SQUARE-INTEGRABILITY OF SOLUTIONS OF THE YAMABE EQUATION

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Abstract. We show that solutions of the Yamabe equation on certain n-dimensional non-compact Riemannian manifolds which are bounded and L^p for $p = 2n/(n-2)$ are also L^2. This L^p-L^2-implication provides explicit constants in the surgery-monotonicity formula for the smooth Yamabe invariant in our article [4]. As an application we see that the smooth Yamabe invariant of any 2-connected compact 7-dimensional manifold is at least 74.5. Similar conclusions follow in dimension 8 and in dimensions ≥ 11.

Contents

1. Introduction

The goal of this article is to show that bounded positive solutions of the Yamabe equation on certain n-dimensional non-compact Riemannian manifolds $\mathbb{M}_c^{n,k}$ with finite L^{p_n}-norm, $p_n = 2n/(n-2)$, also have finite L^2-norm. For $c \neq 0$ these spaces $\mathbb{M}_c^{n,k}$ are products of rescaled hyperbolic space with a standard sphere, while $\mathbb{M}_0^{n,k} = \mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1}$. The integer k satisfies $0 \leq k \leq n-3$. The goal is achieved in some cases, we then say that the L^{p_n}-L^2-implication holds. In particular, it holds for $0 \leq k \leq n-4$. If $k = n-3$ and if n is sufficiently large we will find counterexamples, see Section 4.

In the favourable cases, the proof of the L^{p_n}-L^2-implication is obtained by a combination of tricky integration and suitable estimates, and there is not much hope to generalize our technique to a much larger class of non-compact manifolds. The reader will thus probably ask why we develop such estimates for these special spaces.

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The reason is that these spaces appear naturally as limit spaces in the surgery construction of our article [4], and this limit construction also provides L^{p_n}-solutions of the Yamabe equation. The main result of [4] is a surgery formula for the smooth Yamabe invariant $\sigma(M)$, see Subsection 2.3 for the definition. The surgery formula states that if M is a compact manifold of dimension n and if N is obtained from M through k-dimensional surgery, then the smooth Yamabe invariants $\sigma(M)$ and $\sigma(N)$ satisfy

$$\sigma(N) \geq \min(\sigma(M), \Lambda_{n,k}), \tag{1}$$

provided that $k \leq n-3$. The definition of the numbers $\Lambda_{n,k}$ is quite involved, see Subsection 2.6, but they are proven to be positive and to depend only on n and k. The L^{p_n}-L^2-implication helps to derive explicit lower bounds for these numbers, see Subsections 2.4 to 2.6.

Using estimates for product manifolds from the article [3] we obtain explicit positive lower bounds for $\Lambda_{n,k}$ in the case $2 \leq k \leq n-4$, see Corollary 5.1 for details. This leads directly to a uniform positive lower bound for the smooth Yamabe invariant of 2-connected manifolds that are boundaries of compact spin manifolds, see Corollary 6.3. In dimensions $n \in \{7, 8\}$ and $n \geq 11$ we obtain an explicit positive lower bound $\sigma(M)$, provided that M is 2-connected with vanishing index $\alpha(M) \in KO_n(\mathrm{pt})$, see Corollary 6.7. Note that S^5 is the only 2-connected closed 5-manifold. Further, a 2-connected closed 6-manifold is diffeomorphic to the connected sum of a number of copies of $S^3 \times S^3$ and the smooth Yamabe invariant can be estimated using results of [8] and [3]. It is also known that $\alpha(M) \neq 0$ implies $\sigma(M) = 0$.

Using results from [2], [11], and [12] an explicit positive lower bound can also be obtained in the cases $(n,k) \in \{(4,1), (5,1), (5,2), (6,3), (9,1), (10,1)\}$, see [2] for details about these dimensions.

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2. Preliminaries and Notation

2.1. Spaces of constant curvature and the model spaces $\mathbb{M}_c^{n,k}$.

Here we fix notation for the spaces of constant curvature and we define the spaces $\mathbb{M}_c^{n,k}$.

We denote the Euclidean metric on $\mathbb{R}^n$ by ξ^n. The sphere $S^n \subset \mathbb{R}^{n+1}$ equipped with its standard round metric ρ^n is denoted by $\mathbb{S}^n$. We set $\omega_\ell := \mathrm{vol}(\mathbb{S}^\ell)$. For $c \in \mathbb{R}$ let $\mathbb{H}_c^{k+1}$ be the simply connected, connected, complete Riemannian manifold of constant sectional curvature $-c^2$. Its metric will be denoted as η_c^{k+1}. Polar coordinates around a point x_0 gives an identification of $\mathbb{H}_c^{k+1} \setminus \{x_0\}$ with $(0,\infty) \times S^k$ under which

$$\eta_c^{k+1} = dr^2 + \mathrm{sh}_c(r)^2 \rho^k$$

where

$$\mathrm{sh}_c(r) := \begin{cases} \frac{1}{c}\sinh(cr) & \text{if } c \neq 0, \\ r & \text{if } c = 0. \end{cases}$$

We denote the product metric on $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ by

$$G_c := \eta_c^{k+1} + \rho^{n-k-1}$$

and we define the model space $\mathbb{M}_c^{n,k}$ through

$$\mathbb{M}_c^{n,k} := \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}.$$

The scalar curvature of $\mathbb{M}_c^{n,k}$ is $s^{G_c} = -c^2 k(k+1) + (n-k-1)(n-k-2)$. Note that $\mathbb{M}_c^{n,k} = \mathbb{M}_{-c}^{n,k}$. As a consequence all following infima over $c \in [-1,1]$ could be taken over $c \in [0,1]$. From the conformal point of view the case $c = \pm 1$ is special since $\mathbb{M}_{\pm 1}^{n,k} = \mathbb{H}_{\pm 1}^{k+1} \times \mathbb{S}^{n-k-1}$ is conformal to $\mathbb{S}^n \setminus \mathbb{S}^k$, see [4, Proposition 3.1].

2.2. **The conformal Yamabe constant.** For integers $n \geq 3$ we set $a_n := \frac{4(n-1)}{n-2}$ and $p_n := \frac{2n}{n-2}$. For a Riemannian manifold (M,g) we denote the scalar curvature by s^g, the Laplace operator Δ^g, and the volume form dv^g. The conformal Laplacian is defined as $L^g := a_n \Delta^g + s^g$. In general the dependence on the Riemannian metric is indicated by adding the metric as a superscript.

Let $C_c^\infty(M)$ denote the space of compactly supported smooth functions on M. For a Riemannian manifold (M,g) of dimension $n \geq 3$ we define the Yamabe functional by

$$\mathcal{F}^g(u) := \frac{\int_M \left(a_n |du|_g^2 + s^g u^2\right)\, dv^g}{\left(\int_M |u|^{p_n}\, dv^g\right)^{\frac{2}{p_n}}},$$

where $u \in C_c^\infty(M)$ does not vanish identically. The *conformal Yamabe constant* $\mu(M,g)$ of (M,g) is defined by

$$\mu(M,g) := \inf_{u \in C_c^\infty(M), u \not\equiv 0} \mathcal{F}^g(u).$$

Here M is allowed to be compact or non-compact.

If M is compact, then the infimum is attained by a positive smooth function. It thus satisfies the associated Euler-Lagrange equation, called the *Yamabe equation*,

$$L^g u = \mu u^{p_n - 1} \tag{2}$$

for a suitable constant μ. The invariant is actually a conformal invariant: If g_1 is conformal to g_2, then $\mu(M,g_1) = \mu(M,g_2)$.

