A SPINORIAL ANALOGUE OF AUBIN'S INEQUALITY

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ABSTRACT. Let (M, g, σ) be a compact Riemannian spin manifold of dimension ≥ 2. For any metric $\tilde{g}$ conformal to g, we denote by $\tilde{\lambda}$ the first positive eigenvalue of the Dirac operator on $(M, \tilde{g}, \sigma)$. We show that

$$\inf_{\tilde{g}\in[g]} \tilde{\lambda}\,\mathrm{Vol}(M, \tilde{g})^{1/n} \leq (n/2)\,\mathrm{Vol}(S^n)^{1/n}.$$

This inequality is a spinorial analogue of Aubin's inequality, an important inequality in the solution of the Yamabe problem. The inequality is already known in the case $n \geq 3$ and in the case $n = 2$, $\ker D = \{0\}$. Our proof also works in the remaining case $n = 2$, $\ker D \neq \{0\}$. With the same method we also prove that any conformal class on a Riemann surface contains a metric with $2\tilde{\lambda}^2 \leq \tilde{\mu}$, where $\tilde{\mu}$ denotes the first positive eigenvalue of the Laplace operator.

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Contents

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. We assume that M is spin, and we fix a spin structure σ on M. For any metric $\tilde{g}$ in the conformal class $[g]$ of g, we write $\lambda_1^+(\tilde{g})$ for the smallest positive eigenvalue of the Dirac operator with respect to $(M, \tilde{g}, \sigma)$. We define

$$\lambda^+_{\min}(M, g, \sigma) = \inf_{\tilde{g}\in[g]} \lambda_1^+(\tilde{g})\mathrm{Vol}(M, \tilde{g})^{1/n}.$$

If (M, g) is the round sphere $\mathbb{S}^n$ equipped with the unique spin structure on $\mathbb{S}^n$, we simply write $\lambda^+_{\min}(\mathbb{S}^n)$. It was proven in [Lot86] ($\ker D = \{0\}$) and [Am03b] ($\ker D \neq \{0\}$) that

$$\lambda^+_{\min}(M, g, \sigma) > 0.$$

Several articles have been devoted to the study of this spin-conformal invariant. A non-exhaustive list is [Hij86, Lot86, Bär92, Am03a]. In this article we will prove the following.

Theorem 1.1. *Let (M, g, σ) be a compact spin manifolds of dimension $n \geq 2$. Then,*

$$\lambda^+_{\min}(M, g, \sigma) \leq \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\,\omega_n^{\frac{1}{n}} \tag{1}$$

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where ω_n stands for the volume of the standard sphere $\mathbb{S}^n$.

A similar inequality, established by Aubin, played a prominent role in the solution of the Yamabe problem, see [LP87] for a good overview. We define

$$Y(M,[g]) := \inf_{\tilde{g}\in[g]} \lambda_1(L_{\tilde{g}})\mathrm{Vol}(M,\tilde{g})^{2/n} \in \{-\infty\}\cup\mathbb{R},$$

where $L_{\tilde{g}} := 4\frac{n-1}{n-2}\Delta_{\tilde{g}} + \mathrm{Scal}_{\tilde{g}}$ denotes the conformal Laplacian of $(M,\tilde{g})$. The number $Y(M,[g])$ is called Yamabe constant of $(M,[g])$ if $Y(M[g]) \geq 0$. The definition of the Yamabe constant in the negative case is slightly different. For the sphere one has $Y(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$. Aubin has shown in [Aub76] that $Y(M,[g]) \leq Y(\mathbb{S}^n)$ for any n-dimensional compact manifold M. Furthermore if strict inequality holds, then he showed using previous work by Yamabe and Trudinger that g is conformal to a metric of constant scalar curvature. If M is not conformally flat and of dimension at least 6, then strict inequality was proven in [Aub76] as well. The idea of his proof is to construct a good test function. For all other conformal manifolds (except the sphere $\mathbb{S}^n$, of course!) the strict version of Aubin's inequality $Y(M,[g]) < Y(\mathbb{S}^n)$ follows from work of Schoen and the positive mass theorem.

The proof of our theorem relies on constructing a suitable test spinor, and hence both the inequality and the construction are inspired by Aubin's work together with spinorial techniques provided by [BG92]. The main idea of our construction is to start with a Killing spinor on the round sphere. Under stereographic projection this spinor then yields a solution to the equation $D\psi = c|\psi|^{2/(n-1)}\psi$ on flat $\mathbb{R}^n$. This solution will be rescaled, cut off and finally transplanted to a neighborhood of a given point p of the manifold M. For this transplantation we carry out several calculations in a well-adapted trivialization of the spinor bundle.

The first steps in our proof are common in all dimensions. However, in some final estimates one has to distinguish between the cases $n \geq 3$ and $n = 2$.

In dimension $n \geq 3$ two other proofs for the theorem have already been published: a geometric construction [Am03b, Theorem 3.1] and a proof using an invariant for non-compact spin manifolds [Gro06]. In these dimensions, it is mostly the method of proof that is interesting and helpful: the trivialization presented here has less terms in the Taylor expansion than the trivialization by using parallel transport along radial geodesics. Some formulae of our article also enter in [Gro06]. The calculations of our article also provide helpful formulae used in [AHM03], [AH03] and [Rau06].

The main interest of the theorem however lies in the case $n = 2$. The easier subcase $n = 2$, $\ker D = \{0\}$ could be dealt with by a modification of the geometric proof [Am03b, Theorem 3.2], but the subcase $n = 2$ and $\ker D \neq \{0\}$ remained open for longtimes. Große's method fails as well for $n = 2$ as the contribution of a cut-off function in [Gro06, Lemma 2.1(ii)] is too large. We assume that her method can be adapted by using a logarithmic cut-off function, but the details have not been worked out yet.

Our method of proof in dimension 2 actually admits applications to other problems as well. For example, one obtains the following proposition that provides a negative answer to a question raised in [AAF99].

Proposition 1.2 (See Corollary 7.2)**.** *Let (M,g) be a Riemann surface with fixed spin structure σ. For any metric $\bar{g}$ in the conformal class $[g]$, let $\mu_1(\bar{g})$ be the first positive eigenvalue of the Laplacian, and let $\lambda_1^+(\bar{g})$ be the first positive eigenvalue of the Dirac operator on $(M,\bar{g},\sigma)$. Then*

$$\inf_{\bar{g}\in[g]} \frac{\lambda_1^+(\bar{g})^2}{\mu_1(\bar{g})} \leqslant \frac{1}{2}.$$

Spinors and Dirac operator also appear in many other problems in modern physics. Some associated analytical problems as e.g. the analysis of Dirac-harmonic maps might also profit from the techniques developed in our article. Dirac-harmonic maps are supersymmetric anologues of harmonic maps. Although considerable progress was achieved recently (see [CJLW06] and other articles by the same authors), many interesting questions remain open, e.g. efficient criteria for the existence of solutions on generic manifolds.

The article is organized as follows: in Section 2, we recall that $\lambda^+_{\min}(M,g,\sigma)$ has a variational characterization. Then, in Section 3 we introduce a well-adapted local trivialization of the spinor bundle, called the Bourguignon-Gauduchon-trivialization. In Section 4 we calculate the first terms of the Taylor development of the Dirac operator in this trivialization. In the following, i.e. in Section 5, we construct a good test spinor using a Killing spinor on $\mathbb{S}^n$, and then in Section 6, we set this spinor in the functional to get Theorem 1.1 in dimension $n \geq 3$. In the last section, i.e. in Section 7, we describe the modifications for the case $n = 2$ and prove the proposition.

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2. A variational formulation for the spin conformal invariant

For a section $\psi \in \Gamma(\Sigma M)$ we define

$$J(\psi) = \frac{\left(\int_M |D\psi|^{\frac{2n}{n+1}} v_g\right)^{\frac{n+1}{n}}}{\int_M \langle D\psi, \psi\rangle v_g}.$$

At some places we will wirte J_g instead of J inorder to indicate, that the functional is defined with respect to g. Based on some idea from [Lot86], Ammann proved in [Am03a] that

$$\lambda^+_{\min}(M,g,\sigma) = \inf_{\psi} J(\psi) \tag{2}$$

where the infimum is taken over the set of smooth spinor fields for which

$$\left(\int_M \langle D\psi, \psi\rangle v_g\right) > 0.$$

Hence, to prove Theorem 1.1, we are reduced to find a smooth spinor field ψ satisfying the condition below and such that $J(\psi) \leq \lambda^+_{\min}(\mathbb{S}^n) + \varepsilon$ where $\varepsilon > 0$ is arbitrary small.

3. The Bourguignon-Gauduchon-trivialization

As already explained before, the proof of our main theorem is based on a the construction of a suitable test spinor. We first construct a "good" spinor field of $\mathbb{R}^n$ and then transpose it on the manifold. In order to carry this out, we need to locally identify spinor fields on $(\mathbb{R}^n, g_{\mathrm{eucl}})$ and spinor fields on (M,g). Such an identification will be provided by a well-adapted local trivialization of the spinor bundle of $\Sigma(M,g)$.

If a spin manifold N carries two metrics g_1 and g_2, then it is a priori unclear how to identify spinors on (N,g_1) and spinors on (N,g_2). Bourguignon and Gauduchon [BG92] constructed a convenient map from the spinor bundle of (N,g_1) to the spinor bundle of (N,g_2) that allows us to identify spinors, and it is this identification that will provide the necessary identification to us. The trivialization will be called *Bourguignon-Gauduchon-trivialization.*

This trivialization is more efficient than the commonly used "trivialization by parallel transport along radial geodesics": with respect to the Bourguignon-Gauduchon-trivialization less terms appear in the Taylor expansion in Section 4.

