Minimal Geodesics and Nilpotent Fundamental Groups

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March 1996

Abstract. Hedlund [18] constructed Riemannian metrics on n-tori, $n \geq 3$ for which minimal geodesics are very rare. In this paper we construct similar examples for every nilpotent fundamental group. These examples show that Bangert's existence results of minimal geodesics [4] are optimal for nilpotent fundamental groups.

AMS Subject Classification (1991): Primary 53C22, Secondary 22E25, 20F18:

Key words: minimal geodesics, stable norm, first Betti number, nilpotent Lie groups, cocompact discrete subgroups, nilmanifolds, Hedlund metrics

1. Introduction

DEFINITION 1.1. Let $(M, \langle\, .\, , .\, \rangle)$ be a Riemannian manifold. A non-constant geodesic $c: \mathbb{R} \to M$ is called *minimal* if it satisfies for all $t_1 < t_2$:

$$\mathcal{L}\left(c|_{[t_1,t_2]}\right) \leq \mathcal{L}(\gamma)$$

for all curves γ homotopic to $c|_{[t_1,t_2]}$ with fixed endpoints.

Suppose we fix $\pi_1(M)$. Then there are several known results that guarantee the existence of minimal geodesics.

The simplest one is that $(M, \langle\, .\, , .\, \rangle)$ carries a minimal geodesic if and only if $\pi_1(M)$ is infinite.

For some classes of differentiable manifolds certain existence properties of minimal geodesics do not depend on the choice of Riemannian metric: the bestknown cases are compact manifolds M with hyperbolic fundamental groups. Here one can compactify the universal cover $\widetilde{M}$ of M by a "boundary at infinity" $\widetilde{M}_\infty$. For every Riemannian metric on M the lift of a minimal geodesic to $\widetilde{M}$ converges for $t \to \pm\infty$ to two different points on $\widetilde{M}_\infty$ and, conversely, for each pair of different points on $\widetilde{M}_\infty$ there exists such a minimal geodesic ([23],[8],[19],[17]7.5).

The situation is similar on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ where for every straight line in $\mathbb{R}^2$ and every Riemannian metric on T^2 one finds a minimal geodesic whose lift stays at finite distance from the straight line ([18],[7],[3]).

Surprisingly, the situation is completely different for an n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ if $n \geq 3$. Here existence properties of minimal geodesics depend very much on the choice of the Riemannian metric: for flat metrics every geodesic is minimal (and lifts to a straight line in $\mathbb{R}^n$). On the other hand, there are the Hedlund metrics [18] on T^n, discussed in [4], where one has only n periodic minimal ones. So in these Hedlund examples minimal geodesics are very rare. Using the language of dynamical systems one would say that the set of unit tangent vectors to minimal geodesics consists of $2n$ periodic orbits of the geodesic flow and (countably many) heteroclinic and homoclinic connections between them.

These Hedlund metrics contrast to a theorem of V. Bangert ([4],[5]). He proves that the number of "directions" of minimal geodesics on an arbitrary Riemannian manifold $(M, \langle\,.\,,.\,\rangle)$ is at least the first Betti-number $b_1 := \operatorname{rank} \pi_1/[\pi_1, \pi_1]$. This bound is optimal for the Hedlund metric on $T^n, n \geq 3$.

In this paper we will construct Riemannian manifolds with only b_1 different "directions" of minimal geodesics for arbitrary nilpotent fundamental groups. Therefore Bangert's bound is optimal for arbitrary nilpotent groups.

If one tries to construct such Riemannian manifolds using analogous methods to Hedlund's, one has to prove a group theoretical property for the fundamental group. The groups having this property will be called *groups of bounded minimal generation.* Any finitely generated abelian group is of bounded minimal generation, and every group of bounded minimal generation is virtually nilpotent, i.e. it has a nilpotent subgroup of finite index. Unfortunately, there are only few non abelian groups that are known to be of bounded minimal generation, e.g. discrete subgroups of Heisenberg groups (see section 7). So this type of construction seems to fail for general fundamental groups.

Therefore we will use a different method that will give us examples for any finitely generated nilpotent fundamental group.

The Riemannian manifolds we construct have a universal covering $\widetilde{M} = G \times \tilde{S}$ where G is a nilpotent Lie-group and $\tilde{S}$ is a simply-connected compact manifold. The commutator group $[G, G]$ acts isometrically on $\widetilde{M}$ via left multiplication on the first component. In analogy to the Hedlund metrics on tori we will find two types of minimal geodesics on M: the *left-translated-periodic type* and the *connection type.* There are b_1 periodic minimal geodesics $c_1, \ldots, c_{b_1}$ on M with lifts $\tilde{c}_1, \ldots, \tilde{c}_{b_1}$ on $\widetilde{M}$ with the following property:

for every minimal geodesic c of left-translated-periodic type we can find a lift $\tilde{c}$ to $\widetilde{M}$ and $i \in \{1, \dots, b_1\}$, $a, b \in \mathbb{R}$, $g \in [G, G]$ with

$$\tilde{c}(t) = g \cdot \tilde{c}_i(at + b).$$

On the other hand every $c\colon \mathbb{R} \to M$ satisfying this property is a minimal geodesic.

A minimal geodesic of connection type will always be a homoclinic or heteroclinic connection between two geodesics of left-translated-periodic type.

Additionally, the main theorem for this construction (Theorem 2.1) is useful for other applications. For example, we will be able to determine all minimal geodesics on nilmanifolds with left-invariant metrics. Here minimal geodesics are exactly the horizontal lifts of straight lines on the associated (flat) Jacobi variety T^{b_1}.

Acknowledgement
This paper resulted from the author's Diplomarbeit at the University of Freiburg, Germany. The author wants to thank his supervisor Victor Bangert for many interesting discussions.

2. N-leftinvariant metrics

In this section we will look at metrics on $G \times \tilde{S}$ as above. Applying theorem 2.1, we will be able to reduce the classification problem of minimal geodesic on $G \times \tilde{S}$ to the classification of minimal geodesics on $\mathbb{R}^{b_1} \times \tilde{S}$, or to be more precise:

the minimal geodesics on $G \times \tilde{S}$ are exactly the horizontal lifts of minimal geodesics on $\mathbb{R}^{b_1} \times \tilde{S}$ via the canonical Riemannian submersion

$$G \times \tilde{S} \to \mathbb{R}^{b_1} \times \tilde{S}.$$

We will formulate the theorem in a more general setting.

Let G be a simply connected, nilpotent Lie-group. The Lie-group exponential map exp is a global diffeomorphism from the Lie-algebra $\mathfrak{g}$ to the Lie-group G, and the Formula of Baker, Campbell and Hausdorff states that the pullback of the multiplication on G is a Lie-bracket polynomial on $\mathfrak{g}$. Connected subgroups of G correspond to Lie-subalgebras of $\mathfrak{g}$ and are therefore closed subsets. Normal connected subgroups correspond to ideals of $\mathfrak{g}$. For details and further results about nilpotent Lie-groups look for example at [10] and [25].

We fix now a normal connected subgroup N of G with Lie-algebra $\mathfrak{n} \subset \mathfrak{g}$. We will assume that N is contained in the commutator group

$[G,G]$ of G. This is equivalent to the condition that $\mathfrak{n}$ is in the commutator Lie-algebra $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$.