2.3. **The smooth Yamabe invariant.** Let M be a compact manifold of dimension $n \geq 3$. The smooth Yamabe invariant of M defined as

$$\sigma(M) := \sup \mu(M,g)$$

where the supremum is taken over the set of all Riemannian metrics on M. It is known that $\sigma(S^n) = \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$.

2.4. **The role of the model spaces in the surgery formula.** To give some background and motivation we will now briefly explain the role of the model spaces $\mathbb{M}_c^{n,k}$ in [4], and we want to give a rough idea why the invariants $\Lambda_{n,k}$ defined in the following subsections appear. The discussion in the present subsection is not needed in the proofs of the following results. We thus try to avoid technical details, and do not aim for logical completeness.

Assume that (M,g) is a compact Riemannian manifold of dimension $n \geq 3$ and that N is obtained by k-dimensional surgery from M, where $0 \leq k \leq n-3$. In [4] a sequence of metrics g_i is constructed on N. To prove the surgery formula (1) we show

$$\limsup_{i\to\infty} \mu(N,g_i) \geq \min(\mu(M,g),\Lambda_{n,k}) \tag{3}$$

for a suitable positive constant $\Lambda_{n,k}$.

The solution of the Yamabe problem on (N,g_i) provides positive smooth functions $u_i \in C^\infty(N)$ satisfying the Yamabe equation

$$L^{g_i} u = \mu_i u_i^{p_n-1}$$

where $\mu_i = \mu(N,g_i)$ and $\|u_i\|_{L^{p_n}} = 1$.

In order to prove (3), one analyses the "limits" of the functions u_i in various cases. In some cases these functions u_i "converge" to a nontrivial solution of the Yamabe equation $L^g u = \mu u^{p_n-1}$ on (M,g) with $\mu = \limsup \mu_i$, and it follows that $\limsup_{i\to\infty} \mu(N,g_i) \geq \mu(M,g)$. In other cases the functions concentrate in some points, and $\limsup_{i\to\infty} \mu(N,g_i) \geq \mu(\mathbb{S}^n)$ follows by "blowing-up" such points. This means that one suitably rescales normal coordinates centered in the blow-up point, and then the solutions of the Yamabe equation converge to a solution on $\mathbb{R}^n$, which then yields a solution of the Yamabe equation on a sphere. This phenomenon is also often described in the literature by saying that "a sphere bubbles off".

However, it can also happen that the functions u_i converge to a solution of the Yamabe equation on a model space $\mathbb{M}_c^{n,k}$ with $|c| \leq 1$, this corresponds to Subcases II.1.1 and II.2 in the proof of Theorem 6.1 in [4]. In this case one obtains points $x_i \in N$, such that the pointed Riemannian manifolds (N,g_i,x_i) "converge" to $(\mathbb{M}_c^{n,k},\bar{x})$. Here "convergence" means that balls of arbitrary radius R around x_i in (N,g_i) converge for fixed R and $i\to\infty$ in the C^∞-sense to a ball of radius R around $\bar{x}$ in $\mathbb{M}_c^{n,k}$. The functions u_i will then converge to a positive solution $\bar{u}$ of the Yamabe equation (2) on $\mathbb{M}_c^{n,k}$. The L^{p_n}-norm of the limit function does not increase, that is $\|\bar{u}\|_{L^{p_n}} \leq 1$.

This raises the following question.

Question 2.1. *Assume that $\bar{u} \in C^\infty(\mathbb{M}_c^{n,k})$ is a positive solution of the Yamabe equation $L^{G_c}\bar{u} = \lambda \bar{u}^{p_n-1}$ with $0 < \|\bar{u}\|_{L^{p_n}} \leq 1$. Does this imply $\lambda \geq \mu(\mathbb{M}_c^{n,k})$?*

If $\bar{u}$ is in L^2, then it is already in the Sobolev space $H^{1,2}$ and integration by parts $\int \bar{u}\Delta\bar{u}\,dv = \int |d\bar{u}|^2\,dv$ is allowed on $\mathbb{M}_c^{n,k}$, see Appendix A. It then easily follows that that the answer to the question is positive. In this case we will say that the L^{p_n}-L^2-implication holds.

In turn the L^{p_n}-L^2-implication implies that (3) holds for

$$\Lambda_{n,k} := \inf_{c\in[-1,1]} \mu(\mathbb{M}_c^{n,k}).$$

We will find conditions under which the L^{p_n}-L^2-implication holds. But we will also obtain examples where it is violated, see Section 4.

The fact that the L^{p_n}-L^2-implication is violated in some cases led to a technical difficulty in [4] which was solved by introducing the constant $\Lambda^{(2)}_{n,k}$ into the definition of $\Lambda_{n,k}$ in [4, Definition 3.2]. This is sufficient for proving the *positivity* of the constant $\Lambda_{n,k}$. However, in order to obtain an explicit positive lower bound for $\Lambda_{n,k}$ one would like to avoid the constant $\Lambda^{(2)}_{n,k}$. The possibility to prove the L^{p_n}-L^2-implication in some cases was mentioned in [4, Remark 3.4]. As a consequence finding a positive lower bound for $\Lambda_{n,k}$ reduces to finding a positive lower bound for the constants $\mu(\mathbb{M}^{n,k}_c)$, uniform in $c \in [0,1]$.

In the meantime new results for the explicit lower bounds for $\mu(\mathbb{M}^{n,k}_c)$ were obtained in [3] and [2], and a proof of [4, Remark 3.4] is needed. The goal of the present article is to provide this proof.

2.5. **Modified conformal Yamabe constants.** The technical difficulty described in the previous subsection required the introduction of a modified conformal Yamabe constant. In fact, two different subcases require two versions of modified constants, namely the modified conformal Yamabe constants $\mu^{(1)}(N,h)$ and $\mu^{(2)}(N,h)$, defined below. Our article aims to give some clarification of the relation between these invariants for the model spaces $\mathbb{M}^{n,k}_c$.

Let (N,h) be a Riemannian manifold of dimension n. For $i = 1,2$ we let $\Omega^{(i)}(N,h)$ be the set of non-negative C^2 functions u on N which solve the Yamabe equation

$$L^h u = \mu u^{p_n - 1} \tag{4}$$

for some $\mu = \mu(u) \in \mathbb{R}$. We also require that the functions $u \in \Omega^{(i)}(N,h)$ satisfy

(a) $u \not\equiv 0$,
(b) $\|u\|_{L^{p_n}(N)} \le 1$,
(c) $u \in L^\infty(N)$,

together with

(d1) $u \in L^2(N)$, for $i = 1$,

or

(d2) $\mu(u)\|u\|^{p_n-2}_{L^\infty(N)} \ge \frac{(n-k-2)^2(n-1)}{8(n-2)}$, for $i = 2$.

For $i = 1,2$ we set

$$\mu^{(i)}(N,h) := \inf_{u \in \Omega^{(i)}(N,h)} \mu(u).$$

In particular $\mu^{(i)}(N,h) = \infty$ if $\Omega^{(i)}(N,h)$ is empty. If N is compact then the solution of the Yamabe problem trivially implies $\mu(N,h) = \mu^{(1)}(N,h) \le \mu^{(2)}(N,h)$.