Let (M,g) be a Riemannian manifold with a spin structure $\sigma : \mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)$. Let $(x_1, \dots x_n)$ be the Riemannian normal coordinates given by the exponential map at $p \in M$:

$$\begin{array}{rcl} \exp_p : U \subset T_pM \cong \mathbb{R}^n & \longrightarrow & V \subset M \\ (x_1, \dots, x_n) & \longmapsto & m \end{array}$$

Let

$$\begin{array}{rcl} G: V & \longrightarrow & S^2_+(n,\mathbb{R}) \\ m & \longmapsto & G_m := (g_{ij}(m))_{ij} \end{array}$$

denote the smooth map which associates to any point $m \in V$, the matrix of the coefficients of the metric g at this point, expressed in the basis $(\partial_i := \frac{\partial}{\partial x^i})_{1\leq i\leq n}$. Since G_m is symmetric and positive definite, there is a unique symmetric and positive definite matrix B_m such that

$$B_m^2 = G_m^{-1} .$$

Since

$$^t(B_m X) G_m (B_m Y) = g_{\mathrm{eucl}}(X,Y) , \quad \forall X, Y \in \mathbb{R}^n ,$$

where g_{eucl} stands for the Euclidean scalar product, we get the following isometry defined by

$$\begin{array}{rcl} B_m : (T_{\exp_p^{-1}(m)} U \cong \mathbb{R}^n, g_{\mathrm{eucl}}) & \longrightarrow & (T_m V, g_m) \\ (a^1, \dots, a^n) & \longmapsto & \displaystyle\sum_{i,j} b_i^j(m) a^i \partial_j(m) \end{array}$$

for each point $m \in V$, where $b_i^j(m)$ are the coefficients of the matrix B_m (from now on, we use Einstein's summation convention). As the matrix B_m depends smoothly on m, we can identify the following SO_n-principal bundles:

$$\begin{tikzcd} \mathrm{SO}(U, g_{\mathrm{eucl}}) \arrow[r, "\eta"] \arrow[d] & \mathrm{SO}(V, g) \arrow[d] \\ U \subset T_p M \arrow[r, "\exp_p"] & V \subset M \end{tikzcd}$$

where η is given by the action of B on each component vector of a frame in $\mathrm{SO}(U, g_{\mathrm{eucl}})$. The map η commutes with the right action of SO_n, therefore the map η can be lifted to the spin structures

$$\begin{tikzcd} \mathrm{Spin}(n) \times U = \mathrm{Spin}(U, g_{\mathrm{eucl}}) \arrow[r, "\bar\eta"] \arrow[d] & \mathrm{Spin}(V, g) \subset \mathrm{Spin}(M, g) \arrow[d] \\ U \subset T_p M \arrow[r, "\exp_p"] & V \subset M \end{tikzcd}$$

Hence, we obtain a map between the spinor bundles ΣU and ΣV in the following way:

$$\begin{array}{rcl} \Sigma U = \mathrm{Spin}(U, g_{\mathrm{eucl}}) \times_\rho \Sigma_n & \longrightarrow & \Sigma V = \mathrm{Spin}(V, g) \times_\rho \Sigma_n \\ \psi = [s, \varphi] & \longmapsto & \overline{\psi} = [\bar\eta(s), \varphi] \end{array} \tag{3}$$

where (ρ, Σ_n) is the complex spinor representation, and where $[s, \varphi]$ denotes the equivalence class of (s, φ) under the diagonal action of $\mathrm{Spin}(n)$.

We now define

$$e_i := b_i^j \partial_j ,$$

so that $(e_1, \dots, e_n)$ is an orthonormal frame of (TV, g). Denote by ∇ (resp. $\bar\nabla$) the Levi-Civita connection on (TU, g_{eucl}) (resp. (TM, g)) as well as its lift to the spinor bundle ΣU (resp. ΣV). The Christoffel symbols of the second kind $\widetilde{\Gamma}_{ij}^k$ are defined by

$$\widetilde{\Gamma}_{ij}^k := \langle \bar\nabla_{e_i} e_j, e_k \rangle ,$$

hence $\widetilde{\Gamma}_{ij}^k = -\widetilde{\Gamma}_{ik}^j$.

Remark 3.1. To distinguish the Clifford multiplications on these two spinor bundles, one should use different notations (for instance $\cdot$ and $\tilde{\cdot}$) but in the rest of the paper, we prefer to write $\cdot$ in both cases to make the paper easier to read. With this convention, if $\psi \in \Sigma_x U$ for some x in U, we have

$$e_i \cdot \overline{\psi} = \overline{\partial_i \cdot \psi}. \tag{4}$$

Proposition 3.2. *If D and $\bar{D}$ denote the Dirac operators acting respectively on $\Gamma(\Sigma U)$ and $\Gamma(\Sigma V)$, then we have*

$$\bar{D}\bar{\psi} = \overline{D\psi} + \mathbf{W} \cdot \bar{\psi} + \mathbf{V} \cdot \bar{\psi} + \sum_{ij} (b_i^j - \delta_i^j) \overline{\partial_i \cdot \nabla_{\partial j} \psi}\ , \tag{5}$$

where $\mathbf{W} \in \Gamma(\mathrm{Cl}\ TV)$ and $\mathbf{V} \in \Gamma(TV)$ are defined by

$$\mathbf{W} = \frac{1}{4} \sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}} b_i^r (\partial_r b_j^l)(b^{-1})_l^k e_i \cdot e_j \cdot e_k \tag{6}$$

and

$$\mathbf{V} = \frac{1}{4} \sum_{i,k} \left(\widetilde{\Gamma}_{ik}^i - \widetilde{\Gamma}_{ii}^k \right) e_k = \frac{1}{2} \sum_{i,k} \widetilde{\Gamma}_{ik}^i\, e_k \tag{7}$$

where, for any point $m \in V$, and the coefficients of the inverse matrix of B_m are denoted by $(b^{-1})_l^k(m)$.

Proof. For all spinor field $\psi \in \Gamma(\Sigma U)$, since $\bar{\psi} \in \Gamma(\Sigma V)$ and by definition of $\bar{\nabla}$ (see e.g. [LM89, Theorem 4.14], [Bär91, I Lemma 4.1]), we have

$$\bar{\nabla}_{e_i} \bar{\psi} = \overline{\nabla_{e_i}(\psi)} + \frac{1}{4} \sum_{j,k} \widetilde{\Gamma}_{ij}^k\, e_j \cdot e_k \cdot \bar{\psi}\ . \tag{8}$$

Taking Clifford multiplication by e_i on each member of (8) and summing over i yields

$$\bar{D}\bar{\psi} = \sum_i e_i \cdot \overline{\nabla_{e_i} \psi} + \frac{1}{4} \sum_{i,j,k} \widetilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \bar{\psi}\ .$$

Now, using that $e_i = \sum_j b_i^j \partial_j$ a and that by (4), $e_i \cdot \overline{\nabla_{e_i} \psi} = \overline{\partial_i \cdot \nabla_{e_i} \psi}$, we obtain that

$$\bar{D}\bar{\psi} = \sum_{ij} b_i^j \overline{\partial_i \cdot \nabla_{\partial_j} \psi} + \frac{1}{4} \sum_{i,j,k} \widetilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \bar{\psi}$$

and hence,

$$\bar{D}\bar{\psi} = \overline{D\psi} + \sum_{ij} (b_i^j - \delta_i^j) \overline{\partial_i \cdot \nabla_{\partial_j} \psi} + \frac{1}{4} \sum_{i,j,k} \widetilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \bar{\psi}.$$

See also [Pfa02] for a similar formula, worked out in more detail.

Note that by the definition of e_k, we have

$$\widetilde{\Gamma}_{ij}^k e_k = \widetilde{\Gamma}_{ij}^k b_k^l \partial_l\ .$$

On the other hand, we compute the Christoffel symbols of the second kind

$$\widetilde{\Gamma}_{ij}^k e_k = \bar{\nabla}_{e_i} e_j = b_i^r \bar{\nabla}_{\partial_r} (b_j^s \partial_s) = b_i^r (\partial_r b_j^s) \partial_s + b_i^r b_j^s \Gamma_{rs}^l \partial_l\ ,$$

where as usually the Christoffel symbols of the first kind Γ_{rs}^l are defined by

$$\Gamma_{rs}^l \partial_l = \bar{\nabla}_{\partial_r} \partial_s\ .$$

Therefore we have

$$\widetilde{\Gamma}_{ij}^k b_k^l = b_i^r (\partial_r b_j^l) + b_i^r b_j^s \Gamma_{rs}^l\ ,$$

and hence

$$\widetilde{\Gamma}_{ij}^k = \left(b_i^r (\partial_r b_j^l) + b_i^r b_j^s \Gamma_{rs}^l \right) (b^{-1})_l^k\ . \tag{9}$$

Now, we can write

$$\frac{1}{4}\sum_{i,j,k}\widetilde{\Gamma}_{ij}^{k} e_i \cdot e_j \cdot e_k = \mathbf{W} + \mathbf{V}$$

where $\mathbf{W} \in \Gamma(\Lambda^3 TV)$ and $\mathbf{V} \in \Gamma(TV)$ are defined by

$$\mathbf{W} = \frac{1}{4}\sum_{\substack{i,j,k \\ i\neq j\neq k\neq i}} \widetilde{\Gamma}_{ij}^{k} e_i \cdot e_j \cdot e_k$$

and

$$\begin{aligned}
\mathbf{V} &= \frac{1}{4}\Big(\sum_{i=j\neq k}\widetilde{\Gamma}_{ij}^{k} e_i\cdot e_j\cdot e_k + \overbrace{\sum_{i\neq j=k}\widetilde{\Gamma}_{ij}^{k} e_i\cdot e_j\cdot e_k}^{=0} + \sum_{j\neq i=k}\widetilde{\Gamma}_{ij}^{k} e_i\cdot e_j\cdot e_k + \overbrace{\sum_{i=j=k}\widetilde{\Gamma}_{ij}^{k} e_i\cdot e_j\cdot e_k}^{=0}\Big)\\
&= \frac{1}{4}\sum_{i,k}\left(\widetilde{\Gamma}_{ik}^{i} - \widetilde{\Gamma}_{ii}^{k}\right)e_k
\end{aligned}$$

which is (7).

