HOLONOMY RIGIDITY FOR RICCI-FLAT METRICS

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Abstract. On a closed connected oriented manifold M we study the space $\mathcal{M}_{\parallel}(M)$ of all Riemannian metrics which admit a non-zero parallel spinor on the universal covering. Such metrics are Ricci-flat, and all known Ricci-flat metrics are of this form. We show the following: The space $\mathcal{M}_{\parallel}(M)$ is a smooth submanifold of the space of all metrics, and its premoduli space is a smooth finite-dimensional manifold. The holonomy group is locally constant on $\mathcal{M}_{\parallel}(M)$. If M is spin, then the dimension of the space of parallel spinors is a locally constant function on $\mathcal{M}_{\parallel}(M)$.

1. Overview over the results

Let M be a compact connected oriented manifold without boundary, and let $\pi : \widetilde{M} \to M$ be its universal covering. We assume throughout the article that $\widetilde{M}$ is spin. We define $\mathcal{M}_{\parallel}(M)$ to be the space of all Riemannian metrics on M, such that $(\widetilde{M}, \tilde{g})$, $\tilde{g} \coloneqq \pi^* g$, carries a (non-zero) parallel spinor. This implies that g is a stable Ricci-flat metric on M, see [7]. It is still an open question, whether $\mathcal{M}_{\parallel}(M)$ contains all Ricci-flat metrics on M.

In the past the space of such metrics was studied in much detail in the irreducible and simply-connected case, see e.g. [19], [20]. Much less is known already in the non-simply-connected irreducible case, see e.g. [21]. However, the general reducible case has virtually not been addressed at all. For example there is no classification of the full holonomy groups for metrics in $\mathcal{M}_{\parallel}(M)$ if the restricted holonomy is reducible.

The second named author recently found an efficient method to describe deformations of products of stable Ricci-flat manifolds [13]. In the present article we are using this method to see that the moduli space $\mathcal{M}_{\parallel}(M)$ is well-behaved.

In fact we show that $\mathcal{M}_{\parallel}(M)$ is a smooth submanifold of the space of all metrics, and its premoduli space is a smooth finite-dimensional manifold, see Corollary 5. If we view the dimension of the space of parallel spinors as a function $\mathcal{M}_{\parallel}(M) \to \mathbb{N}_0$, then this function is locally constant, see Corollary 4. Similarly, the full holonomy group, viewed as a function from $\mathcal{M}_{\parallel}(M)$ to conjugacy classes of subgroups in $\mathrm{GL}(n, \mathbb{R})$ is locally constant, see Theorem 1.

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2. Ricci-flat metrics

Conventions. All our manifolds will be connected and of dimension $n \geq 3$ unless stated otherwise. A *spin* manifold is a manifold together with a fixed spin structure. We will assume completeness of all Riemannian metrics without explicitly stating it.

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A Riemannian manifold (M,g) is *Ricci-flat* if its Ricci tensor Ric^g vanishes identically. There are various reasons to study this class of metrics.
It is already a challenging problem to decide whether a compact manifold admits a Ricci-flat metric. Compact manifolds with positive Ricci curvature must have finite fundamental group due to the Bonnet-Myers theorem. A similar obstruction is also available for compact manifolds with nonegative Ricci curvature: as a consequence of the splitting theorem by Cheeger and Gromoll [6], one obtains the following theorem, see also Fischer and Wolf [8].

Theorem A (Structure theorem for Ricci-flat manifolds)**.** *Let (M,g) be a compact Ricci-flat manifold. Then there exists a finite normal Riemannian covering $\overline{M} \times T^q \to M$ with $(\overline{M},\bar{g})$ a compact simply-connected Ricci-flat manifold and (T^q, g_{fl}) a flat torus.*

