manifold with boundary and bounded geometry is provided by Lie manifolds with boundary [7, 8].

Recall the definition of the sets $U_R(N)$ of Equation (5). For the Poincaré inequality, we shall also need to assume that $M \subset U_R(\partial_D M)$, for some $R > 0$, and hence, in particular, that $\partial_D M \neq \emptyset$. We formalize this in the concept of “manifold with finite width.”

Definition 2.7. Let (M, g) be a Riemannian manifold with boundary ∂M and $A \subset \partial M$. We say that (M, A) has *finite width* if:

- (i) (M, g) is a manifold with boundary and bounded geometry, and
- (ii) $M \subset U_R(A)$, for some $R < \infty$.

If $A = \partial M$, we shall also say that M has *finite width*.

Note that axiom (ii) implies that A intersects all connected components of M , as $\text{dist}(x, y) := \infty$ if x and y are in different components of M . Also, recall that the meaning of the assumption that $M \subset U_R(A)$ in this definition is that every point of M is at distance *at most* R to A .

Remark 2.8 (Hausdorff distance). Recall that for subsets A and B of a metric space (X, d) one defines the *Hausdorff distance* between A and B as

$$d_H(A, B) := \inf\{R > 0 \mid A \subset U_R(B) \text{ and } B \subset U_R(A)\} \in [0, \infty].$$

(See Equation (6) for notation.) The assignment $(A, B) \mapsto d_H(A, B)$ defines an extended metric on the set of all closed subsets of X . Again, the word “extended” indicates that d_H satisfies the usual axioms of a metric, but that it takes values in $[0, \infty]$ instead of $[0, \infty)$. See [13, 9.11] or [60, §7/14] for a definition and discussion of the Hausdorff distance on the set of compact subsets. In this language, a pair (M, A) as above has finite width if, and only if, $d_H(M, A) < \infty$. The definition of the Hausdorff distance implies $d_H(\emptyset, M) = \infty$.

Example 2.9. A very simple example of a manifold with finite width is obtained as follows. We consider $M := \Omega \times K$ where Ω is a smooth, compact Riemannian manifold with smooth boundary and K is a Riemannian manifold with bounded geometry (K has no boundary). Then M is a manifold with boundary and bounded geometry. Let $\partial_D \Omega$ be a union of connected components of $\partial \Omega$ that intersects every connected component of Ω . Let $\partial_D M := \partial_D \Omega \times K$. Then $(M, \partial_D M)$ is a manifold with finite width.

On the other hand, let $\Omega \subset \mathbb{R}^N$ be an open subset with smooth boundary. We assume that the boundary of Ω coincides with the boundary of a cone outside some compact set. Then $(\Omega, \partial \Omega)$ is a manifold with boundary and bounded geometry, but it does not have finite width. See also Example 2.23.

An alternative characterization of manifolds with boundary and bounded geometry is contained in the following theorem, that will be proven in Section 2.3 using some preliminaries on the injectivity radius that are recalled in Section 2.2

Theorem 2.10. *Let (M, g) be a Riemannian manifold with smooth boundary and curvature $R^M = (\nabla^M)^2$ and let \mathbb{I} be the second fundamental form of the boundary ∂M in M . Assume that:*

(N) *There is $r_\partial > 0$ such that*

$$\partial M \times [0, r_\partial) \rightarrow M, (x, t) \mapsto \exp^\perp(x, t) := \exp_x(t\nu_x)$$

is a diffeomorphism onto its image.

(I) *There is $r_{\text{inj}}(M) > 0$ such that for all $r \leq r_{\text{inj}}(M)$ and all $x \in M \setminus U_r(\partial M)$, the exponential map $\exp_x: B_r^{T_x M}(0) \subset T_x M \rightarrow M$ defines a diffeomorphism onto its image.*

(B) *For every $k \geq 0$, we have*

$$\|\nabla^k R^M\|_{L^\infty} < \infty \quad \text{and} \quad \|(\nabla^{\partial M})^k \mathbb{I}\|_{L^\infty} < \infty.$$

Then (M, g) is a Riemannian manifold with boundary and bounded geometry in the sense of Definition 2.5.

Remark 2.11. The theorem implies in particular, that our definition of a manifold with boundary and bounded geometry coincides with the one given by Schick in [64, Definition 2.2]. According to Schick’s definition, a manifold with boundary

has bounded geometry if it satisfies (N), (I) and (B) and if the boundary itself has positive injectivity radius. One of the statements in the theorem is that (N), (I) and (B) imply that the boundary has bounded geometry.

Remark 2.12. In [15] Botvinnik and Müller defined manifolds with boundary with (c, k) -bounded geometry. Their definition differs from our definition of manifolds with boundary and bounded geometry in several aspects, in particular they only control k derivatives of the curvature.

2.2. Preliminaries on the injectivity radius. We continue with some technical results on the injectivity radius. Let (M, g) be a Riemannian manifold without boundary and $p \in M$. We write $r_{\text{inj}}(p)$ for the injectivity radius of (M, g) at p (that is the supremum of all r such that $\exp: B_r^{T_p M}(0) \subset T_p M \rightarrow M$ is injective, as before).

We define the *curvature radius* of (M, g) at p by

$$\rho := \sup\{r > 0 \mid \exp_p \text{ is defined on } B_{\pi r}^{T_p M}(0) \text{ and } |\text{sec}| \leq 1/r^2 \text{ on } B_{\pi r}^M(p)\},$$

where sec is the sectional curvature and “ $|\text{sec}| \leq 1/r^2$ on $B_{\pi r}^M(p)$ ” is the short notation for “ $|\text{sec}(E)| \leq 1/r^2$ for all planes $E \subset T_q M$ and all $q \in B_{\pi r}^M(p)$ ”. It is clear that the set defining $\rho = \rho_p$ is not empty, and hence $\rho = \rho_p \in (0, \infty]$ is well defined. Standard estimates in Riemannian geometry, see e.g. [26], show that the exponential map is an immersion on $B_{\pi\rho}^M(p)$. In other words, no point in $B_{\pi\rho}^M(p)$ is conjugate to p with respect to geodesics of length $< \pi\rho$.

Let us notice that if there is a geodesic γ from p to p of length $2\delta > 0$, then it is obvious that $r_{\text{inj}}(p) \leq \delta$. The converse is true if δ is small:

Lemma 2.13 ([61, Proposition III.4.13]). *If $r_{\text{inj}}(p) < \pi\rho$, then there is geodesic of length $2r_{\text{inj}}(p)$ from p to p .*

We will use the following theorem, due to Cheeger, see [22]. We also refer to [57, Sec. 10, Lemma 4.5] for an alternative proof, which easily generalizes to the version below.

Theorem 2.14 (Cheeger). *Let (M, g) be a Riemannian manifold with curvature satisfying $|R^M| \leq K$. Let $A \subset M$. We assume that there is an $r_0 > 0$ such that \exp_p is defined on $B_{r_0}^{T_p M}(0)$ for all $p \in A$. Let $\rho > 0$. Then $\inf_{p \in A} r_{\text{inj}}(p) > 0$ if, and only if, $\inf_{p \in A} \text{vol}(B_\rho^M(p)) > 0$.*

2.3. Controlled submanifolds are of bounded geometry. The goal of this subsection is to prove Theorem 2.10. For further use, some of the Lemmata below are formulated not just for hypersurfaces, such as ∂M , but also for submanifolds of higher codimension.

Let again N^n be a submanifold of a Riemannian manifold M^m with second fundamental form II and normal exponential map \exp^\perp as recalled in (7) and (8). For $X \in T_p N$ and $s \in \Gamma(\nu_N)$, one can decompose $\nabla_X s$ in the $T_p N$ component (which is given by the second fundamental form) and the normal component $\nabla_X^\perp s$. This definition gives us a connection ∇^\perp on the bundle $\nu_N \rightarrow N$. We write R^\perp for the associated curvature.

Definition 2.15. Let M be a Riemannian manifold without boundary. We say that a closed submanifold $N \subset M$ with second fundamental form II is *controlled* if

$\|(\nabla^N)^k \mathbb{I}\|_{L^\infty} < \infty$ for all $k \geq 0$ and there is $r_\partial > 0$ such that

$$\exp^\perp : V_{r_\partial}(\nu_N) \rightarrow M$$

is injective, where $V_r(\nu_N)$ is the set of all vectors in ν_N of length $< r$.

We will show that the geometry of a controlled submanifold is “bounded.” Thus, “controlled” should be seen here as an auxiliary label and we will change the name “controlled submanifold” to “*bounded geometry submanifold*.”

Lemma 2.16. *For every Riemannian manifold (M, g) with boundary and bounded geometry, there is a complete Riemannian manifold \widehat{M} without boundary such that:*

- (i) $\|\nabla^k R^{\widehat{M}}\|_{L^\infty} < \infty$, for all $k \geq 0$;
- (ii) $M \rightarrow \widehat{M}$ is an isometric embedding, and
- (iii) ∂M is a controlled submanifold of \widehat{M} .

Proof. The metric $(\exp^\perp)^*g$ on $\partial M \times [0, r_\partial)$ is of the form $(\exp^\perp)^*g = h_r + dr^2$ where $r \in [0, r_\partial)$ and where h_r , $r \in [0, r_\partial)$, is a family of metrics on ∂M such that $(\partial/\partial r)^k h_r$ is a bounded tensor for any $k \in \mathbb{N}$. Using a cut-off argument, it is possible to define h_r also for $r \in (-1 - r_\partial, 0)$ in such a way that all $(\partial/\partial r)^k h_r$ are bounded tensors and $h_r = h_{-1-r}$ for all $r \in (-1 - r_\partial, r_\partial)$. An immediate calculation shows that then $(\partial M \times (-1 - r_\partial, r_\partial), h_r + dr^2)$ has bounded curvature R , and all derivatives of R are bounded. We then obtain \widehat{M} by gluing together two copies of M together along $\partial M \times (-1 - r_\partial, r_\partial)$. Obviously the curvature and all its derivatives are bounded on \widehat{M} . \square

Lemma 2.17. *Let (M, g) be a complete Riemannian manifold with boundary ∂M satisfying conditions (N), (I), and (B) introduced in Theorem 2.10. Then the injectivity radius of the manifold \widehat{M} as constructed in the proof of the last lemma is positive.*