Let S be a compact manifold; here we do not assume that S is simply connected but we want $\pi_1(S)$ to be finite. G acts on $G \times S$ via left multiplication on the first component. Now we take a Riemannian metric $\langle\,.\,,.\,\rangle_G$ on $G \times S$ that is N*-leftinvariant*, i.e. the subgoup N of G acts isometrically. Then there is a unique Riemannian metric $\langle\,.\,,.\,\rangle_{G/N}$ on $(G/N)\times S$ such that the canonical projection $p\colon G\times S \to (G/N)\times S$ is a Riemannian submersion. Vice versa, for every Riemannian metric on $(G/N)\times S$ there is a (non unique) N-leftinvariant metric on $G\times S$ such that p is Riemannian.

Additionally we suppose that $\langle\,.\,,.\,\rangle_G$ is bi-Lipschitz to a left invariant metric $\langle\,.\,,.\,\rangle_l$ on $G \times S$, i.e. there are constants $c_1, c_2 > 0$ with

$$c_1\langle v,v\rangle_l \le \langle v,v\rangle_G \le c_2\langle v,v\rangle_l \quad \forall v \in T(G\times S).$$

This condition is independent of the choice of the left invariant metric $\langle\,.\,,.\,\rangle_l$.

A vector $v \in T_x(G\times S)$ is called *horizontal* if $v \perp \ker T_x p$.

THEOREM 2.1. *Let N be a normal connected subgroup of the simply connected, nilpotent Lie-group G with $N \subset [G,G]$. We suppose that $G \times S$ carries a Riemannian metric that is N-left-invariant and bi-Lipschitz to a G-left-invariant metric and that*

$$p\colon G\times S \to \frac{G}{N}\times S$$

is a Riemannian submersion. Then $c\colon \mathbb{R}\to G\times S$ is a minimal geodesic on $(G\times S, \langle\,.\,,.\,\rangle_G)$ if and only if

1. *$\dot c(t)$ is horizontal for all $t\in\mathbb{R}$ and*
2. *$p\circ c\colon \mathbb{R}\to (G/N)\times S$ is a minimal geodesic on $\Big((G/N)\times S, \langle\,.\,,.\,\rangle_{G/N}\Big)$.*

Proof. Let $Z_1(G) := \{x\in G \mid xyx^{-1}y^{-1} = e\,\forall y\in G\}$ be the center of G and define inductively $Z_{i+1}(G) := \{x\in G\mid xyx^{-1}y^{-1}\in Z_i(G)\,\forall y\in G\}$.

We will prove the theorem for the case $N\subset Z_1(G)$. By a straightforward induction on i we then get the theorem for $N\subset Z_i(G)$ and therefore the general case.

To prove "$\Leftarrow$" we suppose that c is not minimal. If $\dot c(t)$ is horizontal for all t, then $c|_{[s,t]}$ has the same length as $p\circ c|_{[s,t]}$ and therefore $p\circ c$ cannot be minimal.

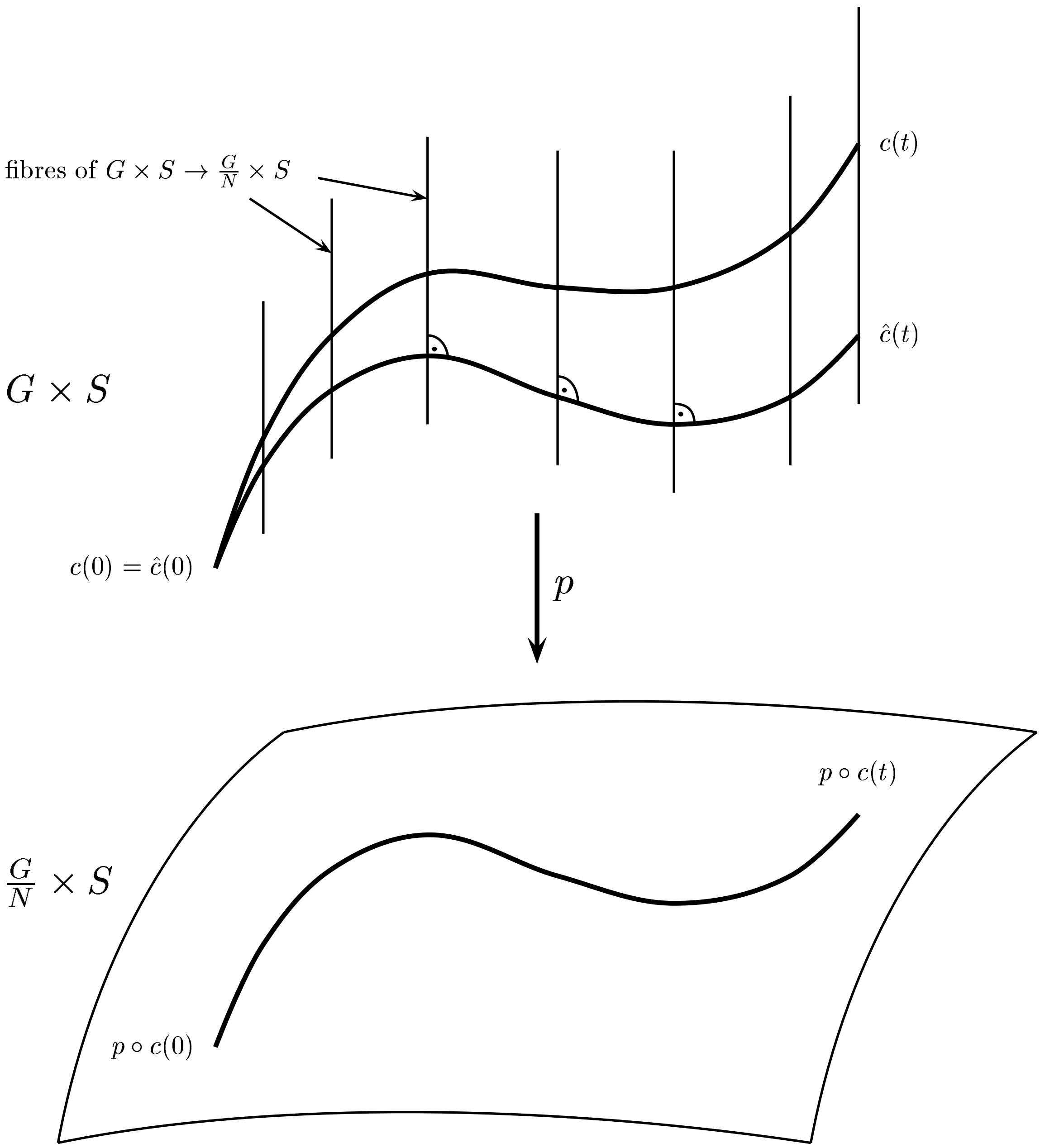

Figure 1. Minimal Geodesics are horizontal

For "$\Rightarrow$" we suppose that $c\colon \mathbb{R} \to G \times S$ is a minimal geodesic, parametrized by arclength. Without loss of generality we can assume that S is simply connected.

For any $v \in \mathfrak{z}$ let $r_{\exp v}$ be the right-translation of $G \times S$ by $\exp v$ acting trivially on S. Then $\mathcal{V}_v := \frac{\partial}{\partial \alpha}|_{\alpha=0} r_{\exp \alpha v}$ is a left-invariant vector field with vanishing S-component.

If $n \in \mathfrak{n}$ then $r_{\exp \alpha n}$ acts isometrically on $G \times S$, since $N \subset Z_1(G)$, so $\mathcal{V}_n$ is a Killing field. Noether's theorem ([2] 4.20) implies that

$$\mathcal{P}_n := \langle \dot{c}(t), \mathcal{V}_n(c(t)) \rangle_G$$

is constant in t. We argue by contradiction to show that $\mathcal{P}_n = 0$.

