L^p-SPECTRUM OF THE DIRAC OPERATOR ON PRODUCTS WITH HYPERBOLIC SPACES

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Abstract. We study the L^p-spectrum of the Dirac operator on complete manifolds. One of the main questions in this context is whether this spectrum depends on p. As a first example where p-independence fails we compute explicitly the L^p-spectrum for the hyperbolic space and its product with compact spaces.

1. Introduction

The L^p-spectrum of the Laplacian and its p-(in)dependence was and still is studied by many authors, e.g. in [17], [18], [22]. On closed manifolds one easily sees that the spectrum is independent of $p \in [1, \infty]$. For open manifolds, independence only holds under additional geometric conditions. Hempel and Voigt [22], [23] proved such results for Schrödinger operators in $\mathbb{R}^n$ with potentials admitting certain singularities. Then Kordyukov [27] generalized this result to uniformly elliptic operators with uniformly bounded smooth coefficients on a manifold of bounded geometry with subexponential volume growth. Independently, Sturm [34] showed the independence of the L^p-spectrum for a class of uniformly elliptic operators in divergence form on manifolds with uniformly subexponential volume growth and Ricci curvature bounded from below. Both results include the Laplacian acting on functions. Later the Hodge-Laplacian acting on k-forms was considered. E.g. under the assumptions of the result by Sturm from above, Charalambous proved the L^p-independence for the Hodge-Laplacian in [14, Proposition 9]. The machinery used to obtain these independence results uses estimates for the heat kernel as in [32].

In contrast, the L^p-spectrum of the Laplacian on the hyperbolic space does depend on p [16, Theorem 5.7.1]. Its L^p-spectrum is the convex hull of a parabola in the complex plane, and this spectrum degenerates only for $p = 2$ to a ray on the real axis, cf. Remark 10.1.

In addition to the intrinsic interest of the p-independence of the L^p-spectrum, such results were used to get information on the L^2-spectrum by considering the L^1-spectrum, as in particular examples the L^1-spectrum can be easier to control. The result of Sturm was used for example by Wang [37, Theorem 3] to prove that the spectrum of the Laplacian acting on functions on complete manifolds with asymptotically non-negative Ricci curvature is $[0, \infty)$.

Some explicit calculations for the L^p spectrum of the Laplace-Beltrami operator on symmetric spaces were already carried by Lahoué and Rychener in the 1980's [29], and Taylor [36] was able to determine the L^p-spectrum of this operator for all symmetric spaces of non-compact type. For locally symmetric spaces important progress was made recently by Ji and Weber, see e.g. [24, 25, 26, 39].

About the L^p-spectrum of the Dirac operator much less is known. As before, on closed manifolds the spectrum is independent on $p \in [0, \infty]$. Kordyukov's methods [27] do not apply directly to the Dirac operator D, but following a remark of [27, Page 224] his methods generalize to suitable systems, and thus also to the square D^2. Unfortunately, the system case is not completely worked out, but it seems to us, that the case of systems is completely analogous to the case of operators on functions. Assuming this, Kordyukov has shown

Date: September 16, 2019.

2010 *Mathematics Subject Classification.* 58J50, 34B27.

Key words and phrases. Dirac operator, L^p-spectrum, Green function, hyperbolic space, product spaces.

that the spectrum of D^2 is p-independent for $1 \leq p < \infty$ on manifolds with bounded geometry and subexponential volume growth. For many such manifolds (e.g. for all such manifolds of even dimension or all manifolds of dimension $4k+1$), this already implies the p-independence of the L^p-spectrum of D, see our Lemma B.8 together with the following symmetry considerations.
Many of the results and techniques that were constructed up for Laplace operators are not yet developed for Dirac operators. It is interesting to compare the properties of the Dirac and Laplace operator. Furthermore, the L^p-spectrum of the Dirac operator is important for non-linear variational problems based on the Dirac operator, e.g., for (classical) Dirac operators certain L^p-spaces and L^p-spectral gaps naturally occur when considering a spinorial Yamabe-type problem which was our motivation to enter into this subject, see [5] and below.

In this paper we determine explicitly the L^p-spectrum for a special class of complete manifolds – products of compact spaces with hyperbolic spaces. More precisely, we study the following manifolds:
Let (N^n, g_N) be a closed Riemannian spin manifold. Let $M = \mathbb{M}_c$ be the product manifold $(\mathbb{M}_c^{m,k} = \mathbb{H}_c^{k+1} \times N^n, g_M = g_{\mathbb{H}_c^{k+1}} + g_N)$ where $\mathbb{H}_c^{k+1}$ is the $(k+1)$-dimensional hyperbolic space scaled such that its scalar curvature is $-c^2k(k+1)$ for $c \neq 0$ and $\mathbb{H}_0^{k+1}$ is the $(k+1)$-dimensional Euclidean space. For those manifolds we obtain the following result which is also illustrated in Figure 1:

Theorem 1.1. *We use the notions from above. Let $p \in [1, \infty]$, $c \geq 0$, and $k \in \mathbb{N} \cup \{0\}$. The L^p-spectrum of the Dirac operator on $\mathbb{M}_c^{m,k} = \mathbb{H}_c^{k+1} \times N^n$ is given by the set*

$$\sigma_p := \left\{ \mu \in \mathbb{C} \;\middle|\; \mu^2 = \lambda_0^2 + \kappa^2, |\mathrm{Im}\,\kappa| \leq ck \left| \frac{1}{p} - \frac{1}{2} \right| \right\}$$

where λ_0^2 is the lowest eigenvalue of $(D^N)^2$, $\lambda_0 \geq 0$, and D^N is the Dirac operator on (N, g_N). In particular, the Dirac operator $D\colon H_1^p \to L^p$ on $\mathbb{M}_c^{m,k}$ has a bounded inverse if and only if $\lambda_0 > ck\left|\frac{1}{p} - \frac{1}{2}\right|$.

Note that we also allow the case $n = 0$, i.e. the case that N is a finite set. In this case we define $\lambda_0 := 0$.
For an overview of the structure of the proof, see the end of the introduction.
From the Theorem 1.1 one can directly read off the L^p-spectrum of D^2 and compare it to the known spectrum of the Laplacian acting on functions which is done in Remark 10.1.
As mentioned above our interest in the L^p-spectrum for the Dirac operator on such spaces comes from an application in conformal spin geometry which we want to sketch very briefly: In the article [3] the first named author studied the behavior of a spinorial analogue of the Yamabe invariant under surgeries. In these Gromov-Lawson type constructions an important role was played by spinorial Yamabe-type invariants on the spaces $\mathbb{H}_c^{k+1} \times \mathbb{S}^n$ where c is in $[0, 1]$ and where $\mathbb{S}^n$ is the standard sphere. When these invariants were studied in [21], the important role of the L^p-spectrum of the Dirac operator on such spaces became apparent. The results of the present article are preliminaries for some theorems about spinorial Yamabe-type invariants in [5].
The paper is structured as follows: Notations and preliminaries are collected in Section 2. Results on the Green function of the Dirac operator acting on L^2-spinors can be found in Section 3. General remarks and results for the Dirac operator acting on L^p-sections are given in Appendix B.
In Section 4, the Dirac operator on the model spaces $\mathbb{M}_c^{m,k}$ is written in polar coordinates and the action of $\mathrm{Spin}(k+1)$ on $\mathbb{M}_c^{m,k}$ is studied. This is used in Section 5 to prove a certain symmetry property of the Green function on $\mathbb{M}_c^{m,k}$ and in Section 6 to study its decay.
After these preparations we are ready to prove the main theorem (Theorem 1.1). In Section 7 we decompose the Green function into a singular part and the integral kernel of a smoothing

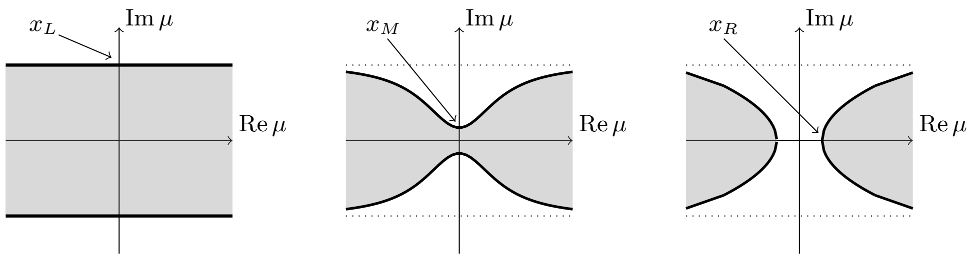

FIGURE 1. The shaded region (including the boundary) illustrates the L^p-spectrum of the Dirac operator on $\mathbb{M}_c^{m,k} = \mathbb{H}_c^{k+1} \times N^n$, $k \geq 1$, cf. Theorem 1.1.
Left: $\lambda_0 = 0$ $\left(x_L = ck\left|\frac{1}{p} - \frac{1}{2}\right|\right)$.
Middle: $0 < \lambda_0 < ck\left|\frac{1}{p} - \frac{1}{2}\right|$ $\left(x_M = \left(c^2k^2\left(\frac{1}{p} - \frac{1}{2}\right)^2 - \lambda_0^2\right)^{\frac{1}{2}}\right)$.
Right: $\lambda_0 > ck\left|\frac{1}{p} - \frac{1}{2}\right|$ $\left(x_R = \left(\lambda_0^2 - c^2k^2\left(\frac{1}{p} - \frac{1}{2}\right)^2\right)^{\frac{1}{2}}\right)$.

operator. Using the homogeneity of the hyperbolic space we show in Proposition 7.1 that the singular part gives rise to a bounded operator from L^p to itself for all $p \in [1, \infty]$. In Proposition 7.2 we show that under certain assumptions on the decay of the Green function also the smoothing part gives rise to a bounded operator from L^p to L^p for certain p. Using the decay estimate obtained in Section 6 we then see in Section 8 that the L^p-spectrum of $\mathbb{M}_c^{m,k}$ is contained in the set σ_p given in Theorem 1.1.
Thus, it only remains to show that each element of σ_p is already in the L^p-spectrum of $\mathbb{M}_c^{m,k}$. For that we construct test spinors on $\mathbb{H}_c^{k+1}$ in Section 9 and finish the proof for product spaces in Section 10.

Acknowledgements. We thank an anonymous referee for many helpful comments.

2. PRELIMINARIES

2.1. **Notations and conventions.** In the article we will use the convention that a spin manifold is a manifold which admits a spin structure together with a fixed choice of spin structure.
Let (M, g) be a spin manifold and Σ_M the corresponding spinor bundle, see Section 2.3.

$\Gamma(\Sigma_M)$ denotes the space of spinors, i.e., sections of Σ_M. The space of smooth compactly supported sections is denoted by $C_c^\infty(M, \Sigma_M)$, or shortly $C_c^\infty(\Sigma_M)$. The hermitian metric on fibers of Σ_M is denoted by $\langle .,. \rangle$, the corresponding norm by $|.|$. For $s_1, s_2 \in \Gamma(M, \Sigma_M)$ we define the L^2-scalar product

$$(s_1, s_2)_{L^2(g)} := \int_M \langle s_1, s_2 \rangle \,\mathrm{dvol}_g.$$

For $s \in [1, \infty]$ $\|.\|_{L^s(g)}$ is the L^s-norm on (M^n, g). In case the underlying metric is clear from the context we abbreviate shortly by $\|.\|_s$.
$\mathrm{Spec}_{L^s}^M(D)$ denotes the spectrum of the Dirac operator on M viewed as an operator from L^s to L^s, cf. Appendix B.
We denote by $\pi_i \colon M \times M \to M$, $i = 1, 2$, the projection to the i-th component. Moreover, we set $\Sigma_M \boxtimes \Sigma_M^* := \pi_1^*(\Sigma_M) \otimes \pi_2^*(\Sigma_M^*)$.
$C^i(M)$ denotes the space of i-times continuously differentiable functions on M.
$B_\varepsilon(x) \subset M$ is the ball around $x \in M$ of radius ε w.r.t. the metric given on M.

A Riemannian manifold is of bounded geometry, if its injectivity radius is positive and the curvature tensor and all derivatives are bounded.
The metric on the k-dimensional sphere $\mathbb{S}^k$ with constant sectional curvature 1 will be denoted by σ^k. For $\mathbb{S}^k$ with metric $r^2\sigma^k$ we write $\mathbb{S}^k_r$.

2.2. **Coordinates and notations for $\mathbb{H}_c^{k+1}$ and its product spaces.** We use polar coordinates on $\mathbb{H}_c^{k+1}$. Namely, we fix a point $p_0 \in \mathbb{H}_c^{k+1}$ and an identification of $T_{p_0}\mathbb{H}_c^{k+1}$ with $\mathbb{R}^{k+1}$. Often we also use the exponential map to identify $T_{p_0}\mathbb{H}_c^{k+1} \cong \mathbb{R}^{k+1}$ with $\mathbb{H}_c^{k+1}$. In polar coordinates the metric is then $g_{\mathbb{H}_c^{k+1}} = \mathrm{d}r^2 + f(r)^2\sigma^k$ where σ^k is the standard metric on $\mathbb{S}^k$ and

$$f(r) := \sinh_c(r) := \begin{cases} \frac{1}{c}\sinh(cr) & \text{if } c \neq 0 \\ r & \text{if } c = 0. \end{cases}$$

In particular, the distance $r = \mathrm{dist}_{\mathbb{H}_c^{k+1}}(y, p_0)$ of y to $p_0 \cong 0$ w.r.t. $g_{\mathbb{H}_c^{k+1}}$ coincides with the euclidean one on $\mathbb{R}^{k+1}$. The subset $\{y \in \mathbb{H}_c^{k+1} \mid \mathrm{dist}_{\mathbb{H}_c^{k+1}}(y, 0) = r\}$ is isometric to $\mathbb{S}^k_{f(r)}$ and its (unnormalized) mean curvature is given by

$$\vec{H}_{\mathbb{S}^k_{f(r)}} = -k\frac{\partial_r f(r)}{f(r)}\partial_r = -k\coth_c(r)\partial_r \quad \text{where } \coth_c r := \begin{cases} c\coth(cr) & \text{if } c \neq 0 \\ \frac{1}{r} & \text{if } c = 0. \end{cases}$$

Let N be a closed Riemannian spin manifold. Note that we include the case where N is just a point. Set $\mathbb{M}_c^{m,k} := \mathbb{H}_c^{k+1} \times N$, and $\pi_{\mathbb{H}}$ shall denote the projection of $\mathbb{M}_c^{m,k}$ onto its $\mathbb{H}_c^{k+1}$-coordinates.

2.3. **General preliminaries about spin geometry.** The following can e.g. be found in [19]. A spin structure on M^m is a pair $(P_{\mathrm{Spin}}(M), \alpha)$ where $P_{\mathrm{Spin}}(M)$ is a principal $\mathrm{Spin}(m)$-bundle and where $\alpha\colon P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$ is a fiber map over the identity of M that is compatible with the double covering $\Theta\colon \mathrm{Spin}(m) \to \mathrm{SO}(m)$ and the corresponding group actions, i.e., the following diagram commutes

$$\begin{tikzcd} \mathrm{Spin}(m) \times P_{\mathrm{Spin}}(M) \arrow[r] \arrow[d, "\Theta\times\alpha"] & P_{\mathrm{Spin}}(M) \arrow[d, "\alpha"] \arrow[dr] & \\ & & M \\ \mathrm{SO}(m) \times P_{\mathrm{SO}}(M) \arrow[r] & P_{\mathrm{SO}}(M) \arrow[ur] & \end{tikzcd}$$

Let Σ_m be an irreducible representation of Cl_m. In case m is odd there are two such irreducible representations. Both of them coincide if considered as $\mathrm{Spin}(m)$-representations. For m odd we fix one of the two representations for the whole article. If m is even, there is only one irreducible Cl_m-representation of Σ_m, but it splits into non-equivalent subrepresentations $\Sigma_m^{(+)}$ and $\Sigma_m^{(-)}$ as $\mathrm{Spin}(m)$-representations.
Define $\omega_M = \mathrm{i}^{[\frac{m+1}{2}]}e_1 \cdot e_2 \cdot \ldots \cdot e_m$ with $(e_i)_i$ being a positively oriented orthonormal frame on M. We have $\omega_M^2 = 1$. If m is odd, then ω_M commutes with the Clifford multiplication and thus acts as $+1$ or -1 on Σ_m. If m is even, the it acts as ± 1 on $\Sigma_m^{(\pm)}$. Sections of $\Sigma_m^{(+)}$ (resp. $\Sigma_m^{(-)}$) are called positive (resp. negative) spinors.
In the article it will be convenient to use the notation $\Sigma_m^{(\varepsilon)}$, $\varepsilon \in \{0+, -\}$ also in the case m odd as well, by setting $\Sigma_m^{(\varepsilon)} = \Sigma_m$ for both choices of ε.
The spinor bundle Σ_M is defined as $\Sigma_M = P_{\mathrm{Spin}}(M) \times_{\rho_m} \Sigma_m$ where $\rho_m\colon \mathrm{Spin}(m) \to \mathrm{End}(\Sigma_m)$ is the complex spinor representation. Moreover, the spinor bundle is endowed with a Clifford multiplication, denoted by '$\cdot$', $\cdot\colon TM \to \mathrm{End}(\Sigma_M)$. Then, the Dirac operator acting on the space of smooth sections of Σ_M is defined as the composition of the

connection ∇ on Σ_M (obtained as a lift of the Levi-Civita connection on TM) and the Clifford multiplication. Thus, in local coordinates this reads as

$$D = \sum_{i=1}^{m} e_i \cdot \nabla_{e_i}$$

where $(e_i)_{i=1,\dots,m}$ is a local orthonormal basis of TM. The Dirac operator is formally self-adjoint as an operator on L^2, i.e., for $\psi \in C^\infty(M,\Sigma_M)$ and $\varphi \in C_c^\infty(M,\Sigma_M)$ we have $(\varphi, D\psi) = (D\varphi, \psi)$.

As M is complete, the Dirac operator is not only formally self-adjoint, but actually has a unique self-adjoint extension that is a densely defined unbounded operator $D\colon H_1^2(\Sigma_M) \subset L^2(\Sigma_M) \to L^2(\Sigma_M)$, see [41]. Here $H_1^2(\Sigma_M)$ is the subset of all $\varphi \in L^2(\Sigma_M)$ such that $\nabla\varphi$, defined in the distributional sense, is again in L^2. The notation above indicates that $H_1^2(\Sigma_M)$ is the domain of the operator, and that "densely defined" and "self-adjoint" should be understood as an unbounded operator from $L^2(\Sigma_M)$ to $L^2(\Sigma_M)$. From the spectral theorem it follows that $D - \mu\colon H_1^2(\Sigma_M) \subset L^2(\Sigma_M) \to L^2(\Sigma_M)$ is invertible for all $\mu \notin \mathbb{R}$, in the sense that the inverse is a bounded operator from $L^2(\Sigma_M)$ to $L^2(\Sigma_M)$. See also Appendices A and B for more details.