Proof. We use (I) for $r := \frac{1}{2} \min\{r_{\text{inj}}(M), r_\partial\}$ to obtain that $\inf_{q \in M \setminus U_r(\partial M)} r_{\text{inj}}(q) > 0$. It thus remains to show that $\inf_{q \in \partial M \times [-1-r, r]} r_{\text{inj}}(q) > 0$. Let $q = (x, t) \in \partial M \times [-1-r, r]$. We define the diffeomorphism $f_q : \partial M \times (t-r, t+r) \rightarrow \partial M \times (0, 2r) \subset M$, $(y, s) \mapsto (y, s-t+r)$. Then the operator norms $\|(df_q)\|$ and $\|(df_q)^{-1}\|$ are bounded on their domains by a bound $C_1 \geq 1$ that only depends on (M, g) and the chosen extension \widehat{M} , but not on q . Theorem 2.14 for $\rho = r/C_1$ gives $v > 0$ such that $\text{vol}(B_{r/C_1}^M(z)) > v$ for all $z \in M \setminus U_r(\partial M)$. Together with $B_{r/C_1}^M(f_q(q)) \subset f_q(B_r^{\widehat{M}}(q)) \subset \partial M \times (0, 2r)$ we get

$$\text{vol}(B_r^{\widehat{M}}(q)) \geq C_1^{-m} \text{vol}(f_q(B_{r/C_1}^M(q))) \geq C_1^{-m} \text{vol}(B_{r/C_1}^M(q)) \geq C_1^{-m} v.$$

Using again Theorem 2.14 for $\rho = r$ we obtain the required statement. \square

Lemma 2.18. *Let (M, g^M) be a Riemannian manifold without boundary and with totally bounded curvature (Definition 2.2). Let $N \subset M$ be a submanifold with $\|(\nabla^N)^k \mathbb{I}\|_{L^\infty} < \infty$ for all $k \geq 0$. Then*

- (i) $\|(\nabla^N)^k R^N\|_{L^\infty} < \infty$ for all $k \geq 0$.
- (ii) *The curvature of the normal bundle of N in M and all its covariant derivatives are bounded. In other words $\|(\nabla^N)^k R^\perp\|_{L^\infty} \leq c_k$ for all $k \in \mathbb{N}_0$.*

Proof. (i) This is [34, Lemma 4.5].

(ii) For a normal vector field η let W_η be the Weingarten map for η , i.e., for $X \in T_p N$, let $W_\eta(X)$ be the tangential part of $-\nabla_X \eta$. Thus $g^M(\mathbb{I}(X, Y), \eta) = g^M(W_\eta(X), Y)$ for the vector-valued second fundamental form \mathbb{I} . The Ricci equation [14, p. 5] states that the curvature R^\perp of the normal bundle is

$$(9) \quad g^M(R^\perp(X, Y)\eta, \zeta) = g^M(R^M(X, Y)\eta, \zeta) - g^M(W_\eta(X), W_\zeta(Y)) \\ + g^M(W_\eta(Y), W_\zeta(X)),$$

where $X, Y \in T_p N$, and where $\eta, \zeta \in T_p M$ are normal to N . The boundedness of $R^M(X, Y)$ and \mathbb{I} thus implies the boundedness of R^\perp . In order to bound $(\nabla^N)^k R^\perp$, one has to differentiate (9) covariantly. The difference between ∇ and ∇^N provides then additional terms that are linear in \mathbb{I} . Thus, one iteratively sees that $(\nabla^N)^k R^\perp$ is a polynomial in the variables $\mathbb{I}, \nabla^N \mathbb{I}, \dots, (\nabla^N)^k \mathbb{I}, R, \nabla R^M, \dots, \nabla^k R^M$, and thus that it is bounded. \square

The total space of the normal bundle ν_N carries a natural Riemannian metric. Indeed, the connection ∇^\perp defines a splitting of the tangent bundle of this total space into the vertical tangent space (which is the kernel of the differential of the base point map that maps every vector to its base point) and the horizontal tangent space (given by the connection). The horizontal space of ν_N inherits the Riemannian metric of N and the vertical tangent spaces of ν_N are canonically isomorphic to the fibers of the normal bundle ν_N and they carry the canonical metric. Let $\pi: \nu_N \rightarrow N$ be the base point map. Then π is a Riemannian submersion.

Let $F := \exp^\perp|_{V_{r_1}(\nu_N)}$ (see Definition 2.15 for notation). It is clear from the definition of the normal exponential map that the bounds on $\nabla^k R^M$ and on $(\nabla^N)^k \mathbb{I}$ imply that there is an $r_1 \in (0, r_\partial)$ such that the derivative dF and its inverse $(dF)^{-1}$ are uniformly bounded.

Lemma 2.19. *Let M^m be a Riemannian manifold without boundary and of bounded geometry. Let N^n be a controlled submanifold of M (Definition 2.15). Then the injectivity radius of N is positive.*

Proof. The proof is similar to the one of Lemma 2.17. Let $F := \exp^\perp|_{V_{r_1}(\nu_N)}$ be as above. Assume $\|dF\| \leq C_2$ and $\|(dF)^{-1}\| \leq C_2$ with $C_2 \geq 1$ on $V_{r_1}(\nu_N)$. For $q \in N$, we define $B(q) := V_{r_1}(\nu_N) \cap \pi^{-1}(B_{r_1}^N(q))$, using the notations of Definition 2.15. Then, $B_{r_1/C_2}^M(q) \subset F(B(q))$ by the boundedness of $(dF)^{-1}$. By Theorem 2.14 applied to M and to $\rho = r_1/C_2$, there is a $v > 0$, independent of q , such that $\text{vol}(B_{r_1/C_2}^M(q)) \geq v$, and thus $\text{vol}(F(B(q))) \geq v$. Thus, $\text{vol}(B(q)) \geq C_2^{-n}v$ where the volume on $B(q)$ is taken with respect to the natural metric on ν_N . Then by Lemma 2.18 (compare to the proof of [34, Lemma 4.4]), $\text{vol}(B(q)) \leq C_3 r_1^{m-n} \text{vol}(B_{r_1}^N(q))$. Thus, $\text{vol}(B_{r_1}^N(q))$ is bounded from below independently on $q \in N$. Theorem 2.14 applied for $\rho = r_1$ then yields that N has a positive injectivity radius. \square

This sequence of lemmata gives us the desired result.

Corollary 2.20. *Let N be a controlled submanifold in a Riemannian manifold of bounded geometry. Then, N is a manifold of bounded geometry.*

The next corollary, together with Lemma 2.16, implies then Theorem 2.10.

Corollary 2.21. *Let M be a Riemannian manifold with boundary satisfying (N), (I) and (B). Then, ∂M is a manifold of bounded geometry.*

As we have seen, controlled submanifolds have bounded geometry in the sense of Theorem 2.10. This suggests the following definition.

Definition 2.22. A controlled submanifold $N \subset M$ of a manifold with bounded geometry will be called a *bounded geometry submanifold* of M .

See also Eldering's papers [28, 29]. In particular a hypersurface with unit normal field is controlled if, only if, it is a bounded geometry hypersurface as in Definition 2.4. The following example is sometimes useful.

Example 2.23. Let (M_0, g_0) be a manifold with bounded geometry (thus, without boundary). Let $f: M_0 \rightarrow \mathbb{R}$ be a smooth function such that df is totally bounded (that is, all covariant derivatives $\nabla^k df$ are uniformly bounded for $k \geq 0$). The function f itself does not have to be bounded. Let us check that

$$\Omega_{\pm}(f) := \{(x, t) \in M := M_0 \times \mathbb{R} \mid \pm(t - f(x)) \geq 0\}$$

is a manifold with boundary and bounded geometry using Theorem 2.10. Most of the conditions of Theorem 2.10 are already fulfilled since (M_0, g_0) is of bounded geometry; only condition (N) and the part of condition (B) including the second fundamental form remain to be checked. We start with the second fundamental form. The second fundamental form of N is defined by $\mathbb{I}(X, Y) = -g(\nabla_X^M \nu, Y)\nu$ for all vector fields X, Y tangent to N where $g = g_0 + dt^2$ where a unit normal vector field of $N := \partial\Omega_{\pm}(f)$ is given by

$$\nu_{(x, f(x))} = (1 + |\text{grad}^g f(x)|^2)^{-\frac{1}{2}}(\text{grad}^g f(x) - \partial_t).$$

A vector field tangent to N has the form $X = (X_1, X_2\partial_t)$ with $X_1 \in \Gamma(TM_0)$ and $X_2 \in C^\infty(\mathbb{R})$ with $df(X_1) = X_2$. Since the metric g has product structure on $M_0 \times \mathbb{R}$, the Levi-Civita connection splits componentwise. More precisely, given the vector fields $X = X_1 + X_2\partial_t$ and $Y = Y_1 + Y_2\partial_t$ tangent to N we get

$$\mathbb{I}(X, Y) = -\langle Y_1 + Y_2\partial_t, (\nabla_{X_1}^{M_0} + X_2\partial_t)\nu \rangle \nu.$$

Since $\nabla^k df$ is bounded for all $k \geq 0$ and (M, g) has bounded geometry, we get that \mathbb{I} and all its covariant derivatives are uniformly bounded. In particular the mean curvature of N is bounded.

In order to verify condition (N) of Theorem 2.10 we first note that due to the product structure on $M = M_0 \times \mathbb{R}$, it suffices to prove that there are constant $r_{\partial}, \delta > 0$ such that $\exp^\perp|_{V_x}$ is a diffeomorphism onto its image for all $x = (y, t) \in N$ and $V_x := \{(y', t') \in N \mid d_{g_0}(y, y') < \delta\}$. Since f is bounded, there is $\hat{\delta} > 0$ such that $V_x \subset B_{\hat{\delta}}(x) \subset M$ and together with the bounded geometry of M_0 the existence of positive constants r_{∂} and δ follow. Hence, altogether we get that $\Omega_{\pm}(f)$ is a manifold with boundary and bounded geometry.

Let us now assume additionally that $g: M_0 \rightarrow \mathbb{R}$ has the same properties (i.e. dg is totally bounded). Assume $g < f$. Then

$$\Omega(f, g) := \Omega_-(f) \cap \Omega_+(g) = \{(x, t) \in M_0 \times \mathbb{R} \mid g(x) \leq t \leq f(x)\}$$

is a manifold with boundary. Denote $M := \Omega(f, g)$ and let $\partial_D M$ be any non-empty union of connected components of ∂M . Then $(M, \partial_D M)$ has bounded geometry if, and only if, there exists $\epsilon > 0$ such that $f - g \geq \epsilon$. It has finite width if, and

only if, there exists also $R > 0$ such that $R \geq f - g \geq \epsilon$. In [17], manifolds with boundary and bounded geometry were considered in the particular case when they were subsets of \mathbb{R}^n . Our criteria for $\Omega(f, g)$ to be of bounded geometry thus helps reconcile the definitions in [17] and [64].

The following corollary of Theorem 2.10, when used in conjunction with Example 2.23, yields many examples of manifolds with boundary and bounded geometry.