First note that by (9) we have

$$\mathbf{W} = \frac{1}{4}\sum_{\substack{i,j,k \\ i\neq j\neq k\neq i}}\left(b_i^r(\partial_r b_j^l)(b^{-1})_l^k + b_i^r b_j^s \Gamma_{rs}^l (b^{-1})_l^k\right) e_i\cdot e_j\cdot e_k \,.$$

However,

$$\sum_{\substack{i,j,k \\ i\neq j\neq k\neq i}} b_i^r b_j^s \Gamma_{rs}^l (b^{-1})_l^k \; e_i\cdot e_j\cdot e_k = 0$$

since $\Gamma_{rs}^l = \Gamma_{sr}^l$ and $e_i \cdot e_j = -e_j \cdot e_i$. Therefore we obtain (6). $\square$

4. Development of the metric at the point p

In this section we give the development of the coefficients $\widetilde{\Gamma}_{ij}^{k}$ in the coordinates $(x_1, \ldots, x_n)$ at the fixed point $p \in M$.

For any point $m \in M$, r denotes the distance from p to m. Recall that in the neighborhood of p, we have the following development of the metric g (see for example [LP87]):

$$\begin{aligned}
g_{ij} &= \delta_{ij} + \frac{1}{3}R_{i\alpha\beta j}(p)x^\alpha x^\beta + \frac{1}{6}R_{i\alpha\beta j;\gamma}(p)x^\alpha x^\beta x^\gamma \\
&\quad + \left(\frac{1}{20}R_{i\alpha\beta j;\gamma\lambda}(p) + \frac{2}{45}\sum_m R_{i\alpha\beta m}(p)R_{j\gamma\lambda m}(p)\right)x^\alpha x^\beta x^\gamma x^\lambda + O(r^5)
\end{aligned} \tag{10}$$

where

$$R_{ijkl} = \langle \nabla_{e_j}\nabla_{e_i} e_k, e_l\rangle - \langle \nabla_{e_i}\nabla_{e_j} e_k, e_l\rangle - \langle \nabla_{[e_j,e_i]} e_k, e_l\rangle$$

and where

$$R_{ijkl;m} = (\nabla R)_{mijkl} \qquad\qquad R_{ijkl;mn} = (\nabla^2 R)_{nmijkl}$$

are the covariant derivatives of R_{ijkl} in direction of e_m (and e_p). Therefore we write

$$G_m = \mathrm{Id} + G_2 + G_3 + O(r^4)$$

with

$$\Big(G_2\Big)_{ij} = \frac{1}{3}R_{i\alpha\beta j}(p)x^\alpha x^\beta$$

and

$$\Big(G_3\Big)_{ij} = \frac{1}{6}R_{i\alpha\beta j;\gamma}(p)x^\alpha x^\beta x^\gamma$$

Writing

$$B_m = \mathrm{Id} + B_1 + B_2 + B_3 + O(r^4)$$

with

$$\Big(B_1\Big)_{ij} = B_{ij\alpha}x^\alpha \ ,$$

$$\Big(B_2\Big)_{ij} = B_{ij\alpha\beta}x^\alpha x^\beta$$

and

$$\Big(B_3\Big)_{ij} = B_{ij\alpha\beta\gamma}x^\alpha x^\beta x^\gamma \ ,$$

the relation $B_m^2 G_m = \mathrm{Id}$ yields $B_1 = 0$ and

$$0 = \Big(2B_2 + G_2\Big) + \Big(2B_3 + G_3\Big) \ ,$$

hence

$$\left\{ \begin{array}{rcl} b_i^j &=& \delta_i^j - \frac{1}{6}R_{i\alpha\beta j}x^\alpha x^\beta - \frac{1}{12}R_{i\alpha\beta j;\gamma}x^\alpha x^\beta x^\gamma + O(r^4) \\ (b^{-1})_i^j &=& \delta_i^j + \frac{1}{6}R_{i\alpha\beta j}x^\alpha x^\beta + \frac{1}{12}R_{i\alpha\beta j;\gamma}x^\alpha x^\beta x^\gamma + O(r^4) \end{array} \right. \tag{11}$$

We also have

$$\partial_l b_i^j = -\frac{1}{6}\Big(R_{il\alpha j} + R_{i\alpha lj}\Big)x^\alpha - \frac{1}{12}\Big(R_{il\alpha j;\beta} + R_{i\alpha lj;\beta} + R_{i\alpha\beta j;l}\Big)x^\alpha x^\beta + O(r^3) \ . \tag{12}$$

4.1. **Development of Γ_{ij}^k, V and W.**

$$\begin{array}{rcl} \Gamma_{ij}^k &=& \frac{1}{2}g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big) \\ &=& \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\Big) + O(r^2) \\ &=& \frac{1}{6}\Big(R_{ji\alpha k} + R_{j\alpha ik} + R_{ij\alpha k} + R_{i\alpha jk} - R_{ik\alpha j} - R_{i\alpha kj}\Big)x^\alpha + O(r^2) \end{array}$$

Using the relations $R_{ij\alpha k} + R_{ji\alpha k} = 0$, $R_{j\alpha ik} - R_{ik\alpha j} = -2R_{ik\alpha j}$ and $R_{i\alpha jk} - R_{i\alpha kj} = -2R_{i\alpha kj}$ we then have

$$\Gamma_{ij}^k \quad = \quad -\frac{1}{3}\Big(R_{ik\alpha j} + R_{i\alpha kj}\Big)x^\alpha + O(r^2) \tag{13}$$

On the other hand, since $\partial_r b_j^l$ and Γ_{rs}^l have no constant term, Formula (9) yields

$$\widetilde{\Gamma}_{ij}^k = \Big(\delta_i^r(\partial_r b_j^l) + \delta_i^r\delta_j^s\Gamma_{rs}^l\Big)\delta_l^k + O(r^2) \ ,$$

and hence

$$\widetilde{\Gamma}_{ij}^k = \partial_i b_j^k + \Gamma_{ij}^k + O(r^2) \ .$$

We have

$$\begin{array}{rcl} \mathbf{V} &=& \frac{1}{4}\sum_{i,k}\Big(\widetilde{\Gamma}_{ik}^i - \widetilde{\Gamma}_{ii}^k)e_k \\ &=& \frac{1}{4}\sum_{i,k}\Big(\partial_i b_k^i + \Gamma_{ik}^i - \partial_i b_i^k - \Gamma_{ii}^k\Big)e_k + O(r^2) \\ &=& \frac{1}{4}\sum_{i,k}\Big(\Gamma_{ik}^i - \Gamma_{ii}^k\Big)e_k + O(r^2) \end{array}$$

since $\partial_i b_k^i = \partial_i b_i^k$.

Moreover, we have

$$\begin{aligned}\sum_i \Big(\Gamma_{ik}^i - \Gamma_{ii}^k\Big) &= -\frac{1}{3}\sum_i \Big(R_{ii\alpha k} + R_{i\alpha ik}\Big)x^\alpha + \frac{1}{3}\sum_i \Big(R_{ik\alpha i} + R_{i\alpha ki}\Big)x^\alpha + O(r^2)\\ &= -(\mathrm{Ric})_{\alpha k} + O(r^2)\end{aligned}$$

Therefore we proved that

$$\boxed{\mathbf{V} = \Big(-\frac{1}{4}(\mathrm{Ric})_{\alpha k}\, x^\alpha + O(r^2)\Big)e_k\ .} \tag{14}$$

The aim now is to show that

$$\mathbf{W} = \frac{1}{4}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} b_i^r(\partial_r b_j^l)(b^{-1})_l^k e_i \cdot e_j \cdot e_k$$

is $O(r^3)$. First note that by Equations (11) and (12) b_i^r has no term of order 1 and $\partial_r b_j^l$ has no term of order 0. Hence, any term in $\mathbf{W}$ of order < 3 is a product of the 0-order term of b_i^r and of a term of order 1 or 2 of $\partial_r b_j^l$.

Therefore $\mathbf{W}$ has no term of order 0. To compute the terms of order 1 and 2, we write

$$\mathbf{W} = \frac{1}{4}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} \Big(\delta_i^r(\partial_r b_j^l)\delta_l^k + O(r^3)\Big)e_i \cdot e_j \cdot e_k\ .$$

We have

$$\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} \partial_i b_j^k e_i \cdot e_j \cdot e_k = 0$$

since

$$\partial_i b_j^k = \partial_i b_k^j \qquad \text{and} \qquad e_j \cdot e_k = -e_k \cdot e_j\ .$$

Therefore $\mathbf{W}$ has no term of order 1 and 2. We proved that

$$\mathbf{W} = O(r^3) \tag{15}$$

Remark 4.1. Similar calculations yield

$$\boxed{\mathbf{V} = -\Big(\frac{1}{4}(\mathrm{Ric})_{\alpha k}\, x^\alpha + \frac{1}{6}(\mathrm{Ric})_{\alpha k,\beta}\, x^\alpha x^\beta + O(r^3)\Big)e_k\ .}$$

$$\boxed{\mathbf{W} = -\frac{1}{144}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} R_{l\beta\gamma k}\Big(R_{ji\alpha l} + R_{jl\alpha i}\Big)\, x^\alpha x^\beta x^\gamma\, e_i \cdot e_j \cdot e_k + O(r^4)\ .}$$

We do not give details here because we do not need explicit computations of terms of order 2 for $\mathbf{V}$ and terms of order 3 for $\mathbf{W}$ in the proof of Theorem 1.1.