We will use this for $(N,h) = \mathbb{M}^{n,k}_c$. In [4, Lemma 3.5] we already showed $\mu^{(1)}(\mathbb{M}^{n,k}_c) \ge \mu(\mathbb{M}^{n,k}_c)$ if $0 \le k \le n-3$. In the present article we will show that (b) implies (d1) in many cases. As a consequence we will obtain

$$\mu^{(2)}(\mathbb{M}^{n,k}_c) \ge \mu^{(1)}(\mathbb{M}^{n,k}_c) \ge \mu(\mathbb{M}^{n,k}_c)$$

in theses cases. We refer to Theorem 3.1 and Corollary 3.2 for details.

2.6. **The numbers $\Lambda_{n,k}$.** For integers $n \ge 3$ and $0 \le k \le n-3$ set

$$\Lambda^{(i)}_{n,k} := \inf_{c \in [-1,1]} \mu^{(i)}(\mathbb{M}^{n,k}_c)$$

and

$$\Lambda_{n,k} := \min\left\{\Lambda^{(1)}_{n,k}, \Lambda^{(2)}_{n,k}\right\}.$$

It is not hard to show that $\Lambda_{n,0} = \mu(\mathbb{S}^n)$, see [4, Subsection 3.5]. The following positivity result for $\Lambda_{n,k}$ is proved in [4, Theorem 3.3].

Theorem 2.2. *For all $n \geq 3$ and $0 \leq k \leq n-3$, we have $\Lambda_{n,k} > 0$.*

Furthermore, the following surgery result is concluded in [4, Corollary 1.4].

Theorem 2.3. *Inequality (1) holds for $0 \leq k \leq n-3$ and the numbers $\Lambda_{n,k} > 0$ defined above.*

3. Main Theorem

Theorem 3.1. *Let $c \in [-1, 1]$ and let $u \in L^\infty(\mathbb{M}_c^{n,k}) \cap L^{p_n}(\mathbb{M}_c^{n,k}) \cap C^2(\mathbb{M}_c^{n,k})$ be a uniformly continuous, positive solution of*

$$L^{G_c} u \leq \mu u^{p_n - 1}. \tag{5}$$

Assume that

$$2k\, |c| < n(n-k-2), \tag{6}$$

then $u \in L^2(\mathbb{M}_c^{n,k})$.

Inequality (6) holds when

- $n \leq 5$, $k \in \{0, \cdots, n-3\}$, and $|c| \leq 1$; or
- $n \geq 6$, $k \in \{0, \cdots, n-4\}$, and $|c| \leq 1$; or
- $n = 6$, $k = n-3$, and $|c| < 1$.

This follows from the fact that $2k \leq n(n-k-2)$ is equivalent to $k \leq n - 4 + \frac{8}{n+2}$.

Non-negative L^∞-solutions of (4) on manifolds with bounded geometry will be uniformly continuous due to standard local regularity and Sobolev embeddings (one even can show boundedness in $C^{3,\alpha}$ for small $\alpha > 0$). Thus the conditions in Theorem 3.1 are satisfied for all L^∞-solutions of (4) on $\mathbb{M}_c^{n,k}$, in particular for all $u \in \Omega^{(i)}(\mathbb{M}_c^{n,k})$, $i = 1, 2$.

Also note that a non-negative solution of (4) will be either identically zero or everywhere positive, due to the maximum principle.

Remark. We expect that a similar theorem for weak solutions of (5) holds as well. Such a version for weak solutions is required in current work [5] of Nadine Große in collaboration with the first author. This project will relate the spinorial analogue of $\Lambda_{n,k}$ defined in [1] to the constants $\Lambda_{n,k}$ defined above.

More precisely, we expect that Theorem 3.1 also holds for uniformly continuous, non-negative functions $u \in L^\infty(\mathbb{M}_c^{n,k}) \cap L^{p_n}(\mathbb{M}_c^{n,k})$ satisfying (5) in the distributional sense, without the conditions of positivity and $u \in C^2$. Unfortunately the proof presented below is no longer valid in the distributional setting, as square roots of distributions and products of distributions would appear and it remains unclear how to handle these objects.

However, we worked out a distributional version of the proof below for non-negative functions $u \in L^\infty(\mathbb{M}_c^{n,k}) \cap L^{p_n}(\mathbb{M}_c^{n,k})$ that are C^2 on Ω for $\Omega := \{x \in \mathbb{M}_c^{n,k} \mid u(x) > 0\}$, that satisfy (5) on Ω, and that satisfy $u^2 \in C^1(\mathbb{M}_c^{n,k})$. Under the above conditions on c, n and k we then get the conclusion $u \in L^2(\mathbb{M}_c^{n,k})$, as well. The non-$C^2$ functions in [5] satisfy all these conditions. In the proof of the generalized version of Theorem 3.1 one has to replace integrals over F_r by integrals over $\{x \in F_r \mid u(x) \geq \varepsilon\}$. The boundary terms thus obtained can be estimated by 0.

Corollary 3.2. *We have*

$$\mu^{(2)}(\mathbb{M}_c^{n,k}) \geq \mu(\mathbb{M}_c^{n,k})$$

for all $k \leq n-4$ and $|c| \leq 1$. The same statement holds for $k = n-3$, $n \in \{4,5\}$ and $|c| \leq 1$. And it also holds for $k = 3$, $n = 6$, $|c| < 1$.

Proof of Corollary 3.2. Under the conditions of the corollary Assumption (6) holds, and hence Theorem 3.1 implies

$$\Omega^{(2)}(\mathbb{M}_c^{n,k}) \subset \Omega^{(1)}(\mathbb{M}_c^{n,k})$$

and as a consequence we get

$$\mu^{(2)}(\mathbb{M}_c^{n,k}) \geq \mu^{(1)}(\mathbb{M}_c^{n,k}).$$

On the other hand it is proved in [4, Lemma 3.5] that

$$\mu^{(1)}(\mathbb{M}_c^{n,k}) \geq \mu(\mathbb{M}_c^{n,k})$$

for all $k \in \{0, \cdots, k-3\}$. The corollary follows. $\square$

Proof of Theorem 3.1. We will now give the proof of Theorem 3.1 in five steps. Let u be as in the statement of this theorem.

Step 1. *The function u tends to* 0 *at infinity.*

We proceed by contradiction and assume that there is an $\varepsilon > 0$ and a sequence of points $(x_j)_{j\in\mathbb{N}}$ tending to infinity (as a sequence in j) such that $u(x_j) \geq \varepsilon$. Denote by $\overline{B(x,1)}$ the closed ball of radius 1 around a point x. By taking a subsequence of (x_j) which tends fast enough to infinity, we can assume that the balls $\overline{B(x_j,1)}$ are all disjoint. Since u is in L^{p_n} we have

$$\lim_{j\to\infty} \int_{\overline{B(x_j,1)}} u^{p_n}\, dv^{G_c} = 0.$$

Since $\mathbb{M}_c^{n,k}$ is homogeneous, there are isometries $\varphi_j : \overline{B(x_j,1)} \to \overline{B(O,1)}$ where O is any fixed point in $\mathbb{M}_c^{n,k}$. Obviously $v_j := u \circ \varphi_j^{-1}$ satisfy

$$\lim_{j\to\infty} \int_{\overline{B(O,1)}} v_j^{p_n}\, dv^{G_c} = 0, \tag{7}$$

and

$$v_j(O) \geq \varepsilon. \tag{8}$$

Since u is assumed to be uniformly continuous and bounded, the sequence v_j is equicontinuous and bounded. A subsequence of (v_j) converges uniformly, i.e. in $C^0(\overline{B(O,1)}, \mathbb{R})$, to a continuous function $v \in C^0(\overline{B(O,1)}, \mathbb{R})$.