We write

$$\dot{c}(t) = \lambda(t)\mathcal{V}_n(c(t)) + c_\perp(t)$$

with $c_\perp(t) \perp \mathcal{V}_n(c(t))$.

Let $\|\,.\,\|_G$ be the norm of tangential vectors induced by $\langle\,.\,,.\,\rangle_G$. If we assume that $\mathcal{P}_n \neq 0$ we can use the Lipschitz constants between $\langle\,.\,,.\,\rangle_G$ and a left-invariant metric to obtain constants $K_1, K_2 > 0$ in the inequalities:

$$|\lambda(t)| = \frac{\left|\left\langle \dot{c}(t), \mathcal{V}_n(c(t)) \right\rangle_G\right|}{\|\mathcal{V}_n(c(t))\|_G^2} = \frac{|\mathcal{P}_n|}{\|\mathcal{V}_n(c(t))\|_G^2} < K_1$$

$$\|\lambda(t)\mathcal{V}_n(c(t))\|_G = \frac{|\mathcal{P}_n|}{\|\mathcal{V}_n((c(t))\|_G} > K_2 > 0$$

Then the curve $\hat{c}$ defined by

$$\hat{c}(t) := c(t) \cdot \exp\left(\left(\int_0^t -\lambda(t')dt'\right)n\right)$$

satisfies $p \circ \hat{c} = p \circ c$, $\hat{c}(0) = c(0)$ and $\dot{\hat{c}}(t) \perp \mathcal{V}_n$ (see also Figure 1). After identification of $T_{\hat{c}(t)}M$ and $T_{c(t)}M$ via left translation, $\dot{\hat{c}}(t)$ is equal to $c_\perp(t)$.

So we know that $\|\dot{\hat{c}}\|_G \leq \sqrt{1 - K_2{}^2}$. Writing d_G for the distance induced by $\langle\,.\,,.\,\rangle_G$ we obtain

$$t = d_G(c(0), c(t)) \leq \underbrace{d_G(\hat{c}(0), \hat{c}(t))}_{\leq \left(\sqrt{1-K_2{}^2}\right)t} + d_G(\hat{c}(t), c(t)) \quad \forall t > 0. \tag{1}$$

We use a result of Pansu ([24]) to state that there is a constant $K_3(n)$ not depending on $\alpha > 0$, $s \in S$ and $g \in G$ such that

$$d_G((g \exp \alpha n, s), (g, s)) \leq K_3(n)\,(\sqrt{\alpha} + 1). \tag{2}$$

Pansu did not prove exactly this statement, but the proof of it is completely analogous to the proof of [24] no. (62) if we use the fact that $\exp \alpha n$ is in the commutator group. Another proof using more elementary methods can be found in [1].

Together with $|\lambda(t)| < K_1$ inequality (2) contradicts (1), so we get $\mathcal{P}_n = 0$ for every n in the Lie algebra of N, i.e. $\dot{c}(t)$ is horizontal. This implies that $p \circ c$ is parametrized by arclength.

It remains to show that $p \circ c$ is minimal. In order to prove it we assume the opposite, i.e.

$$\Delta := t_2 - t_1 - d_{G/N}(p \circ c(t_1), p \circ c(t_2)) > 0.$$

Now take a shortest geodesic $\bar{k}: [t_1, t_2] \to (G/N) \times S$ from $p \circ c(t_1)$ to $p \circ c(t_2)$ (see also Figure 2). This shortest geodesic has a unique horizontal lift $k: [t_1, t_2] \to G \times S$ with $c(t_1) = k(t_1)$.

As the Lie exponential map is a diffeomorphism there is a unique $n \in \mathfrak{n}$, such that $k(t_2) = c(t_2) \cdot \exp n$. For $\mu > 0$ we now extend k continuously by

$$k(t) := c(t) \cdot \exp([1 - \mu(t - t_2)]n) \qquad t_2 \leq t \leq t_\mu := t_2 + 1/\mu.$$

So $k: [t_1, t_\mu] \to G \times S$ is also a curve from $c(t_1)$ to $c(t_\mu)$. We will prove that k is shorter than $c|_{[t_1, t_\mu]}$ for small $\mu > 0$. There is a unique $c' : \mathbb{R} \to \mathfrak{n}^\perp \subset \mathfrak{g}$ such that the G-component of $\dot{c}(t)$ is $\mathcal{V}_{c'(t)}(c(t))$. On (t_2, t_μ) the G-component of $\dot{k}(t)$ is

$$\mathcal{V}_{c'(t)}(k(t)) - \mu \mathcal{V}_n(k(t)),$$

whereas the S-components of $\dot{k}(t)$ and $\dot{c}(t)$ are equal up to left (or right) translation.

So as c is horizontal

$$\begin{aligned} \|\dot{k}(t)\|_G &= \sqrt{\|\dot{c}(t)\|_G^2 + \mu^2 \|\mathcal{V}_n(k(t))\|_G^2} \\ &\leq 1 + \frac{1}{2}\mu^2 \|\mathcal{V}_n(k(t))\|_G^2. \end{aligned}$$

For μ small enough we get

$$\mathcal{L}(c|_{[t_1, t_\mu]}) - \mathcal{L}(k) \geq \Delta - \frac{1}{\mu} \frac{\mu^2}{2} \sup_{\substack{g \in G \\ s \in S}} \|\mathcal{V}_n(g, s)\|_G^2 > 0,$$

which contradicts the minimality of c. Therefore $p \circ c$ is minimal. □

Using Theorem 2.1 we now know the minimal geodesics on any nilpotent Lie-group with a left-invariant metric.