2.4. **Dual spinors.** The hermitian metric induces a natural isomorphism from Σ_M^* to $\bar{\Sigma}_M$. In this way we obtain a metric connection and a Clifford multiplication on Σ_M^* and this allows us to define a Dirac operator $D^{\mathrm{t}}\colon C^\infty(\Sigma_M^*) \to C^\infty(\Sigma_M^*)$. Locally $D^{\mathrm{t}} f = \sum_i e_i \cdot \nabla_{e_i} f$ where $f \in C^\infty(\Sigma_M^*)$ and e_i is a local orthonormal frame on M. Completely analogously to the proof that the usual Dirac operator is formally self-adjoint, one proves that for $f \in C^\infty(\Sigma_M^*)$, $\varphi \in C^\infty(\Sigma_M)$ such that $\operatorname{supp} f \cap \operatorname{supp}\varphi$ is relatively compact we have

$$\int D^{\mathrm{t}} f(\varphi)\mathrm{dvol}_g = \int f(D\varphi)\mathrm{dvol}_g.$$

2.5. **Spinors on product manifolds.** In this subsection our notation is close to [8]. Let $(P^{m+n} = M^m \times N^n,\ g_P = g_M + g_N)$ be a product of Riemannian spin manifolds (M, g_M) and (N, g_N). We have

$$P_{\mathrm{Spin}}(M \times N) = (P_{\mathrm{Spin}}(M) \times P_{\mathrm{Spin}}(N)) \times_\xi \Sigma_{m+n}$$

where $\xi\colon \mathrm{Spin}(m) \times \mathrm{Spin}(n) \to \mathrm{Spin}(m+n)$ is the Lie group homomorphism lifting the standard embedding $\mathrm{SO}(m) \times \mathrm{SO}(n) \to \mathrm{SO}(m+n)$. Note that ξ is not an embedding, its kernel is $(-1,-1)$, where -1 denotes the non-trivial element in the kernel of $\mathrm{Spin}(m) \to \mathrm{SO}(m)$ resp. $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

The spinor bundle can be identified with

$$\Sigma_P = \begin{cases} \Sigma_M \otimes (\Sigma_N \oplus \Sigma_N) & \text{if both } m \text{ and } n \text{ are odd} \\ \Sigma_M \otimes \Sigma_N & \text{else,} \end{cases}$$

and the Levi-Civita connection acts as $\nabla^{\Sigma_M \otimes \Sigma_N} = \nabla^{\Sigma_M} \otimes \mathrm{Id}_{\Sigma_N} + \mathrm{Id}_{\Sigma_M} \otimes \nabla^{\Sigma_N}$. This identification can be chosen such that for $X \in TM$, $Y \in TN$, $\varphi \in \Gamma(\Sigma_M)$, and $\psi = (\psi_1, \psi_2) \in \Sigma_N \oplus \Sigma_N$ for both n and m odd and $\psi \in \Gamma(\Sigma_N)$ otherwise, we have

$$(X, Y) \cdot_P (\varphi \otimes \psi) = (X \cdot_M \varphi) \otimes (\omega_N \cdot_N \psi) + \varphi \otimes (Y \cdot_N \psi)$$

where for both n and m odd we set $\omega_N \cdot_N (\psi_1, \psi_2) := \mathrm{i}(\psi_2, -\psi_1)$ and $Y \cdot_N (\psi_1, \psi_2) := (Y \cdot_N \psi_2, Y \cdot_N \psi_1)$.

The Dirac operator is then given by

$$D^P(\varphi \otimes \psi) = (D^M \varphi \otimes \omega_N \cdot_N \psi) + (\varphi \otimes \tilde{D}^N \psi)$$

where $\tilde{D}^N = \mathrm{diag}(D^N, -D^N)$ if both m and n are odd and $\tilde{D}^N = D^N$ otherwise.

Since $\omega_N \cdot$ and $\tilde{D}^N$ anticommute, $D^M \otimes \omega_N$ and $\mathrm{id} \otimes \tilde{D}^N$ anticommute as well. Thus

$$(D^P)^2 = (D^M)^2 \otimes \mathrm{id} + \mathrm{id} \otimes (\tilde{D}^N)^2. \tag{1}$$

If we additionally assume that M and N are compact, then this equation implies together with the spectral theorem that the L^2-spectrum of the square of the Dirac operator on the product space $M \times N$ is given by

$$\{\lambda^2 + \mu^2 \mid \lambda^2 \in \mathrm{Spec}^M_{L^2}((D^M)^2), \mu^2 \in \mathrm{Spec}^N_{L^2}((D^N)^2)\}.$$

2.6. **The L^2-spectrum of the Dirac operator.** The simplest case of Theorem 1.1 is when $p = 2$ and when N is a point. In this case we have by definition $\lambda_0 = 0$. The theorem then states that the L^2-spectrum of the Dirac operator on $\mathbb{H}^{k+1}_c$ is the whole real line $\mathbb{R} \subset \mathbb{C}$. This fact was already known before, by scaling it suffices to consider the Euclidean case $c = 0$ and the hyperbolic case $c = 1$.

On Euclidean space $\mathbb{R}^{k+1} = \mathbb{H}^{k+1}_0$ it is an easy calculation. The spinor bundle is trivialized by parallel orthonormal sections $s_1, \ldots, s_t$ with t the rank of the spinor bundle. The Dirac operator preserves $V_i := C^\infty(\mathbb{R}^{k+1}) \cdot s_i$, anticommutes with the Clifford action by parallel vector fields, and its square maps $f \cdot s_i$ to $(\Delta f) \cdot s_i$. As Δ has L^2 spectrum $[0, \infty)$ on $\mathbb{R}^{k+1}$, the statement follows.

On hyperbolic space, the L^2-spectrum of the Dirac operator was calculated by representation theoretical methods in [11]. Unfortunately [11] contains a small error, and the value 0 is not an eigenvalue of the Dirac operator in contrast to what is claimed in [11]. The argument can be repaired, and it follows that the spectrum is as claimed. A complete proof was given in [13], using similar techniques as in the present article.

We will not give a proof of this special case of Theorem 1.1, instead we will use this special case for the proof of the full version of Theorem 1.1.

Using the results of the previous subsection, we also get the L^2-spectrum of the Dirac operator on $\mathbb{M}^{m,k}_c = \mathbb{H}^{k+1}_c \times N$: With Lemma B.8 we get

$$\mathrm{Spec}^{\mathbb{M}_c}_{L^2}(D^2) = [\lambda_0^2, \infty).$$

Together with Lemma B.11, Example B.12 and Lemma B.8 this yields

$$\mathrm{Spec}^{\mathbb{M}_c}_{L^2}(D) = (-\infty, -\lambda_0] \cup [\lambda_0, \infty).$$

2.7. **A covering lemma.**

Lemma 2.1 (Covering lemma). *Let (M, g) be a Riemannian manifold of bounded geometry, and let $R > 0$. Then there are points $(x_i)_{i \in I} \subset M$ where I is a countable index set such that*

(i) *the balls $B_R(x_i)$ are pairwise disjoint and*
(ii) *$(B_{2R}(x_i))_{i \in I}$ and $(B_{3R}(x_i))_{i \in I}$ are both uniformly locally finite covers of M.*

Proof. Choose a maximal family of points $(x_i)_{i \in I}$ in M such that the sets $B_R(x_i)$ are pairwise disjoint. Then $\bigcup_{i \in I} B_{2R}(x_i) = M$. For $y \in M$ let $L(y) = \{i \in I \mid y \in B_{3R}(x_i)\}$. For $i \in L(y)$ we have $B_R(x_i) \subset B_{4R}(y)$ and, thus,

$$\bigsqcup_{i \in L(y)} B_R(x_i) \subset B_{4R}(y),$$

where $\sqcup$ denotes disjoint union. Comparing the volumes of both sides and using the bounded geometry of M we see that there exists a number L_R such that $|L(y)| \leq L_R$ for all $y \in M$. Thus, the covering by sets $B_{3R}(x_i)$, and hence the one by $B_{2R}(x_i)$, is uniformly locally finite. $\square$

2.8. **Interpolation theorems.**

Theorem 2.2 (Riesz-Thorin Interpolation Theorem, [40, Theorem II.4.2]). *Let T be an operator defined on a domain $\mathcal{D}$ that is dense in both L^q and L^p. Assume that $Tf \in L^q \cap L^p$ for all $f \in \mathcal{D}$ and that T is bounded in both norms. Then, for any r between p and q the operator T is a bounded operator from L^r to L^r.*

Theorem 2.3 (Stein Interpolation Theorem, [16, Section 1.1.6], [30, Theorem IX.21])**.** *Let $p_0, q_0, p_1, q_1 \in [1, \infty]$, $0 < t < 1$, and $S = \{z \in \mathbb{C} \mid 0 \le \operatorname{Re} z \le 1\}$. Let A_z be linear operators from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} + L^{q_1}$ for all $z \in S$ with the following properties*

(i) *$z \mapsto \langle A_z f, g\rangle$ is uniformly bounded and continuous on S and analytic in the interior of S whenever $f \in L^{p_0} \cap L^{p_1}$ and $g \in L^{r_0} \cap L^{r_1}$ where r_i is the conjugate exponent to q_i.*

(ii) *There is $M_0 > 0$ such that $\|A_{\mathrm{i}y} f\|_{q_0} \le M_0 \|f\|_{p_0}$ for all $f \in L^{p_0} \cap L^{p_1}$ and $y \in \mathbb{R}$.*

(iii) *There is $M_1 > 0$ such that $\|A_{1+\mathrm{i}y} f\|_{q_1} \le M_1 \|f\|_{p_1}$ for all $f \in L^{p_0} \cap L^{p_1}$ and $y \in \mathbb{R}$.*

Then, for $1/p = t/p_1 + (1-t)/p_0$ and $1/q = t/q_1 + (1-t)/q_0$

$$\|A_t f\|_q \le M_1^t M_0^{1-t} \|f\|_p$$

for all $f \in L^{p_0} \cap L^{p_1}$. Hence, A_t can be extended to a bounded operator from L^p to L^q with norm at most $M_1^t M_0^{1-t}$.

3. The Green function

In this section, we collect results on existence and properties of the Green function of the Dirac operator D and its shifts $D - \mu$, $\mu \in \mathbb{C}$. They are obvious applications of standard methods, but a suitable reference does not exist yet. Unless otherwise stated we only assume in this section that the Riemannian spin manifold (M, g) is complete. Let $\Delta := \{(x, x) \mid x \in M\} \subset M \times M$ be the diagonal.

Definition 3.1. [6, Definition 2.1] A smooth section $G_{D-\mu} \colon M \times M \setminus \Delta \to \Sigma_M \boxtimes \Sigma_M^*$ that is locally integrable on $M \times M$ is called a *Green function* of the shifted Dirac operator $D - \mu$ if

$$(D_x - \mu)(G_{D-\mu}(x, y)) = \delta_y \operatorname{Id}_{\Sigma_M|_y} \tag{2}$$

in the sense of distributions, i.e., for any $y \in M$, $\psi_0 \in \Sigma_M|_y$, and $\varphi \in C_c^\infty(\Sigma_M)$

$$\int_M \langle G_{D-\mu}(x, y)\psi_0, (D - \bar{\mu})\varphi(x)\rangle \mathrm{d}x = \langle \psi_0, \varphi(y)\rangle$$

and $G_{D-\mu}(., y) \in L^2(M \setminus B_r(y))$ for any $r > 0$.

In case that the operator $D - \mu$ is clear from the context, we shortly write $G = G_{D-\mu}$.

Proposition 3.2. *If M is a closed Riemannian spin manifold with invertible operator $D - \mu \colon H_1^2(\Sigma_M) \subset L^2(\Sigma_M) \to L^2(\Sigma_M)$, then a unique Green function exists. This Green function is the integral kernel of the inverse of $D - \mu \colon H_1^2(\Sigma_M) \subset L^2(\Sigma_M) \to L^2(\Sigma_M)$.*

To prove the well-known proposition, one usually starts by showing the existence of a parametrix.

Lemma 3.3. [28, III.§4] *Let M be a closed Riemannian spin manifold. Then there is a smooth section $P_{D-\mu} \colon M \times M \setminus \Delta \to \Sigma_M \boxtimes \Sigma_M^*$, called parametrix, which is L^1 on $M \times M$ and which satisfies*

$$(D_x - \mu)(P_{D-\mu}(x, y)) = \delta_y Id_{\Sigma_M|_y} + R(x, y)$$

in the distributional sense for a smooth section $R \colon M \times M \to \Sigma_M \boxtimes \Sigma_M^*$.

Let $\varepsilon > 0$ be smaller than the injectivity radius. We choose $\chi \colon [0, \infty) \to [0, 1]$ smooth with support in $[0, \varepsilon]$ and constant to 1 in a neighborhood of 0. The statement of the lemma is still true is we replace $P_{D-\mu}(x, y)$ by $\chi(\operatorname{dist}(x, y)) P_{D-\mu}(x, y)$ and an appropriately modified R. We thus can assume without loss of generality that the support $P_{D-\mu}(x, y)$ and thus the support of R is in $(M \times M)_\varepsilon := \{(x, y) \in M \times M \mid \operatorname{dist}(x, y) < \varepsilon\}$.
Convolution with $P_{D-\mu}$ defines an operator $\mathtt{P}_{D-\mu}$ by

$$(\mathtt{P}_{D-\mu}\psi, \varphi) = \int_M \int_M \langle P_{D-\mu}(x, y)\psi(y), \varphi(x)\rangle \, \mathrm{d}x \, \mathrm{d}y$$

for all $\psi, \varphi \in C_c^\infty(\Sigma_M)$. Then, $\mathtt{P}_{D-\mu}$ is a right inverse to $D-\mu$ up to infinitely smoothing operators. We thus call it a right parametrix. The existence of such a right parametrix follows using the symbol calculus from the fact that D is an elliptic operator. An efficient and very readable overview over how to construct a right parametrix for an elliptic differential operator on a compact manifold can be found e.g. in [28, III.§4], although the reader should pay attention to the fact that it is not so obvious that the different notions of infinitely smoothing operators used in there are in fact all equivalent. The latter fact follows from standard techniques used in the theory of pseudo differential operators, see e.g. [1] or [35] for textbooks on this subject.

Proof of Proposition 3.2. From the last Lemma we have the existence of a parametrix $P_{D-\mu}(x,y)$. We will use the notations of that Lemma. Since $D-\mu$ is assumed to be invertible, there is a section $P'_{D-\mu}\colon M\times M \to \Sigma_M \boxtimes \Sigma_M^*$ with $(D_x-\mu)P'_{D-\mu}(x,y) = R(x,y)$. By elliptic regularity $P'_{D-\mu}(x,y)$ is smooth in x for each $y \in M$. Again by elliptic regularity $(D-\mu)^{-1}$ is a continuous linear operator $C_c^\infty(\Sigma_M) \to C^\ell(\Sigma_M)$ for any $\ell \in \mathbb{N}$. Further $P'_{D-\mu}(\,\cdot\,, y)$ is the composition

$$\Sigma_M \longrightarrow C_c^\infty(\Sigma_M) \overset{(D-\mu)^{-1}}{\longrightarrow} C^\ell(\Sigma_M)$$

where the first map is the smooth map that maps $\varphi \in \Sigma_M|_y$ to $R(\,\cdot\,, y)\varphi$. This implies that $P'_{D-\mu}(x,y)$ is smooth as function in both x and y. We set $G_{D-\mu}(x,y) = P_{D-\mu}(x,y) - P'_{D-\mu}(x,y)$ and obtain $(D_x-\mu)(G_{D-\mu}(x,y)) = \delta_y \operatorname{Id}_{\Sigma_M|_y}$. Moreover, since $P_{D-\mu}$ is L^1 on $M\times M$ and $P'_{D-\mu}$ is smooth in both entries the Green function $G_{D-\mu}$ is L^1 as well. Furthermore, $P_{D-\mu}(.,y)$ is smooth on $M\setminus B_r(y))$ for any $r>0$ and, hence, the same is true for $G_{D-\mu}(.,y)$. In particular, $G_{D-\mu}(.,y) \in L^2(M\setminus B_r(y))$. If $\tilde{G}_{D-\mu}$ is a possibly different Green function of $D-\mu$ then $(D-\mu)(G_{D-\mu}(.,y) - \tilde{G}_{D-\mu}(.,y)) = 0$ for all $y\in M$. As $D-\mu$ is invertible we have $G_{D-\mu} = \tilde{G}_{D-\mu}$. □

As for $P_{D-\mu}$, convolution with $G_{D-\mu}$ defines an operator $\mathtt{G}_{D-\mu}$ by

$$(\mathtt{G}_{D-\mu}\psi, \varphi) = \int_M \int_M \langle G_{D-\mu}(x,y)\psi(y), \varphi(x)\rangle \mathrm{d}x\mathrm{d}y$$

for all $\psi, \varphi \in C_c^\infty(\Sigma_M)$. By construction $\mathtt{G}_{D-\mu}$ is the right inverse of $D-\mu$, and is thus even defined on L^2. Since the inverse of $D-\mu$ exists by assumption, $\mathtt{G}_{D-\mu} = (D-\mu)^{-1}$, and $\mathtt{G}_{D-\mu}$ is in particular also a left inverse of $D-\mu$.

Lemma 3.4. *Let M be a closed Riemannian spin manifold, and let $D-\mu$ be invertible. Then $G_{D-\mu}(x,y)$ is the adjoint of $G_{D-\bar{\mu}}(y,x)$, i.e. $G_{D-\bar{\mu}}(y,x)$ is the integral kernel of the adjoint operator of $\mathtt{G}_{D-\mu}$.*

Proof. Using the definitions and discussions from above and Lemma B.3(ii) we have $\mathtt{G}^*_{D-\mu} = ((D-\mu)^{-1})^* = (D-\bar{\mu})^{-1} = \mathtt{G}_{D-\bar{\mu}}$. In particular, we get for all $\psi, \varphi \in L^2(\Sigma_M)$ that

$$\begin{aligned}(\psi, \mathtt{G}^*_{D-\mu}\varphi) =& (\mathtt{G}_{D-\mu}\psi, \varphi) = ((D-\mu)^{-1}\psi, \varphi) = (\psi, (D-\bar{\mu})^{-1}\varphi) \\ =& \int\int \langle \psi(y), G_{D-\bar{\mu}}(y,x)\varphi(x)\rangle \mathrm{d}y\mathrm{d}x.\end{aligned}$$

□

Moreover, we have

Lemma 3.5. *In the situation of Lemma 3.4 we have $(D_y^{\mathrm{t}} - \mu)G_{D-\mu}(x,y) = \delta_x \operatorname{Id}_{\Sigma_M^*|_x}$, i.e., for $f_0 \in \Gamma(\Sigma_M^*|_x)$, $\varphi \in C_c^\infty(\Sigma_M)$*

$$\int ((D_y^{\mathrm{t}} - \mu)G_{D-\mu}(x,y)f_0)(\varphi(y))\mathrm{d}y = f_0(\varphi(x)).$$

Proof.
$$\begin{aligned}\int((D_y^{\mathrm{t}}-\mu)G_{D-\mu}(x,y)f_0)(\varphi(y))\mathrm{d}y=&\int(G_{D-\mu}(x,y)f_0)((D_y-\mu)\varphi(y))\mathrm{d}y\\ =&\int f_0(G_{D-\mu}(x,y)(D_y-\mu)\varphi(y))\mathrm{d}y\\ =&f_0(\varphi(x)).\end{aligned}$$
where the last step uses that $\mathtt{G}_{D-\mu}$ is also the left inverse of $D-\mu$.

□

Now, M has no longer to be closed, but we assume bounded geometry.