Corollary 2.24. *Let $M = \bigcup_{i=1}^N W_i$ be a Riemannian manifold with boundary with W_i open subsets in M . We assume that*

(N') *For each $1 \leq i \leq N$, there exists $r_i > 0$ such that*

$$(\partial M \cap W_i) \times [0, r_i) \rightarrow W_i, (x, t) \mapsto \exp^\perp(x, t) := \exp_x(t\nu_x)$$

is well defined and a diffeomorphism onto its image.

(I') *There is $r_{\text{inj}}(M) > 0$ such that for all $r \leq r_{\text{inj}}(M)$ and all $x \in M \setminus U_r(\partial M)$, there exists $1 \leq i \leq N$ such that $x \in W_i$ and the exponential map $\exp_x: B_r^{T_x W_i}(0) \rightarrow W_i$ is well-defined and a diffeomorphism onto its image.*

(B) *For every $k \geq 0$ and $i = 1, \dots, N$, we have*

$$\|\nabla^k R^{W_i}\|_{L^\infty} < \infty \quad \text{and} \quad \|(\nabla^{\partial W_i})^k \mathbf{II}\|_{L^\infty} < \infty.$$

Then (M, g) is a Riemannian manifold with boundary and bounded geometry.

In applications, the subsets W_i will be open subsets of some other manifolds with boundary and bounded geometry (say of some manifold of the form $\Omega(f, g)$). In general, the subsets W_i will not be with bounded geometry. If, moreover, the boundary of M is partitioned: $\partial M = \partial_D M \amalg \partial_N M$ and $d_H(W_i, W_i \cap \partial_D M) < \infty$ for all i , then $(M, \partial_D M)$ will have finite width.

2.4. Sobolev spaces via partitions of unity. *In this subsection, M will be a manifold with boundary and bounded geometry, unless explicitly stated otherwise. This will be the case in most of the rest of the paper.*

We will need local descriptions of the Sobolev spaces using partitions of unity. To this end, it will be useful to think of manifolds with bounded geometry in terms of coordinate charts. This can be done by introducing *Fermi coordinates* on M as in [34]. See especially Definition 4.3 of that paper, whose notation we follow here. Recall that $r_{\text{inj}}(M)$ and $r_{\text{inj}}(\partial M)$ denote, respectively, the injectivity radii of M and ∂M . Also, let $r_\partial := \delta$ with δ as in Definition 2.4.

Let $p \in \partial M$ and consider the diffeomorphism $\exp_p^{\partial M}: B_r^{T_p \partial M}(0) \rightarrow B_r^{\partial M}(p)$, if r is smaller than the injectivity radius of ∂M . Sometimes, we shall identify $T_p \partial M$ with \mathbb{R}^{m-1} using an orthonormal basis, thus obtaining a diffeomorphism $\exp_p^{\partial M}: B_r^{m-1}(0) \rightarrow B_r^{\partial M}(p)$, where $B_r^{m-1}(0) \subset \mathbb{R}^{m-1}$ denotes the Euclidean ball with radius r around $0 \in \mathbb{R}^{m-1}$. Also, recall the definition of the normal exponential map $\exp^\perp: \partial M \times [0, r_\partial) \rightarrow M$, $\exp^\perp(x, t) := \exp_x^M(t\nu_x)$. These two maps combine with the exponential \exp_p^M to define maps

$$(10) \quad \begin{cases} \kappa_p: B_r^{m-1}(0) \times [0, r) \rightarrow M, & \kappa_p(x, t) := \exp^\perp(\exp_p^{\partial M}(x), t), & \text{if } p \in \partial M \\ \kappa_p: B_r^m(0) \rightarrow M, & \kappa_p(v) := \exp_p^M(v), & \text{otherwise.} \end{cases}$$

We let

$$(11) \quad W_p(r) := \begin{cases} \kappa_p(B_r^{m-1}(0) \times [0, r)) \subset M & \text{if } p \in \partial M \\ \kappa_p(B_r^m(0)) = \exp_p^M(B_r^m(0)) & \text{otherwise.} \end{cases}$$

In the next definition, we need to consider only the case $p \in \partial M$, however, the other case will be useful when considering partitions of unity.

Definition 2.25. Let $p \in \partial M$ and $r_{FC} := \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_{\partial} \}$. Fix $0 < r \leq r_{FC}$. The map $\kappa_p: B_r^{m-1}(0) \times [0, r) \rightarrow W_p(r)$ is called a *Fermi coordinate chart* and the resulting coordinates $(x^i, r): W_p(r) \rightarrow \mathbb{R}^{m-1} \times [0, \infty)$ are called *Fermi coordinates (around p)*.

Figure 1 describes the Fermi coordinate chart.

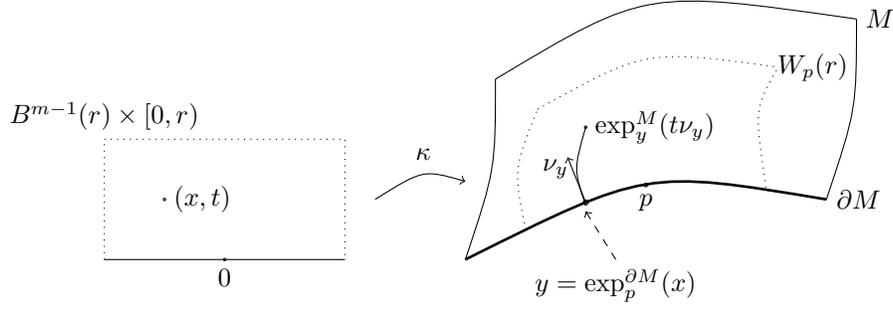


FIGURE 1. Fermi coordinates

Remark 2.26. Let (M, g) be a manifold with boundary and bounded geometry. Then the coefficients of g all their derivatives are uniformly bounded in Fermi coordinates charts, see, for instance, [34, Definition 3.7, Lemma 3.10 and Theorem 4.9].

For the sets in the covering that are away from the boundary, we will use geodesic normal coordinates, whereas for the sets that intersect the boundary, we will use Fermi coordinates as in Definition 2.25. This works well for manifolds with bounded geometry. Note that in this subsection, we *do not* assume that M has finite width.

Recall the notation of Equation (11).

Definition 2.27. Let M^m be a manifold with boundary and bounded geometry. Assume as in Definition 2.25 that $0 < r \leq r_{FC} := \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_{\partial} \}$. A subset $\{p_\gamma\}_{\gamma \in I}$ is called an *r -covering subset of M* if the following conditions are satisfied:

- (i) For each $R > 0$, there exists $N_R \in \mathbb{N}$ such that, for each $p \in M$, the set $\{\gamma \in I \mid \text{dist}(p_\gamma, p) < R\}$ has at most N_R elements.
- (ii) For each $\gamma \in I$, we have either $p_\gamma \in \partial M$ or $d(p_\gamma, \partial M) \geq r$, so that $W_\gamma := W_{p_\gamma}(r)$ is defined.
- (iii) $M \subset \cup_{\gamma=1}^\infty W_\gamma$.

Remark 2.28. It follows as in [34, Remark 4.6] that if $0 < r < r_{FC}$ then an r -covering subset of M always exists, since M is a manifold with boundary and bounded geometry. The picture below shows an example of an r -covering set,

where, the p_β 's denote the points $p_\gamma \in \partial M$ and the p_α 's denote the rest of the points of $\{p_\gamma\}$.

Remark 2.29. Let (M, g) be a manifold with boundary and bounded geometry. Let $\{p_\gamma\}_{\gamma \in I}$ be an r -covering set and $\{W_\gamma\}$ be the associated covering of M . It follows from (i) of Definition 2.27 that the coverings $\{W_\gamma\}$ of M and $\{W_\gamma \cap \partial M\}$ of ∂M are *uniformly locally finite*, i.e. there is an $N_0 > 0$ such that no point belongs to more than N_0 of the sets W_γ .

We shall need the following class of partitions of unity defined using r -covering sets. Recall the definition of the sets W_γ from Definition 2.27(ii).

Definition 2.30. A partition of unity $\{\phi_\gamma\}_{\gamma \in I}$ of M is called an *r -uniform partition of unity associated to the r -covering set $\{p_\gamma\} \subset M$* (see Definition 2.27) if

- (i) The support of each ϕ_γ is contained in W_γ .
- (ii) For each multi-index α , there exists $C_\alpha > 0$ such that $|\partial^\alpha \phi_\gamma| \leq C_\alpha$ for all γ , where the derivatives ∂^α are computed in the normal geodesic coordinates, respectively Fermi coordinates, on W_γ .

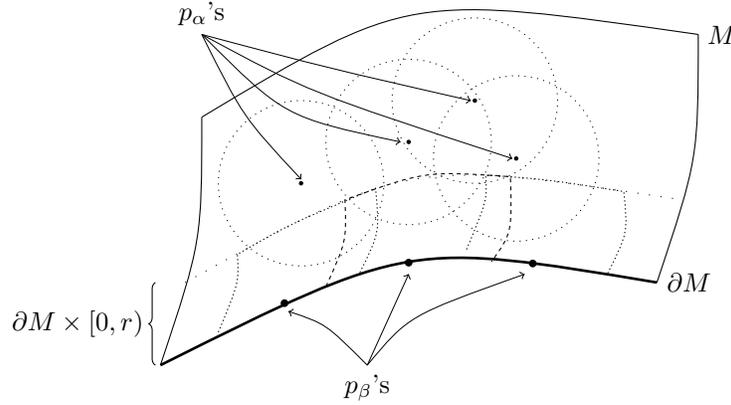


FIGURE 2. A uniformly locally finite cover by Fermi and geodesic coordinate charts, compare with Remark 2.28.

Remark 2.31. Given an r -covering set S with $r \leq r_{FC}/4$, an r -uniform partition of unity associated to $S \subset M$ always exists, since M is a manifold with boundary and bounded geometry [34, Lemma 4.8].

We have then the following proposition that is a consequence of Remark 3.5 and Theorem 3.9 in [34]. See also [3, 4, 7, 10, 11, 30, 46, 65, 69] for related results, in particular, for the use of partitions of unity.

Proposition 2.32. *Let M^m be a manifold with boundary and bounded geometry. Let $\{\phi_\gamma\}$ be a uniform partition of unity associated to the r -covering set $\{p_\gamma\} \subset M$ and let $\kappa_\gamma = \kappa_{p_\gamma}$ be as in Equation 2.25. Then*

$$\| \|u\| \|^2 := \sum_{\gamma} \|(\phi_\gamma u) \circ \kappa_\gamma\|_{H^k}^2$$

defines a norm equivalent to the usual norm $\|u\|_{H^k(M)}^2 := \sum_{i=0}^k \|\nabla^i u\|_{L^2(M)}^2$ on $H^k(M)$, $k \in \mathbb{N}$. Here $\|\cdot\|_{H^k}$ is the H^k norm on either \mathbb{R}^m or on the half-space \mathbb{R}_+^m .