5. The test spinor

5.1. **The explicit spinor.** In this section we construct a good test spinor on $\mathbb{R}^n$. The spinor bundle on $\mathbb{R}^n$ is trivial, so we can identify the fibers. Let $\psi_0 \in \Sigma_0\mathbb{R}^n$ with $|\psi_0| = 1$. We set $f(x) := \frac{2}{1+r^2}$, where $r := |x|$, hence $\partial_i f = -x_i f^2$. Then we define

$$\psi(x) = f^{\frac{n}{2}}(x)(1-x)\cdot\psi_0. \tag{16}$$

One calculates

$$\nabla_{\partial_i}\psi = -f^{\frac{n}{2}}\partial_i \cdot \psi_0 - \frac{n}{2}f^{\frac{n}{2}+1}x_i(1-x)\cdot\psi_0, \tag{17}$$

and hence

$$\begin{aligned} D\psi &= \frac{n}{2} f\psi && (18)\\ |\psi| &= f^{\frac{n-1}{2}} && (19)\\ |D\psi| &= \frac{n}{2} f^{\frac{n+1}{2}}. && (20) \end{aligned}$$

5.2. **Conformal change of metrics.** In order to explain a geometric interpretation of this spinor, we have to recall the behavior of spinors and the Dirac operators under conformal changes. See e.g. [Hit74, Hij01] for proofs.

Let (N, h) be a spin manifold of dimension n. Consider a conformal change of metric $\widetilde{h} = F^{-2}h$ for any positive real function F on (N, h). The map $TN \to TN$, $X \mapsto \tilde{X} = FX$ induces an isomorphism of principal bundles from $\mathrm{SO}(N, h)$ to $\mathrm{SO}(N, \widetilde{h})$. It lifts to a bundle isomorphism between the $\mathrm{Spin}(n)$-principal bundles $\mathrm{Spin}(N, h)$ and $\mathrm{Spin}(N, \widetilde{h})$, and passing to the associated bundles one obtains a map

$$\begin{array}{rcl} \Sigma_h N = \mathrm{Spin}(N, h) \times_\rho \Sigma & \to & \Sigma_{\widetilde{h}} N = \mathrm{Spin}(N, \widetilde{h}) \times_\rho \Sigma \\ \varphi & \mapsto & \widetilde{\varphi} \end{array}$$

between the spinor bundles, which is a fiberwise isometry and we have

$$\widetilde{X} \,\tilde{\cdot}\, \widetilde{\varphi} = \widetilde{X \cdot \varphi}$$

(see [Hij01] for more details on this construction).

By conformal covariance of the Dirac operator, we have, for $\varphi \in \Gamma(\Sigma N)$,

$$\widetilde{D}\Big(F^{\frac{n-1}{2}}\ \widetilde{\varphi} \Big) = F^{\frac{n+1}{2}}\ \widetilde{D\varphi}, \tag{21}$$

5.3. **Geometric interpretation.** We apply this formula to a particular case: let p be any point of the round sphere $\mathbb{S}^n$. Then $\mathbb{S}^n \backslash \{p\}$ is isometric to $\mathbb{R}^n$ with the metric

$$g_S = f^2 g_{\mathrm{eucl}}\ , \tag{22}$$

with

$$f(x) = \frac{2}{1 + r^2}\ .$$

Hence we set $N := \mathbb{R}^n$, $h = g_{\mathrm{eucl}}$, $F = f^{-1}$. One calculates with (18) and (21) that $\Phi := F^{\frac{n-1}{2}} \tilde{\psi}$ satisfies $D\Phi = \frac{n}{2}\Phi$ on $\mathbb{S}^n \setminus \{p\}$, and $|\Phi| = 1$. Hence, the possible singularity at p can be removed (see e.g. the Removal of singularity theorem [Am03c, Theorem 5.1]), and one sees that Φ is an eigenspinor to the eigenvalue $n/2$ on the round sphere $\mathbb{S}^n$. The equality discussion in Friedrich's inequality [Fri80] implies that Φ is a Killing spinor to the constant $-1/2$, i.e. it satisfies

$$\nabla_X \Phi = -\frac{1}{2} X \cdot \Phi.$$

Hence we have seen that our spinor ψ is the "conformal image" of a Killing spinor on $\mathbb{S}^n$.

6. The proof of Theorem 1.1 for $n \geq 3$

We begin with the following Proposition.

Proposition 6.1. *The metric g on M can be chosen such that*

$$\mathrm{Ric}_g(p) = 0 \qquad \textit{and} \qquad \Delta_g(\mathrm{Scal}_g)(p) = 0\ .$$

Proof. Consider a conformal change of the metric $\widetilde{g} = e^{2u}g$ for any real function u on (M,g). Then it is well known that the Ricci curvature $(2,0)$-tensor $\mathrm{Ric}_{\tilde{g}}$, the scalar curvature $\mathrm{Scal}_{\tilde{g}}$ and the Laplacian $\Delta_{\tilde{g}}$ corresponding to the metric $\tilde{g}$ satisfy (see for example Hebey [Heb97] or Aubin [Aub76])

$$\begin{aligned}\mathrm{Ric}_{\tilde{g}} &= \mathrm{Ric}_g - (n-2)\nabla^2 u + (n-2)\nabla u \otimes \nabla u + (\Delta_g u - (n-2)|\nabla u|_g^2)g \ , \\ \mathrm{Scal}_{\tilde{g}} &= e^{-2u}\Big(\mathrm{Scal}_g + 2(n-1)\Delta_g u - (n-1)(n-2)|\nabla u|_g^2\Big) \ , \end{aligned} \tag{23}$$

As a first step, we can assume that $\mathrm{Scal}_g(p) = 0$. Then, let us choose u such that

$$u(x) = \frac{1}{2(n-2)}\Big(\mathrm{Ric}_g(p)_{ij} - \frac{\mathrm{Scal}_g(p)}{n} g_{ij}(p)\Big)\, x^i x^j - \frac{\Delta_g(\mathrm{Scal}_g)(p)}{48(n-1)}(x^1)^4$$

in a neighborhood of the point p. Since $u(p) = 0$ and $(\nabla u)(p) = 0$, it is straightforward to see that $\mathrm{Ric}_{\tilde{g}}(p) = 0$. Moreover, taking the Laplacian of both members of Equation (23), a simple computation shows that $\Delta_{\tilde{g}}\mathrm{Scal}_{\tilde{g}}(p) = 0$. □

Let $\bar{\varphi} \in \Sigma_U M$ where U is the open neighborhood of a point $p \in M$ as defined in the previous sections. With the help of formulas (14) and (15), we have the following

Corollary 6.2. *For any metric g on M chosen as in Proposition 6.1, we have*

$$\bar{D}\bar{\varphi} = \overline{D\varphi} + \sum_{\substack{ijk\alpha\beta\gamma \\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma} x^\alpha\, x^\beta\, x^\gamma\, e_i \cdot e_j \cdot e_k \cdot \bar{\varphi} + \mathbf{W}' \cdot \bar{\varphi} + \mathbf{V} \cdot \bar{\varphi} + \sum_{ij}(b_i^j - \delta_i^j)\overline{\partial_i \cdot \nabla_{\partial j}\psi} \tag{24}$$

where $A_{ijk\alpha\beta\gamma} \in \mathbb{R}$ and where $\mathbf{W}' \in \Gamma(\Lambda^3 TV)$, $\mathbf{V} \in \Gamma(TV)$, $|\mathbf{W}'| \leq C\, r^4$ and $|\mathbf{V}| \leq C'\, r^2$ (C and C' being positive constants independent of φ).

Remark 6.3. Using the formulae in Remark 4.1, we obtain the formula

$$A_{ijk\alpha\beta\gamma} = -\frac{1}{144} R_{l\beta\gamma k}\Big(R_{ji\alpha l} + R_{jl\alpha i}\Big) \ ,$$

Assume now that ψ is the test spinor constructed in Section 5. Let $\varepsilon > 0$ be a small positive number. We set

$$\varphi(x) := \eta\psi(\frac{x}{\varepsilon}) =: \psi_\varepsilon(x)$$

where $\eta = 0$ on $\mathbb{R}^n \setminus B_p(2\delta)$ and $\eta = 1$ on $B_p(\delta)$, and that ψ, defined as in (16) satisfies the following relations (17), (18), (19) and (20) where f is again defined by

$$f(x) = \frac{2}{1+r^2} \ .$$

We now prove some lemmas which will be useful in the proof of Theorem 1.1.

Lemma 6.4. *We have*

$$\left|\sum_{ij}(b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}(\psi(\frac{x}{\varepsilon}))\right| \leq C\frac{r^3}{\varepsilon} f^{\frac{n}{2}}(\frac{x}{\varepsilon}) \tag{25}$$

where $f = \frac{2}{1+r^2}$;.

Proof. At first, we prove that:

$$\sum_{ij\alpha\beta} R_{i\alpha\beta j} x^\alpha x^\beta \partial_i \cdot \Big(\nabla_{\partial_j}\psi(\frac{x}{\varepsilon})\Big) = 0. \tag{26}$$

Indeed, using (17), we compute that

$$(\nabla_{\partial_j}\psi)(\frac{x}{\varepsilon}) = -\frac{f^{\frac{n}{2}}(\frac{x}{\varepsilon})}{\varepsilon}\partial_j \cdot \psi_0 - \frac{n f^{\frac{n+2}{2}}(\frac{x}{\varepsilon})}{2\varepsilon} x^j (1 - \frac{x}{\varepsilon}) \cdot \psi_0.$$

and obtain

$$\sum_{ij\alpha\beta} R_{i\alpha\beta j}x^\alpha x^\beta \partial_i\cdot(\nabla_{\partial_j}\psi)(\frac{x}{\varepsilon}) = -\frac{f^{\frac{n}{2}}(\frac{x}{\varepsilon})}{\varepsilon}\sum_{ij\alpha\beta} R_{i\alpha\beta j}x^\alpha x^\beta \partial_i\cdot\partial_j\cdot\psi_0 - \frac{nf^{\frac{n+2}{2}}(\frac{x}{\varepsilon})}{2\varepsilon}\sum_{ij\alpha\beta} R_{i\alpha\beta j}x^\alpha x^\beta x^j \partial_i\cdot(1-\frac{x}{\varepsilon})\cdot\psi_0.$$