Proposition 3.6. *Let (M,g) be a Riemannian spin manifold of bounded geometry. Let $D-\mu\colon H_1^2(\Sigma_M)\subset L^2(\Sigma_M)\to L^2(\Sigma_M)$ be invertible. Then there exists a unique Green function.*

Proof. We choose $R>0$ such that $3R$ is smaller than the injectivity radius. Let $(x_i)_{i\in I}$ be as in the Covering Lemma 2.1. Recall that $(M\times M)_\varepsilon:=\{(x,y)\in M\times M\,|\,\mathrm{dist}(x,y)<\varepsilon\}$. Because of $M=\bigcup_{i\in I}B_{2R}(x_i)$ we have
$$(M\times M)_R\subset\bigcup_{i\in I}B_{3R}(x_i)\times B_{3R}(x_i).$$
We embed each ball $B_{3R}(x_i)$ isometrically into a closed connected manifold M_{x_i}, which is diffeomorphic to a sphere and $D^{M_{x_i}}-\mu$ is invertible. This can always be achieved by local metric deformation on $M_{x_i}\setminus B_{3R}(x_i)$, see Proposition C.1.
Thus, by Proposition 3.2 the operator $D^{M_{x_i}}-\mu$ possesses a Green function $G^{x_i}(x,y)$ with $(D_x^{M_{x_i}}-\mu)G^{x_i}(x,y)=\delta_y\,\mathrm{Id}_{\Sigma_y}$. By abuse of notation we will view $G^{x_i}(x,y)$ for $x,y\in B_{3R}(x_i)$ also as a partially defined section of $\Sigma_M\boxtimes\Sigma_M^*\to M\times M$, which is defined on $B_{3R}(x_i)\times B_{3R}(x_i)$.
Now we choose smooth functions a_i on $M\times M$ such that $\operatorname{supp}a_i\subset B_{3R}(x_i)\times B_{3R}(x_i)\subset(M\times M)_{6R}$ and such that $\sum_{i\in I}a_i$ equals to 1 on $(M\times M)_{R/2}$. Now we set
$$H(x,y)=\sum_{i\in I}a_i(x,y)G^{x_i}(x,y).$$
This implies $\operatorname{supp}H\subset(M\times M)_{6R}$. Moreover, $H(.,y)\in L^2(M\setminus B_r(y))$ for all $r>0$ since this is true for each summand.
Our next goal is to prove that $(D_x-\mu)H(x,y)-\delta_y\,\mathrm{Id}_{\Sigma_y}$ is smooth. Note that $G^{x_i}(x,y)$ and $G^{x_j}(x,y)$ are both defined for $(x,y)\in(B_{3R}(x_i)\times B_{3R}(x_i))\cap(B_{3R}(x_j)\times B_{3R}(x_j))$, but they will not coincide in general. On the other hand their defining property and the locality of the differential operator D (cp. Lemma 3.5) imply that
$$(D_x-\mu)\left(G^{x_i}(x,y)-G^{x_j}(x,y)\right)=(D_y^{\mathrm{t}}-\mu)\left(G^{x_i}(x,y)-G^{x_j}(x,y)\right)=0.$$
Thus,
$$\underbrace{\left((D_x-\mu)^2+(D_y^{\mathrm{t}}-\mu)^2\right)}_{=:P}\left(G^{x_i}(x,y)-G^{x_j}(x,y)\right)=0.$$
Since P is an elliptic operator, elliptic regularity implies that $G^{x_i}(x,y)-G^{x_j}(x,y)$ viewed as a difference of distributions is a smooth function on $(B_{3R}(x_i)\times B_{3R}(x_i))\cap(B_{3R}(x_j)\times B_{3R}(x_j))$, and thus $a_j(x,y)\left(G^{x_i}(x,y)-G^{x_j}(x,y)\right)$ as well. On $B_{3R}(x_j)\times B_{3R}(x_j)$ we rewrite
$$H(x,y)=G^{x_j}(x,y)+\sum_{i\in I\setminus\{j\}}a_i(x,y)\left(G^{x_i}(x,y)-G^{x_j}(x,y)\right),$$
and we conclude that $(D_x-\mu)H(x,y)=\delta_y\,\mathrm{Id}_{\Sigma_y}+F(x,y)$ where $F(x,y)$ is a smooth section of $\Sigma_M\boxtimes\Sigma_M^*$ with support in $(M\times M)_{6R}$.
There is a unique section H' of $\Sigma_M\boxtimes\Sigma_M^*$ such that $(D_x-\mu)H'(x,y)=F(x,y)$ and such that $H'(.,y)$ is L^2 for all y. This follows for each y from the assumption that $D-\mu$ is invertible.

As $D-\mu$ is a linear operator with continuous inverse and by elliptic regularity H' is smooth in x and y.
We set $G(x,y) = H(x,y) - H'(x,y)$, and this gives a Green function for $D-\mu$. Smoothness of G follows by smoothness of all G^{x_i}, and smoothness of F and H'.
Assume that G and $\tilde{G}$ are two Green functions for D. We set $\psi_y(x) := (G-\tilde{G})(x,y))$. For almost all $y \in M$ the properties of the Green function imply $(D-\mu)\psi_y = 0$ in the sense of distributions, $\psi_y \in L^1_{\mathrm{loc}}(M)$ and $\psi_y|_{M\setminus B_r(y)} \in L^2(M \setminus B_r(y))$. Local regularity implies $\psi_y \in L^2(M)$. By the invertibility of $D-\mu$ we btain $\psi_y = 0$ for almost all $y \in M$ and thus $G = \tilde{G}$. □

Note that due to the last Proposition Lemmata 3.4 and 3.5 also hold true for manifolds M of bounded geometry.

We finish this section by stating another property of the Green function:

Lemma 3.7. *Let (M,g) be a Riemannian spin manifold of bounded geometry, and let $D-\mu$ be invertible. Then the Green function also decays in L^2 in the second entry, i.e., $G_{D-\mu}(x,.) \in L^2(M \setminus B_r(x))$ for all $r > 0$.*

Proof. The Green function $G_{D-\bar{\mu}}(.,x)$ is in $L^2(M \setminus B_r(x))$ in the first component. Then the claim follows from Lemma 3.4 in the extended version to manifolds M of bounded geometry. □

4. The Dirac operator on hyperbolic space and its products

In this section we examine the Dirac operator on the model spaces $\mathbb{M}_c^{m,k} = \mathbb{H}_c^{k+1} \times N$. Note that we also allow the case where N is zero dimensional. First, we introduce polar coordinates on $\mathbb{H}_c^{k+1}$ and write the Dirac operator in these coordinates. For this purpose it is convenient to introduce three foliations on the space $M := (\mathbb{H}_c^{k+1} \times N) \setminus (\{p_0\} \times N)$, and to relate the associated Dirac operator with the Dirac operator on the total space. Then, we study the canonical action of $\mathrm{Spin}(k+1)$ on $\mathbb{M}_c^{m,k}$ and its spinor bundle.

4.1. **The Dirac operator on a foliation.** In this subsection we want to keep the notation rather general to allow other applications in future work. So we recall some well-known facts about Dirac operators on foliations, and describe them in a way suitable for our needs.
We assume that M is a Riemannian spin manifold with the Levi-Civita connection ∇^M. To define foliations on M, we will choose our terminology close to Warner's book, see [38, 1.56–1.64].
We assume that $\mathcal{F} = \bigcup_{x\in M} \mathcal{F}_x$ is an ℓ-dimensional *distribution* on M, so for each $x \in M$, $\mathcal{F}_x$ is an ℓ-dimensional subspace of T_xM, depending smoothly on x. In particular a distribution $\mathcal{F}$ on M defines a vector bundle over M, denoted by $\mathcal{F} \to M$. A vector field $X \in C^\infty(M,TM)$ is called *tangential* to $\mathcal{F}$ if for all $x \in M$ we have $X_x \in \mathcal{F}_x$, and we then simply write $X \in C^\infty(M,\mathcal{F})$. The distribution $\mathcal{F}$ is called *involutive* if $X, Y \in C^\infty(M,\mathcal{F})$ implies $[X,Y] \in C^\infty(M,\mathcal{F})$. A *foliation* is defined as an involutive distribution.
It follows from Frobenius' Theorem that there exists a family of disjoint connected ℓ-dimensional submanifolds $(Z_b)_{b\in B}$, the *leaves* of the foliations, such that $M = \bigcup_{b\in B} Z_b$ and such that $T_xZ_b = \mathcal{F}_x$ for all $x \in Z_b \subset M$ and all $b \in B$.
Note that in general B is just a set. In the cases considered in the following subsections, we can achieve that B is a smooth manifold with Z_b depending smoothly on b and such that all manifolds Z_b are diffeomorphic to each other, but we will not assume this in this subsection.
For each $x \in M$ let ν_x be the orthogonal complement of $\mathcal{F}_x$ in T_xM. We obtain a vector bundle $\nu := \bigcup_{x\in M} \nu_x$ over M, the *normal bundle.* As vector bundles with scalar product we have $TM = \mathcal{F} \oplus \nu$.
Each leaf will be equipped with the Riemannian metric induced from M. Then the Levi-Civita connection provides a connection, called the *intrisic connection* ∇^{int}, on the bundle $TZ_b \to Z_b$ for every $b \in B$. Also the restriction of ν to a leaf Z_b carries a natural connection,

also denoted by ∇^{int}; for $X \in C^\infty(Z_b, TZ_b)$ and $W \in C^\infty(Z_b, \nu|_{Z_b})$ we define $\nabla^{\mathrm{int}}_X W$ as the normal component of $\nabla^M_X W$.

Remark 4.1. *These connections actually yield a* partial connections *on M, i.e. a bilinear maps $\nabla^{\mathrm{int}}\colon C^\infty(M,\mathcal{F}) \times C^\infty(M,\mathcal{F}) \to C^\infty(M,\mathcal{F})$ resp. $\nabla^{\mathrm{int}}\colon C^\infty(M,\mathcal{F}) \times C^\infty(M,\nu) \to C^\infty(M,\nu)$ which are $C^\infty(M)$-linear in the first argument and which satisfy the usual product rule for $C^\infty(M)$-multiplication in the second argument. The partial connection on $C^\infty(M,\mathcal{F})$ is characterized by being torsionfree and metric. However both partial connections are not a connections in the ordinary sense on M as ∇^{int}_X is not defined if X is not tangential to $\mathcal{F}$.*

Although $TM = \mathcal{F} \oplus \nu$ holds in the sense of vector bundles, it does not hold for vector bundles with partial connection. The difference between the (partial) connection on TM and the partial connection of the sum $\mathcal{F}\oplus\nu$ is the second fundamental form II. More exactly for $X \in T_x Z_b$, $Y \in C^\infty(TZ_B)$ and $W \in C^\infty(\nu_B)$ we have

$$\nabla^M_X Y - \nabla^{\mathrm{int}}_X Y = \mathrm{II}_{Z_b}(X,Y),\ \langle \underbrace{\nabla^M_X W - \nabla^{\mathrm{int}}_X W}_{\in T_x Z_b}, Y\rangle = -\langle \mathrm{II}_{Z_b}(X,Y), W\rangle.$$

The bundle of positively oriented orthornormal frames of M will be denoted by $P_{\mathrm{SO}}(M)$, and let $P_{\mathrm{Spin}}(M) \to P_{\mathrm{SO}}(M)$ be a spin structure on M. We restrict these principal bundles to Z_b and obtain principal bundles $P_{\mathrm{Spin}}(M)|_{Z_b}$ and $P_{\mathrm{SO}}(M)|_{Z_b}$ over Z_b. On $TM|_{Z_b} = TZ_b \oplus \nu|_{Z_b}$ we have a connection defined by ∇^M and another connection defined by ∇^{int}. Both connections define connection-1-forms on $P_{\mathrm{SO}}(M)|_{Z_b}$ and they lift to connection-1-forms on $P_{\mathrm{Spin}}(M)|_{Z_b}$. Finally both connection-1-forms yield a connection on the associated spinor bundle again denoted by ∇^M and ∇^{int}. Note that the associated spinor bundle coincides with $\Sigma_M|_{Z_b}$, and ∇^M is just the restriction of the standard connection on Σ_M to Z_b. These connections are in fact partial connections on the bundle Σ_M.
A calculation shows for all $X \in C^\infty(M,\mathcal{F})$ and all spinors $\varphi \in C^\infty(M,\Sigma_M)$

$$\nabla^M_X \varphi = \nabla^{\mathrm{int}}_X \varphi + \frac{1}{2}\sum_i e_i \cdot \mathrm{II}_{Z_b}(X, e_i)\cdot\varphi \tag{3}$$

where $(e_i)_i$ is a local positively oriented orthonormal frame on $\mathcal{F}$, cp. [8, around (9)].

Remark 4.2. *In* [8] *a slightly different notation is used, as can be seen in the following dictionary of notations*

Bär [8]	Q	M	∇^Q *and* $\nabla^{\Sigma Q}$	$\nabla^M \oplus \nabla^N$ *and* $\nabla^{\Sigma M}\otimes \mathrm{id} + \mathrm{id}\otimes\nabla^{\Sigma N}$	$\widehat{D}$	$\tilde{D}$
Our article	$M \subset \mathbb{H}^{k+1}_c \times N$	Z	∇^M	$\approx \nabla^{\mathrm{int}}$	D^Z_∂	D^Z_{int}

Furthermore, in [8] *one single submanifold is considered whereas we foliated the space. But this is not an essential difference.*
In order to avoid missunderstandings, let us mention that a slightly different formalism is used in [9]*, it introduces an internal and an ambient Clifford multiplication, which differ by multiplication with a unit normal field. This is convenient if the submanifold is a hypersurface. However as we need arbitrary codimensions this is not helpful for our purpose.*
In fact, in [9] *the Clifford multiplication of the ambient manifold coincides with the Clifford multiplication on the hypersurface only up to Clifford multiplication with the normal vector field. In contrast to this in our notation the Clifford multiplication of the ambient space M coincides with the one on the submanifolds Z_b.*

The partial Dirac operator $D^{\mathcal{F}}_\partial$ is now defined as $D^{\mathcal{F}}_\partial = \sum_{i=1}^\ell e_i \cdot \nabla^M_{e_i}$, and the intrinsic Dirac operators are given by $D^{\mathcal{F}}_{\mathrm{int}} = \sum_{i=1}^\ell e_i \cdot \nabla^{\mathrm{int}}_{e_i}$. As this definition does not depend on the choice of frame, it yields a global definition. Note that $(D^{\mathcal{F}}_\partial\varphi)|_{Z_b}$ and $(D^{\mathcal{F}}_{\mathrm{int}})|_{Z_b}$ only depend on $\varphi|_{Z_b}$, each of them defines a family of Dirac-type operators on the leaves. On each leaf the intrinsic Dirac operator is locally a twisted Dirac operator on the submanifold N, the twist bundle is either the normal bundle if $\ell(\dim M-\ell)$ is even, or two copies of the normal bundle $\ell(\dim M-\ell)$ is odd.

In the applications in the following subsections all normal bundles have a parallel trivialization, hence, in this case the intrinsic Dirac operator coincides on each leaf with several copies of the Dirac operator on this leaf. As multiplicities are irrelevant for our discussion we have chosen the name 'intrinsic Dirac operator' for D_{int}, slightly abusing the language.

By (3), the intrinsic Dirac operator $D_{\mathrm{int}}^{\mathcal{F}}$ is related to the partial Dirac operator $D_{\partial}^{\mathcal{F}}$ via

$$D_{\partial}^{\mathcal{F}}\varphi = D_{\mathrm{int}}^{\mathcal{F}}\varphi - \frac{1}{2}\vec{H}_{\mathcal{F}}\cdot\varphi,$$

where $\vec{H}_{\mathcal{F}} = \operatorname{tr}\mathrm{II}_{\mathcal{F}} \in C^\infty(M,\nu)$ is the unnormalised mean curvature vector field of the leaves Z_b in M, see [8, Lemma 2.1].

4.2. **The Dirac operator in polar coordinates.** The goal of this subsection is to study spinors in suitable polar coordinates on $\mathbb{M}_c^{m,k} = \mathbb{H}_c^{k+1}\times N$.

We express the hyperbolic metric in polar normal coordinates centered in a fixed point p_0 which will be sometimes identified with 0. In these polar coordinates $M = (\mathbb{H}_c^{k+1}\setminus\{p_0\})\times N$ is parametrized by $\mathbb{R}^+\times\mathbb{S}^k\times N$.

Define the following foliations on M:

- The radial foliation $\mathcal{F}_r$ where the leaves are the 1-dimensional submanifolds $\mathbb{R}^+\times\{x\}\times\{y\}$. The radial vector field ∂_r on hyperbolic space generates $\mathcal{F}_r$ pointwise.
- The spherical foliation $\mathcal{F}_{\mathbb{S}^k}$ where the leaves are the k-dimensional manifolds $\{r\}\times\mathbb{S}^k\times\{y\}$.
- The N-folation $\mathcal{F}_N$, where the leaves are the n-dimensional manifolds $\{r\}\times\{x\}\times N^n$.

On an open subset of M we choose an orthonormal frame $e_1,\ldots,e_m$, $m=n+k+1=\dim M$, such that $e_{k+2},\ldots,e_m$ is an orthonormal frame for $\mathcal{F}_N$, and $e_2,\ldots,e_{k+1}$ is an orthonormal frame for $\mathcal{F}_{\mathbb{S}^k}$ and where $e_1:=\partial_r$. The notation should be read such that $\frac{\nabla}{dr}$ and ∂_r denote essentially the same (radial) vector, but ∂_r is viewed as a vector which acts via Clifford multiplication whereas $\frac{\nabla}{dr}$ acts as a covariant derivative. The three foliations provide three partial Dirac operators $\partial_r\cdot\frac{\nabla}{dr} = D_{\partial}^{\mathcal{F}_r}$, $D_{\partial}^{\mathbb{S}^k} := D_{\partial}^{\mathcal{F}_{\mathbb{S}^k}}$ and $D_{\partial}^{N} := D_{\partial}^{\mathcal{F}_N}$. The partial Dirac operators along N and $\mathbb{S}^k$ are locally defined as

$$D_{\partial}^{N}\varphi := \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}^{M}\varphi,\quad D_{\partial}^{\mathbb{S}^k}\varphi := \sum_{i=n+1}^{n+k} e_i\cdot\nabla_{e_i}^{M}\varphi,$$

for $\varphi\in C^\infty(\Sigma_M)$. Note that we allow $k=0$ which simply gives $D_{\partial}^{\mathbb{S}^0}=0$. The Dirac operator D on $(r_0,\infty)\times\mathbb{S}^k\times N$ is the sum of partial Dirac operators

$$D = \partial_r\cdot\frac{\nabla}{dr} + D_{\partial}^{\mathbb{S}^k} + D_{\partial}^{N}.$$

The intrinsic Dirac operators along N and $\mathbb{S}^k$ are given by

$$D_{\mathrm{int}}^{N}\varphi := \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}^{\mathrm{int}}\varphi,\quad D_{\mathrm{int}}^{\mathbb{S}^k}\varphi := \sum_{i=n+1}^{n+k} e_i\cdot\nabla_{e_i}^{\mathrm{int}}\varphi.$$

We denote the second fundamental form of $\mathbb{S}^k$ in $\mathbb{H}_c^{k+1}$ as $\mathrm{II}_{\mathbb{S}^k}$ and set $\vec{H}_{\mathbb{S}^k} := \operatorname{tr}\mathrm{II}_{\mathbb{S}^k}$. Then using natural identifications $\mathrm{II}_{\mathbb{S}^k}$ and $\vec{H}_{\mathbb{S}^k}$ do not depend on whether they represent the second fundamental form and the mean curvature field of $\mathbb{S}^k$ in $\mathbb{H}_c^{k+1}$, or of $\mathbb{S}^k$ in $\mathbb{H}_c^{k+1}\times N$ or of $\mathbb{S}^k\times N$ in $\mathbb{H}_c^{k+1}\times N$.

Using $\vec{H}_N=0$ and $f(r)=\sinh_c(r)$, cp. Section 2.2,

$$\vec{H}_{\mathbb{S}^k\times N} = \vec{H}_{\mathbb{S}^k} = -k\frac{\partial_r f(r)}{f(r)}\partial_r = -k\coth_c(r)$$

we obtain $D^N := D^N_\partial = D^N_{\mathrm{int}}$ and $D^{\mathbb{S}^k}_\partial = D^{\mathbb{S}^k}_{\mathrm{int}} + \frac{k}{2}\coth_c(r)\partial_r\cdot$.
We set $D^{\mathbb{S}^k} := f(r)D^{\mathbb{S}^k}_{\mathrm{int}}$ which is on each spherical submanifold up to multiplicity the standard Dirac operator on $\mathbb{S}^k$ and obtain

$$D = \frac{1}{\sinh_c(r)}D^{\mathbb{S}^k} + \partial_r \cdot \frac{\nabla}{dr} + \frac{k}{2}\coth_c(r)\partial_r \cdot + D^N. \tag{4}$$

Lemma 4.3. *The following operators anticommute: D^N with $D^{\mathbb{S}^k}$, D^N with $\partial_r\cdot$, D^N with $\partial_r \cdot \frac{\nabla}{dr}$, $D^{\mathbb{S}^k}$ with $\partial_r\cdot$, and $D^{\mathbb{S}^k}$ with $\partial_r \cdot \frac{\nabla}{dr}$. However $\partial_r\cdot$ commutes with $\partial_r \cdot \frac{\nabla}{dr}$, and $(D^{\mathbb{S}^k})^2$ commutes with D.*

Proof. Let $P_{\mathrm{Spin}}(\mathbb{H}^{k+1}_c) \to P_{\mathrm{SO}}(\mathbb{H}^{k+1}_c)$ and $P_{\mathrm{Spin}}(N) \to P_{\mathrm{SO}}(N)$ be the fixed spin structures on $\mathbb{H}^{k+1}_c$ and N. Then we write as in Subsection 2.5

$$\Sigma_{\mathbb{H}^{k+1}_c \times N} = \underbrace{(P_{\mathrm{Spin}}(\mathbb{H}^{k+1}_c) \times P_{\mathrm{Spin}}(N))}_{P} \times_\zeta \Sigma_m \tag{5}$$

where ζ is the composition $\mathrm{Spin}(k+1) \times \mathrm{Spin}(n) \xrightarrow{\xi} \mathrm{Spin}(m) \xrightarrow{\rho_m} \mathrm{End}(\Sigma_m)$. The bundle P carries the Levi-Civita connection-1-form $\alpha^{LC}_{\mathbb{M}_c}$ and another connection-1-form α^{int} as explained before.
We obtain a connection preserving bundle homomorphism I_c, which is fiberwise an isometric isomorphism, and

$$\begin{array}{ccc} \Sigma_{\mathbb{H}^{k+1}_c \setminus \{p_0\} \times N}, \nabla^{\mathrm{int}} & \xrightarrow{I_c} & \Sigma_{\mathbb{R}^+ \times \mathbb{S}^k \times N}, \nabla^{LC} \\ \downarrow & & \downarrow \\ \mathbb{H}^{k+1}_c \setminus \{p_0\} \times N & \xrightarrow{\mathrm{id}} & \mathbb{R}^+ \times \mathbb{S}^k \times N \end{array} \tag{6}$$

commutes. Note that I_c is also compatible with the Clifford multiplication in the sense that for $X \in TZ$ we have

$$I_c(X \cdot \varphi) = \begin{cases} X \cdot I_c(\varphi) & \text{for } Z = \mathbb{R}^+ \times \{x\} \times \{y\} \text{ or } \{r\} \times \{x\} \times N \\ \frac{f(r)}{r} X \cdot I_c(\varphi) & \text{for } Z = \{r\} \times \mathbb{S}^k \times \{y\}. \end{cases}$$

Then the lemma follows immediately by the corresponding statements for $\Sigma_{\mathbb{R}^+ \times \mathbb{S}^k \times N}$. $\square$

We will also use the map $\hat{I}_c := I_0^{-1} \circ I_c : \Sigma_{\mathbb{H}^{k+1}_c \setminus \{p_0\} \times N} \to \Sigma_{\mathbb{R}^{k+1} \setminus \{0\} \times N}$ which allows to identify $\Sigma_{\mathbb{H}^{k+1}_c \times N}|_{(x,y)}$ with $\Sigma_{\mathbb{R}^{k+1} \times N}|_{(x,y)}$ and thus with $\Sigma_{\mathbb{R}^{k+1} \times N}|_{(0,y)}$, $0 \cong p_0$.