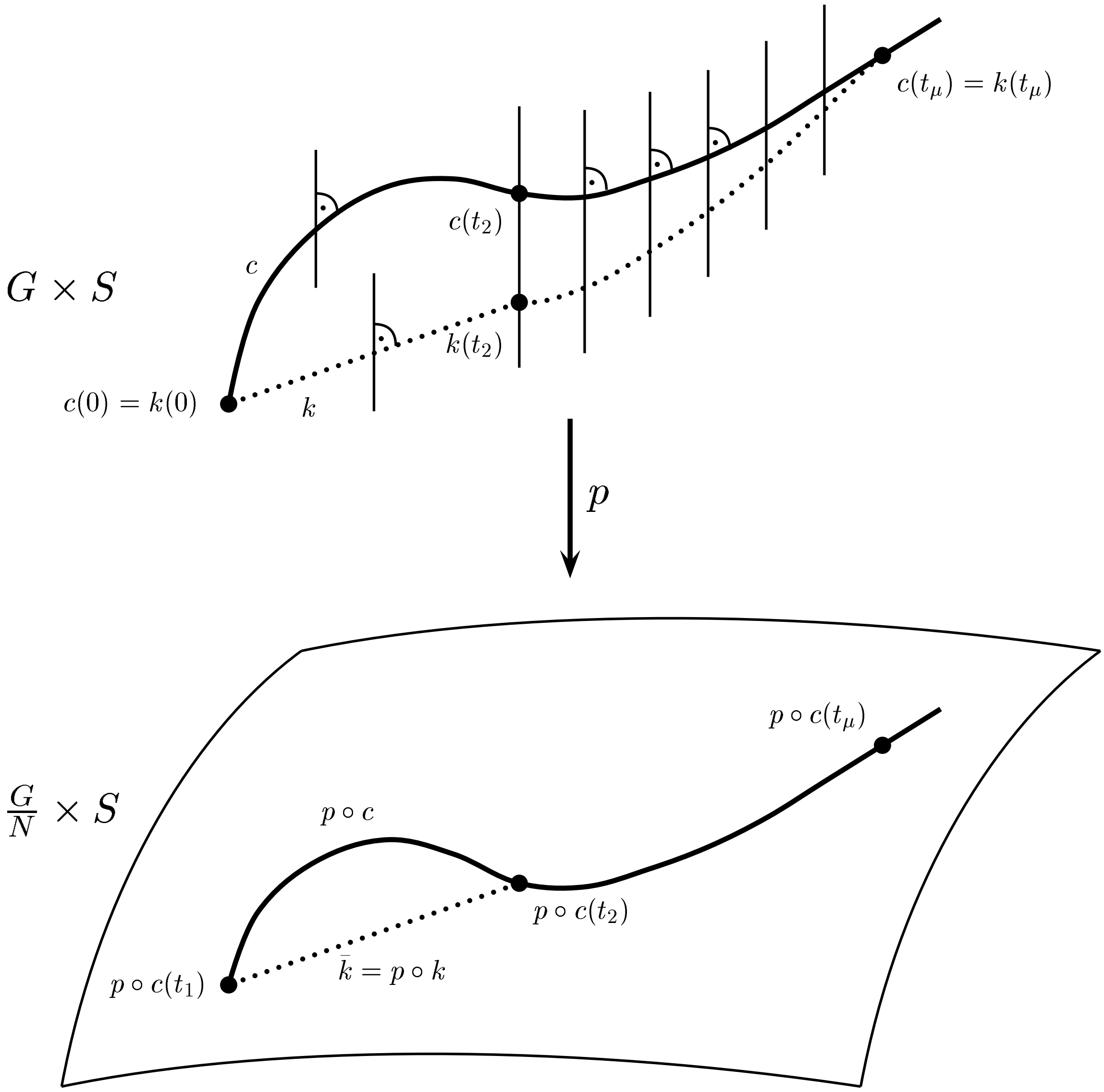

Figure 2. Minimal Geodesics project to Minimal Geodesics

COROLLARY 2.2. *Let G be a nilpotent Lie-group with a left-invariant metric. The minimal geodesics on G are exactly the curves of the form*

$$c(t) = g \cdot \exp tv$$

with $g \in G$ and $v \in \mathfrak{g}$ and $v \perp [\mathfrak{g}, \mathfrak{g}]$.

Remark. A similar type of orthogonality relation was discovered by Patrick Eberlein, Ruth Gornet and Dorothee Schüth when they investigated the following problem. A geodesic c on a nilpotent Lie-group G with left-invariant metric is called *periodic* if there are $g \in G$, $\lambda > 0$ with $c(t+\lambda) = g \cdot c(t)\ \forall t$. A necessary condition for periodicity is that periodic geodesics are orthogonal to certain terms built by commutators ([11]Cor. 4.4, [14]3.1).

Now we try to lift asymptotic behavior from $(G/N) \times S$ to $G \times S$. Here we have to pay attention to the following fact: if $\dim N > 0$ then there are curves $\bar{\gamma}_1, \bar{\gamma}_2 : \mathbb{R} \to (G/N) \times S$ that are asymptotic to each other but do not have horizontal lifts that are asymptotic to each other.

The situation is different if we replace "asymptotic" by "exponentially asymptotic".

DEFINITION 2.3. The (parametrized) curves γ_1 and γ_2 are *exponentially asymptotic* for $t \to \infty$ if there are constants $K_a, K_b > 0$ such that

$$d\left(\gamma_1(t), \gamma_2(t)\right) < K_a e^{-K_b t} \text{ for large } t.$$

Here d is the Riemannian distance. This definition is invariant under bi-Lipschitz change of the metric. The definition of exponentially asymptotic for $t \to -\infty$ is analogous.

PROPOSITION 2.4. *Let $G \times S$ carry a Riemannian metric that is N-left-invariant and invariant under left-action of a lattice Γ of G. Choose a Riemannian metric on the quotient such that $p : G \times S \to (G/N) \times S$ becomes a Riemannian submersion. Furthermore let $\gamma_1, \gamma_2 : \mathbb{R} \to G \times S$ be piecewise C^1-curves with bounded $\|\dot{\gamma}_i\|_G$ and horizontal with respect to p. Then $p \circ \gamma_1$ is exponentially asymptotic to $p \circ \gamma_2$ if and only if there is an $n_\infty \in N$ such that $n_\infty \cdot \gamma_1$ is exponentially asymptotic to γ_2.*

Proof. We only have to prove the "only if". And it is sufficient to prove this for $N \subset Z_1(G)$ as the general case follows by induction. The cases $t \to +\infty$ and $t \to -\infty$ are totally symmetric, so the case $t \to -\infty$ will be omitted.

Because of the action of Γ and the compactness of S the injectivity radius of $G/N \times S$ is positive.

At first we can choose t_0 with

$$d(p\circ\gamma_1(t), p\circ\gamma_2(t)) < \frac{1}{2}\,\mathrm{inj\,rad}(\frac{G}{N}\times S) \qquad \forall t\geq t_0.$$

We glue a surface $A\colon[1,2]\times[t_0,\infty)\to(G/N)\times S$ between the $p\circ\gamma_i|_{[t_0,\infty]}$ such that $A(i,.)=p\circ\gamma_i \quad (i=1,2)$ and $A(.,t)$ is the shortest curve from $p\circ\gamma_1(t)$ to $p\circ\gamma_2(t)$.

As $p\circ\gamma_1(t)$ and $p\circ\gamma_2(t)$ are exponentially asymptotic, there are constants $K_a, K_b>0$ with

$$\mathrm{Area}\, A|_{[1,2]\times[t_1,t_2]} < K_a e^{-K_b t_1} \quad (t_0\leq t_1\leq t_2\leq\infty).$$

Now lift A to $\tilde{A}\colon[1,2]\times[t_0,\infty)\to G\times S$ such that $\tilde{A}(1,.)=\gamma_1$ and $\tilde{A}(.,t)$ is horizontal $\forall t$. There is an $n\colon[t_0,\infty)\to N$ with

$$\tilde{A}(2,t)\cdot n(t)=\gamma_2(t).$$

In general n is non-constant and therefore $\tilde{A}(2,.)$ is non-horizontal, but we will show that $n(t)$ converges for $t\to\infty$.

Note that $G\times S\to(G/N)\times S$ is a principal N-bundle. As N acts isometrically, the horizontal planes determine a connection-1-form $\omega\colon T(G\times S)\to\mathfrak{n}$. (For details on connection-1-forms see [20], Chapter II.) Then $d\omega$ is the curvature of the connection and $\|d\omega\|$ is uniformly bounded on $G\times S$.

As γ_i and $\tilde{A}(.,t)$ are horizontal

$$\int_{\gamma_i|_{[t_1,t_2]}}\omega=0 \quad\text{and}\quad \int_{\tilde{A}(.,t)}\omega=0$$

$$\int_{\partial\left(\tilde{A}|_{[1,2]\times[t_1,t_2]}\right)}\omega=\int_{\tilde{A}(2,.)|_{[t_1,t_2]}}\omega=n(t_1)n(t_2)^{-1}$$

Using Stoke's Theorem we get

$$\begin{aligned} d\left(n(t_1)n(t_2)^{-1},e\right) \leq \int_{\tilde{A}|_{[1,2]\times[t_1,t_2]}}\|d\omega\| &\leq \mathrm{Area}\left(A|_{[1,2]\times[t_1,t_2]}\right)\sup_{G\times S}\|d\omega\| \\ &\leq \sup\|d\omega\|K_a e^{-K_b t_1}. \end{aligned}$$

So $n(t)$ converges exponentially and $n_\infty:=\lim_{t\to\infty}n(t)$ gives the proposition. $\square$

3. Hedlund examples

In this chapter we will give a slight generalisation of the Hedlund examples presented by Bangert in [4], section 5. The proofs are only small variations of Bangert's proofs, so we will skip them.