Now, since if $i \neq j$, $\partial_i \cdot \partial_j = -\partial_j \cdot \partial_i$ and since

$$\sum_{\alpha\beta} R_{i\alpha\beta j}x^\alpha x^\beta = \sum_{\alpha\beta} R_{i\beta\alpha j}x^\alpha x^\beta = \sum_{\alpha\beta} R_{j\alpha\beta i}x^\alpha x^\beta,$$

(we have used that $R_{j\alpha\beta i} = R_{\beta ij\alpha} = R_{i\beta\alpha j}$), we get that

$$\sum_{ij\alpha\beta} R_{i\alpha\beta j}x^\alpha x^\beta \partial_i \cdot \partial_j \cdot \psi_0 = -\sum_{i,\alpha\beta} R_{i\alpha\beta i}x^\alpha x^\beta \psi_0 = 0$$

since $\mathrm{Ric}(p) = 0$. The first summand vanishes.
The second summand vanishes as $\sum_{\beta j} R_{i\alpha\beta j}x^\beta x^j = 0$.
This proves (26). Now, by the development of b_i^j (11), we easily obtain that

$$\left|\sum_{ij}(b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}(\psi(\frac{x}{\varepsilon}))\right| \leq C\frac{r^3}{\varepsilon}|\nabla\psi|(\frac{x}{\varepsilon}).$$

Differentiating expression (16), we see that

$$|\nabla\psi| \leq C(f^{\frac{n}{2}} + rf^{\frac{n+2}{2}}).$$

Together with $rf(r) = \frac{2r}{1+r^2} = 1 - \frac{(1-r)^2}{1+r^2} \leq 1$ we obtain the lemma. $\square$

Now, we can start the proof of Theorem 1.1. We have, with the notations of Corollary 6.2:

$$\begin{aligned}
\bar{D}\bar{\psi}_\varepsilon(x) &= \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \eta\,\bar{D}(\bar{\psi}(\frac{x}{\varepsilon}))\\
&= \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \frac{\eta}{\varepsilon}\overline{D\psi}(\frac{x}{\varepsilon}) + \eta\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\, x^\beta\, x^\gamma\, e_i\cdot e_j\cdot e_k\cdot\bar{\psi}(\frac{x}{\varepsilon})\\
&\quad +\eta\,\mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \eta\mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \eta\sum_{ij}(b_i^j - \delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))}\,.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\bar{D}\bar{\psi}_\varepsilon(x) &= \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \frac{\eta}{\varepsilon}\,\frac{n}{2}\,f(\frac{x}{\varepsilon})\,\bar{\psi}(\frac{x}{\varepsilon}) + \eta\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\, x^\beta\, x^\gamma\, e_i\cdot e_j\cdot e_k\cdot\bar{\psi}(\frac{x}{\varepsilon})\\
&\quad +\eta\,\mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \eta\mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}) + \eta\sum_{ij}(b_i^j - \delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))}\,.
\end{aligned}$$

We write that

$$\begin{aligned}
|\bar{D}\bar{\psi}_\varepsilon|^2(x) &= \mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV}+\mathbf{V}+\mathbf{VI}+\mathbf{VII}+\mathbf{VIII}+\mathbf{IX}+\mathbf{X}+\mathbf{XI}+\mathbf{XII}+\mathbf{XIII}+\mathbf{XIV}+\mathbf{XV}\\
&\quad +\mathbf{XVI}+\mathbf{XVII}+\mathbf{XVIII}+\mathbf{XIX}+\mathbf{XX}+\mathbf{XXI}
\end{aligned}$$

where

$$
\begin{aligned}
\mathbf{I} &= |\bar{\nabla}\eta|^2\,|\bar{\psi}|^2(\frac{x}{\varepsilon})\\
\mathbf{II} &= \frac{\eta^2}{\varepsilon^2}\,\frac{n^2}{4}\,|\bar{\psi}|^2(\frac{x}{\varepsilon})\,f^2(\frac{x}{\varepsilon})\\
\mathbf{III} &= \eta^2|\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\,x^\beta\,x^\gamma\,e_i\cdot e_j\cdot e_k\cdot\bar{\psi}|^2(\frac{x}{\varepsilon})\\
\mathbf{IV} &= \eta^2\,|\mathbf{W}'|^2|\bar{\psi}(\frac{x}{\varepsilon})|^2\\
\mathbf{V} &= \eta^2|\mathbf{V}|^2|\bar{\psi}(\frac{x}{\varepsilon})|^2\\
\mathbf{VI} &= 2\,\Re e < \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}),\frac{\eta}{\varepsilon}\,\frac{n}{2}\,f(\frac{x}{\varepsilon})\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{VII} &= 2\,\Re e < \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}),\eta\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\,x^\beta\,x^\gamma\,e_i\cdot e_j\cdot e_k\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{VIII} &= 2\,\Re e < \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}),\eta\,\mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{IX} &= 2\,\Re e < \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}),\eta\,\mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{X} &= \frac{\eta^2}{\varepsilon}\,n\,f(\frac{x}{\varepsilon})\eta\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\,x^\beta\,x^\gamma\,\Re e < e_i\cdot e_j\cdot e_k\cdot\bar{\psi},\bar{\psi} > (\frac{x}{\varepsilon})\\
\mathbf{XI} &= \frac{\eta^2}{\varepsilon}\,n\,f(\frac{x}{\varepsilon})\Re e < \bar{\psi}(\frac{x}{\varepsilon}),\mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{XII} &= \frac{\eta^2}{\varepsilon}\,n\,f(\frac{x}{\varepsilon})\Re e < \bar{\psi}(\frac{x}{\varepsilon}),\mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{XIII} &= 2\,\eta^2\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\,x^\beta\,x^\gamma\,\Re e < e_i\cdot e_j\cdot e_k\cdot\bar{\psi}(\frac{x}{\varepsilon}),\mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{XIV} &= 2\,\eta^2\sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\,x^\beta\,x^\gamma\,\Re e < e_i\cdot e_j\cdot e_k\cdot\bar{\psi}(\frac{x}{\varepsilon}),\mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{XV} &= 2\eta^2\Re e < \mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}),\mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}) >\\
\mathbf{XVI} &= 2\Re e < \bar{\nabla}\eta\cdot\bar{\psi}(\frac{x}{\varepsilon}),\eta\sum_{ij}(b_i^j-\delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))} >\\
\mathbf{XVII} &= \frac{n\eta^2}{\varepsilon}f(\frac{x}{\varepsilon})\Re e < \bar{\psi}(\frac{x}{\varepsilon}),\sum_{ij}(b_i^j-\delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))} >\\
\mathbf{XVIII} &= 2\eta^2\Re e < \sum_{\substack{ijk\alpha\beta\gamma\\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma}x^\alpha\,x^\beta\,x^\gamma\,e_i\cdot e_j\cdot e_k\cdot\bar{\psi}(\frac{x}{\varepsilon}),\sum_{ij}(b_i^j-\delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))} >\\
\mathbf{XIX} &= 2\eta^2\,\Re e < \mathbf{W}'\cdot\bar{\psi}(\frac{x}{\varepsilon}),\sum_{ij}(b_i^j-\delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))} >\\
\mathbf{XX} &= 2\eta^2\,\Re e < \mathbf{V}\cdot\bar{\psi}(\frac{x}{\varepsilon}),\sum_{ij}(b_i^j-\delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))} >\\
\mathbf{XXI} &= \eta^2\left|\sum_{ij}(b_i^j-\delta_i^j)\overline{\partial_i\cdot\nabla_{\partial j}(\psi(\frac{x}{\varepsilon}))}\right|^2.
\end{aligned}
$$

Since $\mathbf{V}$ is a vector field, we have

$$\mathbf{XII} = 0$$

Assume now that $x \in B_p(2\delta)$. Using the fact that $|\bar{\nabla}\eta| \leq Cr^4$ (C being a constant independent of ε) and since $r \leq \delta \leq 1$, we have:

$$\mathbf{I} + \mathbf{III} + \mathbf{IV} + \mathbf{V} + \mathbf{VII} + \mathbf{VIII} + \mathbf{IX} + \mathbf{XIII} + \mathbf{XIV} + \mathbf{XV} \leq C\, r^4\, f^{n-1}(\frac{x}{\varepsilon}) \, .$$

and

$$\mathbf{VI} + \mathbf{XI} \leq \frac{C}{\varepsilon}\, r^4\, f^n(\frac{x}{\varepsilon}) \, .$$

Since $f \leq 2$ and since $r^2 \leq C$ on $B_p(2\delta)$, we obtain that

$$\mathbf{VI} + \mathbf{XI} \leq C\frac{r^2}{\varepsilon}\, f^{n-\frac{1}{2}}(\frac{x}{\varepsilon}) \, .$$

In the same way, using relation (25) and the fact that for all ε, $\frac{r}{\varepsilon} f(\frac{x}{\varepsilon}) \leq 1$, we have also

$$\mathbf{X} + \mathbf{XVI} + \mathbf{XVII} + \mathbf{XVIII} + \mathbf{XIX} + \mathbf{XX} + \mathbf{XXI} \leq C\frac{r^2}{\varepsilon}\, f^{n-\frac{1}{2}}(\frac{x}{\varepsilon}) \, .$$

Therefore we obtain that

$$\begin{aligned} |\bar{D}\bar{\psi}_\varepsilon|^2(x) &\leq \frac{n^2}{4\varepsilon^2} f^{n+1}(\frac{x}{\varepsilon}) + C\, r^4\, f^{n-1}(\frac{x}{\varepsilon}) + \frac{C}{\varepsilon}\, r^2\, f^{n-\frac{1}{2}}(\frac{x}{\varepsilon}) \\ &\leq \frac{n^2}{4\varepsilon^2} f^{n+1}(\frac{x}{\varepsilon})\, [1 + \Delta] \end{aligned}$$

where

$$\Delta = C\, \varepsilon^2 r^4\, f^{-2}(\frac{x}{\varepsilon}) + C\varepsilon\, r^2\, f^{-\frac{3}{2}}(\frac{x}{\varepsilon}).$$