4.3. **The action of** $\mathrm{Spin}(k+1)$ **on** $\mathbb{M}^{m,k}_c = \mathbb{H}^{k+1}_c \times N$**.** We identify $T_{p_0}\mathbb{H}^{k+1}_c$ with $\mathbb{R}^{k+1}$. The left action a_1 of the spin group $\mathrm{Spin}(k+1)$ on $\mathbb{R}^{k+1}$ obtained by composing the double covering $\mathrm{Spin}(k+1) \to \mathrm{SO}(k+1)$ with the standard representation yields a left action on $\mathbb{H}^{k+1}_c$ via the exponential map $\exp_{p_0} : \mathbb{R}^{k+1} \to \mathbb{H}^{k+1}_c$ which is a diffeomorphism. As this action is isometric it yields a left action on $P_{\mathrm{Spin}}(\mathbb{H}^{k+1}_c)$ – also called a_1. Thus, we obtain a $\mathrm{Spin}(k+1)$-action on $P_{\mathrm{Spin}}(\mathbb{H}^{k+1}_c) \times P_{\mathrm{Spin}}(N) \times \Sigma_m$ as $\hat{a}_1 = a_1 \times \mathrm{id} \times \mathrm{id}$. Since a_1 and the principal $\mathrm{Spin}(k+1)$-action which acts from the right commute, the $\hat{a}_1$-action descends to a $\mathrm{Spin}(k+1)$-action from the left – denoted by a_2 – on the spinor bundle $\Sigma_{\mathbb{H}^{k+1}_c \times N} = (P_{\mathrm{Spin}}(\mathbb{H}^{k+1}_c) \times P_{\mathrm{Spin}}(N)) \times_\zeta \Sigma_m$ (for ζ as in (5)) such that

$$\begin{array}{ccc} \Sigma_{\mathbb{H}^{k+1}_c \times N} & \xrightarrow{a_2(\gamma)} & \Sigma_{\mathbb{H}^{k+1}_c \times N} \\ \downarrow & & \downarrow \\ \mathbb{H}^{k+1}_c \times N & \xrightarrow{a_1(\gamma) \times \mathrm{id}} & \mathbb{H}^{k+1}_c \times N \end{array}$$

commutes.
By construction, the action a_1 does not depend on c. Thus, Diagram (6) commutes with this $\mathrm{Spin}(k+1)$-action.

Moreover, note that a_1 preserves the spheres $\mathbb{S}^k_{r,y} := \{r\} \times \mathbb{S}^k \times \{y\} \subset \mathbb{H}^{k+1}_c \times N$; for $k > 0$ this action is even transitive on $\mathbb{S}^k_{r,y}$. Hence, the diagram above can be restricted to this submanifold. Furthermore, (p_0, y) is a fixed point of $a_1 \times \mathrm{id}$ for all $y \in N$. Thus, the a_2-action can be restricted to an action that maps $\Sigma_{\mathbb{H}^{k+1}_c \times N}|_{(p_0,y)}$ to itself.

4.4. **Spinors on $\mathbb{S}^k \subset \mathbb{R}^{k+1}$.** We will now analyse the special case $N = \{y\}$ and $c = 0$, thus $\mathbb{H}^{k+1}_c = \mathbb{R}^{k+1}$. This well-known case is not only important as an example, but will also be used to derive consequences for the general case.
We obtain immediately from (3) and $\mathrm{II}_{\mathbb{S}^k_r} = -\frac{1}{r} g|_{\mathbb{S}^k_r} \partial_r$ where $\mathbb{S}^k_r$ is the sphere of radius r canonically embedded in $\mathbb{R}^{k+1}$:

Lemma 4.4. *Assume that φ is a parallel spinor on $\mathbb{R}^{k+1}$. Then for any $X \in T\mathbb{S}^k_r$ we have*

$$\nabla^{\mathrm{int}}_X \varphi = -\frac{1}{2r} \partial_r \cdot X \cdot \varphi \text{ and } \nabla^{\mathrm{int}}_X (\partial_r \cdot \varphi) = \frac{1}{2r} \partial_r \cdot X \cdot (\partial_r \cdot \varphi).$$

In particular, we have

$$D^{\mathbb{S}^k} \varphi = r D^{\mathbb{S}^k}_{\mathrm{int}} \varphi = -\frac{k}{2} \partial_r \cdot \varphi \text{ and } D^{\mathbb{S}^k} (\partial_r \cdot \varphi) = -\frac{k}{2} \partial_r \cdot (\partial_r \cdot \varphi).$$

Using Lemma 4.3 and $\nabla^{\mathrm{int}}_X \partial_r = 0$ this implies

$$(D^{\mathbb{S}^k})^2 \varphi = \frac{k^2}{4} \varphi \text{ and } (D^{\mathbb{S}^k})^2 (\partial_r \cdot \varphi) = \frac{k^2}{4} (\partial_r \cdot \varphi).$$

5. Modes of $\mathrm{Spin}(k+1)$-equivariant maps

In this section we assume $k \geq 1$. We now have a $\mathrm{Spin}(k+1)$-action on $\Sigma_{\mathbb{R}^{k+1}}|_0 \cong \Sigma_{k+1}$, $\{r\} \times \mathbb{S}^k$ and $\Sigma_{\mathbb{R}^{k+1}}|_{\{r\}\times\mathbb{S}^k}$, and thus one on $C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}}|_{\{r\}\times\mathbb{S}^k})$ given by $(\gamma \cdot f)(x) = a_2(\gamma) f(a_1(\gamma)^{-1} x)$. To simplify notations we mostly write $\mathbb{S}^k$ for $\{r\} \times \mathbb{S}^k$.
We now have to classify $\mathrm{Spin}(k+1)$-equivariant functions $\Sigma_{\mathbb{R}^{k+1}}|_0 \to C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}}|_{\mathbb{S}^k})$.
For $\psi_0 \in \Sigma_{\mathbb{R}^{k+1}}|_0$ let the parallel spinor on $\mathbb{R}^{k+1}$ with value ψ_0 at 0 be denoted by Ψ_0. For k odd, the positive and negative parts of Ψ_0 are denoted by $\Psi_0^{(\pm)}$.

Lemma 5.1. *Let $F\colon \Sigma_{k+1} \to C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}}|_{\mathbb{S}^k})$ be a linear $\mathrm{Spin}(k+1)$-equivariant map. Then for k even F has the form*

$$\psi_0 \mapsto (a_1 \Psi_0 + a_2 \partial_r \cdot \Psi_0)|_{\mathbb{S}^k}$$

and for k odd F has the form

$$\psi_0 \mapsto (a_{11} \Psi_0^{(+)} + a_{22} \Psi_0^{(-)} + a_{21} \partial_r \cdot \Psi_0^{(+)} + a_{12} \partial_r \cdot \Psi_0^{(-)})|_{\mathbb{S}^k}$$

for suitable constants $a_i, a_{ij} \in \mathbb{C}$.

The lemma is in fact a special case of Frobenius reciprocity as partially explained in the proof.

Proof. In the case k even, one easily checks that $\psi_0 \mapsto \Psi_0|_{\mathbb{S}^k}$ and $\psi_0 \mapsto \partial_r \cdot \Psi_0|_{\mathbb{S}^k}$ are linear $\mathrm{Spin}(k+1)$-equivariant maps. The same statement holds for k odd for $\psi_0 \mapsto \Psi_0^{(+)}|_{\mathbb{S}^k}$, $\psi_0 \mapsto \partial_r \cdot \Psi_0^{(+)}|_{\mathbb{S}^k}$ $\psi_0 \mapsto \Psi_0^{(-)}|_{\mathbb{S}^k}$ and $\psi_0 \mapsto \partial_r \cdot \Psi_0^{(-)}|_{\mathbb{S}^k}$.
The converse will be checked using Frobenius reciprocity. Since ∂_r is a $\mathrm{Spin}(k+1)$-equivariant vector field, we obtain a $\mathrm{Spin}(k+1)$-equivariant inclusion $P_{\mathrm{SO}}(\mathbb{S}^k) \hookrightarrow P_{\mathrm{SO}}(\mathbb{R}^{k+1})|_{\mathbb{S}^k}$, $(e_1, \cdots, e_k) \mapsto (e_1, \cdots, e_k, \partial_r)$. Via the inclusion $\mathrm{SO}(k) \to \mathrm{SO}(k+1)$ we can view $P_{\mathrm{SO}}(\mathbb{R}^{k+1})|_{\mathbb{S}^k}$ as an $\mathrm{SO}(k)$-principal bundle, then the inclusion commutes with the $\mathrm{SO}(k)$ action of the $\mathrm{SO}(k)$-principle bundles. It follows from the construction of the spin structure on $\mathbb{S}^k$ that this inclusion lifts to a $\mathrm{Spin}(k+1)$-equivariant inclusion $P_{\mathrm{Spin}}(\mathbb{S}^k) \hookrightarrow P_{\mathrm{Spin}}(\mathbb{R}^{k+1})|_{\mathbb{S}^k}$ of $\mathrm{Spin}(k)$-principal bundles. Note that the $\mathrm{Spin}(k+1)$ action is the one described above which commutes with

the $\mathrm{Spin}(k)$-actions from the principal bundles. Thus we have the $\mathrm{Spin}(k+1)$-equivariant isomorphism of $\mathrm{Spin}(k+1)$-principal bundles.

$$P_{\mathrm{Spin}}(\mathbb{S}^k) \times_{\mathrm{Spin}(k)} \mathrm{Spin}(k+1) \xrightarrow{\cong} P_{\mathrm{Spin}}(\mathbb{R}^{k+1})|_{\mathbb{S}^k}.$$

We apply Frobenius reciprocity [33, Theorem 7.47] for the homogeneous space $\mathbb{S}^k = G/H$, $G = \mathrm{Spin}(k+1)$, $H = \mathrm{Spin}(k)$ which states that

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \cong \mathrm{Hom}_H(W|_H, V)$$

for any G-representation W and any H-representation V. Here $W|_H$ denotes the restriction of the G action on W to an H action on W. Further, $\mathrm{Ind}_H^G(V)$ is the induced representation, defined as

$$\mathrm{Ind}_H^G(V) := \{\text{smooth maps } G \to V \mid f(gh) = h^{-1}f(g) \quad \forall g \in G,\, h \in H\}$$

and the action of G on $f \in \mathrm{Ind}_H^G(V)$ is given by

$$(\tilde{g}f)(g) := f(\tilde{g}^{-1}g) \quad \forall g, \tilde{g} \in G.$$

So let $W = V = \Sigma_{k+1}$ be the standard spinor representation with the standard action of G, resp. its restriction to H. Then

$$\Sigma_{\mathbb{R}^{k+1}}|_{\mathbb{S}^k} = P_{\mathrm{Spin}}(\mathbb{R}^{k+1})|_{\mathbb{S}^k} \times_{\mathrm{Spin}(k+1)} \Sigma_{k+1} \cong P_{\mathrm{Spin}}(\mathbb{S}^k) \times_{\mathrm{Spin}(k)} \Sigma_{k+1} \cong G \times_H V$$

as a G-equivariant bundle over $\mathbb{S}^k = G/H$.
Thus its space of sections $C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}}|_{\mathbb{S}^k})$ equals $\mathrm{Ind}_H^G(\Sigma_{k+1})$, see also [33, Theorem 7.46]. Hence, $\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V))$ is the space of all F satisfying the assumptions of the lemma. The lemma follows by calculating

$$\dim \mathrm{Hom}_H(W|_H, V) = \dim \mathrm{Hom}_{\mathrm{Spin}(k)}(\Sigma_{k+1}, \Sigma_{k+1}) = \begin{cases} 2 & \text{for } k \text{ even}, \\ 4 & \text{for } k \text{ odd}. \end{cases}$$

This follows from Schur's theorem as V is the sum of two non-isomorphic irreducible representations if k is even and as V is the sum of two isomorphic irreducible representations if k is odd. $\square$

Then using Lemma 4.4 we obtain immediately

Corollary 5.2. *Let $F\colon \Sigma_{k+1} \to C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}}|_{\mathbb{S}^k})$ be a linear $\mathrm{Spin}(k+1)$-equivariant map. Let $\psi_0 \in \Sigma_{k+1}$ and $\varphi = F\psi_0$. Then $(D^{\mathbb{S}^k})^2\varphi = \frac{k^2}{4}\varphi$.*

We say that φ is in the spherical mode $\frac{k^2}{4}$, and thus φ is in the mode of lowest energy on the sphere.

Now we want to carry over the last result to $\mathbb{M}_c^{m,k}$. In the following $p_0 \in \mathbb{H}_c^{k+1}$ denotes again the fixed point of the $\mathrm{Spin}(k+1)$-action, and let $y_0, y \in N$.

Lemma 5.3. *Let $F\colon \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{(p_0,y_0)} \to C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{\mathbb{S}^k\times\{y\}})$ be a linear $\mathrm{Spin}(k+1)$-equivariant map. Let $\psi_0 \in \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{(p_0,y_0)}$ and $\varphi = F\psi_0$. Then $(D^{\mathbb{S}^k})^2\varphi = \frac{k^2}{4}\varphi$.*

Proof. Note that the composition $\hat{I}_c := I_0^{-1} \circ I_c$ where I_c is defined as in (6) maps the spinor bundle over $(\mathbb{H}_c^{k+1} \setminus \{p_0\}) \times N$ to the spinor bundle over $(\mathbb{R}^{k+1} \setminus \{0\}) \times N$. This map preserves the intrinsic connection ∇^{int} and uniquely extends into $p_0 \cong 0$. Via pullback we then obtain a $\mathrm{Spin}(k+1)$-equivariant vector space isomorphism

$$C^\infty(\{r\} \times \mathbb{S}^k, \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{\{r\}\times\mathbb{S}^k\times\{y\}}) \overset{J_{r,y}}{\to} C^\infty(\{r\} \times \mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}\times N}|_{\{r\}\times\mathbb{S}^k\times\{y\}}),\ \psi \mapsto \hat{I}_c \circ \psi.$$

Moreover, we can write in the sense of $\mathrm{Spin}(k+1)$-modules $\Sigma_{\mathbb{R}^{k+1}\times N}|_{(x,y)} \cong \Sigma_m \cong \Sigma_{k+1} \otimes V$ if k is even or $\Sigma_{\mathbb{R}^{k+1}\times N}|_{(x,y)} \cong \Sigma_{k+1}^{(+)} \otimes V^{(+)} \oplus \Sigma_{k+1}^{(-)} \otimes V^{(-)}$ if k is odd, where $V^{(\varepsilon)} := \mathrm{Hom}_{\mathrm{Spin}(k+1)}(\Sigma_{k+1}^{(\varepsilon)}, \Sigma_{\mathbb{R}^{k+1}\times N}|_{(x,y)})$ is a vector space which is independent of $x \in \mathbb{R}^{k+1}$.

Let now k be odd. Then any $\alpha \in (V^{(\varepsilon)})^*$ defines a map $\Sigma_{\mathbb{H}_c^{k+1}\times N}|_{(x,y)} \to \Sigma_{k+1}^{(\varepsilon)}$.
Let $A\colon \Sigma_{k+1}^{(\delta)} \to \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{(p_0,y_0)}$ be a $\mathrm{Spin}(k+1)$-equivariant map. By composition we obtain for fixed A, α and $\delta, \varepsilon \in \{+,-\}$ a $\mathrm{Spin}(k+1)$-equivariant map

$$\begin{aligned}\Sigma_{k+1}^{(\delta)} &\xrightarrow{A} \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{(p_0,y_0)} \xrightarrow{F} C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{\mathbb{S}^k\times\{y\}})\\ &\xrightarrow{J_{r,y}} C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}_c^{k+1}\times N}|_{\mathbb{S}^k\times\{y\}}) \cong C^\infty(\mathbb{S}^k, \Sigma_{\mathbb{R}^{k+1}}|_{\mathbb{S}^k}\otimes V) \xrightarrow{\alpha} C^\infty(\mathbb{S}^k, \Sigma_{k+1}^{(\varepsilon)}).\end{aligned}$$

Let now k be even. Then the argumentation is analogous to the one above when replacing $V^{(\varepsilon)}$ by V and $\Sigma_{k+1}^{(\varepsilon)}$ by Σ_{k+1}.
Then the Lemma follows from Corollary 5.2 together with the identification by $J_{r,y}$. ☐

Corollary 5.4. *Let $G(q,p)$ be the Green function of the operator $D-\mu$, $\mu \notin \mathrm{Spec}_{L^2}^{\mathbb{M}_c}(D)$. Let $q=(r,x,y) \in \mathbb{M}_c^{m,k}$ be the polar coordinates when using p_0 as the origin, $r>0$. Let $\psi_0 \in \Sigma_{\mathbb{M}_c^{m,k}}|_{(p_0,y_0)}$, $y_0 \in N$. Set $\varphi(q) := G(q,(p_0,y_0))\psi_0$. Then*

$$(D^{\mathbb{S}^k})^2\varphi|_{\{r\}\times\mathbb{S}^k\times\{y\}} = \frac{k^2}{4}\varphi|_{\{r\}\times\mathbb{S}^k\times\{y\}}.$$

Proof. Now we consider the Green function of the shifted Dirac operator $D-\mu$ for $\mu \notin \mathrm{Spec}_{L^2}^{\mathbb{M}_c}(D)$ as a map

$$G(\,\cdot\,,(p_0,y_0))\colon \Sigma_{\mathbb{H}_c^{k+1}\times N}|_{(p_0,y_0)} \to \Gamma(\Sigma_{\mathbb{H}_c^{k+1}\times N\setminus\{(p_0,y_0)\}}).$$

It follows from the definition of G, in particular from its uniqueness, and from the equivariance of D under $\mathrm{Spin}(k+1)$ that $G(\,\cdot\,,(p_0,y_0))$ is $\mathrm{Spin}(k+1)$-equivariant as well. In particular, $G(\,\cdot\,,(p_0,y_0))|_{\mathbb{S}^k\times\{y\}}$ is a $\mathrm{Spin}(k+1)$-equivariant map as considered in Lemma 5.3. Thus, together with Lemma 5.3 the corollary follows. ☐