In this section we construct similar metrics, which we will also call "Hedlund metrics". These metrics are defined on manifolds of the form $M = T^{b_1} \times S$ where $T^{b_1} = \mathbb{R}^{b_1}/\mathbb{Z}^{b_1}, b_1 \geq 1$ is the torus and S is an arbitrary compact connected manifold with finite $\pi_1(S)$. We exclude the case $M = T^2$ by assuming $\dim S > 0$ or $b_1 \neq 2$. Note that b_1 is the first Betti number of M.

We denote the standard flat metric on T^{b_1} by $\langle\,.\,,.\,\rangle_T$ and we choose a metric $\langle\,.\,,.\,\rangle_S$ on S. The product metric on M will be called $\langle\,.\,,.\,\rangle_{T\times S}$. The vectors of the canonical basis $e_1, \ldots, e_{b_1}$ of $\mathbb{R}^{b_1}$ induce $\langle\,.\,,.\,\rangle_{T\times S}$-orthonormal vector fields $E_1, \ldots, E_{b_1}: M \to TM$.

The Hedlund metrics $\langle\,.\,,.\,\rangle_H$ will be defined in Definition 3.1. In this definition we use b_1 closed curves $c_1, \ldots, c_{b_1}$ on M that will become the only geodesics that are minimal and closed. To define them, we have to distinguish two cases.

In the case "$b_1 \neq 2$" choose $s \in S$ and define

$$c_i(t) := \left(\frac{\frac{1}{2}e_i + te_{i+1}}{\mathbb{Z}^{b_1}},\, s \right)$$

for all $t \in \mathbb{R}$, where $e_{b_1+1} := e_1$.

In the case "$b_1 = 2$" we have assumed that $\dim S > 0$, so we can choose different $s_1, s_2 \in S$ and define

$$c_i(t) := \left(\frac{te_i}{\mathbb{Z}^{b_1}},\, s_i \right)$$

for all $t \in \mathbb{R}$.

In both cases let L_i be the trace of c_i. The fact that L_i and L_j are disjoint for $i \neq j$ plays an important role in the proofs. The construction of Hedlund type metrics on T^2 fails because such c_i and L_i do not exist on T^2.

Now define $U_\epsilon(L_i)$ to be the ϵ-neighborhood of L_i with respect to $\langle\,.\,,.\,\rangle_{T\times S}$.

For $\epsilon > 0$ (that will be chosen very small) we define in analogy to Definition 5.1 of [4]:

DEFINITION 3.1. $\langle\,.\,,.\,\rangle_H$ is an ϵ-Hedlund metric on M iff there are $\epsilon_1, \ldots, \epsilon_{b_1} \in (0, \epsilon]$ such that for $i = 1, \ldots, b_1$:

$$\begin{array}{lll}
(P1) & \langle v,v\rangle_H \leq (1+\epsilon)^2 \langle v,v\rangle_{T\times S} & \forall v \in TM \\
(P2) & \langle E_i(x), E_i(x)\rangle_H = \epsilon_i^2 & \forall x \in L_i \\
 & \langle v,v\rangle_H \geq \epsilon_i^2 \langle v,v\rangle_{T\times S} & \forall v \in T_xM, x \in L_i \\
 & \langle v,v\rangle_H > \epsilon_i^2 \langle v,v\rangle_{T\times S} & \forall v \in T_xM \setminus \{0\}, x \in U_\epsilon(L_i) \setminus L_i \\
(P3) & \langle v,v\rangle_H \geq \langle v,v\rangle_{T\times S} & \forall v \in T_xM, x \notin \bigcup_j U_\epsilon(L_j)
\end{array}$$

The following propositions 3.2, 3.4 and 3.5 are analogues to Propositon 5.2, Proposition 5.3 and Corollary 5.4 of [4]. Because of the definition of ϵ-Hedlund metric it is clear that any statement in these propositions that holds for $\epsilon > 0$ also holds for any $\epsilon' \in (0, \epsilon)$.

PROPOSITION 3.2. *There is an $\epsilon > 0$ and a $K_1 \in \mathbb{R}$ such that for any ϵ-Hedlund metric on M and any arclength-parametrized minimal geodesic c with respect to this metric the length of*

$$A := c^{-1}(M \setminus \bigcup_i U_\epsilon(L_i)) \subset \mathbb{R}$$

is bounded by K_1.

That means that c "stays out of $\bigcup_i U_\epsilon(L_i)$ only for a bounded time". As an immediate consequence c cannot change its "tube" too often. To make this precise we define:

DEFINITION 3.3. Let $\epsilon > 0$ be so small that the $U_\epsilon(L_i)$ $(i = 1, \ldots, b_1)$ are disjoint and let $c\colon \mathbb{R} \to M$ be a minimal geodesic. We define the change number $\mathcal{C}(c) \in \mathbb{N} \cup \{0, \infty\}$ to be the supremum of all $n \in \mathbb{N}$ such that we find $t_0 < t_1 < \ldots < t_n$ and $i_j \in \{1, \ldots, b_1\}$ with $c(t_j) \in U_\epsilon(L_{i_j})$ and $i_j \neq i_{j+1}$.

PROPOSITION 3.4. *There is an $\epsilon > 0$ and $K_2 \in \mathbb{N}$ such that $\mathcal{C}(c) \leq K_2$ for any minimal geodesic c with respect to any ϵ-Hedlund metric on M.*

Remark. If $S = \{\text{one point}\}$ Bangert proved in [4] that we can even find $\epsilon > 0$ with $K_2 := b_1$. For general S this statement does not hold.

PROPOSITION 3.5. *There is an $\epsilon > 0$ such that every minimal geodesic on an ϵ-Hedlund metric on M is asymptotic in each of its senses to one of the L_i's.*

In section 5 we will need a stronger version, so we formulate a supplement.

SUPPLEMENT 3.6. If the c_i are even hyperbolic closed geodesics, e.g. if

$$A_{jk}(x) := E_k(E_j(\langle E_i, E_i \rangle_H))(x) \qquad j, k \neq i$$

is positive definite for all $x \in L_i$, then any minimal geodesic c is exponentially asymptotic to one of the L_i in each of its senses (see Definition 2.3).

Remark. It is also possible to formulate analogues to the propositions 5.6, 5.7 and 5.8 from [4].

4. Lattices in nilpotent Lie-groups

Here we will summarize some facts used in the next section. For the discrete group or Lie-group G we define the descending central series $(G^i)_{i\in\mathbb{N}}$ inductively by $G^1 := G$ and $G^{i+1} := [G, G^i]$. Then G is nilpotent iff $G^i = \{e\}$ for sufficiently big $i \in \mathbb{N}$.