Similarly, we have the following extension of the trace theorem to the case of a manifold M with boundary and bounded geometry, see Theorem 5.14 in [34]. (See also [7] for the case of Lie manifolds.)

Theorem 2.33 (Trace theorem). *Let M be a manifold with boundary and bounded geometry. Then, for every $s > 1/2$, the restriction to the boundary $\text{res}: \Gamma_c(M) \rightarrow \Gamma_c(\partial M)$ extends to a continuous, surjective map*

$$\text{res}: H^s(M) \rightarrow H^{s-\frac{1}{2}}(\partial M).$$

Notations 2.34. We shall denote by $H_0^1(M)$ the closure of $C_c^\infty(M \setminus \partial M)$. Then it coincides with the kernel of the trace map $H^1(M) \rightarrow L^2(\partial M)$. We also have that $H^{-1}(M)$ identifies with the (complex conjugate) dual of $H_0^1(M)$ with respect to the duality map given by the L^2 -scalar product. In general, we shall denote by $H_D^1(M)$ the kernel of the restriction map $H^1(M) \rightarrow L^2(\partial_D M)$ and by $H_D^1(M)^*$ the (complex conjugate) dual of $H_D^1(M)$ (see Equation (2) for the definition of these spaces).

3. THE POINCARÉ INEQUALITY

Recall that the boundary of M is partitioned, that is, that we have fixed a boundary decomposition $\partial M = \partial_D M \amalg \partial_N M$ into open and closed disjoint subsets $\partial_D M$ and $\partial_N M$. We assume in this section that $(M, \partial_D M)$ has **finite width**. In particular, in this section, M is a manifold with boundary and bounded geometry and $\partial_D M$ has a non-empty intersection with each connected component of M .

3.1. Uniform generalization of the Poincaré inequality. The following seemingly more general statement is in fact equivalent to the Poincaré inequality (Theorem 1.1).

Theorem 3.1. *Let M_α , $\alpha \in I$, be a family of m -dimensional smooth Riemannian manifolds with partitioned smooth boundaries $\partial M_\alpha = \partial_D M_\alpha \amalg \partial_N M_\alpha$. Let us assume that the disjoint union $M := \coprod M_\alpha$ is such that $(M, \partial_D M)$ has finite width, where $\partial_D M := \coprod \partial_D M_\alpha$. Then, for every $p \in [1, \infty]$, there exists $0 < c_{\text{unif}} < \infty$ (independent of α) such that*

$$\|f\|_{L^p(M_\alpha)} \leq c_{\text{unif}} (\|f\|_{L^p(\partial_D M_\alpha)} + \|df\|_{L^p(M_\alpha)})$$

for all $\alpha \in I$ and all $f \in W_{\text{loc}}^{1,p}(M_\alpha)$.

The main point of this reformulation of the Poincaré inequality is, of course, that c_{unif} is independent of $\alpha \in I$. Here $W^{1,p}$ refers to the norm $\|f\|_{W^{1,p}}^p = \|f\|_{L^p}^p + \|df\|_{L^p}^p$ (with the usual modification for $p = \infty$) and $W^{1,2} = H^1$. Moreover, $W_{\text{loc}}^{1,p}$, respectively L_{loc}^p , consists of distributions that are locally in $W^{1,p}$, respectively in L^p . Note that $f \in W_{\text{loc}}^{1,p}(M)$ implies $f|_{\partial M} \in L_{\text{loc}}^p(\partial M)$ by standard local trace inequalities [17]. In particular, if $\|f\|_{L^p(\partial M)} + \|df\|_{L^p(M)}$ is finite, then $\|f\|_{L^p}$ is also finite and the inequality holds.

Theorem 3.1 follows from Theorem 1.1 with the following arguments. Assume that a constant $c_{\text{unif}} > 0$ as above does not exist, then there is a sequence $\alpha_i \in I$ and a sequence of non-vanishing $f_i \in W_{\text{loc}}^{1,p}(M_{\alpha_i})$ with

$$\frac{(\|f_i\|_{L^p(\partial_D M_{\alpha_i})} + \|df_i\|_{L^p(M_{\alpha_i})})}{\|f_i\|_{L^p(M_{\alpha_i})}} \rightarrow 0$$

By extending f_i by zero to a function defined on all of M (which is the *disjoint* union of all the manifolds M_α), we would get a counterexample to Theorem 1.1.

Note that in the case that I is uncountable, M is not second countable. Many textbooks do not allow such manifolds, see Remark 2.1. If one wants to restrict to second countable manifolds, we can only allow countable sets I above.

Using Kato's inequality, we obtain that the Poincaré inequality immediately extends to sections in vector bundles equipped with Riemannian or hermitian bundle metrics and compatible connections. In Example 4.8, we will give an example that illustrates the necessity of the finite width assumption.

3.2. Idea of the proof of the Poincaré inequality. We shall now prove the version of the Poincaré inequality stated in the Introduction (Theorem 1.1). We will do that for all $p \in [1, \infty]$, although we shall use only the case $p = 2$ in this paper. The proof of the Poincaré inequality will be split into several steps and will be carried out in the next subsections. For the benefit of the reader, we first explain in Remark 3.2 the idea of the proof of the Poincaré inequality by Sakurai for the case $\partial M = \partial_D M$. In Remark 3.3 we comment on the differences of that proof and ours.

Remark 3.2 (Idea of the proof for $\partial M = \partial_D M$). The case $\partial M = \partial_D M$ of Theorem 1.1 was proven by Sakurai in [62, Lemma 7.3] (for real valued functions and $p = 1$) under weaker assumptions than ours on the curvature tensor. We recall the main ideas of Sakurai's proof: As in the classical case of bounded domains, one starts with the one-dimensional Poincaré estimate on geodesic rays emanating from ∂M and perpendicular to ∂M up to the point where those geodesics no longer minimize the distance to the boundary. The main point of the proof is that every point is covered exactly once by one of these minimizing geodesic perpendiculars, except for a zero measure set (the cut locus of the boundary). Another important point is that the constant in each of these one-dimensional Poincaré estimates is given in terms of the lengths of those minimal geodesics and, thus, is uniformly bounded in terms of the width of the manifold M . The global Poincaré inequality then follows by integrating over ∂M the individual one-dimensional Poincaré inequalities. Here one needs to take care of the volume elements since M is not just a product of ∂M with a finite interval – but this can be done by comparison arguments using the assumptions on the geometry.

See also [20, 36, 63] for Poincaré type inequalities on manifolds without boundary and bounds on the Ricci tensor.

Remark 3.3 (Idea of proof in the general case). Now, we no longer require $\partial_D M = \partial M$. In this case, we use a similar strategy, with the difference that now we are using the one-dimensional Poincaré inequality *only* for geodesics emanating perpendicularly from $\partial_D M$. One can see then that an additional technical problem occurs since there may be a subset of M with non-zero measure that cannot be reached by such geodesics. This phenomenon is illustrated in Figure 3. Nevertheless, this problem can be circumvented by first assuming that the metric has a product structure near the boundary. The necessary preliminaries for this part will be given in Section 3.3 and the proof of our Poincaré inequality in this case is carried out in Subsection 3.4, following the method of [62]. The case of a general metric is then obtained by comparing the given metric with a metric that has a product structure

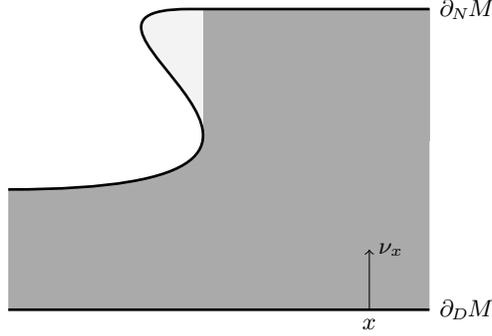


FIGURE 3. Manifold $M \subset \mathbb{R}^2$ with boundary $\partial_D M \cup \partial_N M$ where only a subset (in gray) can be reached by geodesics perpendicular to $\partial_D M$.

near the boundary. We thus begin by examining the case of a product metric (near the boundary). This is done in Subsection 3.5 (next). See also [43, 44].

3.3. Geometric preliminaries for manifolds with product metric. In the following subsection, we will *furthermore assume* that there exists $r_\partial > 0$ such that the metric g is a product metric near the boundary, that is, it has the form

$$(12) \quad g = g_\partial + dt^2, \text{ on } \partial M \times [0, r_\partial),$$

where g_∂ is the metric induced by g on ∂M .

We identify in what follows $\partial M \times \mathbb{R}$ with the normal bundle to ∂M in M as before for hypersurfaces, i.e. we identify $(x, t) \in \partial M \times \mathbb{R}$ with $t\nu_x$ in the normal bundle of ∂M in M . Also, we shall identify $(x, t) \in \partial M \times [0, r_\partial)$ with $\exp^\perp(x, t) = \exp^\perp(t\nu_x) \in M$.

Our proof of the Poincaré inequality under the product metric assumption on $\partial M \times [0, r_\partial)$ is based on several intermediate results. By decreasing r_∂ we may assume that δ in Definition 2.4.(iv) and r_∂ in Equation (12) are the same.

First, note that, by the product structure of g near the boundary, the submanifolds $\partial M \times \{t\}$ with $t \in [0, r_\partial)$ are totally geodesic submanifolds of M and that a geodesic in $\partial M \times [0, r_\partial)$ always has the form $c(t) = \exp^\perp(c_\partial(t), at)$ for some $a \in \mathbb{R}$ and some geodesic c_∂ in $(\partial M, g_\partial)$. This implies that a geodesic $c: [a, b] \rightarrow M$ with $c(a) \notin \partial M$ and $c(b) \in \partial M$ cannot be extended to $[a, b + \epsilon]$ for any $\epsilon > 0$.

For a subset $A \subset M$ we define

$$\text{dist}(y, S) := \inf_{x \in S} \{\text{dist}(x, y)\}.$$

Let $y \in M$. A *shortest curve joining y to $S \subset M$* is by definition a rectifiable curve $\gamma: [a, b] \rightarrow M$ from y to S (that is, $\gamma(a) = y$, $\gamma(b) \in S$) such that no other curve from y to S is shorter than γ . If a shortest curve is parametrized proportional to arc length and its interior does not intersect the boundary, then it is a geodesic. Such geodesics will be called *length minimizing geodesics*.

Remark 3.3 shows that the following proposition is less obvious than one might first think. In particular, the assumption that we have a product metric near the boundary is crucial if $\partial_D M \neq \partial M$.