6. Decay estimates for a fixed mode

Let $\mu \notin \mathrm{Spec}_{L^2}^{\mathbb{M}_c^{m.k}}(D)$. Then, by Theorem 3.6 there exists a unique Green function for $D-\mu$. The goal of this section is to estimate the decay of this Green function at infinity. For that, let $y=(p_0,y_N) \in \mathbb{H}_c^{k+1}\times N$ and $\psi_0 \in \Sigma_{\mathbb{M}_c}|_y$ be fixed. Set $\varphi(x) := G(x,y)\psi_0$. The Definition of the Green function, cf. (2), implies that φ is an L^2-eigenspinor of D to the eigenvalue μ outside a neighbourhood of y. Moreover, by Corollary 5.4 we know that φ is in the spherical mode $\frac{k^2}{4}$.
Recall from Subsection 2.6 that

$$\mathrm{Spec}_{L^2}^{\mathbb{M}_c}(D) = (-\infty,-\lambda_0]\cup[\lambda_0,\infty).$$

In the following the complement of this spectrum is denoted by $I_{\lambda_0} := (\mathbb{C}\setminus\mathbb{R})\cup(-\lambda_0,\lambda_0)$. Now we decompose the space of spinors restricted to $\{r_1\}\times\mathbb{S}^k\times N$ into complex subspaces of minimal dimensions which are invariant under D^N, $\partial_r\cdot$, $D^{\mathbb{S}^k}$. For $k>0$ such spaces have a basis of the form ψ, $\partial_r\cdot\psi$, $P\psi$, and $\partial_r\cdot P\psi$, where ψ satisfies $D^N\psi=\lambda\psi$, $(D^{\mathbb{S}^k})^2\psi=\rho^2\psi$, $\rho\in\frac{k}{2}+\mathbb{N}_0$, and $P := D^{\mathbb{S}^k}/\rho$. All these operations commute with parallel transport in r-direction, so by applying parallel transport in r-direction we obtain spinors ψ, $\partial_r\cdot\psi$, $P\psi$, and $\partial_r\cdot P\psi$ on $\mathbb{R}^+\times\mathbb{S}^k\times N$ with similar relations, and the space of all spinors of the form

$$\varphi = \varphi_1(r)\psi + \varphi_2(r)\partial_r\cdot\psi + \varphi_3(r)P\psi + \varphi_4(r)\partial_r\cdot P\psi \tag{7}$$

is preserved under the Dirac operator D on $\mathbb{M}_c^{m,k}$ because of (4). Then the operators discussed above restricted to such a minimal subspace are represented by the matrices, cp. Lemma 4.3,

$$D^N = \begin{pmatrix}\lambda & 0 & 0 & 0\\ 0 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 0\\ 0 & 0 & 0 & \lambda\end{pmatrix} \qquad D^{\mathbb{S}^k} = \begin{pmatrix}0 & 0 & \rho & 0\\ 0 & 0 & 0 & -\rho\\ \rho & 0 & 0 & 0\\ 0 & -\rho & 0 & 0\end{pmatrix} \qquad \partial_r\cdot = \begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

For $k=0$ we have $D^{\mathbb{S}^0}=0$ and one can proceed analogously with a basis $\psi, \partial_r\cdot\psi$ and obtains

$$D^N=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}\qquad \partial_r\cdot=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Proposition 6.1. *Assume that φ is an L^2-solution to the equation $D\varphi=\mu\varphi$, $\mu\in I_{\lambda_0}$ on $(\mathbb{M}_c^{m,k})_{>r_0} := (\mathbb{H}_c^{k+1}\setminus B_{r_0}(p_0))\times N$. Assume that φ has the form given in (7) with parameters ρ and λ. Let κ satisfy $\kappa^2=\lambda^2-\mu^2$, $\operatorname{Re}\kappa\geq 0$. Then $\operatorname{Re}\kappa>0$. Moreover, let $\kappa_{\lambda_0}^2=\lambda_0^2-\mu^2$. If $\operatorname{Re}\kappa_{\lambda_0}>0$, then there is are positive constants C and r_1 such that*

$$|\varphi(x)|\leq C\|\varphi\|_{L^2((\mathbb{M}_c^{m,k})_{>r_0})}e^{(-ck/2-\operatorname{Re}\kappa_{\lambda_0})d(x_1,p_0)}\ \text{ for all } x=(x_1,x_2)\in(\mathbb{H}_c^{k+1}\backslash B_{r_1}(p_0))\times N$$

where C is a constant that only depends on c,k,r_1,λ_0,μ and ρ but not on λ. For $c=0$ an analogous estimate holds when replacing $e^{-(ck/2)d(x_1,p_0)}$ by $r^{-k/2}$ where $r=d(x_1,p_0)$.

Proof. We prove first the case $k>0$: By assumption φ can be written as in (7). We view the components of φ as a vector in $\mathbb{C}^4$, i.e., $\Phi(r):=(\varphi_1(r),\varphi_2(r),\varphi_3(r),\varphi_4(r))$. So by (4) the following equation is equivalent to $D\varphi=\mu\varphi$:

$$0=\begin{pmatrix}\lambda-\mu & -\frac{k}{2}\coth_c r & \frac{\rho}{\sinh_c r} & 0\\ \frac{k}{2}\coth_c r & -\lambda-\mu & 0 & -\frac{\rho}{\sinh_c r}\\ \frac{\rho}{\sinh_c r} & 0 & -\lambda-\mu & -\frac{k}{2}\coth_c r\\ 0 & -\frac{\rho}{\sinh_c r} & \frac{k}{2}\coth_c r & \lambda-\mu\end{pmatrix}\Phi(r)+\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}\Phi'(r).$$

Thus using $\mathbb{1}$ for the identity matrix and setting

$$A:=\begin{pmatrix}0 & \lambda+\mu & 0 & 0\\ \lambda-\mu & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda+\mu\\ 0 & 0 & -\lambda-\mu & 0\end{pmatrix},\qquad B:=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},$$

we obtain

$$\Phi'(r)=\left(A-\frac{k\coth_c r}{2}\mathbb{1}+\frac{\rho}{\sinh_c r}B\right)\Phi(r).$$

We start with the case $c\neq 0$: We now substitute $t=e^{-cr}$, $\widetilde{\Phi}(t)=\Phi(-c^{-1}\log t)$. Then

$$\frac{d\widetilde{\Phi}}{dt}=\left(-\frac{1}{ct}A+\frac{k(1+t^2)}{2(t-t^3)}\mathbb{1}+\frac{2\rho}{t^2-1}B\right)\widetilde{\Phi}.$$

Such singular ordinary differential equations are well understood, see [15, Chap. 4, Sec. 1–3]. In particular, $t=0$ is a singular point of first kind, and [15, Chap. 4 Thm 2,1] yields that $t=0$ is a so-called "regular singular point", and the associated theory applies. However, in our situation it is more efficient to analyse the equation directly.

We set $h(t):=\big(\log t-\log(t+1)-\log(1-t)\big)k/2$, then $h'(t)=\frac{k(1+t^2)}{2(t-t^3)}$. We define

$$\widehat{\Phi}(t):=e^{-h(t)}t^{A/c}\widetilde{\Phi}(t),$$

and we calculate

$$\frac{d\widehat{\Phi}}{dt}=-\frac{2\rho}{1-t^2}t^{A/c}Bt^{-A/c}\widehat{\Phi}.$$

As B anticommutes with A, we have $t^{A/c}Bt^{-A/c}=t^{2A/c}B$, and as B is an isometry of $\mathbb{C}^4$, we see that

$$\|t^{A/c}Bt^{-A/c}\|=t^{2|\operatorname{Re}\kappa_+|/c}$$

where $\|\,.\,\|$ denotes the operator norm and where

$$\kappa_\pm:=\pm\sqrt{\lambda^2-\mu^2}.$$

are the (complex) eigenvalues of A. It follows that for $0 < t < 1/2$

$$\left|\frac{d}{dt}\log|\widehat{\Phi}(t)|\right| \leq \frac{|\frac{d}{dt}\widehat{\Phi}|}{|\widehat{\Phi}|} \leq \frac{2\rho}{1-t^2}\|t^{A/c}Bt^{-A/c}\| \leq 3\rho t^{2|\mathrm{Re}\,\kappa_+|/c}.$$

Thus the solution extends to $t = 0$, and

$$|\widehat{\Phi}(0)|e^{-3\rho t^{2|\mathrm{Re}\,\kappa_+|/c}} \leq |\widehat{\Phi}(t)| \leq |\widehat{\Phi}(0)|e^{3\rho t^{2|\mathrm{Re}\,\kappa_+|/c}}.$$

This estimate yields explicit asymptotic control for $\widehat{\Phi}(t)$, and thus for φ. Namely, assume $cr_0 \geq 1 > \log 2$, there are two fundamental solutions $\varphi_\pm$ of $D\varphi_\pm = \mu\varphi_\pm$ such that $\widehat{\Phi}_\pm(0)$ is an eigenvector of A to the eigenvalue $\kappa_\pm$ and such that

$$e^{-3\rho e^{-2|\mathrm{Re}\,\kappa_+|r}}e^{\mathrm{Re}\,\kappa_\pm r}e^{h(e^{-cr})} \leq \frac{|\varphi_\pm(x)|}{|\widehat{\Phi}_\pm(0)|} \leq e^{3\rho e^{-2|\mathrm{Re}\,\kappa_+|r}}e^{\mathrm{Re}\,\kappa_\pm r}e^{h(e^{-cr})} \qquad r := d(x_1, p_0) > r_0.$$

This implies that for every $\delta \in (0,1)$ there is $\tilde{r}_0$ such that

$$(1-\delta)e^{(-(ck/2)+\mathrm{Re}\,\kappa_\pm)r} \leq \frac{|\varphi_\pm(x)|}{|\widehat{\Phi}_\pm(0)|} \leq (1+\delta)e^{(-(ck/2)+\mathrm{Re}\,\kappa_\pm)r} \qquad r := d(x_1, p_0) > \tilde{r}_0. \quad (8)$$

From

$$\int_{\tilde{r}_0}^\infty |\Phi(r)|^2(\sinh_c r)^k\,\mathrm{d}r \leq \frac{\|\varphi\|^2_{L^2}}{\mathrm{vol}(\mathbb{S}^k)\mathrm{vol}(N)}$$

and the left inequality of (8) we see that $\varphi_\pm$ is in $L^2((\mathbb{M}_c^{m,k})_{>\tilde{r}_0})$ if and only of $\mathrm{Re}\,\kappa_\pm < 0$. In the following we call this $\kappa_\pm$ just κ_λ and also replace the $\pm$ index by λ in all other occurrences. We note that $|\mathrm{Re}\,\kappa_\lambda|$ is increasing in $|\lambda|$. Thus, δ and $\tilde{r}_0$ from above can be chosen independent on λ.

Next, we multiply the first inequality of (8) by $|\widehat{\Phi}_\lambda(0)|$ and then integrate its square:

$$\frac{\|\varphi\|^2_{L^2}}{\mathrm{vol}(\mathbb{S}^k\times N)} \geq (1-\delta)^2|\widehat{\Phi}_\lambda(0)|^2\int_{\tilde{r}_0}^\infty e^{(-ck+2\mathrm{Re}\,\kappa_\lambda)r}(\sinh_c r)^k\,\mathrm{d}r.$$

Hence, we obtain an upper bound

$$|\widehat{\Phi}_\lambda(0)|^2 \leq C_1^2(1-\delta)^{-2}\|\varphi\|^2_{L^2((\mathbb{M}_c^{m,k})_{>\tilde{r}_0})}\left(\frac{e^{2\mathrm{Re}\,\kappa_\lambda\tilde{r}_0}}{-2\mathrm{Re}\,\kappa_\lambda}\right)^{-1}$$

where C_1 is a constant independent on λ.

Using this again with the right inequality of (8) we get for all x with $r = \mathrm{dist}(x, p_0) > \tilde{r}_0$ that

$$\begin{aligned}|\varphi(x)| \leq& \frac{1+\delta}{1-\delta}C_1\|\varphi\|_{L^2((\mathbb{M}_c^{m,k})_{>\tilde{r}_0})}e^{(-ck/2+\mathrm{Re}\,\kappa_\lambda)r}\left(\frac{e^{2\mathrm{Re}\,\kappa_\lambda\tilde{r}_0}}{-2\mathrm{Re}\,\kappa_\lambda}\right)^{-1/2}\\ \leq& C_1(-2\mathrm{Re}\,\kappa_\lambda)^{\frac{1}{2}}\|\varphi\|_{L^2((\mathbb{M}_c^{m,k})_{>\tilde{r}_0})}e^{-ckr/2+\mathrm{Re}\,\kappa_\lambda(r-\tilde{r}_0)}. \end{aligned} \quad (9)$$

For $r > \tilde{r}_0$ we see that $(-2\mathrm{Re}\,\kappa_\lambda)e^{2\mathrm{Re}\,\kappa_\lambda(r-\tilde{r}_0)}$ is monotonically decreasing in $|\mathrm{Re}\,\kappa_\lambda|$, and we obtain from (9)

$$\begin{aligned}|\varphi(x)| \leq& C_1(-2\mathrm{Re}\,\kappa_{\lambda_0})^{\frac{1}{2}}\|\varphi\|_{L^2((\mathbb{M}_c^{m,k})_{>\tilde{r}_0})}e^{-ckr/2+\mathrm{Re}\,\kappa_{\lambda_0}(r-\tilde{r}_0)}\\ \leq& C\|\varphi\|_{L^2((\mathbb{M}_c^{m,k})_{>\tilde{r}_0})}e^{(-ck/2+\mathrm{Re}\,\kappa_{\lambda_0})r}\end{aligned}$$

for all x with $r = d(x_1, p_0) > \tilde{r}_0$. Here, C can be chosen such that it only depends on c, k, $\tilde{r}_0$, λ_0, μ and ρ but not on λ. Note that the κ in the claim is simply $-\kappa_{\lambda_0}$.

Next we consider the case $c = 0, k > 0$:

$$\Phi'(r) = \left(A - \frac{k}{2r}\mathbb{1} + \frac{\rho}{r}B\right)\Phi(r).$$

Set $\hat{\Phi}(r) = r^{\frac{k}{2}}e^{-Ar}\Phi(r)$. Then, $\hat{\Phi}'(r) = \frac{\rho}{r}e^{-Ar}Be^{Ar}\hat{\Phi} = \frac{\rho}{r}e^{-2Ar}B\hat{\Phi}$. Then we can proceed as above and obtain the claim.

It remains the case $k=0$: Then

$$\begin{pmatrix}\varphi_1(r)\\ \varphi_2(r)\end{pmatrix}' = \begin{pmatrix}0 & \lambda+\mu\\ \lambda-\mu & 0\end{pmatrix}\begin{pmatrix}\varphi_1(r)\\ \varphi_2(r)\end{pmatrix}$$

and direct calculation gives the claim. □

In order to estimate the decay of $\varphi(x) = G(x,y)\psi_0$, $\psi_0 \in \Sigma_{\mathbb{M}_c^{m,k}}|_y$ at infinity we will decompose φ into its modes in $\mathbb{S}^k$ and N direction, respectively. Lemma 6.1 provides an estimate of the decay of each mode which is independent of the mode in direction of N. Moreover, from Corollary 5.4 we know that φ has spherical mode $\frac{k^2}{4}$. Thus, we obtain a decay estimate for φ:

Lemma 6.2. *Let $\mu \notin \mathrm{Spec}_{L^2}^{\mathbb{M}_c}(D)$, and let G be the unique Green function of $D-\mu$. We set $M_y(r) := \{x \in \mathbb{M}_c \mid \mathrm{dist}(x, N^y) = r\}$ where $N^y = \{p_0\}\times N$ and $y = (p_0, y_N) \in \mathbb{H}_c^{k+1}\times N$. Let κ satisfy $\kappa^2 = \lambda_0^2 - \mu^2$ and $\mathrm{Re}\,\kappa \geq 0$. Then for all $\varepsilon > 0$ and r_0 sufficiently large there is a constant $C>0$ independent on y such that*

$$\int_{M_y(r)} |G(x,y)|^2 dx \leq Ce^{-2r\,\mathrm{Re}\kappa} \quad \textit{for all } r > r_0.$$

Proof. Let $\psi_0 \in \Sigma_{\mathbb{M}_c}|_y$. Set $\varphi(x) := G(x,y)\psi_0$. Then, for $k>0$ the spinor φ decomposes into a sum of spinors $\varphi_{\rho^2,\lambda}$ of the form (7) with $(D^{\mathbb{S}^k})^2\varphi_{\rho^2,\lambda} = \rho^2\varphi_{\rho^2,\lambda}$ and $D^N\varphi_{\rho^2,\lambda} = \lambda\varphi_{\rho^2,\lambda}$, respectively. By Corollary 5.4 ρ^2 may only take the value $\frac{k^2}{4}$. For $k=0$ the spinor φ decomposes into a sum of spinors $\varphi_{0,\lambda}$ with $D^N\varphi_{0,\lambda} = \lambda\varphi_{0,\lambda}$. Thus, for all k $\int_{M_y(r)} |\varphi(x)|^2 dx = \sum_\lambda \|\varphi_{k^2/4,\lambda}\|^2_{L^2(M_y(r))}$.
Together with Proposition 6.1 we obtain for $c \neq 0$

$$\begin{aligned}\int_{M_y(r)} |\varphi(x)|^2 dx &\leq \sum_\lambda C\|\varphi_{k^2/4,\lambda}\|^2_{L^2((\mathbb{M}_c^{m,k})_{>r_0})} e^{(-ck-2\mathrm{Re}\,\kappa)r}\sinh_c^k(r)\\ &\leq C'e^{-2r\mathrm{Re}\,\kappa}\sum_\lambda \|\varphi_{k^2/4,\lambda}\|^2_{L^2((\mathbb{M}_c^{m,k})_{>r_0})}\\ &\leq C'e^{-2r\,\mathrm{Re}\kappa}\|\varphi\|^2_{L^2((\mathbb{M}_c^{m,k})_{>r_0})}.\end{aligned}$$

The case $c=0$ follows analogously. □

7. Decomposition of the Green function

We decompose the Green function G of the shifted Dirac operator $D-\mu$ on $M = \mathbb{M}_c^{m,k}$ into a singular part and a smoothing operator. Both operators will be shown to be bounded operators from L^p to L^p for all $p \in [1,\infty]$.
At first we choose a smooth cut-off function $\chi\colon \mathbb{R} \to [0,1]$ with $\mathrm{supp}\,\chi \subset [-R,R]$ and $\chi|_{(-R/2,R/2)} \equiv 1$. Let $\rho\colon M\times M \to [0,1]$ be given by $\rho(x,y) = \chi(\mathrm{dist}_{\mathbb{H}_c^{k+1}}(\pi_{\mathbb{H}}(x), \pi_{\mathbb{H}}(y)))$.
Let now

$$G_1(x,y) := \rho(x,y)G(x,y) \text{ and } G_2(x,y) := G(x,y) - G_1(x,y).$$

Then G_2 is zero on a neighbourhood of the diagonal, and thus smooth everywhere. The singular part is only contained in G_1.

Proposition 7.1. *Let $M = \mathbb{M}_c^{m,k}$ and G_1 be as defined above. Then, for all $1 \leq p \leq \infty$ the map $P_1\colon \varphi \mapsto \int_M G_1(.,y)\varphi(y)\mathrm{d}y$ defines a bounded operator from L^p to L^p.*

Proof. We start with a smooth spinor φ compactly supported in $B_{2R}(0)\times N \subset M$. For such a φ the spinor $P_1\varphi$ is supported in $B_{3R}(0)\times N \subset M$. We embed $B_{3R}(0)$ isometrically into a closed Riemannian manifold M_R. Let $M_R \times N$. The metric on M_R can be chosen such that $D^{M_R\times N} - \mu$ is invertible, cf. Proposition C.1. The norm of $(D^{M_R\times N}-\mu)^{-1}\colon L^p \to L^p$ is denoted by $C_R(p)$.