Finally (7) and (8) tell us that

$$\int_{\overline{B(O,1)}} v^{p_n}\, dv^{G_c} = 0$$

and

$$v(O) \geq \varepsilon,$$

which gives the desired contradiction. This ends the proof of Step 1.

We now work in polar coordinates on the hyperbolic space factor of $\mathbb{M}_c^{n,k} = \mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ as introduced in Subsection 2.1. We thus study the metric $G_c = dr^2 + \mathrm{sh}_c(r)^2\rho^k + \rho^{n-k-1}$ on the manifold $(0,\infty) \times S^k \times S^{n-k-1}$. Using these coordinates, we denote by F_r the set of constant r, that is $F_r := S^k \times S^{n-k-1}$ and we denote the restriction of g to F_r by $g_r = \mathrm{sh}_c(r)^2\rho^k + \rho^{n-k-1}$. We define

$$\omega(r) := \left(\int_{F_r} u^2\, dv^{g_r}\right)^{\frac{1}{2}}$$

for $r > 0$. Next we prove a differential inequality for ω.

Step 2. *For any γ with $0 < \gamma < \frac{n-k-2}{2}$ there is an $r_0(\gamma, u)$ such that*

$$\omega''(r) \geq \gamma^2 \omega(r)$$

for all $r > r_0(\gamma, u)$.

The argument for this step is a modification of the proof of Theorem 5.2 in [4].

The Laplacian operators of the total space Δ^{G_c} and of the fibers Δ^{g_r} are related by

$$\Delta^{G_c} = \Delta^{g_r} - \partial_r^2 + (n-1)H_r\partial_r$$

where H_r denotes the mean curvature of the fiber F_r in $\mathbb{M}_c^{n,k}$. It follows that

$$\begin{aligned}
\int_{F_r} u\Delta^{G_c}u\, dv^{g_r} &= \int_{F_r} \left(u\Delta^{g_r}u - u\partial_r^2 u + (n-1)H_r u\partial_r u\right) dv^{g_r} \\
&= \int_{F_r} \left(|d_{\mathrm{vert}}u|^2 - u\partial_r^2 u + (n-1)H_r u\partial_r u\right) dv^{g_r}.
\end{aligned}$$

where $d_{\mathrm{vert}}u$ denotes the differential of u along the fiber, that is $d_{\mathrm{vert}}u = d(u_{|F_r})$. Using Inequality (5) we get

$$a_n \int_{F_r} \Big(u\partial_r^2 u - (n-1)H_r u\partial_r u\Big)\, dv^{g_r} \geq s^{G_c}\omega(r)^2 - \mu \int_{F_r} u^{p_n}\, dv^{g_r}. \tag{9}$$

Computing the derivative of $\omega(r)^2/2$ we get

$$\begin{aligned}
\omega'(r)\omega(r) &= \frac{1}{2}\frac{d}{dr}\int_{F_r} u^2\, dv^{g_r} \\
&= \int_{F_r} u\partial_r u\, dv^{g_r} - \frac{n-1}{2}H_r\omega(r)^2,
\end{aligned} \tag{10}$$

where we used that H_r is constant on F_r. Differentiating this again and using Inequality (9) we get

$$\begin{aligned}
\omega'(r)^2 + \omega''(r)\omega(r) &= \int_{F_r} (\partial_r u)^2\, dv^{g_r} + \int_{F_r} \Big(u\partial_r^2 u - (n-1)H_r u\partial_r u\Big)\, dv^{g_r} \\
&\quad - \frac{n-1}{2}(\partial_r H_r)\omega(r)^2 - (n-1)H_r\omega'(r)\omega(r) \\
&\geq \int_{F_r} (\partial_r u)^2\, dv^{g_r} + \frac{s^{G_c}}{a_n}\omega(r)^2 - \frac{\mu}{a_n}\int_{F_r} u^{p_n}\, dv^{g_r} \\
&\quad - \frac{n-1}{2}(\partial_r H_r)\omega(r)^2 - (n-1)H_r\omega'(r)\omega(r).
\end{aligned} \tag{11}$$

From the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\omega(r)^2 \int_{F_r} (\partial_r u)^2\, dv^{g_r} &\geq \left(\int_{F_r} u(\partial_r u)\, dv^{g_r}\right)^2 \\
&= \left(\omega'(r)\omega(r) + \frac{n-1}{2}H_r\omega(r)^2\right)^2,
\end{aligned}$$

where we used Equation (10) in the second line. Thus,

$$\int_{F_r} (\partial_r u)^2 \, dv^{g_r} \geq \left(\omega'(r) + \frac{n-1}{2} H_r \omega(r) \right)^2 . \tag{12}$$

Let $\varepsilon > 0$ be a constant to be fixed later. By Step 1 we have

$$\frac{\mu}{a_n} \int_{F_r} u^{p_n} \, dv^{g_r} \leq \varepsilon \omega(r)^2 \tag{13}$$

for all r large enough (depending on ε). Inserting (12) and (13) into (11) we obtain

$$\begin{aligned} \omega'(r)^2 + \omega''(r)\omega(r) \geq \left(\omega'(r) + \frac{n-1}{2} H_r \omega(r) \right)^2 + \frac{s^{G_c}}{a_n} \omega(r)^2 - \varepsilon \omega(r)^2 \\ - \frac{n-1}{2} (\partial_r H_r) \omega(r)^2 - (n-1) H_r \omega'(r) \omega(r), \end{aligned}$$

or after some rearranging,

$$\omega''(r) \geq \underbrace{\left(\frac{(n-1)^2}{4} H_r^2 + \frac{s^{G_c}}{a_n} - \varepsilon - \frac{n-1}{2} (\partial_r H_r) \right)}_{=:\alpha(r)} \omega(r). \tag{14}$$

A computation tells us that

$$H_r = -\frac{k}{n-1} \partial_r \ln \mathrm{sh}_c(r) = \begin{cases} -\frac{k}{n-1} c \coth(cr) & \text{if } c \neq 0, \\ -\frac{k}{n-1} \frac{1}{r} & \text{if } c = 0, \end{cases}$$

so in particular,

$$\lim_{r \to \infty} H_r = -\frac{k}{n-1} |c| \tag{15}$$

and

$$\lim_{r \to \infty} \partial_r H_r = 0. \tag{16}$$

Using (15) and (16) together with $s^{G_c} = -c^2 k(k+1) + (n-k-1)(n-k-2)$ we see that the coefficient $\alpha(r)$ in the right hand side in (14) tends to $\alpha_c - \varepsilon$ where

$$\begin{aligned} \alpha_c &:= \frac{(n-1)^2}{4} \frac{k^2}{(n-1)^2} c^2 + \frac{n-2}{4(n-1)} \Big(- c^2 k(k+1) + (n-k-1)(n-k-2) \Big) \\ &= -(n-k-2) \frac{k}{4(n-1)} c^2 + \frac{(n-2)(n-k-1)(n-k-2)}{4(n-1)} \\ &\geq -(n-k-2) \frac{k}{4(n-1)} + \frac{(n-2)(n-k-1)(n-k-2)}{4(n-1)} \\ &= \frac{(n-k-2)^2}{4}. \end{aligned}$$

Here the inequality comes from the fact the coefficient of c^2 is negative so the smallest value over $c \in [-1, 1]$ is attained for $c = 1$. Choosing ε small enough we have proved Step 2.