Proposition 3.4. *We still assume that $(M, \partial_D M)$ has finite width and the product structure assumption at the beginning of Subsection 3.3. Then, for every $y \in M$, there is a length minimizing smooth geodesic γ from y to $\partial_D M$.*

Note that obviously we need here our requirement that $\partial_D M$ intersects all connected components of M , see Definition 2.7. Also note that, in general, there may be more than one (geometrically distinct) shortest geodesic $\gamma: [a, b] \rightarrow M$ joining y to $\partial_D M$. If $y \notin \partial M$, every such curve satisfies $\gamma([a, b]) \cap \partial M = \emptyset$ and $\gamma'(b) \perp \partial_D M$.

Proof. The proof is analogous to the classical proof of the Hopf-Rinow Theorem (which implies that in a geodesically complete manifold, any two points are joined by a length minimizing geodesic, see Chapter 7 in [26]). We follow Theorem 2.8 (Hopf-Rinow) in Chapter 7 of the aforementioned book. For a given point $y \in M \setminus \partial_N M$ we define $r := \text{dist}(y, \partial_D M)$. We only have to consider the case $y \notin \partial_D M$, and in this case $r > 0$. Let $\delta < r_{\text{inj}}(M)$. It follows from the Gauss lemma that the length of any curve joining y to a point of the “sphere” $S_\delta(y) := \{\exp_y(\delta v) \mid |v| = 1\}$ is at least δ , with the infimum being attained by the image of the straight line under \exp_y , see Chapter 3 in [26] for details. The function $x \mapsto \text{dist}(x, \partial_D M)$ is continuous, and thus we can choose a point $x_0 \in S_\delta(y)$ with

$$\text{dist}(x_0, \partial_D M) = \min\{\text{dist}(x, \partial_D M) \mid x \in S_\delta(y)\}.$$

Every curve from y to $\partial_D M$ will intersect $S_\delta(y)$ somewhere, thus we obtain

$$(13) \quad \text{dist}(y, x_0) + \text{dist}(x_0, \partial_D M) = \text{dist}(y, \partial_D M).$$

Let $|v| = 1$ with $\exp_y(\delta v) = x_0$. We now claim that

$$(14) \quad \exp_y(tv) \text{ is defined and } \text{dist}(\exp_y(tv), \partial_D M) = r - t$$

holds for all $t \in (0, r)$. The proof is again analogous to the proof of the theorem by Hopf-Rinow in [26]. Let $A := \{t \in [0, r] \mid (14) \text{ holds for } t\}$. Obviously A is closed in $[0, r]$, and from the triangle inequality we see that $t \in A$ implies $\tilde{t} \in A$ for all $\tilde{t} \in [0, t]$. So $A = [0, b]$ for some $b \in [0, r]$. Further (13) implies $b \geq \delta$. Due to the product structure near $\partial_N M$ the geodesic $[0, b] \ni t \mapsto \exp_y(tv)$ does not hit $\partial_N M \times [0, d(y))$ for some $d(y)$ small enough, as otherwise this would violate (14). We will show that for any $s_0 \in A$, $s_0 < r$, there is a $\delta' > 0$ with $s_0 + \delta' \in A$. To this end, we repeat the above argument for $y' := \exp_y(s_0 v)$ instead of y . We obtain $\delta' > 0$ and $x'_0 \in S_{\delta'}(y')$ such that $\text{dist}(y', x'_0) + \text{dist}(x'_0, \partial_D M) = \text{dist}(y', \partial_D M)$, and we write $x'_0 = \exp_{y'}(\delta' v')$. This implies

$$(15) \quad \text{dist}(y, y') + \text{dist}(y', x'_0) + \text{dist}(x'_0, \partial_D M) = \text{dist}(y, \partial_D M),$$

and then we get $\text{dist}(y, y') + \text{dist}(y', x'_0) = \text{dist}(y, x'_0)$. We have shown that the curve

$$\gamma(t) := \begin{cases} \exp_y(tv) & 0 \leq t \leq s_0 \\ \exp_{y'}((t - s_0)v') & s_0 \leq t \leq s_0 + \delta' \end{cases}$$

is a shortest curve from y to x'_0 , and thus the geodesic is not broken in y' . In other words

$$\exp_y(tv) = \exp_{y'}((t - s_0)v') \quad s_0 \leq t \leq s_0 + \delta'.$$

Using (15) once again, we see that $s_0 + \delta' \in A$.

We have seen that $b = \max A = r$. We obtain $\exp_y(rv) \in \partial_D M$, which gives the claim. Moreover, the first variation formula implies $\gamma'(r) \perp \partial_D M$. \square

Proposition 3.5. *There is a continuous function $L: \partial_D M \rightarrow (0, \infty]$ such that the restriction of \exp^\perp to*

$$\{(x, t) \mid 0 \leq t \leq L(x), x \in \partial_D M\}$$

is surjective, and such that the restriction of \exp^\perp to

$$\{(x, t) \mid 0 < t < L(x), x \in \partial_D M\}$$

is an embedding. Furthermore

$$\text{dist}(\exp^\perp(x, t), \partial_D M) = t$$

if $0 \leq t \leq L(x)$. The set

$$M_S := \exp^\perp(\{(x, t) \mid t = L(x), x \in \partial_D M\})$$

is of measure zero.

Proof. For $x \in \partial_D M$, let us consider the geodesic $\gamma_x: I_x \subset [0, \infty) \rightarrow M$ with $\gamma_x(0) = x$ and $\gamma'_x(0) = \nu_x$, defined on its maximal domain I_x . We choose $L(x) \in I_x$ as the maximal number such that $\gamma_x|_{[0, L(x)]}$ realizes the minimal distance from x to $\gamma_x(t)$ if $0 \leq t \leq L(x)$. Let $y \in M$ and $d = \text{dist}(y, \partial_D M)$. From the Proposition 3.4 above, we see that a shortest curve from y to $\partial_D M$ exists; in other words, there is $x \in \partial_D M$ with $y = \exp^\perp(x, d)$, where $\exp^\perp(x, t) := \exp_x(t\nu_x)$. Therefore, the restriction of \exp^\perp to $\{(x, t) \mid 0 \leq t \leq L(x), x \in \partial_D M\}$ is surjective. The continuity of L is analogous to [26, Chapter 13, Proposition 2.9]. As the geodesics $t \mapsto \gamma_x(t)$ are minimizing for $0 \leq t \leq L(x)$, it follows similar to [26, Chapter 13, Proposition 2.2] that there is a unique shortest curve from $\gamma_x(t)$ to $\partial_D M$ if $0 \leq t < L(x)$, and that the restriction of \exp^\perp to $\{(x, t) \mid 0 < t < L(x), x \in \partial_D M\}$ is an injective immersion. From the inverse function theorem we see that this injective immersion is a homeomorphism onto its image, thus it is an embedding.

The subset M_S of M is closed and has measure zero as it is the image of the measure zero set $\{(x, L(x)) \mid x \in \partial_D M\}$ under the smooth map \exp^\perp . \square

Remark 3.6. One usually defines the *cut locus* $\mathcal{C}(S)$ of a subset $S \subset M$ as the set of all points x in the interior of M for which there is a geodesic $\gamma: [-a, \epsilon) \rightarrow M$ with $\gamma(-a) \in S$, $\gamma(0) = x$, γ being minimal for all $t \in (-a, 0)$, but no longer minimal for $t > 0$. The name ‘‘cut locus’’ comes from the fact that this is the set where several shortest curves emanating from S will either intersect classically or in an infinitesimal sense. The relevance of this concept is that the set M_S introduced in Proposition 3.5 satisfies $M_S = \partial_N M \cup \mathcal{C}(\partial_D M)$.

If $H: T_p M \rightarrow T_q M$ is an endomorphism, then we express it in an orthonormal basis as a matrix A . The quantity

$$(16) \quad |\det H| := |\det A|$$

is then well-defined and does not depend on the choice of orthonormal base.

For $x \in \partial_D M$, $0 \leq t \leq L(x)$, let $v(x, t)$, be the volume distortion of the normal exponential map, that is,

$$v(x, t) := |\det d_{(x,t)} \exp^\perp|,$$

with the absolute value of the determinant defined using a local orthonormal base, as in (16). Clearly, $v(x, 0) = 1$.

Proposition 3.7. (Special case of [62, Lemma 4.5]) *We assume that M^m is a manifold with boundary and Ricci curvature bounded from below. Assume that the metric is a product near the boundary. We also assume that there exists $R > 0$ with $\text{dist}(x, \partial_D M) < R$ for all $x \in M$. Then there is a constant $C > 0$ such that, for all $x \in \partial M$ and all $0 \leq s \leq t \leq L(x)$, we have*

$$\frac{v(x, t)}{v(x, s)} \leq C.$$

The constant C can be chosen to be $e^{(m-1)R\sqrt{|\text{cl}|}}$, where $(m-1)c$ is a lower bound for the Ricci curvature of M .

This proposition is essentially a special case of the Heintze–Karcher inequality [42]. We also refer to the section on Heintze–Karcher inequalities in [9] for a proof of the full statement, some historical notes, and some similar inequalities.

3.4. Proof in the case of a manifold with product boundary. We keep all the notations introduced in the previous subsection, i. e. Subsection 3.3. In particular, we assume that g is a product metric near the boundary.

Proof of Theorem 1.1 for g a product near the boundary. We set

$$\gamma_x(s) := \exp^-(x, s) := \exp_x^M(s\nu_x),$$

to simplify the notation. Recall that dvol_g denotes the volume element on M associated to the metric g and dvol_{g_∂} denotes the corresponding volume form on the boundary. Let us assume first that $p < \infty$. Since

$$(17) \quad \int_M |u|^p \text{dvol}_g = \int_{\partial_D M} \int_0^{L(x)} |u(\gamma_x(s))|^p v(x, s) ds \text{dvol}_{g_\partial},$$

and $|\nabla u| \geq |\nabla_{\gamma'_x(s)} u|$, it suffices to find a $c > 0$ (independent on x) such that

$$(18) \quad \int_0^{L(x)} |\nabla_{\gamma'_x(s)} u|^p v(x, s) ds \geq c \int_0^{L(x)} |u(\gamma_x(t))|^p v(x, t) dt$$

for all $x \in \partial_D M$. Indeed equations (17) and (18) yield Theorem 1.1 by integration over $\partial_D M$ (recall that for now $p < \infty$).