Since $|\bar{D}\bar{\psi}_\varepsilon|^2 \geq 0$ we have $\Delta \geq -1$. Moreover, if we define

$$g(x) = 1 + \frac{n}{n+1}x - (1+x)^{\frac{n}{n+1}} \, , \qquad \forall x \geq -1 \, ,$$

then

$$g'(x) = \frac{n}{n+1}\Big(1 - (1+x)^{\frac{-1}{n+1}}\Big) \, , \qquad \forall x > -1 \, .$$

Therefore g admits a minimum at 0 on the interval $[-1, +\infty[$. This yields that, $\forall x \geq -1$,

$$(1+x)^{\frac{n}{n+1}} \leq 1 + \frac{n}{n+1}x \, .$$

We then have

$$|\bar{D}\bar{\psi}_\varepsilon|^{\frac{2n}{n+1}}(x) \leq (\frac{n}{2\varepsilon})^{\frac{2n}{n+1}}\, f^n(\frac{x}{\varepsilon})\, [1+\Delta]^{\frac{n}{n+1}} \leq (\frac{n}{2\varepsilon})^{\frac{2n}{n+1}}\, f^n(\frac{x}{\varepsilon}) + \frac{n}{n+1}\, (\frac{n}{2\varepsilon})^{\frac{2n}{n+1}}\, f^n(\frac{x}{\varepsilon})\, \Delta \, .$$

Taking into account the definition of Δ and integrating over M leads to

$$\int_M |\bar{D}\bar{\psi}_\varepsilon|^{\frac{2n}{n+1}} \mathrm{d}v_g \leq \varepsilon^{\frac{-2n}{n+1}}\, [\mathbf{A} + \mathbf{B} + \mathbf{C}] \, , \tag{27}$$

where

$$\begin{aligned} \mathbf{A} &= \int_{B_p(2\delta)} \Big(\frac{n}{2}\Big)^{\frac{2n}{n+1}} f^n(\frac{x}{\varepsilon})\, \mathrm{d}v_g \\ \mathbf{B} &= C\int_{B_p(2\delta)} \varepsilon^2 r^4\, f^{n-2}(\frac{x}{\varepsilon})\, \mathrm{d}v_g \\ \mathbf{C} &= C\int_{B_p(2\delta)} \varepsilon\, r^2\, f^{n-\frac{3}{2}}(\frac{x}{\varepsilon})\, \mathrm{d}v_g. \end{aligned}$$

Since the function f is radially symmetric, we can compute $\mathbf{A}$ with the help of spherical coordinates:

$$\mathbf{A} = \int_{B_p(2\delta)} \left(\frac{n}{2}\right)^{\frac{2n}{n+1}} f^n(\frac{x}{\varepsilon})\, \omega_{n-1}\, G(r)\, r^{n-1}\mathrm{d}r \ ,$$

where ω_{n-1} stands for the volume of the unit sphere $\mathbb{S}^{n-1}$ and

$$G(r) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \sqrt{|g|_{rx}}\, \mathrm{d}\sigma(x) \qquad\qquad |g|_y := \det g_{ij}(y).$$

From Proposition 6.1, Hebey [Heb97] or Lee-Parker [LP87], we know that

$$G(r) \leq 1 + O(r^4)\ .$$

Therefore, we can estimate $\mathbf{A}$ in the following way:

$$\begin{aligned}
\mathbf{A} \ &\leq\ \left(\frac{n}{2}\right)^{\frac{2n}{n+1}} \omega_{n-1} \left[\int_0^{2\delta} f^n(\frac{x}{\varepsilon})\, r^{n-1}\mathrm{d}r + C\int_0^{2\delta} f^n(\frac{x}{\varepsilon})\, r^{n+3}\mathrm{d}r\right] \\
&\leq\ \left(\frac{n}{2}\right)^{\frac{2n}{n+1}} \omega_{n-1}\, \varepsilon^n \left[\int_0^{\frac{2\delta}{\varepsilon}} \frac{2^n r^{n-1}}{(1+r^2)^n}\mathrm{d}r + C\varepsilon^4 \int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{n+3}}{(1+r^2)^n}\mathrm{d}r\right].
\end{aligned}$$

Since

$$\int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{n+3}}{(1+r^2)^n}\mathrm{d}r \leq O\left(\int_1^{\frac{2\delta}{\varepsilon}} r^{3-n}\mathrm{d}r\right)$$

we get that

$$\mathbf{A} \leq \left(\frac{n}{2}\right)^{\frac{2n}{n+1}} \omega_{n-1}\, \varepsilon^n \left[\int_0^{\frac{2\delta}{\varepsilon}} \frac{2^n r^{n-1}}{(1+r^2)^n}\mathrm{d}r + o(1)\right]\ ,$$

and hence

$$\mathbf{A} \leq \left(\frac{n}{2}\right)^{\frac{2n}{n+1}} \omega_{n-1}\, \varepsilon^n \left[\int_0^{\frac{2\delta}{\varepsilon}} \frac{2^n r^{n-1}}{(1+r^2)^n}\mathrm{d}r + o(1)\right]\ . \tag{28}$$

Let us show that

$$\mathbf{B} = o(\varepsilon^n). \tag{29}$$

Since $dv_g \leq Cdx$, setting $y = \frac{x}{\varepsilon}$, we have

$$\begin{aligned}
\int_{B_p(2\delta)} r^4 f^{n-2}(\frac{x}{\varepsilon})\, dv_g \ &\leq\ C\varepsilon^{n+4} \int_{B_p(\frac{2\delta}{\varepsilon})} r^4 f^{n-2}((\frac{y}{\varepsilon})\, dy \\
&\leq\ C\,\varepsilon^{n+4} \int_0^{\frac{2\delta}{\varepsilon}} \frac{r^{n+3}}{(1+r^2)^{n-2}}\, dr. \\
&\leq\ C\,\varepsilon^{n+4} O\left(\int_1^{\frac{2\delta}{\varepsilon}} r^{7-n}\, dr\right).
\end{aligned}$$

It is easy to check that relation (29) follows if $n \geq 3$. In the same way, we can prove that $\mathbf{C} = o(\varepsilon^n)$. Together with Equation (28), we can conclude that

$$\int_M |\bar{D}\bar{\psi}_\varepsilon|^{\frac{2n}{n+1}}\mathrm{d}v_g \quad\leq\quad \varepsilon^{\frac{-2n}{n+1}+n} \left[\left(\frac{n}{2}\right)^{\frac{2n}{n+1}} \omega_{n-1} \int_0^{+\infty} r^{n-1} f^n(r)\mathrm{d}r + o(1)\right]\ ,$$

which yields

$$\left(\int_M |\bar{D}\bar{\psi}_\varepsilon|^{\frac{2n}{n+1}}\mathrm{d}v_g\right)^{\frac{n+1}{n}} \leq \varepsilon^{n-1} \left[\left(\frac{n^2}{4}\right)^{\frac{n}{n+1}} \omega_{n-1} I\right]^{\frac{n+1}{n}} (1+o(1))\ , \tag{30}$$

where

$$I = \int_0^{+\infty} \frac{2^n r^{n-1}}{(1+r^2)^n}\mathrm{d}r\ .$$

We are now going to estimate $\left|\int_M \Re e < \bar{D}\bar{\psi}_\varepsilon, \bar{\psi}_\varepsilon > \mathrm{d}v_g\right|$. We start by computing

$$\left|\int_M \Re e < \bar{D}\bar{\psi}_\varepsilon, \bar{\psi}_\varepsilon > \mathrm{d}v_g\right| \geq \mathbf{A}' - \mathbf{B}' - \mathbf{C}' - \mathbf{D}' - \mathbf{E}' \,,$$

where

$$\begin{aligned}
\mathbf{A}' &= \int_{B_p(\delta)} \frac{n}{2\varepsilon} f^n(\frac{x}{\varepsilon}) \mathrm{d}v_g \\
\mathbf{B}' &= \left|\int_M \Re e < \bar{\nabla}\eta \cdot \bar{\psi}(\frac{x}{\varepsilon}), \eta\bar{\psi}(\frac{x}{\varepsilon}) > \mathrm{d}v_g\right| \,, \\
\mathbf{C}' &= \left|\int_M \eta^2 \sum_{\substack{ijk\alpha\beta\gamma \\ i\neq j\neq k\neq i}} A_{ijk\alpha\beta\gamma} x^\alpha\, x^\beta\, x^\gamma\, \Re e < e_i \cdot e_j \cdot e_k \cdot \bar{\psi}, \bar{\psi} > (\frac{x}{\varepsilon})\, \mathrm{d}v_g\right| \,, \\
\mathbf{D}' &= \left|\int_M \eta^2 \Re e < \mathbf{W}'\bar{\psi}(\frac{x}{\varepsilon}), \bar{\psi}(\frac{x}{\varepsilon}) >\ \mathrm{d}v_g\right| \\
\mathbf{E}' &= \left|\int_M \eta^2 \Re e < \sum_{ij} (b_i^j - \delta_i^j) \overline{\partial_i \cdot \nabla_{\partial_j}(\psi(\frac{x}{\varepsilon}))}, \bar{\psi}(\frac{x}{\varepsilon}) >\ \mathrm{d}v_g\right| \,.
\end{aligned}$$

(The term in $\mathbf{V}$ is zero). Note that $\mathbf{A}' = \frac{1}{\varepsilon}\left(\frac{n}{2}\right)^{1-\frac{2n}{n+1}} \mathbf{A}$ where η has been replaced by 2η. As to obtain (29), we get that

$$\mathbf{B}' + \mathbf{C}' + \mathbf{D}' \leq C \int_{B_p(2\delta)} f^{n-1}(\frac{x}{\varepsilon})\, \mathrm{d}v_g \leq 0(\varepsilon^n) = o(\varepsilon^{-1})$$

and

$$\mathbf{E}' \leq C \int_{B_p(2\delta)} \frac{r^3}{\varepsilon} f^{n-\frac{1}{2}}(\frac{x}{\varepsilon})\, \mathrm{d}v_g \leq o(\varepsilon^{n-1}).$$