By assumption u and thus ω are positive. As a consequence we can define

$$\tau(r) := (\ln \omega(r))'$$

for $r > 0$.

Step 3. *One of the following statements is true,*

$$\liminf_{r\to\infty} \tau(r) \geq (n-k-2)/2, \tag{17a}$$

$$\limsup_{r\to\infty} \tau(r) \leq -(n-k-2)/2. \tag{17b}$$

Assume (17b) is not true. Then there is a $\tilde{r} \geq r_0(\gamma)$ with $\tau(\tilde{r}) > -(n-k-2)/2$. We assume $\tau(\tilde{r})^2 \leq \frac{(n-k-2)^2}{4} - 2\varepsilon$ where $\varepsilon > 0$. Choose $\gamma := \sqrt{\frac{(n-k-2)^2}{4} - \varepsilon}$. Using Step 2 we calculate

$$\tau'(r) = \frac{\omega''(r)}{\omega(r)} - \left(\frac{\omega'(r)}{\omega(r)}\right)^2 \geq \gamma^2 - \tau(r)^2$$

for all $r \geq \tilde{r}$, and thus $\tau'(r) \geq \varepsilon$ as long as $\tau(r)^2 \leq \frac{(n-k-2)^2}{4} - 2\varepsilon$. An easy argument on first order ordinary differential equations yields an $R > \tilde{r}$ such that

$$\tau(r) \geq \sqrt{\frac{(n-k-2)^2}{4} - 2\varepsilon}$$

for all $r \geq R$. As $\varepsilon > 0$ can be chosen arbitrarily small we conclude that (17a) must hold, and Step (3) follows.

Step 4. *If* (17a) *holds then L^{p_n}-boundedness of u contradicts the assumption* (6).

Since $\mathbb{M}^{n,k}_{-c} = \mathbb{M}^{n,k}_c$ we may assume that $c \geq 0$. At first we consider the case $c > 0$. In the following argument we denote by C a positive constant that might change value from line to line. Since

$$\mathrm{vol}^{g_r}(F_r) = \int_{F_r} \left(\frac{\sinh(cr)}{c}\right)^k dv^{\rho^k + \rho^{n-k-1}} \leq C e^{kcr}$$

we get

$$\begin{aligned} \int_0^\infty \omega(r)^{p_n} e^{-\frac{2kc}{n-2}r}\, dr &= \int_0^\infty \left(\int_{F_r} u^2\, dv^{g_r}\right)^{\frac{p_n}{2}} e^{-\frac{2kc}{n-2}r}\, dr \\ &\leq C \int_0^\infty \int_{F_r} u^{p_n}\, dv^{g_r}\, dr \\ &\leq C \int_{\mathbb{M}^{n,k}_c} u^{p_n}\, dv^{G_c} \end{aligned} \tag{18}$$

using the Hölder inequality. This is bounded since u is assumed to be in $L^{p_n}(\mathbb{M}^{n,k}_c)$. If (17a) holds then for any $\gamma \in \left(0, \frac{n-k-2}{2}\right)$ there is an $r_1 = r_1(\gamma)$ so that

$$\omega(r) \geq C e^{\gamma r}$$

for all $r \geq r_1$. Thus

$$\int_0^\infty \omega(r)^{p_n} e^{-\frac{2kc}{n-2}r}\, dr \geq C \int_{r_1}^\infty e^{br}\, dr \tag{19}$$

where

$$b := p_n\gamma - \frac{2kc}{n-2} = \frac{2}{n-2}(n\gamma - kc).$$

If $b \geq 0$ then the right hand side of (19) is infinite, which gives a contradiction to the boundedness of (18). Thus we have $b < 0$, implying $\gamma < kc/n$. Taking $\gamma \to (n-k-2)/2$ yields $(n-k-2)/2 \leq k|c|/n$, which finishes the proof of Step (4) for $c > 0$.

The case $c = 0$ can be solved with similar estimates.

Step 5. *Conclusion.*

It remains to show the L^2-boundedness of u if (17b) of Step 3 holds. We choose any $\gamma \in (0, (n-k-2)/2)$. If (17b) holds we have

$$\omega(r) \leq Ce^{-\gamma r}$$

for all $r \geq r_2(\gamma)$, and by possibly enlarging C this holds for all r. From this estimate we have

$$\int_{\mathbb{M}_c^{n,k}} u^2 \, dv^{G_c} = \int_0^\infty \omega(r)^2 \, dr < \infty,$$

which ends the proof of Theorem 3.1. □

4. A counterexample to Theorem 3.1 for $k = n - 3$

As noted after Theorem 3.1, Assumption (6) holds if $n \leq 6$, $k \in \{0, \dots, n-3\}$, $|c| < 1$ or $n \geq 7$, $k \in \{0, \dots, n-4\}$. It is natural to ask whether the conclusion of Theorem 3.1 holds for all $k \in \{0, \dots, n-3\}$. The following proposition answers this in the negative.

Proposition 4.1. *Let $n \geq 7$. There exists a smooth positive function $u \in L^\infty(\mathbb{H}_1^{n-2} \times \mathbb{S}^2) \cap L^{p_n}(\mathbb{H}_1^{n-2} \times \mathbb{S}^2)$ which satisfies*

$$L^{G_1} u = 0$$

and is not in $L^2(\mathbb{H}_1^{n-2} \times \mathbb{S}^2)$.

Note that the function u given by Proposition 4.1 satisfies Inequality (5) with $\mu = 0$.

Proof. Consider $\mathbb{S}^{n-3}$ as a totally geodesic sphere in $\mathbb{S}^n$. For $y \in \mathbb{S}^n$ let Γ_y be the Green's function of L^{ρ^n} at y. That is Γ_y satisfies $L^{\rho^n}\Gamma_y = \delta_y$ in the sense of distributions, where δ_y is the Dirac distribution at y. It is well known that Γ_y exists and satisfies $\Gamma_y(x) \sim (4(n-1)\omega_{n-1})^{-1} r(x)^{-(n-2)}$ when x tends to y. Here $r(x)$ denotes the geodesic distance from x to y. Define H on $\mathbb{S}^n \setminus \mathbb{S}^{n-3}$ by

$$H(x) := \int_{\mathbb{S}^{n-3}} \Gamma_y(x) \, dv^{\rho^{n-3}}(y).$$

It is straightforward to check that for x tending to $\mathbb{S}^{n-3}$ we have $H(x) \sim c_n' r'(x)^{-1}$ where c_n' depends only on n and where r' is the geodesic distance to $\mathbb{S}^{n-3}$. Hence we have $H \in L^{p_n}(\mathbb{S}^n \setminus \mathbb{S}^{n-3})$ since $n \geq 7$. In [4, Proposition 3.1] it was proven that $\mathbb{S}^n \setminus \mathbb{S}^{n-3}$ and $\mathbb{H}_1^{n-2} \times \mathbb{S}^2$ are conformal. Let f be the conformal factor so that $G_1 = f^{\frac{4}{n-2}} \rho^n$. As explained in [4], $f(r') \sim (r')^{-\frac{2}{n-2}}$ when r' tends to 0. We set $u := f^{-1} H$. By conformal covariance of the conformal Laplacian we have

$$L^{G_1} u = 0.$$

Moreover,

$$\int_{\mathbb{H}_1^{n-2} \times \mathbb{S}^2} u^{p_n} \, dv^{G_1} = \int_{\mathbb{S}^n \setminus \mathbb{S}^{n-3}} H^{p_n} \, dv^{\rho^n} < \infty.$$

In addition, using the asymptotics of f given above, it is easy to check that u is not in L^2 and hence provides the desired counterexample. □

5. Consequences for the surgery formula

The goal of this paper is to find explicit lower bounds for $\Lambda_{n,k}$. We find the following.