Let $f(s) := u(\gamma_x(s))$, for some fixed $x \in M \setminus (M_S \cup \partial_D M)$. Then $f'(s) = \nabla_{\gamma'_x(s)} u$. Fix $t \in [0, L(x)]$ and let q be the exponent conjugate to p , that is, $p^{-1} + q^{-1} = 1$. Using $f(t) = f(0) + \int_0^t f'(s) ds$, we obtain, for $p < \infty$,

$$\begin{aligned} |f(t)|^p v(x, t) &\leq \left(|f(0)| + \int_0^t |f'(s)| ds \right)^p v(x, t) \\ &\leq 2^{p-1} \left[|f(0)|^p + \left(\int_0^t |f'(s)| ds \right)^p \right] v(x, t) \\ &\leq 2^{p-1} |f(0)|^p v(x, t) + 2^{p-1} t^{p/q} \int_0^t |f'(s)|^p v(x, s) ds \\ &\leq C |f(0)|^p + CL(x)^{p/q} \int_0^{L(x)} |f'(s)|^p v(x, s) ds \\ &\leq C |f(0)|^p + CR^{p/q} \int_0^{L(x)} |\nabla_{\gamma'_x(s)} u|^p v(x, s) ds. \end{aligned}$$

We have used here the fact that $v(x, 0) = 1$ and, several times, Proposition 3.7. Hence, integrating once more with respect to t from 0 to $L(x) \leq R$, we obtain

$$\int_0^{L(x)} |f(t)|^p v(x, t) dt \leq CR |f(0)|^p + CR^p \int_0^{L(x)} |\nabla_{\gamma'_x(s)} u|^p v(x, s) ds,$$

which gives (18) by integration with respect to dvol_{g_∂} , and hence our result for $p < \infty$. The case $p = \infty$ is simpler. Indeed, it suffices to use instead

$$|f(t)| \leq |f(0)| + \int_0^t |f'(s)| ds \leq |f(0)| + t \|f'\|_{L^\infty} \leq |f(0)| + R \|du\|_{L^\infty(M)}$$

By taking the ‘sup’ with respect to t and $x \in \partial D M$, we obtain the result. \square

3.5. The general case. We now show how the general case of the Poincaré inequality for a metric with bounded geometry and finite width on M can be reduced to the case when the metric is a product metric in a small tubular neighborhood of the boundary, in which case we have already proved the Poincaré inequality.

The general case of Theorem 1.1 follows directly from the special case where g is product near the boundary and the following lemma. Recall that we identify $\partial M \times [0, r_\partial)$ with its image in M via the normal exponential map \exp^\perp .

Lemma 3.8. *Let (M^m, g) be a manifold with boundary and bounded geometry. Let g_∂ be the induced metric on the boundary ∂M . Then there is a metric g' on M of bounded geometry such that*

- (1) $g' = g_\partial + dt^2$ on $\partial M \times [0, r']$ for some $r' \in (0, r_\partial)$
- (2) g and g' are equivalent, that is, there is $C > 0$ such that $C^{-1}g \leq g' \leq Cg$.

In particular, the norms $|\cdot|_g$ and $|\cdot|_{g'}$ on E -valued one-forms, respectively on the volume forms for g and g' , are equivalent.

Proof of Lemma 3.8. Let $0 < r' < r_{FC}/4 := \frac{1}{4} \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_\partial \}$ be small enough, to be specified later. Here r_{FC} is as in the choice of our Fermi coordinates in Definition 2.25. Let $\eta: [0, 3r'] \rightarrow [0, 1]$ be a smooth function with $\eta|_{[0, r']} = 0$ and $\eta|_{[2r', 3r']} = 1$. We set $g'(x, t) := \eta(t)g(x, t) + (1 - \eta(t))(g_\partial(x) + dt^2)$ for $(x, t) \in \partial M \times [0, 3r']$ and $g' = g$ outside $\partial M \times [0, 3r']$. By construction g' is smooth and is a product metric on $\partial M \times [0, r']$. The rest of the proof is based on the use of Fermi coordinates around any $p \in \partial M$ to prove that g and g' are equivalent for r' small enough.

Then, in the Fermi coordinates $(x, t) := \kappa_p^{-1}$ around $p \in \partial M$, see Equation (10), we have $g_{ij}(x, t) = g_{ij}(x, 0) + tg_{ij,t}(x, 0) + O(t^2)$, $g_{it}(x, 0) = 0$ for $i \neq t$, $g_{tt}(x, t) = 1$, and $(g_\partial)_{ij}(x) = g_{ij}(x, 0)$ for $i, j \neq t$. Thus,

$$(19) \quad g'_{ij}(x, t) - g_{ij}(x, t) = \begin{cases} -(1 - \eta(t))(tg_{ij,t}(x, 0) + O(t^2)) & \text{if } (i, j) \neq (t, t) \\ 0 & \text{otherwise.} \end{cases}$$

Since $g_{ij,t}$ is uniformly bounded by Remark 2.26, we obtain $|g'_{ij}(x, t) - g_{ij}(x, t)| \leq tC$, for $(x, t) \in B_{r'}^{m-1} \times [0, r']$, where the constant C is independent of the chosen p . We note that, in these coordinates, the metric g is equivalent to the Euclidean metric on $B_{r'}^{m-1}(0) \times [0, r'] \subset \mathbb{R}^m$ in such a way that the equivalence constants do not depend on the chosen p . This can be seen from

$$\begin{aligned} |g_{ij}(x, t) - g_{ij}(0, 0)| &= |g_{ij}(x, t) - \delta_{ij}| \\ &\leq \sup |\nabla_{(x,t)} g_{ij}(x, t)| \|(x, t)\| + O(t^2) \leq Cr' + O(t^2), \end{aligned}$$

where C is the uniform bound for $\nabla_{(x,t)}g_{ij}(x,t)$, which is finite by the bounded geometry assumption. Moreover, $O(t^2) \leq ct^2$ with c depending on the uniform bound of the second derivatives on $g_{ij}(x,t)$ and r_{FC} . Let X be a vector in (x,t) . Then

$$|g(X, X) - |X|^2| = \left| \sum_{ij} (g_{ij} - \delta_{ij}) X^i X^j \right| \leq Cr' |X|^2.$$

Thus, for r' such that $Cr' < 1$, it follows that g and the Euclidean metric are equivalent on the chart around p in such a way that the constants do not depend on p . Similarly, we then obtain

$$\begin{aligned} |g'(X, X) - g(X, X)| &= \left| \sum_{ij} (g'_{ij}(x, t) - g_{ij}(x, t)) X^i X^j \right| \\ &\leq r' C |X|^2 \leq r' C (1 - Cr')^{-1} g(X, X). \end{aligned}$$

Hence, g' and g are equivalent for r small.

In particular, $|\det g_{ij}(x, t) - \det g'_{ij}(x, t)| \leq ct$ for a positive c independent of x , p , and t . Since M has bounded geometry, $\det g_{ij}$ is uniformly bounded on all of M both from above and away from zero. Therefore the volume forms for g and g' are equivalent.

An estimate similar to (19) holds for $(g')^{ij}(x, t) - g^{ij}(x, t)$. Together with the relation $|\alpha|_g^2(p) = \sum_{i,j} g^{ij}(p) \alpha_p(e_i) \alpha_p(e_j)$ for a one-form α , this gives the claimed result for the one-forms \square

Remark 3.9. The proof implies that the constant c in the Poincaré inequality can be chosen to only depend on the bounds for R^M and its derivatives, on the bounds for Π , on r_{FC} (Definition 2.25), on $p \in [1, \infty]$, and on the width of $(M, \partial_D M)$. See also [62].

With Lemma 3.8, the proof of Theorem 1.1 is now complete.

4. INVERTIBILITY OF THE LAPLACE OPERATOR

We now proceed to apply our Poincaré inequality to the study of the spectrum of the Laplace operator with mixed boundary conditions and to its invertibility in the standard scale of Sobolev spaces. In Subsection 4.1 M will be an arbitrary Riemannian manifold with boundary. However, in Subsections 4.2 and 4.3, we will resume our assumption that M has bounded geometry.

4.1. Lax–Milgram, Poincaré, and well-posedness in energy spaces. In this subsection, we discuss the relation between the Poincaré inequality and well-posedness in H^1 for the Poisson problem with suitable mixed boundary conditions. We do not assume, in this subsection, that M has bounded geometry, except when stated otherwise. The bounded geometry assumption will be needed, however, for one of the main results of this paper, which is the well-posedness of the Poisson problem with suitable mixed boundary conditions on manifolds with boundary and finite width, see Definition 2.7 and Theorem 4.6 below. We define the semi-norms $|u|_{W^{1,p}(M)} := \|du\|_{L^p(M; T^*M)}$ and $|u|_{H^1(M)} := |u|_{W^{1,2}(M)}$.

Definition 4.1. We say that $(M, \partial_D M)$ satisfies the L^p -Poincaré inequality if there exists $c_p > 0$ such that

$$\|u\|_{L^p(M)} \leq c_p \|du\|_{L^p(M; T^*M)} =: c_p |u|_{W^{1,p}(M)},$$

for all $u \in H_D^1(M)$ (recall Notations 2.34). If $(M, \partial_D M)$ satisfies the L^2 -Poincaré inequality, we define $c_{M, \partial_D M}$ to be the least c_2 with this property, compare (3). Otherwise we set $c_{M, \partial_D M} = \infty$.

We have the following simple lemma.

Lemma 4.2. *The semi-norm $|\cdot|_{H^1(M)}$ is equivalent to the H^1 -norm on $H_D^1(M)$ if, and only if, $(M, \partial_D M)$ satisfies the L^2 -Poincaré inequality. In particular, this is true if $(M, \partial_D M)$ has finite width.*

Proof. For the simplicity of the notation, we omit below the manifold M from the notation of the (semi-)norms. Let us assume that $(M, \partial_D M)$ satisfies the L^2 -Poincaré inequality. By definition we have $|u|_{H^1} \leq \|u\|_{H^1}$, so to prove the equivalence of the norms, it is enough to show that there exists $C > 0$ such that $C|u|_{H^1} \geq \|u\|_{H^1}$ for all $u \in H_D^1$. Indeed, the L^2 -Poincaré inequality gives

$$(20) \quad \underbrace{(c_{M, \partial_D M}^2 + 1)}_{C:=} |u|_{H^1}^2 \geq \|u\|_{L^2}^2 + |u|_{H^1}^2 =: \|u\|_{H^1}^2.$$

Conversely, if the two norms are equivalent, then we have for $u \in H_D^1(M)$

$$\|u\|_{L^2} \leq \|u\|_{H^1} \leq C|u|_{H^1} := C\|du\|_{L^2}.$$

The proof is complete. \square

Clearly, the last lemma holds for functions in $W^{1,p}$, for all $p \in [1, \infty]$. We shall need the Lax–Milgram lemma. Let us recall first the following well-known definition.