For $p < \infty$ we estimate

$$\int_M |P_1\varphi|^p\,\mathrm{d}x = \int_M \left|\int_M G_1(x,y)\varphi(y)\mathrm{d}y\right|^p \mathrm{d}x$$
$$\leq \int_{B_{3R}(p_0)\times N} \left|\int_{B_{2R}(p_0)\times N} G(x,y)\varphi(y)\,\mathrm{d}y\right|^p \mathrm{d}x \leq \int_{M_R\times N} |(D^{M_R\times N}-\mu)^{-1}\varphi|^p\,\mathrm{d}x$$
$$\leq C_R(p)^p \int_{M_R\times N} |\varphi|^p\,\mathrm{d}x = C_R(p)^p\,\|\varphi\|^p_{L^p}.$$

Next we want to consider arbitrary $\varphi\in L^p(M,\Sigma_M)$, $p<\infty$. Then $C_c^\infty(M,\Sigma_M)$ is dense in $L^p(M,\Sigma_M)$, and it suffices to consider $\varphi\in C_c^\infty(M,\Sigma_M)$. Choose points $(x_i)_{i\in I}\subset \mathbb{H}_c^{k+1}$ as in Lemma 2.1. Then $(B_{2R}(x_i)\times N)_{i\in I}$ and $(B_{3R}(x_i)\times N)_{i\in I}$ both cover $\mathbb{M}_c^{m,k}$ uniformly locally finite. We denote the multiplicity of the second cover by L and choose a partition of unity η_i subordinated to $(B_{2R}(x_i)\times N)_{i\in I}$.
Let $\varphi=\sum\varphi_i$ where $\varphi_i=\eta_i\varphi\in C_c^\infty(B_{2R}(x_i)\times N,\Sigma_M)$. Hence, $P_1\varphi_i\in C_c^\infty(B_{3R}(x_i)\times N,\Sigma_M)$. Moreover, let $\bar f_i\colon M\to M$ be given by $\bar f_i=(\mathrm{Id},f_i)$ where f_i is an isometry of $\mathbb{H}_c^{k+1}$ that maps x_i to p_0. We choose a lift of $\tilde f_i$ to an isometry on the spinor bundle. Due to the homogeneity of $\mathbb{H}_c^{k+1}$ we have $P_1(\tilde f_i\circ\varphi\circ\bar f_i^{-1})=\tilde f_i\circ(P_1\varphi)\circ\bar f_i^{-1}$.
Then, by triangle inequality and Hölder inequality and since for fixed x the value $P_1\varphi_i(x)$ is nonzero for at most L spinors φ_i, we have

$$|P_1\varphi(x)|^p = \left|\sum_i P_1\varphi_i(x)\right|^p \leq \left|\sum_i |P_1\varphi_i(x)|\right|^p \leq L^{p-1}\sum_i |P_1\varphi_i(x)|^p.$$

Thus, we obtain

$$\|P_1\varphi\|^p_{L^p(M)} \leq L^{p-1}\sum_i \|P_1\varphi_i\|^p_{L^p(B_{3R}(x_i)\times N)} = L^{p-1}\sum_i \|P_1(\tilde f_i\circ\varphi_i\circ\bar f_i^{-1})\|^p_{L^p(B_{3R}(p_0)\times N)}$$
$$\leq L^{p-1}C_R(p)^p\sum_i \|\tilde f_i\circ\varphi_i\circ\bar f_i^{-1}\|^p_{L^p(B_{3R}(p_0)\times N)}$$
$$= L^{p-1}C_R(p)^p\sum_i \|\varphi_i\|^p_{L^p(B_{3R}(x_i)\times N)}$$
$$\leq L^pC_R(p)^p\|\varphi\|^p_{L^p(M)}.$$

It remains the case $p=\infty$. Let η_i as above, and let $\varphi\in L^\infty$. We decompose again $\varphi=\sum\varphi_i$ where $\varphi_i=\eta_i\varphi$ is compactly supported. Then, we obtain as above that

$$\|P_1\varphi\|_{L^\infty(M)} \leq \sum_i \|P_1\varphi_i\|_{L^\infty(B_{3R}(x_i)\times N)} \leq C\sum_i \|\varphi_i\|_{L^\infty(B_{3R}(x_i)\times N)} \leq CL\|\varphi\|_{L^\infty(M)}.$$

□

We now turn to the off-diagonal part G_2.

Note that $\mathbb{H}_c^{k+1}$ is homogeneous for all c. In particular, the representation of the metric in polar coordinates – $\mathrm{d}r^2+\sinh_c^2(r)\,\sigma^k$ (cf. Section 2.2) – is independent of the chosen origin of the polar coordinates on $\mathbb{H}_c^{k+1}$. We set $M_y(r):=\{x\in\mathbb{M}_c^{m,k}\mid \mathrm{dist}(x,N^y)=r\}$ where $N^y=\{y_1\}\times N$ where $y=(y_1,y_2)\in\mathbb{H}_c^{k+1}\times N$. Then, the volume $\mathrm{vol}(M_y(r))=f(r)^k\mathrm{vol}(N)\mathrm{vol}(S^k)=\sinh_c^k(r)\mathrm{vol}(N)\mathrm{vol}(S^k)$ is independent of y. We will subsequently leave out the y in the notation and write $\mathrm{vol}(M(r))$.

Proposition 7.2. *Using the notations from above, assume that there are constants $C,\rho>0$ with*

$$\int_{M_y(r)} |G_2(x,y)|^2\mathrm{d}x \leq Ce^{-2\rho r} \quad \textit{for all } r>0.$$

Let $p=1$ and $p=\infty$. Then, for $\rho>\frac{ck}{2}$ the operator $P_2\colon\varphi\mapsto\int_M G_2(.,y)\varphi(y)\mathrm{d}y$ from L^p to L^p is bounded.

Proof. We start with $p = 1$ and estimate for $\varphi \in C_c^\infty(M, \Sigma_M)$

$$\begin{aligned}
\int_M |(P_2\varphi)(x)|\,\mathrm{d}x &\le \int_M \int_M |G_2(x,y)||\varphi(y)|\,\mathrm{d}y\,\mathrm{d}x = \int_M \left(\int_M |G_2(x,y)|\mathrm{d}x\right)|\varphi(y)|\,\mathrm{d}y \\
&= \int_M \left(\int_{\mathbb{R}_+}\int_{M_y(r)} |G_2(x,y)|\,\mathrm{d}\tilde{x}\,\mathrm{d}r\right)|\varphi(y)|\mathrm{d}y \\
&\le \int_M \left(\int_{\mathbb{R}_+} \mathrm{vol}(M(r))^{\frac{1}{2}} \left(\int_{M_y(r)} |G_2(x,y)|^2\,\mathrm{d}\tilde{x}\right)^{\frac{1}{2}}\right)|\varphi(y)|\,\mathrm{d}y \\
&\le C' \int_{r\ge r_0} \sinh_c^{\frac{k}{2}}(r)e^{-\rho r}\,\mathrm{d}r\|\varphi\|_{L^1}.
\end{aligned}$$

where $\tilde{x}$ is the angular part and r the radial part of x.
For $\rho > \frac{ck}{2}$ the integral $\int_{r\ge r_0} \sinh_c^{\frac{k}{2}}(r)e^{-\rho r}\,\mathrm{d}r$ is bounded. Hence, $P_2\colon L^1 \to L^1$ is invertible. Next, we consider the other case $p = \infty$. Then for $\varphi \in L^\infty(M, \Sigma_M)$

$$\begin{aligned}
|(P_2\varphi)(x)| &\le \int_{\frac{R}{2}}^\infty \int_{M_x(r)} |G_2(x,y)||\varphi(y)|\,\mathrm{d}\tilde{y}\,\mathrm{d}r \\
&\le \int_{\frac{R}{2}}^\infty \sup_{M_x(r)} |\varphi| \left(\int_{M_x(r)} |G_2(x,y)|\,\mathrm{d}\tilde{y}\right)\,\mathrm{d}r \\
&\le \|\varphi\|_{L^\infty} \int_{\frac{R}{2}}^\infty \|G_2(x,y)\|_{L^2(M_x(r))}\mathrm{vol}(M(r))^{\frac{1}{2}}\,\mathrm{d}r \\
&\le C\|\varphi\|_{L^\infty} \int_{\frac{R}{2}}^\infty e^{-\rho r}\sinh_c^{\frac{k}{2}}(r)\,\mathrm{d}r \le \tilde{C}\|\varphi\|_\infty.
\end{aligned}$$

where for $\rho > \frac{ck}{2}$ the last inequality follows as above. Thus, $\|P_2\varphi\|_\infty \le \tilde{C}\|\varphi\|_\infty$. $\square$

8. σ_p contains the L^p-spectrum on $\mathbb{M}_c^{m,k}$

In this section we prove one direction of Theorem 1.1.

Proposition 8.1. *Let $p \in [1, \infty]$. Let λ_0^2, $\lambda_0 \ge 0$, be the lowest eigenvalue of the Dirac square on the closed Riemannian spin manifold N. The L^p-spectrum of the Dirac operator on $\mathbb{M}_c^{m,k}$ is a subset of*

$$\sigma_p := \left\{\mu \in \mathbb{C} \;\middle|\; \mu^2 = \lambda_0^2 + \kappa^2, |\mathrm{Im}\,\kappa| \le ck\left|\frac{1}{p} - \frac{1}{2}\right|\right\}.$$

Proof. We will show that $D - \mu\colon H_1^p \subset L^p \to L^p$ has a bounded inverse for all $\mu \in \mathbb{C}\setminus\sigma_p$. Fix $\mu \in \mathbb{C}\setminus\sigma_p$, and let $\kappa \in \mathbb{C}$ such that $\mu^2 = \lambda_0^2 + \kappa^2$. For $p = 2$, the lemma follows from Subsection 2.6.
Let now $p \in \{1, \infty\}$ and $\mu \notin \sigma_1 = \sigma_\infty$. Then $\mu \notin \sigma_2$ and $(D-\mu)\colon H_1^2(\mathbb{M}_c^{m,k}) \subset L^2(\mathbb{M}_c^{m,k}) \to L^2(\mathbb{M}_c^{m,k})$ has a bounded inverse given by $P_\mu\colon \varphi \mapsto \int_{\mathbb{M}_c} G_\mu(x,y)\varphi(y)dy$. By Proposition 7.1, 7.2 and Lemma 6.2 the operator $P_\mu\colon L^p \to L^p$ is bounded for $|\mathrm{Im}\,\kappa| > ck\left|\frac{1}{p} - \frac{1}{2}\right| = c\frac{k}{2}$. Hence, the L^1- and the L^∞-spectrum of D on $\mathbb{M}_c^{m,k}$ have to be contained in $\sigma_1 = \sigma_\infty$.
First we deal with the case that $\mathrm{Im}\,\kappa > 0$. For $p \in [1,2]$ we use the Stein Interpolation Theorem 2.3: Fix $\varepsilon > 0$ and $y_0 \in \mathbb{R}$. We set $h(z) := \mu(z)^2 := \lambda_0^2 + \kappa(z)^2 := \lambda_0^2 + (y_0 +$

$\frac{ck}{2}\mathrm{i}z+\mathrm{i}\varepsilon)^2$ and $A_z=(D^2-h(z))^{-1}$. By Subsection 2.6 the operators

$$A_{w+\mathrm{i}y}=\left(D^2-\left[\lambda_0^2+\left(y_0-\frac{ck}{2}y+\mathrm{i}\big(\underbrace{\frac{ck}{2}w+\varepsilon}_{=\mathrm{Im}\,\kappa(z)>0}\big)\right)^2\right]\right)^{-1},$$

for $0\leq w\leq 1$ and $y\in\mathbb{R}$, are bounded as operators from L^2 to L^2. Furthermore

$$A_{1+\mathrm{i}y}=\left(D^2-\left(\lambda_0^2+\left(y_0-\frac{ck}{2}y+\mathrm{i}\left(\frac{ck}{2}+\varepsilon\right)\right)^2\right)\right)^{-1}$$

is bounded from L^1 to L^1 as seen above. Thus – as required to apply the Stein interpolation theorem – $A_{\mathrm{i}y}$ and $A_{1+\mathrm{i}y}$ are bounded operators from $L^1\cap L^2$ to L^1+L^2.
Let now $\varphi\in L^1\cap L^2$ and $\psi\in L^\infty\cap L^2$. Set $S:=\{z\in\mathbb{C}\mid 0\leq\mathrm{Re}\,z\leq 1\}$. We define $b_{\varphi,\psi}(z)=\langle A_z\varphi,\psi\rangle$. The map $b_{\varphi,\psi}$ is analytic in the interior of S, since the resolvent is, see Lemma B.5. Moreover, $|b_{\varphi,\psi}(z)|\leq\|A_z\|\|\varphi\|_{L^2}\|\psi\|_{L^2}\leq(\max_{0\leq\mathrm{Re}\,z\leq 1}\|A_z\|)\|\varphi\|_{L^2}\|\psi\|_{L^2}$ where $\|A_z\|$ denotes the operator norm for $A_z\colon L^2\to L^2$. Thus, $b_{\varphi,\psi}(z)$ is uniformly bounded and continuous on $S:=\{z\in\mathbb{C}\mid 0\leq\mathrm{Re}\,z\leq 1\}$. Thus, we can apply Theorem 2.3 and obtain for $t\in(0,1)$ and $p=\frac{2}{1+t}$ that $A_t=\left(D^2-h\left(\frac{2}{p}-1\right)\right)^{-1}=\left(D^2-\left(\lambda_0^2+\left(y_0+ck\mathrm{i}\left(\frac{1}{p}-\frac{1}{2}\right)+\mathrm{i}\varepsilon\right)^2\right)\right)^{-1}$ is bounded from L^p to L^p.
In the case $\mathrm{Im}\kappa<0$ we set analogously $A_z=(D^2-g(z))^{-1}$ for $g(z)=\lambda_0^2+\left(y_0-\frac{ck}{2}\mathrm{i}z-\mathrm{i}\varepsilon\right)^2$ and obtain that $A_t=\left(D^2-g(\frac{2}{p}-1)\right)^{-1}$ is bounded from L^p to L^p. Since $y_0\in\mathbb{R}$ and $\varepsilon>0$ can be chosen arbitrarily, we get for all $\mu\in\mathbb{C}\setminus\sigma_p$ that μ^2 is not in the L^p-spectrum of D^2. Using Lemma B.8 the claim follows for $p\in[1,2]$ and with Lemma B.3.(i) for $p\in[2,\infty)$. □

9. Construction of test spinors on $\mathbb{H}^{k+1}$

In this section we determine the Dirac L^p-spectrum of the hyperbolic space. The general case for $\mathbb{M}_c$ is given in the next section.

Proposition 9.1. *Let $p\in[1,\infty]$. The L^p-spectrum of the Dirac operator D on the hyperbolic space $\mathbb{H}^{k+1}$ is given by the set*

$$\sigma_p^{\mathbb{H}}:=\left\{\mu\in\mathbb{C}\;\middle|\;|\mathrm{Im}\,\mu|\leq k\left|\frac{1}{p}-\frac{1}{2}\right|\right\}.$$

Proof. From Proposition 8.1 we know that the L^p-spectrum is contained in $\sigma_p^{\mathbb{H}}$. Thus, it remains to show that each element μ of $\sigma_p^{\mathbb{H}}$ is contained in the L^p-spectrum of D. For that we start with a similar ansatz as was used in [18, Lemma 7] for the Laplacian.
Let the hyperbolic space $\mathbb{H}^{k+1}$, $k\geq 0$, be modelled by the space $\{(y,x_1,\ldots,x_k)\mid y>0\}$ equipped with the metric $g=y^{-2}(dx_1^2+\ldots+dx_k^2+dy^2)$. We set $e_i=y\frac{\partial}{\partial x_i}=y\partial_i$ for $i=1,\ldots,k$ and $e_y=y\frac{\partial}{\partial y}=y\partial_y$. Then, $(e_y,e_1,\ldots,e_k)$ forms an orthonormal basis, which can assumed to be positively oriented. Further we have $[e_y,e_i]=e_i=-[e_i,e_y]$. All other commutators vanish. Then, $-\Gamma_{iy}^i=\Gamma_{ii}^y=1$ and all other Christoffel symbols vanish. The orthonormal frame $(e_y,e_1,\ldots,e_k)$ can be lifted to the spin structure $\vartheta\colon P_{\mathrm{Spin}}(\mathbb{H}^{k+1})\to P_{\mathrm{SO}}(\mathbb{H}^{k+1})$, namely we choose a map $E\colon\mathbb{H}^{k+1}\to P_{\mathrm{Spin}}(\mathbb{H}^{k+1})$ with $\vartheta(E)=(e_y,e_1,\ldots,e_k)$. A spinor is by definition a section of the associated bundle $\Sigma_{\mathbb{H}^{k+1}}=P_{\mathrm{Spin}}(\mathbb{H}^{k+1})\times_{\rho_{k+1}}\Sigma_{k+1}$, so every spinor can be written as $x\mapsto[E(x),\varphi(x)]$ for a function $\varphi\colon\mathbb{H}^{k+1}\to\Sigma_{k+1}$.
Hence, identifying $(e_y,e_1,\ldots,e_k)$ with the standard basis of $\mathbb{R}^{k+1}$ we obtain [12, (4.8)], [7, Lemma 4.1]

$$\nabla_{e_i}[E,\varphi]=[E,\partial_{e_i}\varphi+\frac{1}{2}e_i\cdot e_y\cdot\varphi];\quad\nabla_{e_y}[E,\varphi]=[E,\partial_{e_y}\varphi]$$

and

$$\begin{aligned} D[E,\varphi] &= [E, \sum_{i=1}^{k} e_i \cdot \partial_{e_i}\varphi + e_y \cdot \partial_{e_y}\varphi - \frac{k}{2} e_y \cdot \varphi] \\ &= [E, \sum_{i=1}^{k} y e_i \cdot \partial_i \varphi + y e_y \cdot \partial_y \varphi - \frac{k}{2} e_y \cdot \varphi]. \end{aligned} \tag{10}$$

Let $\psi_0 \in \Sigma_{k+1}$ be a unit-length eigenvector of the Clifford multiplication with the vector $e_y = (1, 0, \ldots, 0)^t \in \mathbb{R}^{k+1}$ to the eigenvalue $\pm\mathrm{i}$, i.e. $e_y \cdot \psi_0 = \pm\mathrm{i}\psi_0$. Set $\varphi_n(x, y) = b(x) c_n(\log y) y^\alpha \psi_0$ where $\alpha \in \mathbb{C}$, $b(x)$ is any compactly supported function on $\mathbb{R}^k$, and where $c_n \colon \mathbb{R} \to \mathbb{R}$ is chosen to be a smooth cut-off function compactly supported on $(-4n, -n)$, $c_n|_{[-3n,-2n]} \equiv 1$ and $|c_n'| \leq 2/n$. Then for $p \in [1, \infty)$ one estimates $\|c_n'\|_p^p / \|c_n\|_p^p \leq C n^{-p} \to 0$ as $n \to \infty$. For $p = \infty$ we have $\|c_n'\|_\infty / \|c_n\|_\infty \leq 2/n \to 0$ as $n \to \infty$. Then we set $\Phi_n := [E, \varphi_n]$ and obtain

$$\begin{aligned} (D-\mu)\Phi_n = \Bigg[& E, y c_n(\log y)\, y^\alpha \sum_{i=1}^{k} (\partial_i b)\, e_i \cdot \psi_0 \pm b(x) c_n'(\log y) y^\alpha \mathrm{i}\psi_0 \\ & + b(x) c_n(\log y)(\pm\mathrm{i}\alpha \mp \mathrm{i}\frac{k}{2} - \mu) y^\alpha \psi_0 \Bigg]. \end{aligned} \tag{11}$$