THEOREM 4.1 (Malcev, [25] theorem 2.18). *A group Γ is isomorphic to a lattice in a nilpotent, simply connected Lie-group iff Γ is finitely generated, nilpotent and torsion free.*

THEOREM 4.2. *Let Γ be a lattice in the nilpotent, simply connected Lie-group G and N a closed normal subgroup (not necessarily connected), $p: G \to G/N$.*
If two of the following three conditions are true, the third follows:

1. *$\Gamma \cap N$ is a lattice in N,*
2. *$p(\Gamma)$ is a lattice in G/N,*
3. *Γ is a lattice in G.*

"1. and 2. $\Rightarrow$ 3." and "1. and 3. $\Rightarrow$ 2." are proved in [10] lemma 5.1.4, the proof of "2. and 3. $\Rightarrow$ 1." is straightforward.

THEOREM 4.3. *Let Γ be a lattice in G, then Γ^i is a lattice in G^i for $i \in \mathbb{N}$.*

This follows from the theory of Malcev bases ([10]) and [10] corollary 5.4.5. It is a slight generalisation of [25] corollary 1 of theorem 2.3 saying that $\Gamma \cap G^i$ is cocompact in G^i.

Using the theory of Malcev bases it is also evident that

$$\operatorname{rank} \frac{\Gamma}{[\Gamma, \Gamma]} = \dim \frac{G}{[G, G]}.$$

5. Main construction

In this section we will construct our examples with minimal geodesics in only "few directions" by combining the results we obtained in the previous sections.

THEOREM 5.1. *For any finitely generated nilpotent group Π_1 we find a connected compact Riemannian manifold $(M, \langle\,.\,,.\,\rangle_M)$ satisfying:*

(1) $\pi_1(M) = \Pi_1$

(2) *M has a universal covering $\widetilde{M} = G \times \tilde{S}$ where G is a nilpotent Lie-group and $\tilde{S}$ is a compact manifold.*

(3) *The commutator group $[G, G]$ acts isometrically on the Riemannian covering $\widetilde{M}$ via left multiplication on the first component.*

(4) *There are minimal geodesics $c_i\colon \mathbb{R} \to M$ $(i \in \{1, \dots, b_1\})$ with lifts $\tilde{c}_i\colon \mathbb{R} \to \widetilde{M}$ such that every minimal geodesic $\gamma\colon \mathbb{R} \to M$ is of one of the following types:*

Type I: left-translated-periodic
γ has a lift $\tilde{\gamma}\colon \mathbb{R} \to \widetilde{M}$ such that there are $a, b \in \mathbb{R}$, $g \in [G, G]$, $i \in \{1, \dots, b_1\}$ with $\tilde{\gamma}(t) = g \cdot c_i(at + b)$.

Type II: connection type
γ is not of Type I, but there are minimal geodesics γ_+ and γ_- of Type I such that $\gamma(t)$ is exponentially asymptotic to $\gamma_\pm(t)$ for $t \to \pm\infty$.

Any compact manifold is a finite CW-complex and therefore the fundamental group of any compact manifold is finitely genrated.

The author has presented another version of this construction in [1] and proves that if Π_1 has no torsion and $\Pi_1 \neq \mathbb{Z}^2$ we even get an example as above with $\widetilde{M} = G \times \{\text{one point}\}$ (but without property (3) in the case $b_1 = 2$). It uses the fact that subgroups of the 3-dimensional Heisenberg group are of bounded minimal generation (Theorem 7.4). So T^2 is the only nilmanifold that does not admit a metric of Hedlund type.

To prove theorem 5.1 we will use:

THEOREM 5.2 (K.A. Hirsch, [6] theorem 2.1). *Let Π_1 be a finitely generated nilpotent group. Then Π_1 can be embedded as a subgroup into the direct product $D = \Gamma \times F$ where Γ is a torsionfree nilpotent finitely generated group and F is a finite nilpotent group.*

The elements in Π_1 of finite order form a normal subgroup $T = F \cap \Pi_1$. Looking at Baumslag's proof of the theorem of Hirsch, we immediately see that Γ can be chosen as Π_1/T and that

$$\begin{tikzcd}
\Pi_1 \arrow[r, tail] \arrow[dr] & \Gamma \times F \arrow[d, two heads] \\
& \Gamma = \Pi_1/T
\end{tikzcd}$$

commutes.

Proof of Theorem 5.1.
Suppose Π_1 to be embedded in $D = \Gamma \times F$ as above.

Now embed Γ as a lattice in the nilpotent Lie-group G (Theorem 4.1).

Abelianisation of the above diagram and tensoring with $\mathbb{R}$ shows that

$$\operatorname{rank} \frac{\Gamma}{[\Gamma, \Gamma]} = \operatorname{rank} \frac{\Pi_1}{[\Pi_1, \Pi_1]} (= b_1),$$

so there is a natural isomorphism

$$\frac{G}{[G,G]} \xrightarrow{\sim} \mathbb{R}^{b_1}$$

that maps the image of Γ to $\mathbb{Z}^{b_1}$.

It is well-known that for $n \geq 4$ any finitely presented group is the fundamental group of an n-dimensional compact manifold ([22] p. 114). So let S be a 4-dimensional compact Riemannian manifold with $\pi_1(S) = F$.

Now take an ϵ-Hedlund metric on $T^{b_1} \times S$ with hyperbolic c_i and choose a metric on $(\Gamma \backslash G) \times S$ such that

$$(\Gamma \backslash G) \times S \to T^{b_1} \times S$$

is a Riemannian submersion and such that $[G, G]$ acts isometrically on $G \times S$. As $(\Gamma \backslash G) \times S$ has fundamental group D, we can find a Riemannian covering M with $\pi_1(M) = \Pi_1$. So (1), (2) and (3) are fulfilled.

Using Theorem 2.1, Proposition 3.5, Supplement 3.6 and Proposition 2.4 we also get (4). $\square$

In the remaining part of this section we will discuss some related topics and some additional properties of the above examples.

For arbitrary compact manifolds M with $\Pi_1 := \pi_1(M)$ we get via the Hurewicz map

$$H_1(M,\mathbb{Z}) = \frac{\Pi_1}{[\Pi_1,\Pi_1]} \quad \text{and} \quad H_1(M,\mathbb{R}) = H_1(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Dividing $H_1(M,\mathbb{R})$ by the image of $H_1(M,\mathbb{Z})$ we get a torus T^{b_1} in analogy to the above constructions. This torus is known as the Jacobi variety ([16]4.21). $H_1(M,\mathbb{R})$ also carries a norm, the "stable norm $\|\,.\,\|$", induced by the Riemannian structure of M ([12],[16]4.18,[5]). $\left(\widetilde{M}, \epsilon\langle\,.\,,.\,\rangle\right)$ converges in the Gromov-Hausdorff-sense to $(H_1(M,\mathbb{R}), \|\,.\,\|)$ for $\epsilon \to 0$ if Π_1 is abelian. For nilpotent Π_1 it converges to a Carnot-Caratheodory space ([24]) and the stable norm is essential for measuring distances on this space.

Bangert used the stable norm to prove an existence theorem for minimal geodesics ([4],[5]). As a corollary he proved the existence of at least b_1 different geodesics such that the "rotation set" $\mathcal{R}(c_i)$ of each c_i contains only one vector in $H_1(M,\mathbb{R})$ and $\bigcup_i \mathcal{R}(c_i)$ is a basis of $H_1(M,\mathbb{R})$. In the above examples the geodesics $c_1,\ldots,c_{b_1}$ have the properties of the geodesics whose existence has been shown by Bangert: each $\mathcal{R}(c_i)$ contains only one vector and their union is a basis. For our examples the stable norm written in this basis is just

$$\|x\| = \sum_{i=1}^{b_1} \epsilon_i |x^i|. \tag{3}$$

Equation (3) can be seen from Bangert's theorem and the characterisation of the minimal geodesics or just using the fact that if a Riemannian submersion $p\colon M_1 \to M_2$ of compact manifolds M_i induces an isomorphism $p_\#\colon H_1(M_1,\mathbb{R}) \to H_1(M_2,\mathbb{R})$, then $p_\#$ preserves the stable norm.