Definition 4.3. Let V be a Hilbert space and let $P: V \rightarrow V^*$ be a bounded operator. We say that P is *strongly coercive* if there exists $\gamma > 0$ such that $|\langle Pu, u \rangle| \geq \gamma \|u\|_V^2$ for all $u \in V$. The “best” γ with this property (i.e. the largest) will be denoted γ_P .

Lemma 4.4 (Lax–Milgram lemma). *Let V be a Hilbert spaces and $P: V \rightarrow V^*$ be a strongly coercive map. Then P is invertible and $\|P^{-1}\| \leq \gamma_P^{-1}$.*

For a proof of the Lax–Milgram lemma, see, for example, [32, Section 5.8] or [56, 40]. The following result explains the relation between the Poincaré inequality and the Laplace operator. Note that $H_D^1(M) \subset L^2(M) \subset H_D^1(M)^*$. In particular, $\Delta - \lambda: H_D^1(M) \rightarrow H_D^1(M)^*$, for $\lambda \in \mathbb{C}$, is defined by $\langle (\Delta - \lambda)u, v \rangle = (du, dv) - \lambda(u, v)$, where $\langle \cdot, \cdot \rangle$ is the duality pairing (considered conjugate linear in the second variable).

Recall the constant $c_{M, \partial_D M}$ of Definition 4.1. In particular, $(1 + c_{M, \partial_D M}^2)^{-1} = 0$ precisely when $(M, \partial_D M)$ does not satisfy the L^2 -Poincaré inequality. Also, recall the definition of the spaces H_D^1 from Equation (2).

Proposition 4.5. *Assume that M is a manifold with boundary (no bounded geometry assumption). We have that the map $\Delta: H_D^1(M) \rightarrow H_D^1(M)^*$ is an isomorphism if, and only if, $(M, \partial_D M)$ satisfies the L^2 -Poincaré inequality. Moreover, if $\Re(\lambda) < (1 + c_{M, \partial_D M}^2)^{-1}$, then $\Delta - \lambda: H_D^1(M) \rightarrow H_D^1(M)^*$ is strongly coercive and hence an isomorphism.*

Proof. Indeed, let us assume that $\Delta: H_D^1(M) \rightarrow H_D^1(M)^*$ is an isomorphism. If the L^2 -Poincaré inequality was not satisfied, then there would exist a sequence of

functions $u_n \in H_D^1(M)$ such that $\|u_n\|_{H^1} = 1$, but $\|du_n\|_{L^2}^2 = \langle \Delta u_n, u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, by Lemma 4.2. Since Δ is an isomorphism, there is a $\gamma > 0$ such that $\|\Delta u\|_{(H_D^1)^*} \geq 2\gamma\|u\|_{H^1}$. Therefore, using the definition of the (dual) norm on $H_D^1(M)^*$ we can find a sequence $v_n \in H_D^1(M)$ such that $\|v_n\|_{H^1(M)} = 1$ and $\langle \Delta u_n, v_n \rangle \geq \gamma$. Using the Cauchy-Schwarz inequality for the L^2 -scalar product on 1-forms, we then obtain the following

$$\begin{aligned} \|du_n\|_{L^2}^2 &= \|du_n\|_{L^2}^2 \|v_n\|_{H^1}^2 \geq \|du_n\|_{L^2}^2 \|dv_n\|_{L^2}^2 \\ &\geq |(du_n, dv_n)|^2 = |\langle \Delta u_n, v_n \rangle|^2 \geq \gamma^2 > 0, \end{aligned}$$

which contradicts $\|du_n\|_{L^2} \rightarrow 0$.

Conversely, let $\Re(\lambda) < (1 + c_{M, \partial M}^2)^{-1}$ and $u \in H_D^1(M)$. Using (20) we have

$$\begin{aligned} \Re(\langle (\Delta - \lambda)u, u \rangle) &= (du, du) - \Re(\lambda)(u, u) = |u|_{H^1}^2 - \Re(\lambda)\|u\|_{L^2}^2 \\ &\geq ((1 + c_{M, \partial M}^2)^{-1} - \Re(\lambda)) \|u\|_{H^1}^2. \end{aligned}$$

The operator $\Delta - \lambda: H_D^1(M) \rightarrow H_D^1(M)^*$ thus satisfies the assumptions of the Lax-Milgram lemma, and hence $\Delta - \lambda$ is an isomorphism for $\Re(\lambda) < (1 + c_{M, \partial M}^2)^{-1}$. \square

In our setting this directly gives

Theorem 4.6. *If $(M, \partial M)$ has finite width, then $(1 + c_{M, \partial M}^2)^{-1} > 0$, and hence $\Delta: H_D^1(M) \rightarrow H_D^1(M)^*$ is an isomorphism.*

The converse of this result is not true, the following two examples show that, without the assumption of finite width, $\Delta: H_D^1(M) \rightarrow H_D^1(M)^*$ may fail to be an isomorphism.

Example 4.7. Let M be the n -dimensional hyperbolic space \mathbb{H}^n , $n \geq 2$, whose boundary is empty. Hence $(\mathbb{H}^n, \emptyset)$ does *not* have finite width, but the L^2 -Poincaré inequality holds with

$$c_{(\mathbb{H}^n, \emptyset)} = 2/(n-1)$$

since $\frac{(n-1)^2}{4}$ is the infimum of the L^2 -spectrum of the Laplacian on \mathbb{H}^n , [27, 37]. Thus $\Delta: H^1(\mathbb{H}^n) \rightarrow H_D^1(\mathbb{H}^n)^* = H^{-1}(\mathbb{H}^n)$ is an isomorphism. So the finite width condition is not necessary for the Laplacian on a manifold with boundary and bounded geometry to be invertible. See also [21, 38, 50] for further results on the spectrum of the Laplacian on symmetric spaces.

Example 4.8. Let us assume that in Example 2.23 $M_0 = \mathbb{R}^{m-1}$ with the standard euclidean metric. As in that example, $f > g$ are smooth functions such that df and dg are totally bounded (i.e. bounded and with all their covariant derivatives bounded as well). Assume that the functions f and g satisfy also that $\lim_{|x| \rightarrow \infty} f(x) - g(x) = \infty$. Then, using the notation of Example 2.23, we have that $M := \Omega(f, g)$ is a manifold with boundary and bounded geometry. However, $(M, \partial M)$ does not have finite width. Moreover, it does not satisfy the Poincaré inequality. This can easily be seen as follows: Since 0 is in the essential spectrum of Δ on \mathbb{R}^n , there is sequence $v_i \in C_c^\infty(B_i(0))$ with $\|v_i\|_{L^2} = 1$ and $\|dv_i\|_{L^2} \rightarrow 0$. By translation we can assume that the support of v_i is in $\Omega(f, g)$. Thus, the Poincaré inequality is violated and by Proposition 4.5, the operator Δ (with any kind of mixed boundary conditions) is not invertible.

4.2. Higher regularity. We consider in this subsection the regularity and invertibility of the Laplacian in the scale of Sobolev spaces determined by the metric, which is the main question considered in this paper (see the Introduction). We assume that $(M, \partial_D M)$ is a manifold with boundary and bounded geometry, however, for regularity questions *we do not require* $(M, \partial_D M)$ *to have finite width*.

We shall use the notation introduced in Equations (10) and (11) and in Definition 2.27. We set $P_x := \kappa_x^* \circ \Delta \circ (\kappa_x)_*$, where κ_x is viewed as a diffeomorphism on its image. Thus P_x is a differential operator on the Euclidean ball, respectively cylinder, corresponding to the geodesic normal coordinates, respectively Fermi coordinates, on $W_x := W_x(r)$, see (11). Thus $\langle P_x u, v \rangle = \int_{W_x} (du, dv) \, \text{dvol}_g$. We endow the space of these differential operators with the norm defined by the maximum of the $W^{2,\infty}$ -norms of the coefficients.

Lemma 4.9. *Let $r < r_{FC}$. Then the set $\{P_x \mid \text{dist}(x, \partial M) \geq r\}$ is a relatively compact subset of the set of differential operators on the ball $B_r^m(0) \subset \mathbb{R}^m$. Similarly, the set $\{P_y \mid y \in \partial M\}$ is a relatively compact subset of the set of second order differential operators on $b(r) := B_r^{m-1}(0) \times [0, r) \subset \mathbb{R}^m$.*

Proof. The coefficients of the operators P_x and all their derivatives are uniformly bounded, by assumption. By decreasing r , if necessary, we get that they are bounded on a compact set. The Arzela-Ascoli theorem then yields the result. \square

We get the following result

Lemma 4.10. *For any $k \in \mathbb{N}$ (so $k \geq 1$), there exists $C_k > 0$ such that*

$$\|w\|_{H^{k+1}(M)} \leq C_k (\|\Delta w\|_{H^{k-1}(M)} + \|w\|_{H^1(M)} + \|w\|_{H^{k+1/2}(\partial M)}),$$

for any $w \in H^1(W_\gamma)$ with compact support, where W_γ is a coordinate patch of Definition 2.27.

Any undefined term on the right hand side of the inequality in Lemma 4.10 is set to be ∞ (for instance $\|w\|_{H^{k+1/2}(\partial M)} = \infty$ if $w \notin H^{k+1/2}(\partial M)$), in which case the stated inequality is trivially satisfied. Note also that the condition that w have compact support in W_γ does *not* mean that w vanishes on $W_\gamma \cap \partial M$. (This comment is relevant, of course, only if $W_\gamma \cap \partial M \neq \emptyset$.)

Proof. Elliptic regularity for strongly elliptic equations (see, for instance, Theorem 8.13 in [32], Theorem 9.3.3 in [40], or Proposition 11.10 in [68]) gives, for every $\gamma \in I$ as in Definition 2.27, that there exists C_γ such that

$$\|w\|_{H^{k+1}(M)} \leq C_\gamma (\|\Delta w\|_{H^{k-1}(M)} + \|w\|_{H^1(M)} + \|w\|_{H^{k+1/2}(\partial M)})$$

for any $w \in H^1(W_\gamma)$ with compact support. Of course, if W_γ is an interior set (that is, it does not intersect the boundary), then $\|w\|_{H^{k+1/2}(\partial M)} = 0$. This should be understood in the sense that if the right hand side is finite, then $\|w\|_{H^{k+1}(M)} < \infty$, and hence $w \in H^{k+1}(M)$. We need to show that we can choose C_γ independently of γ . Let us assume the contrary. Then, for a suitable subsequence p_j , we have that there exist $w_j \in H^1(W_j)$ with compact support such that

$$\|w_j\|_{H^{k+1}(M)} > 2^j (\|\Delta w_j\|_{H^{k-1}(M)} + \|w_j\|_{H^1(M)} + \|w_j\|_{H^{k+1/2}(\partial M)}).$$

We can assume that the points p_j are either all at distance at least r to the boundary, or that they are all on the boundary. Let us assume that they are all on the boundary. The other case is very similar (even simpler). Using Fermi coordinates

$\kappa_j: b(r) := B_r^{m-1}(0) \times [0, r] \rightarrow W_j$ we can move to a fixed cylinder, but with the price of replacing Δ with $P_j := P_{p_j}$, a variable coefficient differential operator on $b(r)$. By Lemma 4.9, after passing to another subsequence, we can assume that the coefficients of the corresponding operators P_j converge in the \mathcal{C}^∞ topology on $b(r)$ to the coefficients of a fixed operator P_∞ . In particular, we can assume that

$$(21) \quad \|P_j u\|_{H^{k-1}(b(r))} \geq \|P_\infty u\|_{H^{k-1}(b(r))} - \epsilon_j \|u\|_{H^{k+1}(b(r))},$$

for all $u \in H^{k+1}(b(r))$, where $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$ independent of w .