In the following we will use the notation $(X \cdot\ .\,) \in \mathrm{End}(\Sigma_{k+1})$ for the Clifford multiplication by $X \in \mathbb{R}^{k+1}$, and obviously its operator norm $|(X \cdot\ .\,)|$ equals to the usual norm of X.
Let $\mu = s \pm \mathrm{i}k\left(\frac{1}{p} - \frac{1}{2}\right)$, $s \in \mathbb{R}$. We choose $z = \log y$ and $\alpha = \frac{k}{2} \mp \mathrm{i}\mu = \frac{k}{p} \mp \mathrm{i}s$. Thus, the last summand in (11) vanishes and $p\mathrm{Re}\,\alpha = k$. Then, for $p \in [1, \infty)$ we have

$$\begin{aligned} &\frac{\|(D-\mu)\Phi_n\|_p}{\|\Phi_n\|_p} \\ &\leq \frac{\left(\int_{\mathbb{R}^k} |\sum_i (\partial_i b)(e_i \cdot\ .\,)|^p \int_0^\infty |c_n(\log y)|^p y^{p\mathrm{Re}\,\alpha + p - k - 1}\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^k} |b(x)|^p \int_0^\infty |c_n(\log y)|^p y^{p\mathrm{Re}\,\alpha - k - 1}\right)^{\frac{1}{p}}} \\ &\quad + \frac{\left(\int_{\mathbb{R}^k} |b(x)|^p \int_0^\infty |c_n'(\log y)|^p y^{p\mathrm{Re}\,\alpha - k - 1}\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^k} |b(x)|^p \int_0^\infty |c_n(\log y)|^p y^{p\mathrm{Re}\,\alpha - k - 1}\right)^{\frac{1}{p}}} \\ &\overset{p\mathrm{Re}\,\alpha = k}{\leq} c \left(\frac{\int_{\mathbb{R}^k} \sum_i |\partial_i b|^p \int_0^\infty |c_n(\log y)|^p y^{p-1}}{\int_{\mathbb{R}^k} |b(x)|^p \int_0^\infty |c_n(\log y)|^p y^{-1}} \right)^{\frac{1}{p}} + \left(\frac{\int_0^\infty |c_n'(\log y)|^p y^{-1}}{\int_0^\infty |c_n(\log y)|^p y^{-1}} \right)^{\frac{1}{p}} \\ &\overset{z = \log y}{\leq} c \left(\frac{\int_{\mathbb{R}^k} \sum_i |\partial_i b|^p \int_{-\infty}^0 |c_n(z)|^p e^{zp}}{\int_{\mathbb{R}^k} |b(x)|^q \int_{-\infty}^0 |c_n(z)|^p} \right)^{\frac{1}{p}} + \left(\frac{\int_{-\infty}^0 |c_n'(z)|^p}{\int_{-\infty}^0 |c_n(z)|^p} \right)^{\frac{1}{p}} \\ &\leq C e^{-n} + \left(\frac{\int_{-\infty}^0 |c_n'(z)|^p}{\int_{-\infty}^0 |c_n(z)|^p} \right)^{\frac{1}{p}} \to 0 \end{aligned}$$

where the last inequality uses

$$\int_{-\infty}^0 |c_n(z)|^p e^{zp}\,\mathrm{d}z = \int_{-4n}^{-n} |c_n(z)|^p e^{zp}\,\mathrm{d}z \leq e^{-np} \int_{-4n}^{-n} |c_n(z)|^p\,\mathrm{d}z = e^{-np} \int_{-\infty}^{-0} |c_n(z)|^p\,\mathrm{d}z.$$

For $p = \infty$ we have $\mu = s \pm \mathrm{i}\frac{k}{2}$, $\alpha = \mp s$ and the estimate above is done analogously.
Summarizing, we have shown that $\partial\sigma_p^{\mathbb{H}}$, the boundary of $\sigma_p^{\mathbb{H}}$, is a subset of the Dirac L^p-spectrum for $\mathbb{H}^{k+1}$ for $p \in [1, \infty]$. Note that $\sigma_s^{\mathbb{H}} = \bigcup_{2 \geq r \geq s} \partial\sigma_r^{\mathbb{H}}$ for $s < 2$ and $\sigma_s^{\mathbb{H}} = \bigcup_{2 \leq r \leq s} \partial\sigma_r^{\mathbb{H}}$ for $s > 2$, respectively. Thus, using the Riesz-Thorin interpolation theorem we see that $\sigma_p^{\mathbb{H}}$ is a subset of the L^p-spectrum of D on $\mathbb{H}^{k+1}$ for $p \in [1, \infty]$. □

Remark 9.2. *From* (10) *we obtain*

$$\begin{aligned}
D^2[E,\varphi] =&[E, \sum_{i,j} y^2 e_i\cdot e_j\cdot \partial_i\partial_j\varphi + \sum_i y^2 e_i\cdot e_y\cdot\partial_i\partial_y\varphi - y\frac{k}{2}\sum_i e_i\cdot e_y\cdot\partial_i\varphi \\
&+\sum_i y^2 e_y\cdot e_i\cdot\partial_y\partial_i\varphi + \sum_i y e_y\cdot e_i\cdot\partial_i\varphi - y^2\partial_y\partial_y\varphi - y\partial_y\varphi + y\frac{k}{2}\partial_y\varphi \\
&- y\frac{k}{2}\sum_i e_y\cdot e_i\cdot\partial_i\varphi + y\frac{k}{2}\partial_y\varphi - \frac{k^2}{4}\varphi] \\
=&[E, -y^2\sum_i \partial_i^2\varphi - y^2\partial_y^2\varphi + y(k-1)\partial_y\varphi + \sum_i y e_y\cdot e_i\cdot\partial_i\varphi - \frac{k^2}{4}\varphi].
\end{aligned}$$

We use μ and $\varphi_n = b(x)c_n(\log y)y^\alpha\psi_0$ of the last proposition with b, α, c_n and ψ_0 as therein. For c_n we require additionally $|c_n''|\le 8n^{-2}$. Hence, $\|c_n''\|_p/\|c_n\|_p\to 0$ as $n\to\infty$ for $p\in[1,\infty]$. Then we have

$$\begin{aligned}
(D^2-\mu^2)[E,\varphi_n] =& \Bigg[E, \Bigg(-y^2c_n(\log y)y^\alpha\sum_i\partial_i^2 b - y^2 b\partial_y^2(c_n(\log y)y^\alpha) + y(k-1)b\partial_y(c_n(\log y)y^\alpha) \\
&-(\frac{k^2}{4}+\mu^2)bc_n(\log y)y^\alpha\Bigg)\psi_0 - \mathrm{i}c_n(\log y)y^\alpha\sum_i y(\partial_i b)e_i\cdot\psi_0\Bigg] \\
=&\Bigg[E, -y^2c_n(\log y)y^\alpha\sum_i\partial_i^2 b\psi_0 - \mathrm{i}c_n(\log y)y^\alpha\sum_i y(\partial_i b)e_i\cdot\psi_0 \\
&-y^\alpha b\left(c_n'' + (2\alpha+k-2)c_n' + c_n\left(\alpha(\alpha-1)-(k-1)\alpha+\frac{k^2}{4}+\mu^2\right)\right)\psi_0\Bigg] \\
=&[E, -c_n(\log y)y^{\alpha+2}\sum_i\partial_i^2 b\,\psi_0 - \mathrm{i}c_n(\log y)y^{\alpha+1}\sum_i(\partial_i b)e_i\cdot\psi_0 \\
&-y^\alpha b\,(c_n''+(2\alpha+k-2)c_n')\,\psi_0]\,,
\end{aligned}$$

and by analogous estimates as in Proposition 9.1 we have $\|(D^2-\mu^2)[E,\varphi_n]\|_p/\|[E,\varphi_n]\|_p\to 0$ as $n\to\infty$.

Remark 9.3. *Note that while the L^2-spectrum of the hyperbolic space only consists of continuous spectrum, this is no longer true for the L^p-spectrum for $p\neq 2$ as can be seen by considering $0\in\sigma_p^H$:*
We view the hyperbolic space $(\mathbb{H}^{k+1}, g_{\mathbb{H}})$ modelled on the unit ball $B_1(0)\subset\mathbb{R}^{k+1}$ of the Euclidean space and equipped with the metric $g_{\mathbb{H}} = f^2 g_E$ where $f(x) = \frac{2}{1-|x|^2}$ and $|.|$ denotes the Euclidean norm. Take a constant spinor ψ on $B_1(0)$ normalized such that $\|\psi\|_{L^p(B_1(0),g_E)} = 1$. Then $D^{g_E}\psi = 0$. Using the identification of spinors of conformal metrics set $\varphi := f^{-\frac{k}{2}}\psi$. Then $D^{g_{\mathbb{H}}}\varphi = 0$ and $\|\varphi\|^p_{L^p(g_{\mathbb{H}})} = \int_{B_1(0)} f^{k+1-\frac{k}{2}p}|\psi|^p \mathrm{dvol}_{g_E}$. Thus, φ is an L^p-harmonic spinor if and only if $\int_{B_1(0)}(1-|x|^2)^{-k-1+\frac{k}{2}p}\mathrm{dvol}_{g_E}<\infty$, i.e., if and only if $\int_0^1(1-r^2)^{-1+\frac{k}{2}(p-2)}r^k\mathrm{dr}<\infty$. This is true precisely if $p>2$ and $k>0$. Thus, for all $p>2$ and $k>0$ the L^p-kernel of the Dirac operator on $(\mathbb{H}^{k+1},g_{\mathbb{H}})$ is nontrivial.

10. The L^p-spectrum on $\mathbb{M}_c^{m,k}$ contains σ_p

In this section we complete the proof of Theorem 1.1. In Proposition 8.1 it was shown that the L^p-spectrum on $\mathbb{M}_c^{m,k}$ is contained in σ_p. Thus, the converse remains to be shown. The case $N=\{y\}$ was solved in Proposition 9.1.
Recall that by Lemma B.11 and Example B.12 the Dirac L^p-spectrum on $\mathbb{M}_c^{m,k}$ is point symmetric, i.e., it is symmetric with respect to the reflection $\lambda\mapsto -\lambda$.

Let now $\mu \in \partial\sigma_p$ with $\mu^2 = \lambda_0^2 + \kappa^2$, $|\operatorname{Im}\kappa| = ck\left|\frac{1}{p} - \frac{1}{2}\right|$ be given. By Proposition 9.1 and scaling, we see that κ is in the spectrum of the Dirac operator of $\mathbb{H}_c^{k+1}$. Then, by Lemma B.8 κ^2 is in the L^p-spectrum of $(D^{\mathbb{H}_c^{k+1}})^2$, and by Remark 9.2 there is a sequence $\psi_i \in \Gamma(\Sigma_{\mathbb{H}_c^{k+1}})$ with $\|((D^{\mathbb{H}_c^{k+1}})^2 - \kappa^2)\psi_i\|_{L^p(\mathbb{H}_c^{k+1})} \to 0$ while $\|\psi_i\|_{L^q(\mathbb{H}_c^{k+1})} = 1$. Moreover, by Remark B.7 there is a $\psi \in \Gamma(\Sigma_N)$ with $\|\psi\|_{L^q(N)} = 1$ and $(D^N)^2\psi = \lambda_0^2\psi$.
Assume that at least one of the dimensions of N and $\mathbb{H}_c^{k+1}$ is even. Then $\Sigma_{\mathbb{M}_c} = \Sigma_{\mathbb{H}_c^{k+1}} \otimes \Sigma_N$ and by (1) we have $D^2 = (D^{\mathbb{H}_c^{k+1}})^2 + (D^N)^2$. We set $\varphi_i = \psi_i \otimes \psi$. Then

$$\begin{aligned}\|(D^2 - \mu^2)\varphi_i\|_p =& \|\psi_i \otimes ((D^N)^2 - \lambda_0^2)\psi + ((D^{\mathbb{H}_c^{k+1}})^2 - \kappa^2)\psi_i \otimes \psi\|_p \\ =& \|((D^{\mathbb{H}_c^{k+1}})^2 - \kappa^2)\psi_i \otimes \psi\|_p \to 0.\end{aligned}$$

Thus, μ^2 is in the L^p-spectrum of D^2. By the point symmetry of the spectrum and by Lemma B.8 both μ and $-\mu$ are in the L^p-spectrum of D.
Similarly we obtain the result if both the dimensions of N and $\mathbb{H}_c^{k+1}$ are odd by setting $\varphi_i := \psi_i \otimes (\psi, \psi)$ in notation of Section 2.5.
Up to now we have shown that all $\mu \in \partial\sigma_p$ are in the L^p-spectrum of the Dirac operator on $\mathbb{M}_c$. Following the same arguments as in the last lines of the proof of Proposition 9.1 the proof of Theorem 1.1 is completed.

Remark 10.1. *From Theorem 1.1 and Lemma B.8, we can immediately read off the L^p-spectrum of D^2 on $\mathbb{M}_c^{m,k}$. This consists of the closed parabolic region bounded by*

$$s \in \mathbb{R} \mapsto \lambda_0^2 - c^2k^2\left(\frac{1}{p} - \frac{1}{2}\right)^2 + s^2 + 2\mathrm{i}sck\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Let us compare the L^p-spectrum for D^2 on $\mathbb{M}_c^{k+1,k} = \mathbb{H}^{k+1}$ ($c = 1$ and $\lambda_0 = 0$)

$$s \in \mathbb{R} \mapsto -k^2\left(\frac{1}{p} - \frac{1}{2}\right)^2 + s^2 + 2\mathrm{i}sk\left(\frac{1}{2} - \frac{1}{p}\right),$$

with the one of the Laplacian on functions whose L^p-spectrum is given by the closed parabolic region bounded by [18, (1.5)]

$$s \in \mathbb{R} \mapsto k^2\frac{1}{p}\left(1 - \frac{1}{p}\right) + s^2 + 2\mathrm{i}sk\left(\frac{1}{2} - \frac{1}{p}\right).$$

Up to a shift in the real direction this is the same spectrum. However the qualitative difference is that for $p \neq 2$ the spectrum of D^2 contains negative real numbers, in contrast to the Laplacian.

Appendix A. Function spaces

We want to recall some analytical facts which are helpful to define spinorial function spaces on manifolds.
Let (M^n, g) be an n-dimensional Riemannian spin manifold with the classical Dirac operator $D\colon H_1^2(M, \Sigma_M) = \operatorname{dom} D \subset L^2(M, \Sigma_M) \to L^2(M, \Sigma_M)$. The set of compactly supported spinors $C_c^\infty(M, \Sigma_M)$ is a core of D, i.e., D is the closure of $D|_{C_c^\infty(M,\Sigma_M)}$ w.r.t. the graph norm H_1^2.
A distributional spinor (or distribution with spinor values) is a linear map $C_c^\infty(M, \Sigma_M) \to \mathbb{C}$ with the usual continuity properties of distributions. Any spinor with regularity L^1_{loc} defines a distributional spinor by using the standard L^2-scalar product on spinors.
Then $D\varphi$ can be defined in the sense of distributions. Let $H_1^s(M, \Sigma_M)$ be the set of distributional spinors φ, such that φ und $D\varphi$ are in L^s, $s \in [1, \infty]$. Equipped with the norm $\|\varphi\|_{H_1^s} := \|\varphi\|_s + \|D\varphi\|_s$ this is a Banach space. This norm is the graph norm of D viewed as an operator in L^s to L^s. If we consider the operator $D\colon H_1^s \subset L^s \to L^s$, we will shortly write D_s in the following.

Lemma A.1. *Let $1 \leq s < \infty$. $C_c^\infty(M, \Sigma_M)$ is dense in $H_1^s(M, \Sigma_M)$.*

Proof. Assume that $\varphi \in H_1^s(M, \Sigma_M)$, $s < \infty$, is given. For a given point $p \in M$ and for any $R > 0$ one can find a compactly supported smooth function $\eta_R \colon M \to [0, 1]$ such that $\eta_R \equiv 1$ on $B_R(p)$ and such that $|\nabla \eta_R| \leq R^{-1}$. Then one easily sees $\lim_{R\to\infty} \|\varphi - \eta_R \varphi\|_s = 0$. Further we calculate

$$\|D(\varphi - \eta_R\varphi)\|_s \leq \|\nabla \eta_R \cdot \varphi\|_s + \|(1 - \eta_R)D\varphi\|_s \to 0 \quad \text{as } R \to \infty.$$

Thus the elements with compact support are dense in $H_1^s(M, \Sigma_M)$. Now if $\psi \in H_1^s(M, \Sigma_M)$ has compact support, it follows from standard results that it can be approximated by smooth compactly supported spinors. $\square$

Thus, for $s < \infty$, $H_1^s(M, \Sigma_M)$ is equal to the completion of $C_c^\infty(M, \Sigma_M)$ with respect to the graph norm of $D \colon L^s \to L^s$. In particular this implies that for $s < \infty$ the operator D_s is a closed extension of $D|_{C_c^\infty(M,\Sigma_M)}$ with core $C_c^\infty(M, \Sigma_M)$. Note that D_∞ is a closed extension of $D|_{C_c^\infty(M,\Sigma_M)}$ as well but $C_c^\infty(M, \Sigma_M)$ is in general no longer a core for this operator. Moreover, in the standard literature for L^p-theory of the Laplacian, e.g. [18], the operator for $s = \infty$ is directly defined to be the adjoint operator for $s = 1$.

Lemma A.2. *Let $1 < s < \infty$. On manifolds with bounded geometry, the H_1^s-norm is equivalent to the norm $\|\varphi\|_s + \|\nabla\varphi\|_s$.*

The proof of the lemma relies on local elliptic estimates which follow from the Calderon-Zygmund inequality, e.g. [20, Theorem 9.9], see also [2, Lemma 3.2.2] for the geometric adaptation.

Appendix B. General notes on the L^p-spectrum

In this section we collect general facts on the L^p-spectrum of the Dirac operator. Unless stated otherwise, we only assume that (M, g) is complete.
We examine the adjoint of the operator $D_s \colon L^s \to L^s$ with respect to the duality pairing $\langle ., . \rangle \colon L^s \times (L^s)^* \to \mathbb{C}$ whose restriction to compactly supported spinors coincides with the hermitian L^2-product. We use the convention that this pairing is antilinear in the second component. The adjoint D_s^* is an operator in $(L^s)^*$. For $1 \leq s < \infty$ and $s^{-1} + (s^*)^{-1} = 1$, $(L^s)^* = L^{s^*}$ whereas $(L^\infty)^*$ is larger than L^1. From the formal self-adjointness of D we see, that $D_{s^*}|_{C_c^\infty(M,\Sigma_M)} = D_s^*|_{C_c^\infty(M,\Sigma_M)}$. Moreover, we have

Lemma B.1. *For all $\varphi \in H_1^s$ and $\psi \in H_1^{s^*}$, $1 \leq s \leq \infty$, we have*

$$(D_s\varphi, \psi) = (\varphi, D_{s^*}\psi).$$

Proof. For $1 < s < \infty$, let $\varphi_i, \psi_j \in C_c^\infty(M, \Sigma_M)$ with $\varphi_i \to \varphi$ in H_1^s and $\psi_j \to \psi$ in $H_1^{s^*}$. Then,

$$\int_M \langle D_s\varphi, \psi\rangle \mathrm{dvol}_g \leftarrow \int_M \langle D_s\varphi_i, \psi_j\rangle \mathrm{dvol}_g = \int_M \langle \varphi_i, D_{s^*}\psi_j\rangle \mathrm{dvol}_g \to \int_M \langle \varphi, D_{s^*}\psi\rangle \mathrm{dvol}_g$$

as $i, j \to \infty$.
Let now $s = 1$. For $\varphi \in C_c^\infty(M, \Sigma_M)$ the equality follows from the distributional definition of D_∞. The rest follows since $C_c^\infty(M, \Sigma_M)$ is dense in H_1^1. The remaining case $s = \infty$ just follows from the last one by interchanging s and s^*. $\square$

Lemma B.2. *For all $1 \leq s < \infty$ the operators D_{s^*} and D_s^* coincide.*

Proof. For $\psi \in H_1^{s^*}$ Lemma B.1 yields $(D_s\varphi, \psi) = (\varphi, D_{s^*}\psi)$ for all $\varphi \in H_1^s = \mathrm{dom}\, D_s$. This implies $\psi \in \mathrm{dom}\, D_s^*$ and $D_s^*\psi = D_{s^*}\psi$. Hence, $H_1^{s^*} \subset \mathrm{dom}\, D_s^*$ and $D_s^*|_{H_1^{s^*}} = D_{s^*} \colon H_1^{s^*} \subset L^{s^*} \to L^{s^*}$. It remains to show that $\mathrm{dom}\, D_s^* \subset H_1^{s^*}$: Let $\psi \in \mathrm{dom}\, D_s^* \subset (L^s)^* = L^{s^*}$. Then there is a $\rho \in L^{s^*}$ such that for all $\varphi \in \mathrm{dom}\, D_s$ it holds $(D_s\varphi, \psi) = (\varphi, \rho)$. In particular, this is true for all $\varphi \in C_c^\infty(M, \Sigma_M)$. In other words $D_s^*\psi = \rho$ in the sense of distributions. Thus, $\psi \in H_1^{s^*}$. $\square$

Since $\varphi \in H_1^s \cap H_1^r$ implies $D_s\varphi = D_r\varphi$ we often denote all those Dirac operators in the following just by D.
Moreover, a closed operator $P\colon \operatorname{dom} P \subset V_1 \to V_2$ between Banach spaces V_i, and with dense domain $\operatorname{dom} P$, will be called invertible if there exists a bounded inverse $P^{-1}\colon V_2 \to V_1$. We will use the phrase "P has a bounded inverse" synonymously.