Corollary 5.1. *Assume that $k \in \{2, \cdots, n-4\}$, then*

$$\Lambda_{n,k} \geq \underline{\Lambda}_{n,k}$$

where

$$\underline{\Lambda}_{n,k} := \frac{n a_n}{((k+1)a_{k+1})^{\frac{k+1}{n}}((n-k-1)a_{n-k-1})^{\frac{n-k-1}{n}}} \mu(\mathbb{S}^{k+1})^{\frac{k+1}{n}} \mu(\mathbb{S}^{n-k-1})^{\frac{n-k-1}{n}}.$$

Note that we have $\Lambda_{n,0} = \mu(\mathbb{S}^n)$ from [4, Section 3.5], and hence the only cases not covered by this corollary are $k = 1$ and $k = n-3$. Further,

$$\underline{\Lambda}_{n,2} = \underline{\Lambda}_{n,n-4} = n a_n \left(\frac{\mu(\mathbb{S}^3)}{3a_3}\right)^{\frac{3}{n}} \left(\frac{\mu(\mathbb{S}^{n-3})}{(n-3)a_{n-3}}\right)^{\frac{n-3}{n}} = n a_n \left(\frac{\pi^4}{16}\right)^{\frac{1}{n}} \nu_{n-3}^{1/n},$$

where we defined

$$\nu_\ell := \left(\frac{\mu(\mathbb{S}^\ell)}{\ell a_\ell}\right)^\ell = \omega_\ell^2 \left(\frac{\ell-2}{4}\right)^\ell$$

and it holds that

$$\omega_\ell = \mathrm{vol}(\mathbb{S}^\ell) = \frac{2\pi^{(\ell+1)/2}}{\Gamma\left(\frac{\ell+1}{2}\right)}.$$

We define

$$\underline{\Lambda}_{n,2+} := \min\{\underline{\Lambda}_{n,2}, \ldots, \underline{\Lambda}_{n,n-4}\}.$$

Some values for $\underline{\Lambda}_{n,2+}$ are listed in Figure 1. Numerically we calculated $\underline{\Lambda}_{n,2+} = \underline{\Lambda}_{n,2}$ for $n \leq 3000$, and it seems reasonable to conjecture this for all n, but we do not have a proof.

Proof of Corollary 5.1. The conformal Yamabe invariant $\mu(\mathbb{M}_c^{n,k})$ as defined in Section 2 for non-compact manifolds is, by virtue of [3, Theorem 2.3], bounded from below by $\underline{\Lambda}_{n,k}$. To see this we just have to notice that $\mu(\mathbb{H}_c^{k+1}) = \mu(\mathbb{S}^{k+1})$, which holds since $\mathbb{H}_c^{k+1}$ is conformal to a subset of $\mathbb{S}^{k+1}$. Thus Corollary 5.1 is a direct consequence of Corollary 3.2. $\square$

6. Topological applications

In this section we derive some topological consequences of our main theorem. Recall that by definition a manifold M is k-connected, $k \geq 1$, if it is connected and if $\pi_1(M) = \pi_2(M) = \cdots = \pi_k(M) = 0$.

Proposition 6.1. *Let M_0 and M_1 be non-empty compact 2-connected manifolds of dimension $n \geq 7$, and assume that M_0 is spin bordant to M_1. Then M_1 can be obtained from M_0 by a sequence of surgeries of dimensions ℓ, $3 \leq \ell \leq n-4$.*

Note that 2-connected manifolds are orientable and spin, and they carry a unique spin structure.

The proposition is well-known, but for the sake of being self-contained we include a proof following the lines of [10, Lemma 4.2]). As a first step we prove a lemma.

Lemma 6.2. *Let M_0 and M_1 compact spin manifolds of dimension $n \geq 7$ and assume that M_0 is spin bordant to M_1. Then there is a 3-connected spin bordism W from M_0 to M_1.*

Proof of the Lemma. We start with a given spin bordism W_0 from M_0 to M_1. From this bordism we construct a bordism W which is 3-connected.

By performing 0-dimensional surgeries at the bordism, one can modify the original bordism W_0 to be connected. This can be done such that the bordism W_1 thus obtained is again orientable, and W_1 then carries a spin structure.

We now perform 1-dimensional surgeries to reduce the fundamental group to the trivial group. Assume that $[\gamma] \in \pi_1(W_1)$. We can assume that $\gamma : S^1 \to W_1$ is an embedding. Its normal bundle is trivial as W_1 is orientable. Performing a 1-dimensional surgery along γ using a trivialization ν of this normal bundle yields a new bordism $W_1^{\gamma,\nu}$ which depends both on γ and ν. This bordism is orientable. For any γ one can choose a trivialization ν such that the bordism $W_1^{\gamma,\nu}$ carries a spin structure that coincides with the spin structure of W_1 outside a tubular neighborhood of the image of γ. The van Kampen Theorem gives a surjective homomorphism $\pi_1(W_1) \to \pi_1(W_1^{\gamma,\nu})$ such that $[\gamma]$ is in the kernel. The fundamental group $\pi_1(W)$ is finitely generated, let γ_i be disjoint embedded circles such that $[\gamma_1], \dots, [\gamma_\ell]$ is a set of generators $\pi_1(W)$. Performing 1-dimensional surgeries along the γ_i with suitable trivializations of their normal bundles then yields a simply-connected spin bordism W_2 from M_0 to M_1.

Next we perform 2-dimensional surgeries to remove $\pi_2(W_2)$. Assume that $[\sigma] \in \pi_2(W_2)$ is given and assume that $\sigma : S^2 \to W_2$ is an embedding. Since W_2 is spin the normal bundle of the image of σ is trivial. Performing a 2-dimensional surgery along σ yields a new spin bordism W_2^σ which depends on the choice of σ. However, it is independent of the choice of trivialization as different trivializations are homotopic. After a finite number of 2-dimensional surgeries we obtain a 2-connected spin bordism W_3 from M_0 to M_1.

In a similar way one can also remove $\pi_3(W_3)$. The Whitney embedding theorem implies that any class in $\pi_3(W_3)$ can be represented by an embedding $\tau : S^3 \to W_3$, as $n \geq 6$. The normal bundle of the image of τ is trivial, as $\pi_2(O(n-3)) = 0$. A surgery along τ with any trivialization ν will then kill $[\tau]$, and since $n \geq 7$ the spin bordism $W_3^{\tau,\nu}$ thus obtained is again 2-connected and will have $\pi_3(W_3^{\tau,\nu}) \cong \pi_3(W_3)/[\tau]$. After finitely many surgery steps we obtain a 3-connected bordism W as claimed in the lemma. □

Proof of the Proposition. Assume that W is 3-connected spin bordism from M_0 to M_1. Then $H_i(W, M_j) = 0$ for $i = 0, 1, 2, 3$, in particular $b_i(W, M_0) = 0$ for these numbers i. We can apply [9, VIII Theorem 4.1] for $k = 4$ and $m = n + 1$. One obtains that there is a presentation of the bordism such that for any $i < 4$ and any $i > n - 3$ the number of i-handles is given by $b_i(W, M_0)$. Any i-handle corresponds to a surgery of dimension $i - 1$. It remains to show that $b_i(W, M_0) = 0$ for $i \in \{0, 1, 2, 3, n + 1, n, n - 1, n - 2\}$. For $i \in \{0, 1, 2, 3\}$ this was discussed above. By Poincaré duality $H^{n+1-i}(W, M_0)$ is dual to $H_i(W, M_1)$ which vanishes for $i = 0, 1, 2, 3$. One the other hand the universal coefficient theorem tells us that the free parts of $H^i(W, M_0)$ and $H_i(W, M_0)$ are isomorphic. Thus $b_i(W, M_0)$ which is by definition the rank of (the free part of) $H_i(W, M_0)$ vanishes for $i \in \{n + 1, n, n - 1, n - 2\}$. □

Corollary 6.3. *Let M be a 2-connected compact manifold of dimension $n \geq 7$ which is a spin boundary. Then*

$$\sigma(M) \geq \underline{\Lambda}_{n,2+}$$

where $\underline{\Lambda}_{n,2+}$ is defined in Section 5.