Moreover, with the same method which was used to obtain (28), we get

$$\mathbf{A}' \geq \frac{n}{2}\, \omega_{n-1}\, \varepsilon^{n-1}\, I\ [1 + o(1)] \,.$$

This proves that

$$\left|\int_M \Re e < \bar{D}\bar{\psi}_\varepsilon, \bar{\psi}_\varepsilon > \mathrm{d}v_g\right| \geq \frac{n}{2}\, \omega_{n-1}\, \varepsilon^{n-1}\, I\ [1 + o(1)] \,. \tag{31}$$

Finally, Equations (30) and (31) allow to estimate $J(\bar{\psi}_\varepsilon)$ in the following way:

$$J(\psi_\varepsilon) = \frac{\left(\int_M |\bar{D}\bar{\psi}_\varepsilon|^{\frac{2n}{n+1}} \mathrm{d}v_g\right)^{\frac{n+1}{n}}}{\int_M \Re e < \bar{D}\bar{\psi}_\varepsilon, \bar{\psi}_\varepsilon > \mathrm{d}v_g} \leq \frac{n}{2}\, \omega_{n-1}^{\frac{1}{n}}\, I^{\frac{1}{n}}\ [1 + o(1)] \ .$$

By (22), we have

$$w_{n-1} I = \int_{\mathbb{R}^n} f^n dx = \omega_n$$

Therefore, we proved that for the test spinor φ, we have

$$J(\bar{\psi}_\varepsilon) \leq \lambda^+_{\min}(\mathbb{S}^n)\ [1 + o(1)] \ . \tag{32}$$

Hence Theorem 1.1 is proven.

Remark 6.5. There is a variant of this proof which needs less calculations. As a first step, one proves that for any $\varepsilon > 0$ there is a test spinor φ_ε on $\mathbb{R}^n$ with support in $B_0(1)$ such that $J^{\mathbb{R}^n}_{g_{\mathrm{eucl}}}(\varphi_\varepsilon) \leq \lambda^+_{\min}(\mathbb{S}^n) + \varepsilon$ where $\varepsilon > 0$. The argument for this coincides with the above proof, but the terms **IV** to **XXI** vanish, as $\mathbb{R}^n$ is flat.

In a second step, one transplants this compactly supported spinor φ_ε to the arbitrary compact spin manifold $(M, \Lambda^2 g)$, where $\Lambda > 0$ is constant, and one obtains a spinor $\overline{\varphi_\varepsilon}$ on $(M, \Lambda^2 g)$. The terms **IV** to **XXI** reappear. However, from our Taylor expansion worked before, it is easy to see that for $\Lambda \to \infty$ these terms dissapear.

One concludes that there for any $\varepsilon > 0$ there is a $\Lambda_\varepsilon > 0$ and a spinor $\overline{\varphi_\varepsilon}$ on $(M, \Lambda_\varepsilon^2 g)$ such that

$$J_{\Lambda_\varepsilon^2 g}(\overline{\varphi_\varepsilon}) < \lambda^+_{\min}(\mathbb{S}^n) + 2\varepsilon.$$

Together with

$$\lambda^+_{\min}(M, g, \sigma) = \lambda^+_{\min}(M, \Lambda_\varepsilon^2 g, \sigma) \leq J_{\Lambda_\varepsilon^2 g}(\overline{\varphi_\varepsilon})$$

the theorem follows.

This proof is simpler. We chose the way presented above because of various reasons. One the other hand, as indicated in the introduction, in the case $n \geq 3$ it is not the result, but the method of proof which is interesting. The above formulae enter at several places in the literature, e.g. [AHM03], [AH03] and [Rau06]. Secondly, the simpler proof is close to Große's proof [Gro06] and we refer to her article for the probably most elegant proof in dimension $n \geq 3$. Also in her proof some Taylor expansions from the present article are used.

7. The case $n = 2$

The 2-dimensional case is simpler since g is locally conformally flat. On the other hand, some estimates of the last section are no longer valid in dimension 2, hence some parts have to be modified. These modifications will be carefully carried out in this section.

Let (M, g) be a compact Riemannian surface equipped with a spin structure. If $\bar{g}$ is conformal to g we denote by $\mu_1(\bar{g})$ the smallest positive eigenvalue of $\Delta_{\bar{g}}$. We prove the theorem.

Theorem 7.1. *There exists a family of metrics $(g_\varepsilon)_\varepsilon$ conformal to g for which*

$$\limsup_{\varepsilon\to 0} \lambda_1^+(g_\varepsilon)^2 \mathrm{Vol}_{g_\varepsilon}(M) \leqslant 4\pi$$

$$\liminf_{\varepsilon\to 0} \ \mu_1(g_\varepsilon)\mathrm{Vol}_{g_\varepsilon}(M) \geqslant 8\pi.$$

Theorem 7.1 clearly implies Theorem 1.1.
Roughly, these metrics can be described as follows. At first we choose a metric in the conformal class which is flat in a neighborhood of a point p. We remove a small ball around it and glue in a large truncated sphere. This removal and gluing can be done in such a way that we stay within a conformal class. $\varepsilon \to 0$. In the limit this truncated sphere is getting larger and larger compared to the original part of M.

Agricola, Ammann and Friedrich asked the following question [AAF99]:

Let M be a two-dimensional torus equipped with a trivial spin structure, can we find on M a Riemannian metric $\tilde{g}$ for which $\lambda_1^+(\tilde{g})^2 < \mu_1(\tilde{g})$?

To understand this question, recall that the two-dimensional torus carries 4 spin structures. Three of them (the non-trivial ones) are spin boundaries: for these spin structures it is easy to find flat examples with $\lambda_1^+(\tilde{g})^2 = \frac{1}{4}\mu_1(\tilde{g})$. For the trivial spin structure, one has $\lambda_1^+(\tilde{g})^2 = \mu_1(\tilde{g})$ for all flat metrics and $\lambda_1^+(\tilde{g})^2 > \mu_1(\tilde{g})$ for many S^1-equivariant one's.

Clearly, Theorem 7.1 answers this question but says much more: firstly, the result is true on any compact Riemannian surface equipped with a spin structure and not only when M is a two-dimensional torus. In addition, the metric $\tilde{g}$ can be chosen in a given conformal class. Finally, this metric $\tilde{g}$ can be chosen such that $(2-\delta)\lambda_1^+(g)^2 < \mu_1(g)$ where $\delta > 0$ is arbitrarily small. More precisely Theorem 1.1 shows the corollary

Corollary 7.2 (Proposition 1.2 of the Introduction)**.** *On any compact Riemannian surface (M, g), we have*

$$\inf \frac{\lambda_1^+(\bar{g})^2}{\mu_1(\bar{g})} \leqslant \frac{1}{2}$$

where the infimum is taken over all metrics $\bar{g}$ conformal to g.

7.1. C^0**-metrics.** Let f be a smooth positive function and set $\bar{g} = f^2 g$. Let also for $u \in C^\infty(M)$

$$I_{\bar{g}}(u) = \frac{\int_M |\nabla u|_{\bar{g}} dv_{\bar{g}}}{\int_M u^2 dv_{\bar{g}}}.$$

It is well known that $\mu_1(\bar{g}) = \inf I_{\bar{g}}(u)$ where the infimum is taken over the smooth non-zero functions u for which $\int_M u dv_{\bar{g}} = 0$. Another way to express $\mu_1(\bar{g})$ is

$$\mu_1(\bar{g}) := \inf_V \sup_{u \in V \setminus \{0\}} I_{\bar{g}}(u) \tag{33}$$

where the infimum runs over all 2-dimensional subspaces V of $C^\infty(M)$. We now can write all these expressions in the metric g. We then see that for $u \in C^\infty(M)$, we have

$$I_{\bar{g}}(u) = \frac{\int_M |\nabla u|_g^2 dv_g}{\int u^2 f^2 dv_g}$$

and $\mu_1(\bar{g})$ is characterized in a way analogous to (33). Now if f is only continuous, we can define $\bar{g} = f^2 g$. The symmetric 2-tensor $\bar{g}$ is not really a metric since f is not smooth. We then say that g is a C^0*-metric*. We can define the first eigenvalue $\mu_1(\bar{g})$ of $\Delta_{\bar{g}}$ using the definition above.

Suppose that

$$(1+\rho)^{-1} f \leq \tilde{f} \leq (1+\rho) f. \tag{34}$$

Then

$$(1+\rho)^{-2} I_{\tilde{f}^2 g}(u) \leq I_{f^2 g}(u) \leq (1+\rho)^2 I_{\tilde{f}^2 g}(u).$$

From the variational characterization (33) it the follows that

$$(1+\rho)^{-2}\mu_1(\tilde{f}^2 g) \leq \mu_1(f^2 g) \leq (1+\rho)^2 \mu_1(\tilde{f}^2 g),$$

which is a special case of a result by Dodziuk [Dod82, Proposition 3.3]. In particular, we get

Lemma 7.3. *If (f_n) is a sequence of smooth positive functions that converges uniformily to f, then $\mu_1(f_n^2 g)$ tends to $\mu_1(f^2 g)$.*

In the same way, if $\bar{g} = f^2 g$ is a metric conformal to g where f is positive and smooth, we define

$$\mathcal{J}_{\bar{g}}(\psi) = \frac{\displaystyle\int_M |D_{\bar{g}}\psi|_{\bar{g}}^2 dv_{\bar{g}}}{\displaystyle\int_M \langle D_{\bar{g}}\psi, \psi\rangle_{\bar{g}} dv_{\bar{g}}}.$$

The first eigenvalue of the Dirac operator $D_{\bar{g}}$ is then given by $\lambda_1^+(\bar{g}) = \inf \mathcal{J}_{\bar{g}}(\psi)$ where the infimum is taken over the smooth spinor fields ψ for which $\int_M \langle D_{\bar{g}}\psi, \psi\rangle dv_g > 0$. Now, as explained in paragraph 5.2 we can identify spinors for the metric g and spinors for the metric $\bar{g}$ by a fiberwise isometry. Moreover, using this identification, we have for all smooth spinor field:

$$D_{\bar{g}}(f^{-\frac{1}{2}}\varphi) = f^{-\frac{3}{2}} D_g \varphi.$$

This implies that if we set $\varphi = f^{\frac{1}{2}}\psi$, we have

$$\mathcal{J}'_{\bar g}(\varphi) := \frac{\displaystyle\int_M |D_g\varphi|^2 f^{-1} dv_g}{\displaystyle\int_M \langle D_g\varphi, \varphi\rangle dv_g} = \mathcal{J}_{\bar g}(\psi)$$

and the first eigenvalue of the Dirac operator $D_{\bar g}$ is given by

$$\lambda_1^+(\bar g) = \inf \mathcal{J}'_{\bar g}(\varphi) \tag{35}$$

where the infimum is taken over the smooth spinor fields φ for which $\displaystyle\int_M \langle D_g\varphi, \varphi\rangle dv_g > 0$. Now, when $\bar g = f^2 g$ is no longer smooth, but a C^0-metric, we can use (35) to define $\lambda_1^+(\bar g)$.