This last fact can also be used to construct metrics on nilmanifolds with non-left-invariant metric on the universal covering that have a smooth unit ball of the stable norm. (Just take a suitable lift of a flat metric on the Jacobi variety.)

6. Expressway metrics

In the previous section we proved that for every finitely generated nilpotent group Π_1 there is a Riemannian manifold M with $\pi_1(M) = \Pi_1$

and only few directions of minimal geodesics. On the other hand, many properties concerning minimal geodesics only depend on the fundamental group and an induced distance on it. Therefore it seems likely that we could find a suitable Riemannian metric on every compact manifold with nilpotent fundamental group. Moreover the metrics in the last section admit continuous families of minimal geodesics if $\dim[G,G] > 0$, whereas Hedlund's original examples only admit very few ones. So it would be interesting to generalize Hedlund's methods directly.

In this section we try to use Hedlund's method (generalized by Bangert [4]) directly to construct Hedlund type metrics on arbitrary (compact) manifolds with nilpotent fundamental group Π_1. It turns out that we succeed only if Π_1 has an algebraic property that we call "bounded minimal generation". We can prove that lattices in Heisenberg groups have this property but we do not know if this is true for all finitely generated nilpotent groups or not.

The following construction seems to be very special, but the author thinks that it will be difficult to find a different construction without getting a problem similiar to the bounded-minimal-generation-problem described in the next section. We will omit an explicit definition of the metrics and proofs of the statements concerning these metrics as the exact formulae only give little insight in what happens. (For details see [1]). Instead we will give an informal description.

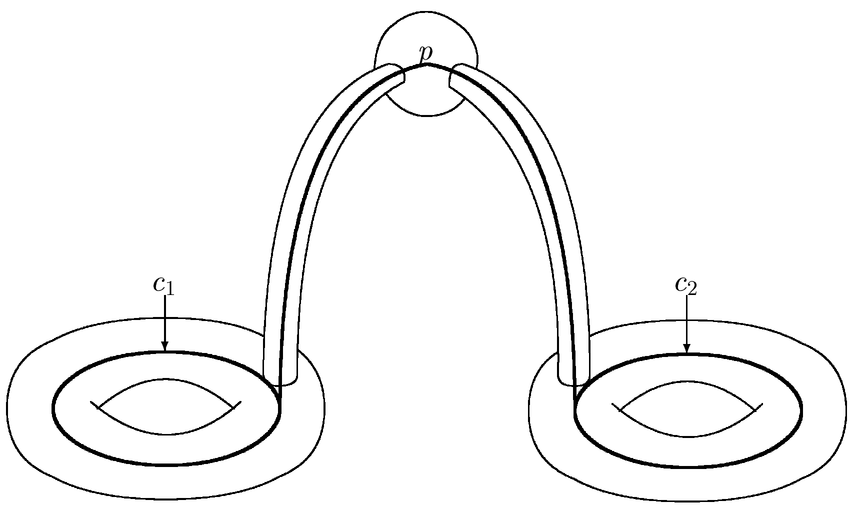

Figure 3. An expressway with two generators

When the author tried to generalise Bangert's construction of Hedlund metrics ([4]) to manifolds with arbitrary fundamental groups, he took closed curves $c_1, \ldots, c_k$ based in $p \in M$, whose homotopy classes

$[c_i]$ generate Π_1. By a small perturbation it is possible to transform the c_i into smooth disjoint embeddings of $S^1 \hookrightarrow M$ passing near p if $\dim M \geq 3$. Now he chose a Riemannian metric that is very small in a small neighborhood of the c_i and small in the neighborhood of certain paths joining the c_i to p, but relatively big outside these neighborhoods. We can assume that the c_i are hyperbolic minimal geodesics of length ϵ.

Roughly speaking, the Riemannian distance looks like the distance, a car driver has in his mind: there are some "expressways" (the neighborhoods of the c_i and the joining paths) where normed curves run very fast and in other regions where they move relatively slow. So minimal geodesics run most of their time on these "expressways". To be more precise the author showed that if c is a minimal geodesic and E the expressway, then the length of each connected component of $c^{-1}(M \setminus E)$ is small. But we do not know whether the total length of $c^{-1}(M \setminus E)$ is bounded.

So for arbitrary nilpotent fundamental groups the author was unable to get analogues of propositions 3.2 and 3.4. The situation is much nicer when Π_1 is of bounded minimal generation with respect to the $[c_i]$ (Definition 7.1). Here we get analogues of propositions 3.2 and 3.4. The classification of minimal geodesics then can be reduced to combinatorial group properties of Π_1, and every minimal geodesic is asymptotic in each of its senses to one of the c_i. So the results are similar as on T^n $(n \geq 3)$. Unfortunately the bounded-minimal-generation-problem seems to be hard.

7. Groups of Bounded Minimal Generation

Let S_Γ be a finite set of generators of the group Γ, i.e. every $\gamma \in \Gamma$ can be written as a *word* $(s_1, s_2, \ldots, s_l)$ with $s_i \in S_\Gamma \dot\cup S_\Gamma^{-1}$. The number $l =: l((s_1, s_2, \ldots))$ is the *length* of the word, furthermore we define the change number

$$\begin{aligned} \mathcal{C}((s_1, s_2, \ldots, s_l)) &:= \#\{i \in \{1, 2, \ldots, l-1\} \mid s_i \neq s_{i+1}\} + 1 \\ &= \sup\{k \mid s_{i_1} \neq \ldots \neq s_{i_k},\ 1 \leq i_1 < \ldots < i_k \leq l\}. \end{aligned}$$

This is a group theoretical analogue of Definition 3.3. For the empty word representing the neutral element we set $l(\emptyset) = \mathcal{C}(\emptyset) = 0$.

The word $(s_1, s_2, \ldots)$ is of minimal length if every $(s_1', s_2', \ldots)$ representing the same $\gamma \in \Gamma$ satisfies:

$$l((s_1, s_2, \ldots)) \leq l((s_1', s_2', \ldots)).$$

DEFINITION 7.1. (Γ, S_Γ) is a *Group of Bounded Minimal Generation (BMG group)* if there is a $B \in \mathbb{N}$ such that every $\gamma \in \Gamma$ can be represented by a word of minimal length $(s_1, s_2, \ldots)$ in $S_\Gamma \dot\cup S_\Gamma^{-1}$ with $\mathcal{C}((s_1, s_2, \ldots)) \leq B$. The minimal such B will be called the *bound*.

Every finitely generated abelian group together with an arbitrary finite set of generators S_Γ is a BMG group with $B \leq \# S_\Gamma$.

Gromov proved in [15] that every finitely generated group of polynomial growth is virtually nilpotent, i.e. it contains a nilpotent subgroup of finite index. If (Γ, S_Γ) is a BMG group, then Γ is of polynomial growth and therefore virtually nilpotent. Yet, it is not clear to the author if the converse holds.

OPEN PROBLEM 7.2. Is every finitely generated virtually nilpotent group a BMG group?

Remark. It is not even clear whether the BMG property is independent of the choice of the set of generators.