Proof. Assume that M is the boundary of a compact spin manifold W of dimension $n+1$. By removing a ball we obtain a spin-bordism from S^n to M. The preceding proposition tells us that M can be obtained by surgeries of dimensions $\ell \in \{3, \dots, n-4\}$ from S^n. By applying the surgery formula (1) and Corollary 5.1 we get the stated lower bound for $\sigma(M)$. □

Theorem 6.4 (Stolz [14, Theorem B]). *Let M be a compact spin manifold of dimension $n \geq 5$. Assume that the index $\alpha(M) \in KO_n(\mathrm{pt})$ vanishes. Then M is spin-bordant to the total space of an $\mathbb{H}P^2$-bundle over a base Q for which the structure group is* $\mathrm{PSp}(3)$.

The base Q has to be understood as a spin manifold, so that it admits a spin structure and the choice of spin structure matters. The theorem includes the fact the every spin manifold of dimension 5, 6, or 7 is a spin boundary, in these cases $Q = \emptyset$.

Proposition 6.5 (Extended Stolz theorem). *In the case $n \geq 9$ one can assume that the base Q is connected, and in the case $n \geq 11$ one can assume that it is simply connected.*

Note that $M := \mathbb{H}P^2 \amalg \mathbb{H}P^2$ is an example of an 8-dimensional manifold where Q cannot be chosen to be connected. This follows from the fact that $\mathbb{H}P^2$ has non-vanishing signature and thus $[\mathbb{H}P^2]$ is an element of infinite order in Ω_8^{spin}. If S^1 carries the spin structure that does not bound a disc, then $\mathbb{H}P^2 \times S^1$ and $\mathbb{H}P^2 \times S^1 \times S^1$ are examples of dimension 9 and 10 where Q cannot be chosen to be simply connected. This is a consequence of the fact that $[\mathbb{H}P^2 \times S^1] \in \Omega_9^{\mathrm{spin}}$ and $[\mathbb{H}P^2 \times S^1 \times S^1] \in \Omega_{10}^{\mathrm{spin}}$ are non-zero elements (of order 2), see [6, Cor. 1.9] or [7, Cor. 2.6].

Proof. Assume that M is spin bordant to a spin manifold N_0 which is the total space of a fiber bundle with fiber $\mathbb{H}P^2$ and structure group $\mathrm{PSp}(3)$ over a base Q_0 of dimension $n-8 \geq 1$. By performing 0-dimensional surgery on Q_0 we obtain a connected space Q_1. The spin bordism from Q_0 to Q_1 which is associated to the 0-dimensional surgeries yields a spin bordism from N_0 to a total space of a fiber bundle with fiber $\mathbb{H}P^2$ and structure group $\mathrm{PSp}(3)$ over Q. This Q is connected, but not necessarily simply-connected.

Now assume $n \geq 11$. Any path $\gamma : S^1 \to Q_1$ is homotopic to an embedding and has a trivial normal bundle as Q_1 is orientable. A tubular neighborhood of the image of γ in Q_1 is diffeomorphic to $S^1 \times B^{n-9}$. Any trivialization of this normal bundle yields the germ of such a diffeomorphism up to isotopy. Because of our condition $n \geq 11$ we can choose the trivialization of the normal bundle such that the induced spin structure on $S^1 \times B^{n-9}$ is the bounding spin structure of $B^2 \times B^{n-9}$. Doing a surgery along γ with respect to such a trivialization we obtain a spin manifold Q_2, and the associated bordism from Q_1 to Q_2 is a spin bordism. As $\mathrm{PSp}(3)$ is connected the $\mathbb{H}P^2$-bundle with structure group $\mathrm{PSp}(3)$ extends to a bundle of the same type over this bordism.

We now perform a sequence of such 1-dimensional surgeries, where γ runs through a generating set of $\pi_1(Q_1)$. The space thus obtained is then simply-connected. □

Combining the previous results we obtain the following.

Corollary 6.6. *Let M be a 2-connected compact manifold of dimension $n = 7$. Then*

$$\sigma(M) \geq \underline{\Lambda}_{7,2+} > 74.5.$$

To derive a similar result for $n = 8$, we remark that the conformal Yamabe constant of $\mathbb{H}P^2$, equipped with the standard metric, is $128\pi/120^{1/4} = 121.4967... > \underline{\Lambda}_{8,2+} = 92.24278....$

Corollary 6.7. *Let M be a 2-connected compact manifold of dimension $n = 8$. Then $\sigma(M) = 0$ if $\alpha(M) \neq 0$, and*

$$\sigma(M) \geq \underline{\Lambda}_{8,2+} > 92.2$$

if $\alpha(M) = 0$.

Proposition 6.8. *Let M_0 be the total space of a bundle with fiber $\mathbb{H}P^2$ and structure group* $\mathrm{PSp}(3)$ *over a base B of dimension $n-8$. Then, if $n \geq 11$*

$$\sigma(M_0) \geq \underline{\lambda}_n := n a_n \left(\frac{3^6 2^{18}}{7^8 5^2}\pi^8\right)^{1/n} \nu_{n-8}^{1/n}$$

Proof. M. Streil has shown in [15] that $\sigma(M_0) \geq \mu(\mathbb{H}P^2 \times \mathbb{R}^{n-8})$, where $\mathbb{H}P^2 \times \mathbb{R}^{n-8}$ carries the product metric of the standard metrics on $\mathbb{H}P^2$ and $\mathbb{R}^{n-8}$. On the other hand it follows from [3, Theorem 2.3] that

$$\mu(\mathbb{H}P^2 \times \mathbb{R}^{n-8}) \geq \frac{n a_n}{(8a_8)^{8/n}((n-8)a_{n-8})^{(n-8)/n}} \mu(\mathbb{H}P^2)^{8/n} \mu(\mathbb{S}^{n-8})^{(n-8)/n}.$$

We also have

$$\left(\frac{\mu(\mathbb{H}P^2)}{8a_8}\right)^8 = \frac{3^6 2^{18}}{7^8 5^2}\pi^8 = 1.32599...\pi^8 = 12581.78...$$

This clearly implies the proposition. □

As an example we study $n = 11$. Then $\nu_3 = \pi^6/8$ and thus $\underline{\lambda}_{11} = 178.23277$. Some further values for $\underline{\lambda}_n$ are listed in Figure 1.