Under the assumption (34), we get

$$(1+\rho)^{-1}\mathcal{J}'_{\tilde f^2 g}(\varphi) \le \mathcal{J}'_{f^2 g}(\varphi) \le (1+\rho)\mathcal{J}'_{\tilde f^2 g}(\varphi),$$

and hence

$$(1+\rho)^{-1}\lambda_1^+(\tilde f^2 g) \le \lambda_1^+(f^2 g) \le (1+\rho)\lambda_1^+(\tilde f^2 g).$$

We have proven a result similar to Lemma 7.3:

Lemma 7.4. *If (f_n) is a sequence of smooth positive functions that converges uniformly to f, then $\lambda_1^+(f_n^2 g)$ tends to $\lambda_1^+(\bar g)$.*

7.2. **The metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$.** In this paragraph, we construct the metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ conformal to g which will satisfy:

$$\limsup_{\varepsilon\to 0} \lambda_1^+(g_{\alpha,\varepsilon})^2 \mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) \leqslant 4\pi \tag{36}$$

and

$$\liminf_{\alpha\to 0} \liminf_{\varepsilon\to 0} \mu_1(g_{\alpha,\varepsilon}) \mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) \geqslant 8\pi. \tag{37}$$

Clearly this implies Theorem 1.1. By Lemmas 7.3 and 7.4, it suffices to construct C^0-metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$. Recall that the volume of M for a C^0-metric is defined by $\mathrm{Vol}_{f^2 g}(M) = \int_M f^2 dv_g$. At first, without loss of generality, we can assume that g is flat near a point $p \in M$. Let $\alpha > 0$ be a small number to be fixed later such that g is flat on $B_p(\alpha)$. We set for all $x \in M$ and $\varepsilon > 0$,

$$f_{\alpha,\varepsilon}(x) = \begin{cases} \frac{\varepsilon^2}{\varepsilon^2 + r^2} & \text{if } \; r \leqslant \alpha \\ \frac{\varepsilon^2}{\varepsilon^2 + \alpha^2} & \text{if } \; r > \alpha \end{cases}$$

where $r = d_g(.,p)$. The function $f_{\alpha,\varepsilon}$ is continuous and positive on M. We then define for all $\varepsilon > 0$, $g_{\alpha,\varepsilon} = f_{\alpha,\varepsilon}^2 g$. The symmetric 2-tensors $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ will be the desired C^0-metrics. For these metrics, we have

$$\mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) = \int_M f_{\alpha,\varepsilon}^2 dv_g = \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g + \int_{M\setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g.$$

Since g is flat on $B_p(\alpha)$, we have

$$\int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = \int_0^{2\pi}\int_0^{\alpha} \frac{\varepsilon^4 r}{(\varepsilon^2 + r^2)^2} dr d\Theta.$$

Substituting $\rho = r^2/\varepsilon^2$ we obtain

$$\int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = \pi\varepsilon^2 \int_0^{\frac{\alpha^2}{\varepsilon^2}} \frac{1}{(1+\rho)^2} dr = \pi\varepsilon^2 + o(\varepsilon^2).$$

Since $f_{\alpha,\varepsilon}^2 \leqslant \frac{\varepsilon^4}{\alpha^4}$ on $M \setminus B_p(\alpha)$, we have $\int_{M\setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = o(\varepsilon^2)$. We obtain

$$\mathrm{Vol}_{g_\varepsilon}(M) = \pi\varepsilon^2 + o(\varepsilon^2). \tag{38}$$

7.3. **Proof of relation (36).** We define on $\mathbb{R}^2$ as in subsection 5.1 the spinor field

$$\psi(x) = f(x)(1-x)\cdot\psi_0$$

where $f(x) = \frac{2}{1+|x|^2}$, $|\psi_0| = 1$. We have

$$D\psi = f\psi \ \text{ and } \ |\psi| = f^{\frac{1}{2}}. \tag{39}$$

Now, we fix a small number $\alpha > 0$ such that g is flat on $B_p(2\alpha)$. Then, let δ be a small number such that we take $0 \leqslant \delta \leqslant \alpha$. Assume that ε tends to 0. Furthermore let η be a smooth cut-off function defined on M by

$$\eta(x) = \left| \begin{array}{cl} 1 & \text{if} \quad r \le \delta^2 \\ \frac{\log(r)}{\log(\delta)} - 1 & \text{if} \quad r \in [\delta^2, \delta] \\ 0 & \text{if} \quad r \ge \delta \end{array} \right.$$

The function η is such that $0 \leqslant \eta \leqslant 1$, $\eta(B_p(\delta)) = \{1\}$, $\eta(\mathbb{R}^n \setminus B_p(2\delta)) = \{0\}$ and

$$\kappa_\delta := \int_M |\nabla\eta|^2 dv_g \to 0 \text{ for } \delta \to 0. \tag{40}$$

Identifying $B_p(2\delta)$ in M with $B_0(2\delta)$ in $\mathbb{R}^2$, we can define a smooth spinor field on M by $\psi_\varepsilon = \eta(x)\psi\left(\frac{x}{\varepsilon}\right)$. Using (39), we have

$$D_g(\psi_\varepsilon) = \nabla\eta\cdot\psi\left(\frac{x}{\varepsilon}\right) + \frac{\eta}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)\psi\left(\frac{x}{\varepsilon}\right). \tag{41}$$

Since $\langle\nabla\eta\cdot\psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon})\rangle \in i\mathbb{R}$ and since $|D_g\psi_\varepsilon|^2 \in \mathbb{R}$, we have

$$\int_M |D_g\psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g = I_1 + I_2 \tag{42}$$

where

$$I_1 = \int_M |\nabla\eta|^2 \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 dx \ \text{ and } I_2 = \int_M \frac{\eta^2}{\varepsilon^2} f^2\left(\frac{x}{\varepsilon}\right)\left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 f_{\alpha,\varepsilon}^{-1} dx.$$

By (39), $\left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 \le 2$ and hence

$$I_1 \leqslant 2\int_M |\nabla\eta|^2 dv_g = 2\kappa_\delta \to 0 \tag{43}$$

for $\delta \to 0$. Now, by (39),

$$I_2 \leqslant \frac{2}{\varepsilon^2}\int_{B_p(2\delta)} f^3(\frac{x}{\varepsilon}) f_{\alpha,\varepsilon}^{-1} dx.$$

Since $f_{\alpha,\varepsilon} = \frac{1}{2} f(\frac{x}{\varepsilon})$ on the support of η, we have

$$I_2 \leqslant \frac{2}{\varepsilon^2}\int_{B_p(2\delta)} f^2(\frac{x}{\varepsilon}) dx.$$

Mimicking what we did to get (38), we obtain that

$$I_2 \leqslant 8\pi + o_\varepsilon(1)$$

where $o_\varepsilon(1)$ denotes a term tending to 0 for $\varepsilon \to 0$. Together with (42) and (43), we obtain

$$\int_M |D_g\psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g \leqslant 8\pi + 2\kappa_\delta + o_\varepsilon(1). \tag{44}$$

In the same way, by (41), since $\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon\rangle dv_g \in \mathbb{R}$ and since $\langle\nabla\eta\cdot\psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon})\rangle \in i\mathbb{R}$, we have

$$\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon\rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)\left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 dv_g.$$

By (39), this gives

$$\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f^2 \left(\frac{x}{\varepsilon}\right) dv_g.$$

With the computations made above, it follows that

$$\int_M \langle D_g(\psi_\varepsilon), \psi_\varepsilon \rangle dv_g = 4\pi\varepsilon + o(\varepsilon).$$

Together with (44) and (38), we obtain

$$\begin{aligned} \lambda_1^+(g_{\alpha,\psi})^2 \mathrm{Vol}_{g_{\alpha,\psi}}(M) &\leqslant \left(\mathcal{J}'_{g_{\alpha,\varepsilon}}(\psi_\varepsilon)\right)^2 \mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) \leqslant \left(\frac{8\pi + 2\kappa_\delta + o_\varepsilon(1)}{4\pi\varepsilon + o(\varepsilon)}\right)^2 (\pi\varepsilon^2 + o(\varepsilon^2)) \\ &= 4\pi + 2\kappa_\delta + \frac{1}{4\pi}\kappa_\delta^2 + o_\varepsilon(1). \end{aligned}$$

Letting ε then δ go to 0, we get Relation (36).

7.4. **Proof of relation (37).** As pointed out by the referee the metrics $g_{\alpha,\varepsilon}$ coincide with metrics contructed in [Tak02], and relation (37) is proven in this article.

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