Lemma B.3. *Let $1 \leq s < \infty$.*

(i) *If $\overline{\mu}$ is in the $L^{s^*}-$spectrum of the Dirac operator where $(s^*)^{-1} + s^{-1} = 1$, then μ is in its L^s-spectrum.*
(ii) *Let $D_s - \mu$ be invertible. Then, $(D_{s^*} - \bar{\mu})^{-1} = ((D_s - \mu)^{-1})^*$ and $\|(D_{s^*} - \bar{\mu})^{-1}\| = \|(D_s - \mu)^{-1}\|$.*

Proof. We prove this for $\mu = 0$. For arbitrary μ this is done analogously.
Assume that 0 is not in the L^s-spectrum of D, i.e., it has a bounded inverse $E = D^{-1}\colon L^s \to L^s$ with range $\operatorname{ran} E = H_1^s$. Let $\varphi \in L^{s^*}$. Since E is bounded, $f\colon L^s \to \mathbb{C}, \rho \mapsto (E\rho, \varphi)$ is a bounded functional and, thus, f is in the dual space of L^s, i.e., there is $\psi \in L^{s^*}$ with $(\rho, \psi) = f(\rho) = (E\rho, \varphi)$ for all $\rho \in L^s$. Hence, $\varphi \in \operatorname{dom} E^*$, i.e., $\operatorname{dom} E^* = L^{s^*}$.
Now we can estimate for all $\varphi \in H_1^s$ and all $\psi \in L^{s^*}$ that $(D\varphi, E^*\psi) = (ED\varphi, \psi) = (\varphi, \psi)$ which implies $E^*\psi \in \operatorname{dom} D^*$ and $D^*E^*\psi = \psi$. Thus, $\operatorname{ran} D^* = L^{s^*}$ and $D^*E^* = \mathrm{Id}\colon L^{s^*} \to L^{s^*}$.
If $\rho \in L^s$ and $\varphi \in \operatorname{dom} D^*$, we get $(\rho, E^*D^*\varphi) = (E\rho, D^*\varphi) = (DE\rho, \varphi) = (\rho, \varphi)$. Hence, $E^*D^* = \mathrm{Id}\colon \operatorname{dom} D^* \to \operatorname{dom} D^*$. Together with the corresponding statement from above this gives that $(D^{-1})^* = (D^*)^{-1}$. Thus, 0 is not in the L^{s^*}-spectrum of D. This proves (i) and the first claim of (ii). The operator norm of an operator and its adjoint coincide, see [31, Thm VI.2]. Thus, the equality of the operator norms follows. □

Corollary B.4. *If $D\colon H_1^q \to L^q$ has a bounded inverse for some $q \in (1, \infty)$. Then as an operator from $H_1^s \to L^s$ it has a bounded inverse for all $s \in [q_1, q_2]$ where $q_1 = \min\{q, q^*\}$, $q_2 = \max\{q, q^*\}$, and $(q^*)^{-1} + q^{-1} = 1$. In particular, the L^2-spectrum of D is a subset of the L^q-spectrum.*

Proof. This Lemma follows directly from the Riesz-Thorin Interpolation Theorem 2.2 (using $\mathcal{D} = C_c^\infty(M, \Sigma_M)$) and Lemma B.3. □

Lemma B.5. *Let $1 \leq s \leq \infty$. Let $R_s = \mathbb{C} \setminus \operatorname{Spec}_{L^s}(D)$ be the resolvent set of $D\colon L^s \to L^s$. Then, the resolvent*

$$\mu \in R_s \mapsto (D - \mu)^{-1} \in \mathcal{B}(L^s)$$

is analytic, i.e., the map is locally given by a convergent power series with coefficients in $\mathcal{B}(L^s)$. Here, $\mathcal{B}(L^s)$ denotes the set of bounded operators from L^s to itself.

See [42, VIII.2 Theorem 1] for a proof.
For rounding up our presentation we will next add a lemma not needed in our context but maybe helpful to other applications.

Lemma B.6.

(1) The operator $D\colon H_1^s \subset L^s \to L^s$, $s \in [1, \infty]$, is an invertible map onto its image if and only if there is a constant $C > 0$ with $\|D\varphi\|_s \geq C\|\varphi\|_s$ for all $\varphi \in H_1^s$.
(2) Under the above conditions the image $D(H_1^s)$ is closed.
(3) Let $s^{-1} + (s^*)^{-1} = 1$, $s < \infty$, and assume the conditions from above. Then D is surjective if and only if there is a $C > 0$ with $\|D\varphi\|_{s^*} \geq C\|\varphi\|_{s^*}$ for all $\varphi \in H_1^{s^*}$.

Proof. (1) The proof is straightforward.
(2) The operator $D\colon H_1^s \to D(H_1^s)$, where the latter space is equipped with the L^s-norm, is a bijective bounded linear map. Hence, $D(H_1^s)$ is a complete subspace of L^s and thus closed.
(3) Suppose that $D(H_1^s)$ is a proper subspace of L^s. Due to Hahn-Banach there is a non-zero continuous functional $\psi\colon L^s \to \mathbb{C}$ vanishing on $D(H_1^s)$. We interpret ψ as an element in

L^{s^*} using the Riesz representation theorem, i.e. $\psi \in L^{s^*}$ is orthogonal on $D(H_1^s)$. Then, $\psi \in \mathrm{dom}\,(D_s)^*$, and we even have $D_s^*\psi = 0$. Hence, by Lemma B.2 $\psi \in H_1^{s*}$. This contradicts the estimate.
Now assume that D is surjective. Then there is a bounded operator $D^{-1}\colon L^s \to L^s$, inverse to D. Thus $(D^{-1})^*\colon L^{s^*} \to L^{s^*}$ is bounded as well, and $(D^{-1})^*$ is the inverse of $D^*\colon H_1^{s^*} \to L^{s^*}$. The fact that the latter map has a bounded inverse is equivalent to the existence of a constant $C > 0$ with $\|D\varphi\|_{s^*} \geq C\|\varphi\|_{s^*}$.

□

Remark B.7. *The L^s-spectrum of the Dirac operator D on a closed manifold (M^m, g) is independent of s. We sketch the proof: Let φ be an L^2-eigenspinor of D. Then regularity theory implies that $\varphi \in C^\infty(M, \Sigma_M)$ and, hence, $\varphi \in L^s$ for all $1 \leq s \leq \infty$. In particular, $\mathrm{Spec}_{L^2}^M(D) \subset \mathrm{Spec}_{L^s}^M(D)$. Let now $\mu \notin \mathrm{Spec}_{L^2}^M(D)$, i.e., $(D-\mu)^{-1}\colon L^2 \to L^2$ is bounded. Let $G(x,y)$ be the unique Green function of $D-\mu$, see Proposition 3.2. Then, $\int_M |G(.,y)|^2 dy$ is bounded uniformly in y. Hölder's inequality implies that also $\int_M |G(.,y)| dy$ is bounded uniformly in y. Hence, $(D-\mu)^{-1}\colon L^1 \to L^1$ is a bounded operator. Then interpolation gives that $(D-\mu)^{-1}\colon L^s \to L^s$ is bounded for all $1 \leq s < 2$. Because of $\mathrm{Spec}_{L^2}(D) \subset \mathbb{R}$ the same is true for $(D-\bar{\mu})^{-1}\colon L^s \to L^s$, and by using Lemma B.3 we get that $(D-\mu)^{-1}\colon L^s \to L^s$ is bounded for all $2 < s < \infty$. It remains $s = \infty$: Let $r > m$. Then by the Sobolev Embedding Theorem $H_1^r \hookrightarrow L^\infty$ is bounded. Moreover, by the discussion above and using the fact that H_1^r carries the graph norm of D we know that $(D-\mu)^{-1}\colon L^r \to H_1^r$ is bounded for $\mu \notin \mathrm{Spec}_{L^2}^M(D)$ the Hölder inequality gives that*

$$(D-\mu)^{-1}\colon\ L^\infty \to L^r \to H_1^r \to L^\infty$$

is bounded.

Lemma B.8. *Let $1 \leq s \leq \infty$, and let $\mathrm{Spec}_{L^s}^M(D) \neq \mathbb{C}$. Then the complex number μ^2 is in the L^s-spectrum of D^2 if and only if μ or $-\mu$ is in the L^s-spectrum of D.*

Proof. We start with the "only if" part. So assume that both μ and $-\mu$ are not in the L^s-spectrum of D. Then we have bounded operators $(D-\mu)^{-1}\colon L^s \to L^s$ and $(D+\mu)^{-1}\colon L^s \to L^s$. It is then easy to verify that $(D-\mu)^{-1} \circ (D+\mu)^{-1}\colon L^s \to L^s$ is a bounded inverse of $D^2 - \mu^2 = (D+\mu) \circ (D-\mu)$. Thus μ^2 is not in the L^s-spectrum of D^2.
In order to prove the "if" statement, we assume that μ^2 is not in the spectrum of D^2. Then $D^2 - \mu^2$ has a bounded inverse $P := (D^2-\mu^2)^{-1}\colon L^s \to L^s$. Let $\psi \in P(L^s)$. Then $\psi \in L^s$ and $D^2\psi \in L^s$. Next we will show that this implies $D\psi \in L^s$. For that we choose $\lambda \notin \mathrm{Spec}_{L^s}^M(D)$. Then $D\psi = (D-\lambda)^{-1}(D^2-\lambda^2)\psi - \lambda\psi$, and hence $D\psi \in L^s$. Thus, $P(L^s) \subset H_1^s$. Hence $Q_1 := (D \pm \mu) \circ P$ is a bounded operator with $\mathrm{dom}\, Q_1 = L^s$, and one easily checks that this a right inverse to $(D \mp \mu)$. Similarly, one shows that $Q_2 := P \circ (D \pm \mu)$ is a left inverse of $(D \mp \mu)$. A priori Q_2 is only defined on H_1^s, but using $Q_1 = Q_1 \circ (D \mp \mu) \circ Q_2 = Q_2$ it is clear that Q_2 and Q_1 coincide on H_1^s. So the integral kernels of Q_1 and Q_2 have to coincide, so Q_1 is a left and right inverse of $(D \mp \mu)$ and thus $\pm\mu$ is not in the spectrum of D. □

Remark B.9. *In the case $1 < s < \infty$ and M of bounded geometry, one can also prove that $\mathrm{Spec}_{L^s}^M(D) = \mathbb{C}$ implies $\mathrm{Spec}_{L^s}^M(D^2) = \mathbb{C}$: As in the proof of the "if" statement from above one has to show that $D\psi \in L^s$. This can be proven using regularity theory on manifolds of bounded geometry.*

Lemma B.10 (Pointwise symmetries)**.** *Let $1 \leq s \leq \infty$. Let (M, g) be an m-dimensional Riemannian spin manifold.*

(i) $m \equiv 0 \mod 2$*: The number μ is in the L^s-spectrum of D if and only if $-\mu$ is in the L^s-spectrum of D if and only if $\bar{\mu}$ is in the L^s-spectrum of D.*
(ii) $m \equiv 1 \mod 4$*: The number μ is in the L^s-spectrum of D if and only if $-\bar{\mu}$ is in the L^s-spectrum of D.*

(iii) $m \equiv 3 \mod 4$*: The number* μ *is in the* L^s*-spectrum of* D *if and only if* $\bar{\mu}$ *is in the* L^s*-spectrum of* D.

Proof. By [19, Prop. p. 31] we have a map $\alpha\colon \Sigma_m \to \Sigma_m$ that is

- a $\mathrm{Spin}(m)$-equivariant real structure that anticommutes with Clifford multiplication if $m \equiv 0, 1 \mod 8$.
- a $\mathrm{Spin}(m)$-equivariant quaternionic structure that commutes with Clifford multiplication if $m \equiv 2, 3 \mod 8$.
- a $\mathrm{Spin}(m)$-equivariant quaternionic structure that anticommutes with Clifford multiplication if $m \equiv 4, 5 \mod 8$.
- a $\mathrm{Spin}(m)$-equivariant real structure that commutes with Clifford multiplication if $m \equiv 6, 7 \mod 8$.

Note that by definition real structure means that $\alpha^2 = \mathrm{Id}$ and $\alpha(\mathrm{i}v) = -\mathrm{i}\alpha(v)$. Moreover, quaternionic structure means that $\alpha^2 = -\mathrm{Id}$ and $\alpha(\mathrm{i}v) = -\mathrm{i}\alpha(v)$.
Due to the $\mathrm{Spin}(m)$-equivariance α induces a fiber preserving map $\tilde{\alpha}$ on the spinor bundle with the same properties as above. Thus,

$$(D-\mu)\circ\tilde{\alpha}(\varphi) = \begin{cases} \tilde{\alpha}\circ(-D-\bar{\mu})(\varphi) & m \equiv 0,1 \mod 4 \\ \tilde{\alpha}\circ(D-\bar{\mu})(\varphi) & m \equiv 2,3 \mod 4. \end{cases}$$

Thus, if μ is in the L^s-spectrum of D then $-\bar{\mu}$ (resp. $\bar{\mu}$) in the L^s-spectrum of D for $m \equiv 0, 1$ (resp. 2,3) $\mod 4$. This gives (ii) and (iii).
If m is even, then $D(\omega_M \cdot \varphi) = -\omega_M \cdot D\varphi$. Thus, the spectrum is symmetric when reflected on the imaginary axis. Together with the symmetries from above, (i) follows. □

Lemma B.11 (Orientation reversing isometry)**.** *Let* $1 \le s \le \infty$. *Assume there is an orientation reversing isometry* $f\colon M^m \to M^m$ *that "lifts" to the spin structure as described in the proof. Then* μ *is in the* L^s*-spectrum of* D *if and only if* $-\mu$ *is in the* L^s*-spectrum of* D.

Proof. The proof follows the lines of [4, Appendix A]. In this reference, f is required to be a reflection at a hyperplane of M. But this doesn't change the part we need: We lift f to the bundle $P_{\mathrm{SO}(m)}M$ of oriented orthonormal frames by mapping the frame $\mathcal{E} = (e_1, \ldots, e_m)$ to $f_*\mathcal{E} = (-\mathrm{d}f(e_1), \mathrm{d}f(e_2), \ldots, \mathrm{d}f(e_m))$, so $f_*\colon P_{\mathrm{SO}(m)}M \to P_{\mathrm{SO}(m)}M$. Since f is an orientation reserving isometry,

$$f_*(\mathcal{E}A) = f_*(\mathcal{E})JAJ \text{ for all } A \in \mathrm{SO}(m)$$

where $J = \mathrm{diag}(-1, 1, 1, \ldots, 1)$. The map f is assumed to lift to the spin structure, i.e., there is a lift $\tilde{f}_*\colon P_{\mathrm{Spin}(m)}(M) \to P_{\mathrm{Spin}(m)}(M)$ with $\vartheta \circ \tilde{f}_* = f_* \circ \vartheta$ where ϑ denotes the double covering $\vartheta\colon P_{\mathrm{Spin}(m)}(M) \to P_{\mathrm{SO}(m)}(M)$. By [4, Lemma A.1 and Lemma A.4], f then lifts to a map $f_\sharp\colon \Sigma_M \to \Sigma_M$ on the spinor bundle which fulfils $f_\sharp(D\varphi) = -D(f_\sharp\varphi)$. □

Example B.12.
(i) Let M^{n+1} be a Riemannian spin manifold with a spin structure ϑ as above. Assume that up to isomorphism this is the unique spin structure on M. Let $f\colon M \to M$ be an orientation reversing isometry. By pulling back the double covering $P_{\mathrm{Spin}}M \to P_{\mathrm{SO}}M$ by f_* we obtain the double covering $f^*\vartheta : f^*P_{\mathrm{Spin}}M \to P_{\mathrm{SO}}M$. We then turn $f^*P_{\mathrm{Spin}}M$ into a $\mathrm{Spin}(n+1)$-principal bundle by conjugating the action of $\mathrm{Spin}(n+1)$ on $P_{\mathrm{Spin}}M$ with Clifford multiplication with e_0. Then $f^*\vartheta$ is a spin structure on M. Thus an isomorphism from ϑ to $f^*\vartheta$ yields a map $f_\sharp$ as above.
(ii) Consider the map $f = f_1 \times \mathrm{id}\colon \mathbb{M}_c^m = \mathbb{H}_c^{k+1} \times N^n \to \mathbb{M}_c^{m,k}$ where f_1 is an orientation reversing isometry as in (i). Then, f is again an orientation reversing isometry. Using $P_{\mathrm{SO}}(\mathbb{H}_c \times N) = (P_{\mathrm{SO}}(\mathbb{H}_c^{k+1}) \times P_{\mathrm{SO}}(N)) \times_{\bar{\xi}} \mathrm{SO}(m)$ where $\bar{\xi}\colon \mathrm{SO}(k+1) \times \mathrm{SO}(n) \to \mathrm{SO}(m)$ is the standard embedding and using the analogous describtion for $P_{\mathrm{Spin}}(\mathbb{H}_c \times N)$, see Section 2.5, one see that also f lifts to the spin structure.

Appendix C. Dirac eigenvalues of generic metrics

Proposition C.1. *Let (M,g) be a closed, connected Riemannian spin manifold, let $\mu \in \mathbb{R}$. Let $U \subset M$ be a nonempty open subset. In case that $\mu = 0$, assume additionally that the α-genus of M is zero. Then, there is a metric $\tilde{g}$ on M with $\tilde{g} = g$ on $M \setminus U$ and* $\ker(D^{\tilde{g}} - \mu) = \{0\}$.

Proof. For $\mu = 0$, the proposition follows from [4, Theorem 1.1]. For $\mu \neq 0$, the proof is a direct consequence of the following lemma. □

Lemma C.2. *Let (M,g) be a closed, connected Riemannian spin manifold, let $\mu \in \mathbb{R} \setminus \{0\}$, and let $U \subset M$ be a nonempty open subset. Then there is a function $f \in C^\infty(M, \mathbb{R}^+)$ with $f|_{M\setminus U} \equiv 1$ such that* $\ker(D^{fg} - \mu) = \{0\}$.

Proof. Choose $f \in C^\infty(M, \mathbb{R}^+)$ with $f|_{M\setminus U} \equiv 1$ such that $d = \dim(E_{f,\mu} := \ker(D^{fg} - \mu))$ is minimal. Assume $d > 0$, and set $g_0 = fg$. For $\alpha \in C^\infty(M)$ with $\operatorname{supp}\alpha \subset U$ and t close to 0 we define $g_t := (1 + t\alpha)fg$. Then by [10] there are real analytic functions $\mu_1, \ldots, \mu_d \colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $\mu_i(0) = \mu$ such that $\mathrm{Spec}^M_{L^2}(D^{g_t}) \cap (\mu - \delta, \mu + \delta) = \{\mu_1(t), \ldots, \mu_d(t)\}$ including multiplicities. It is shown in [10] that there is an orthonormal basis $(\psi^{(1)}, \ldots, \psi^{(d)})$ of $E_{f,\mu}$ depending on the choice of α such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mu_i(t) = -\frac{1}{2}\int_M \langle \alpha g_0, Q_{\psi^{(i)}} \rangle \mathrm{dvol}_{g_0}$$

where $Q_\psi(X,Y) = \frac{1}{2}\mathrm{Re}\,\langle X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi \rangle$. Thus,

$$\langle g_0, Q_{\psi^{(i)}} \rangle = \sum_r \langle e_r \cdot \nabla_{e_r}\psi^{(i)}, \psi^{(i)} \rangle = \mu |\psi^{(i)}|^2.$$

As d is minimal, we see that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mu_i(t) = 0$, and thus for all α as above

$$-\frac{1}{2}\int_M \alpha\mu \sum_{i=1}^d |\psi^{(i)}|^2 \mathrm{dvol}_{g_0} = 0.$$

Note that $\varphi := \sum_{i=1}^d |\psi^{(i)}|^2 \in C^\infty(M)$ does not depend on the choice of α. This can be seen by direct calculation with base change matrices or alternatively by observing that φ is the pointwise trace of the integral kernel of the projection to $E_{f,\mu}$. With $\mu \neq 0$ this implies that φ and thus all $\psi^{(i)}$ vanish on U. The unique continuation principle implies then $\psi^{(i)} \equiv 0$ which gives a contradiction. □

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