The main point of this proof (exploited in greater generality in [33]) is that P_∞ also satisfies elliptic regularity. This is because it is strongly elliptic, since this property is preserved by limits in which the strong ellipticity constant is bounded from below and strongly elliptic operators with Dirichlet boundary conditions satisfy elliptic regularity (see the references above). This means that there exists $C > 0$ such that

$$C(\|P_\infty u\|_{H^{k-1}(b(r))} + \|u\|_{H^1(b(r))} + \|u\|_{H^{k+1/2}(B_r^{m-1}(0))}) \geq \|u\|_{H^{k+1}(b(r))}$$

for all $u \in H^{k+1}(b(r))$ with compact support in $b(r) := B_r^{m-1}(0) \times [0, r]$.

By Proposition 2.32 we also have

$$(22) \quad \frac{1}{1 + C_0} \leq \frac{\|w \circ \kappa_j^{-1}\|_{H^\ell(W_j)}}{\|w\|_{H^\ell(b(r))}} \leq 1 + C_0,$$

for some $C_0 > 0$, for all $w \in H^{k+1}(b(r))$, and for $\ell \leq k + 1$, by the bounded geometry of M .

Equations (21) and (22) then give, first in the boundary-less case

$$\begin{aligned} \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} &\leq C(\|P_\infty(w_j \circ \kappa_j)\|_{H^{k-1}(b(r))} + \|w_j \circ \kappa_j\|_{H^1(b(r))}) \\ &\leq C(\|P_j(w_j \circ \kappa_j)\|_{H^{k-1}(b(r))} + \|w_j \circ \kappa_j\|_{H^1(b(r))} \\ &\quad + \epsilon_j \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))}) \\ &\leq C(1 + C_0)(\|\Delta w_j\|_{H^{k-1}(M)} + \|w_j\|_{H^1(M)}) \\ &\quad + C\epsilon_j \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \\ &\leq 2^{-j} C(1 + C_0) \|w_j\|_{H^{k+1}(M)} + C\epsilon_j \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \\ &\leq C((1 + C_0)^2 2^{-j} + \epsilon_j) \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \end{aligned}$$

and this is a contradiction for large j , since $\|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \neq 0$. For the case when $b(r)$ has a non-empty boundary, we just include the boundary terms in the right hand side. \square

We finally have the following result.

Theorem 4.11. *Let M be a manifold with boundary and bounded geometry (not necessarily with finite width). Let $k \in \mathbb{N}$. Then there exists $c > 0$ (depending on k and $(M, \partial_D M)$) such that*

$$\|u\|_{H^{k+1}(M)} \leq c(\|\Delta u\|_{H^{k-1}(M)} + \|u\|_{H^1(M)} + \|u\|_{H^{k+1/2}(\partial M)}),$$

for any $u \in H^1(M)$.

Proof. The proof is the same as in the case of compact manifolds with boundary [49, 68], but carefully keeping track of the norms of the commutators. Let us review the main points, stressing the additional reasoning that is used in the bounded geometry

(non-compact) case. More to the point, we first use the definition of Sobolev spaces using partitions of unity (ϕ_γ) , Proposition 2.32, to reduce our estimate to uniform estimates of the norms of the terms $\phi_\gamma \Delta u$. Since the derivatives of the partition of unity functions ϕ_γ are bounded, the terms $\phi_\gamma \Delta u$ can be uniformly estimated using estimates of $\Delta(\phi_\gamma u)$ and lower order norms. Lemma 4.10, then gives the result. \square

The meaning of Theorem 4.11 is also that if $\Delta u \in H^{k-1}(M)$, $u \in H^1(M)$, and $u \in H^{k+1/2}(\partial M)$, then, in fact, $u \in H^{k+1}(M)$. This result shows that the domain of Δ^k is contained in H^{2k} .

See [1, 2, 17, 68] and the references therein for the analogous classical results for smooth, bounded domains. In [17], more general regularity results were obtained when M is contained in a flat space. Our method of proof is, however, different. No general results similar to the results on the invertibility of the Laplacian (stated below) appeared in these or other papers, though. Let us recall the following well-known self-adjointness criteria.

Remark 4.12. Recall from an unnumbered corollary in [58], Section X.1, that every *closed*, symmetric (unbounded) operator T that has at least one real value in its resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$ is self-adjoint. Every operator T_1 that has a non-empty resolvent set is closed, as one can see as a trivial exercise. One obtains that if T is symmetric and $\rho(T) \cap \mathbb{R} \neq \emptyset$, then T is self-adjoint. We note that the above mentioned corollary in [58], Section X.1, does not state explicitly that T is required to have dense domain, but this is a running assumption in the quoted book (see the first paragraph of Section VIII.1 in [59]). However, it is again a trivial exercise to show that if T_1 is symmetric and $\rho(T_1) \cap \mathbb{R} \neq \emptyset$, then T_1 is densely defined.

Theorem 4.13. *Assume that M is a manifold with bounded geometry and finite width. Let*

$$\tilde{\Delta}_k - \lambda: H^{k+1}(M) \rightarrow H^{k-1}(M) \oplus H^{k+1/2}(\partial M)$$

be given by $(\tilde{\Delta}_k - \lambda)u := (\Delta u - \lambda u, u|_{\partial M})$, $k \in \mathbb{N}$. Then $\tilde{\Delta}_k - \lambda$ is an isomorphism for $\Re(\lambda) < (1 + c_{M, \partial_D M}^2)^{-1}$. In particular, Δ is self-adjoint on $L^2(M)$ with domain $H^2(M) \cap H_0^1(M)$.

Recall that $c_{M, \partial_D M} := \infty$ if $(M, \partial_D M)$ does not satisfy the L^2 -Poincaré inequality, see (3) and Definition 4.1.

Proof. The first part of the result follows from Theorem 4.11, the trace theorem (Theorem 2.33), and the isomorphism of Theorem 4.6. To prove that Δ is self-adjoint, we notice that, $\Delta - \lambda$ is invertible for $\lambda \in \mathbb{R}$, $\lambda < 0$. Since Δ is symmetric, the result follows from standard characterizations of self-adjoint operators, see Remark 4.12. \square

4.3. Mixed boundary conditions. The case of mixed boundary conditions (thus including the case of pure Neumann boundary conditions, i.e. $\partial M = \partial_N M$) is very similar. The main technical difference arises due to the fact that $(H_D^1(M))^* \not\cong H^{-1}(M)$ if $\partial_N M \neq \emptyset$. To deal with mixed boundary conditions (Dirichlet on $\partial_D M$, Neumann on $\partial_N M$), one has to replace the boundary terms in u with $\partial_\nu u$. Of course, the question arises how to define $\partial_\nu u$ if $u \in H^1(M)$. This is done in a weak sense, noticing that one has an inclusion $j_k: H^{k-1}(M) \oplus H^{k-1/2}(M) \rightarrow H_D^1(M)^*$ by $(f, g)(w) = \int_M f w d \text{vol}_g + \int_{\partial_N M} g w d \text{vol}_g$. If $u \in H_D^1(M)$ and $\Delta u = j_k(f, g)$,

then we say that f is the “classical” Δu and that $g = \partial_\nu u$. These elementary but tedious details pertain more to analysis than geometry, so we skip the simple proofs (the reader can see more details in a more general setting in [33, 6]), which however follow a different line than the one indicated here. A more substantial remark is that strongly elliptic operators with Neumann boundary conditions still satisfy elliptic regularity, so the proof of Lemma 4.10 extends to the case of Neumann boundary conditions.

Theorem 4.14. *Let M be a manifold with boundary and bounded geometry (not necessarily with finite width). Let $k \in \mathbb{N}$. Then there exists $c > 0$ (depending on k and $(M, \partial_D M)$) such that*

$$\|u\|_{H^{k+1}(M)} \leq c(\|\Delta u\|_{H^{k-1}(M)} + \|u\|_{H^1(M)} + \|u\|_{H^{k+1/2}(\partial_D M)} + \|\partial_\nu u\|_{H^{k-1/2}(\partial_N M)}),$$

for any $w \in H^1(M)$. If Δu is in the image of j_k , then $\partial_\nu u$ is defined by duality, as explained above. Otherwise, we set $\|\partial_\nu w\|_{H^{k-1/2}(\partial_N M)} = \infty$.

The following result then follows in the same way from Theorem 4.14 as Theorem 4.13 follows from Theorem 4.11 in the previous subsection.

Theorem 4.15. *Assume that M is a manifold with bounded geometry and boundary, but not necessarily of finite width. Let*

$$\tilde{\Delta}_k - \lambda: H^{k+1}(M) \rightarrow H^{k-1}(M) \oplus H^{k+1/2}(\partial_D M) \oplus H^{k-1/2}(\partial_N M)$$

be given by $(\tilde{\Delta}_k - \lambda)u := (\Delta u - \lambda u, u|_{\partial_D M}, \partial_\nu u|_{\partial_N M})$, $k \in \mathbb{N}$. Then $\tilde{\Delta}_k - \lambda$ is an isomorphism for $\Re(\lambda) < (1 + c_{M, \partial_D M}^2)^{-1}$, where $c_{M, \partial_D M}$ is the best constant in the Poincaré inequality, as in the Introduction. In particular, Δ is self-adjoint on $L^2(M)$ with domain

$$\mathcal{D}(\Delta) := \{u \in H^2(M) \mid u = 0 \text{ on } \partial_D M \text{ and } \partial_\nu u = 0 \text{ on } \partial_N M\}.$$

Several extensions of the results of this paper are possible. Some of them were included in the first version of the associated ArXiv preprint, 1611.00281.v1, which had a slightly different title and will not be published in that form. The results of that preprint that were not already included in this paper, as well as others, will be included in two forthcoming papers [33, 6].

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