Constructing new BMG groups from old ones. It is staightforward to show:

1. If (Γ, S_Γ) and $(\Gamma', S_{\Gamma'})$ are BMG groups with bounds B and B', then $(\Gamma \times \Gamma', S_\Gamma \cup S_{\Gamma'})$ is a BMG group with bound $\leq B + B'$.
2. If (Γ, S_Γ) is a BMG-group with bound B and $h\colon \Gamma \to \Gamma'$ a group homomorphism, then $(h(\Gamma), h(S_\Gamma))$ is a BMG-group with bound $\leq B$.
3. If Γ_2 is the semidirect product of a BMG group (Γ_1, S_{Γ_1}) and a finite group, then there is a generating system S_{Γ_2} of Γ_2 such that (Γ_2, S_{Γ_2}) is a BMG group.

Heisenberg groups. For $m \in \mathbb{N}$, $\vec{p}, \vec{q} \in \mathbb{R}^m$, $z \in \mathbb{R}$ we define the matrix

$$M(\vec{p}, \vec{q}, z) := \begin{pmatrix} 1 & \vec{p}^{\,t} & z \\ 0 & 1_m & \vec{q} \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{H}_m := \{M(\vec{p}, \vec{q}, z) \,|\, \vec{p}, \vec{q} \in \mathbb{R}^m,\ z \in \mathbb{R}\}$ is the $2m+1$-dimensional Heisenberg group.

For $r = (r_1, \ldots, r_m) \in \mathbb{N}^m$ such that r_i divides r_{i+1}, $1 \leq i < m$, we set

$$r\mathbb{Z}^m := \{(r_1 x_1, \ldots, r_m x_m) \,|\, x_j \in \mathbb{Z}\}$$
$$\Gamma_r := \{M(\vec{p}, \vec{q}, z) \,|\, \vec{p} \in r\mathbb{Z}^m, \vec{q} \in \mathbb{Z}^m, z \in \mathbb{Z}\}.$$

These Γ_r are *lattices* in $\mathcal{H}_m$, i.e. discrete, cocompact subgroups.

THEOREM 7.3 ([13], §2.). *For every lattice* Γ *of* $\mathcal{H}_m$ *there exists a unique* r *and an automorphism of* $\mathcal{H}_m$ *mapping* Γ *to* Γ_r.

THEOREM 7.4. *Every lattice* Γ *of* $\mathcal{H}_m$ *has a set of generators* S*, such that* (Γ, S) *is a BMG-group.*

Remark. It is not difficult to show that any discrete subgroup of $\mathcal{H}_m$ is of the form $\Gamma' \times \mathbb{Z}^k$, where Γ' is trivial or a lattice in a Heisenberg group. So Theorem 7.4 immediately generalizes to discrete subgroups.

LEMMA 7.5. $\Gamma_1 = \{M(p,q,z) \,|\, p,q,z \in \mathbb{Z}\} \subset \mathcal{H}_1$ *together with* $S_{\Gamma_1} := \{M(1,0,0), M(0,1,0)\}$ *is a BMG-group.*

Proof of Theorem 7.4. We can assume $\Gamma = \Gamma_r$. We denote the standard basis of $\mathbb{R}^m$ by $e_1, \ldots, e_m$. The mappings

$$\begin{aligned} f_i \colon \mathcal{H}_1 &\to \mathcal{H}_m \\ M(p,q,z) &\mapsto M(pr_i e_i, q e_i, z r_i) \end{aligned}$$

for $i = 1, \ldots, m$ and

$$\begin{aligned} f_0 \colon \mathbb{R} &\to \mathcal{H}_m \\ z &\mapsto M(0,0,z) \end{aligned}$$

define a group epimorphism

$$\begin{aligned} f \colon \mathbb{R} \times \mathop{\times}_{i=1}^{m} \mathcal{H}_1 &\to \mathcal{H}_m \\ (z, h_1, \ldots, h_m) &\mapsto f_0(z) f_1(h_1) f_2(h_2) \ldots f_m(h_m) \end{aligned}$$

that maps $\mathbb{Z} \times \mathop{\times}_{i=1}^{m} \Gamma_1$ to Γ_r. Using lemma 7.5 we know that (Γ_1, S_{Γ_1}) is a BMG-group. Applying the above constructions of new BMG-groups we see that (Γ, S) is a BMG-group for $S := f(\bigcup_{i=0}^m S_i)$, where S_0 is $1 \in \mathbb{R}$ and S_i is S_{Γ_1} in the i-th slot. $\square$

Proof of Lemma 7.5. We set $g_1 := M(1,0,0)$, $g_2 := M(0,1,0)$, then $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1} = M(0,0,1)$. For every word w in the generators g_1 and g_2 we will explain how to construct a word w' of minimal length and with $\mathcal{C}(w') \leq 6$ representing the same $\gamma \in \Gamma$.

In order to construct w' we will give a geometric interpretation of the problem. To every word w we will associate inductively a path $p(w)$ in

$$P := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,|\, x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \right\}.$$

At first we associate to $w = \emptyset$ the constant path in $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and to $w = g_1$ (resp. $w = g_2, g_1^{-1}, g_2^{-1}$) the straight line from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$). Then we associate to the word $w_1 w_2$ the path $p(w_1 w_2)$ that consists of the path $p(w_1)$ and then $p(w_2)$, translated by the endpoint of $p(w_1)$ — we get a path from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to the sum of the endpoints of $p(w_1)$ and $p(w_2)$.

Now we define

$$I(w) = \int_{p(w)} x\, dy.$$

Let (i_1, i_2) be the endpoint of p_w. Then w represents ${g_2}^{i_2} {g_1}^{i_1} [g_1, g_2]^{I(w)}$, $l(w) = \mathcal{L}(p(w))$, the change number $\mathcal{C}(w)$ of w is equal to the number of direction changes of $p(w)$ plus 1.

We get an geometric interpretation of $I(w)$ by applying Stoke's theorem:

$$I(w) = \int_{p(w)} x\, dy = \int_{p\left(w {g_1}^{-i_1} {g_2}^{-i_2}\right)} x\, dy = \int_A dx \wedge dy,$$

where A is the 2-chain whose boundary is $p\left(w {g_1}^{-i_1} {g_2}^{-i_2}\right)$. As $dx \wedge dy$ is the volume element of $\mathbb{R}^2$, we interpret $I(w)$ as the oriented area of A.

We get an isoperimetric problem on P: Finding words of minimal length means finding paths of minimal length in P with given integral $\int x\, dy$.

With this geometric interpretation the lemma is almost obvious. Using symmetries we can assume $i_1, i_2 \geq 0$, $I(w) \geq 0$. Consider the special cases

1. $I(w) \leq i_1 \cdot i_2$,
2. $i_1 i_2 < I(w) \leq \max\{i_1, i_2\}^2$,
3. $\max\{i_1, i_2\}^2 < I(w)$.

In each case we can find a word w' of minimal length, equivalent to w with $\mathcal{C}(w') \leq 6$. □

Remark. Michael Stoll [28] treats the continuous analogue of bounded minimal generation for nilpotent Lie-groups. He proves that every 2-step nilpotent Lie-group fulfills it, but he states that there are counterexamples for 3-step nilpotent Lie-groups.

Remark. Several authors ([9],[29],[26] and [27]) consider definitions ("groups of bounded generation", "groups of finite width") which are formally related to our groups of bounded minimal generation. The main difference is that these notions use arbitrary not only minimal representatives.

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