Proposition 6.9. *Let M be a 2-connected compact manifold of dimension $n \geq 11$. Then $\sigma(M) = 0$ if $\alpha(M) \neq 0$. If $\alpha(M) = 0$, then*

$$\sigma(M) \geq \begin{cases} \underline{\lambda}_{11} > 135.90 & \text{if } n = 11, \\ \underline{\lambda}_{12} > 158.72 & \text{if } n = 12, \\ \underline{\Lambda}_{n,2+} & \text{if } n \geq 13. \end{cases}$$

Proof. If $\alpha(M) = 0$, then we have seen

$$\sigma(M) \geq \min(\underline{\Lambda}_{n,2+}, \underline{\lambda}_n).$$

It remains to compare $\underline{\Lambda}_{n,2+}$ and $\underline{\lambda}_n$. Numerically we calculated $\underline{\lambda}_{11} \leq \underline{\Lambda}_{11,2+}$, $\underline{\lambda}_{12} \leq \underline{\Lambda}_{12,2+}$, and $\underline{\lambda}_n \geq \underline{\Lambda}_{n,2+}$ for $13 \leq n \leq 5000$.

For $n \geq 1100$ we found

$$\underline{\lambda}_n^n \geq 1.43\, \underline{\Lambda}_{n,2}^n \geq 1.43\, \underline{\Lambda}_{n,2+}^n$$

by studying the Γ-function and by using $\Gamma(n)/\Gamma(n-1/2) \geq \sqrt{n-1}$. This yields the required result. □

n	$Y(\mathbb{S}^n)$	$\underline{\Lambda}_{n,2+}$	$\underline{\lambda}_n$
7	113.5272754	74.50435	
8	130.7157953	92.24278367	
9	147.8778709	109.4260421	
10	165.0220642	126.4134026	
11	182.1536061	143.3280094	135.9033973
12	199.2758713	160.2189094	158.7256737
13	216.3911332	177.1071517	178.0562033
14	233.5009793	194.0019409	196.2714765
15	250.6065514	210.9071013	213.9967504
16	267.7086915	227.8239126	231.4689436
17	284.8080344	244.7524346	248.7967717
18	301.9050675	261.6921542	266.0365304

FIGURE 1. Some values for $\underline{\Lambda}_{n,2+}$ and $\underline{\lambda}_n$.

APPENDIX A. DETAILS OF AN ARGUMENT

We here give some details on an argument sketched directly after Question 2.1.

Proposition A.1. *Let (N,h) be a complete n-dimensional Riemannian manifold of bounded geometry and let $u \in L^2(N) \cap L^{p_n}(N)$ be a positive solution of the Yamabe equation (4). Then $\int_N u\Delta u\, dv^h$ and $\int_N |du|^2\, dv^h$ exist and satisfy*

$$\int_N u\Delta u\, dv^h = \int_N |du|^2\, dv^h.$$

It also follows that $u \in H_1^2(N)$ and $\mu \geq \mu(N,h)$.

Proof. Since $uL^h u = \mu u^{p_n}$ it is clear that $\int uL^h u\, dv^h$ exists and as s^h is bounded and because of $u \in L^2$ this also implies the existence of the integral $\int_N u\Delta u\, dv^h \in \mathbb{R}$.

Let $\eta_i : N \to [0,1]$ be a sequence of smooth compactly supported cut-off functions such that $|d\eta_i| \leq 1$ and such that the $\eta_i \equiv 1$ on K_i where K_i is a compact exhaustion of N, that is $N = \bigcup K_i$. In particular $\int_{\mathrm{supp}\, d\eta_i} u^2\, dv^h \to 0$ for $i \to \infty$. We can also assume $\eta_{i+1} \geq \eta_i$.

In the case $(N,h) = \mathbb{M}_c^{n,k}$ such functions can be obtained as $\eta_i(x) := \chi(r(x)-2i)$ where r is the radial function on $\mathbb{H}_c^{k+1}$ and where $\chi : \mathbb{R} \to [0,1]$ is a smooth function with $0 \leq \chi' \leq 1$, $\chi|_{(-\infty,0]} \equiv 0$, $\chi|_{[2,\infty)} \equiv 1$. For general (N,h) of bounded geometry this existence follows similarly using [13, Appendix A1.2., Lemma 2.1., page 70].

We calculate

$$\begin{aligned}\int_N |d(\eta_i u)|^2\, dv^h &= \int_N \left(u^2|d\eta_i|^2 + \eta_i u\langle d\eta_i, du\rangle + \langle du, \eta_i d(\eta_i u)\rangle\right)\, dv^h \\ &= \int_N \left(u^2|d\eta_i|^2 + \eta_i u\langle d\eta_i, du\rangle + \langle du, d(\eta_i^2 u) - \eta_i u d\eta_i\rangle\right)\, dv^h \\ &= \int_N \left(u^2|d\eta_i|^2 + \langle du, d(\eta_i^2 u)\rangle\right)\, dv^h \\ &= \int_N u^2|d\eta_i|^2\, dv^h + \int_N \eta_i^2 u\Delta^h u\, dv^h.\end{aligned} \tag{20}$$

As u is in $L^{p_n} \cap L^2$ and as s^h is bounded, we see that

$$(1-\eta_i^2)u\Delta^h u = (1-\eta_i^2)\underbrace{\left(\frac{\mu}{a_n}u^{p_n} - \frac{s^h}{a_n}u^2\right)}_{\in L^1}$$

converges to 0 in L^1. Thus $\int_N \eta_i^2 u\Delta^h u\, dv^h$ exists and converges to $\int_N u\Delta^h u\, dv^h$.

We also see that

$$\int_N u^2|d\eta_i|^2\, dv^h \le \int_{\operatorname{supp} d\eta_i} u^2\, dv^h \to 0.$$

Thus

$$\int_N |d(\eta_i u)|^2\, dv^h \to \int_N u\Delta^h u\, dv^h \tag{21}$$

and together with $\|\eta_i u\|_{L^{p_n}} \to \|u\|_{L^{p_n}} = 1$ this implies

$$\mathcal{F}^g(\eta_i u) \to \mu$$

from which we directly see that $\mu \ge \mu(N,h)$.

From (21) we also see $\int_N u\Delta^h u\, dv^h \ge 0$ and (21) can be rewritten as

$$\|d(\eta_i u)\|_{L^2} \to \sqrt{\int_N u\Delta^h u\, dv^h}.$$

As a consequence

$$\begin{aligned}\|\eta_i du\|_{L^2(N)} &\le \|d(\eta_i u)\|_{L^2(N)} + \|ud\eta_i\|_{L^2(N)} \\ &\le \|d(\eta_i u)\|_{L^2(N)} + \|u\|_{L^2(\operatorname{supp} d\eta_i)} \\ &\to \sqrt{\int_N u\Delta^h u\, dv^h}\end{aligned} \tag{22}$$

is bounded. As $\eta_i|du|$ is increasing with limit $|du|$, we get $|du| \in L^2$ with $\|du\|_{L^2} \le \sqrt{\int_N u\Delta^h u\, dv^h}$, and thus $u \in H_1^2$.

On the other hand

$$\|d(\eta_i u)\|_{L^2(N)} \le \underbrace{\|u\|_{L^2(\operatorname{supp} d\eta_i)}}_{\to 0} + \|du\|_{L^2(N)},$$

so

$$\int_N |du|^2\, dv^h = \|u\|^2_{L^2(N)} = \int_N u\Delta^h u\, dv^h.$$

□

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