The theorem implies that $\pi_1(M)$ contains a free abelian group $\mathbb{Z}^q$ of rank q of finite index, and q satisfies $b_1(M) \leq q \leq \dim M$, which acts by translations on $\mathbb{R}^q$ and trivially on $\overline{M}$. The number q only depends on the fundamental group: If we choose a finite subset of generators $S = S^{-1} \subset \pi_1(M)$ and if N_r denotes the number of elements of $\pi_1(M)$ that can be represented by a word containing at most r generators in S, then there is a constant $C \in \mathbb{R}$ with $C^{-1}r^q \leq N_r \leq Cr^q$.
Note that on a compact Ricci-flat manifold every Killing field is parallel. Thus compact locally symmetric Ricci-flat manifolds are flat.
A second reason for studying Ricci-flat metrics comes from the intimate relation with the concept of *holonomy.* To fix notation we briefly recall the definition of holonomy groups and some of their main properties. We refer to [3, Chapter 10] for further details. Let us fix a point $x \in M$ and an identification of T_xM with $\mathbb{R}^n$. Up to conjugacy in $\mathrm{GL}(n,\mathbb{R})$ these choices define a subgroup $\mathrm{Hol}(M,g) \subset \mathrm{GL}(n,\mathbb{R})$ as the set of endomorphisms given by parallel transport around a loop in x, called the *(full) holonomy group* of (M,g). The subgroup obtained by taking loops homotopic to the constant path is called the *restricted holonomy group* and is denoted by $\mathrm{Hol}_0(M,g)$. Obviously, $\mathrm{Hol}_0(M,g)$ is a normal and connected subgroup of $\mathrm{Hol}(M,g)$, and we have $\mathrm{Hol}(\widetilde{M},\tilde{g}) = \mathrm{Hol}_0(M,g)$. It is also known that $\mathrm{Hol}_0(M,g)$ is a closed Lie subgroup of $\mathrm{SO}(n)$, in contrast to $\mathrm{Hol}(M,g)$ which might be non-compact, see [22]. Parallel transport induces a group epimorphism $\pi_1(M) \to \mathrm{Hol}(M,g)/\mathrm{Hol}_0(M,g)$, and thus $\mathrm{Hol}(M,g)/\mathrm{Hol}_0(M,g)$ is countable. As a consequence $\mathrm{Hol}_0(M,g)$ is the connected component of $\mathrm{Hol}(M,g)$ containing the identity, and the index of $\mathrm{Hol}_0(M,g)$ in $\mathrm{Hol}(M,g)$ is the number of connected components of $\mathrm{Hol}(M,g)$.
As we explained above locally symmetric, Ricci-flat Riemannian manifolds are flat. A locally irreducible, Ricci-flat Riemannian manifold (M,g) of dimension $n \geq 2$, is never locally symmetric. In this case $\mathrm{Hol}_0(M,g)$ is conjugate to one of the groups occuring in Table 1. The possible full holonomy groups of such manifolds were studied by McInnes [15], see Wang [21] for the spin case, and they derived a list of possible groups. On the other hand, it was shown by Moroianu and Semmelmann [17] that every group in this list is the holonomy group of such a manifold. Thus the possible full holonomy groups of locally irreducible, Ricci-flat Riemannian manifolds are classified.
On the other hand, in the case of a flat (reducible) metric we have $\mathrm{Hol}_0(M,g) = \{1\}$. Then the possible full holonomy groups and their associated representations are studied in the context of Bieberbach groups, e.g. Auslander and Kuranishi proved that every finite group is isomorphic to $\mathrm{Hol}(M,g)$ for some compact flat manifold (M,g), see [5, III.5].
In the reducible non-locally symmetric case, however, less is known about the full holonomy group of than one would expect. For example, it seems to be unknown

whether there are Ricci-flat spin manifolds with reducible restricted holonomy and without flat factors whose full holonomy group is not contained in a product of holonomy groups in McInnes' list. Still, it follows from the structure theorem for Ricci-flat manifolds, that the subgroup $\mathbb{Z}^q \subset \pi_1(M)$ is in the kernel of the map $\pi_1(M) \to \mathrm{Hol}(M,g)/\mathrm{Hol}_0(M,g)$. Thus $\mathrm{Hol}(M,g)/\mathrm{Hol}_0(M,g)$ is finite, see also [6, Theorem 6].

A third reason comes from the connection with spin geometry (see for instance [10] for more background on spinors). A parallel spinor on any Riemannian spin manifold forces the underlying metric to be Ricci-flat with holonomy group strictly contained in $\mathrm{SO}(n)$. Conversely, as we have just recalled, a locally irreducible orientable Ricci-flat manifold is either of holonomy $\mathrm{SO}(n)$, or it has a simply-connected restricted holonomy group. In the latter case case the universal covering of M is spin and there is a non-trivial parallel spinor on the universal covering. The dimension of the space of parallel spinors $\mathcal{P}(M,g)$ is determined by the holonomy group [19], see Table 1. Furthermore, if the compact manifold (M,g) carries a parallel spinor, then it follows from Wang's work [20] that one obtains a parallel spinor for any metric in the connected component of g within the space of Ricci-flat metrics on M.

In the whole article a spinor is a smooth section of the complex spinor bundle and $\dim \mathcal{P}(M,g)$ is the complex dimension of the space of parallel spinors.

Our main theorem studies the holonomy group of a metric with parallel spinor under Ricci-flat deformations.

Theorem 1 (Rigidity of the holonomy group). *Let (M,g) be a compact Riemannian manifold whose universal covering is spin and carries a parallel spinor. If g_t, $t \in I := [0,T]$ is a smooth family of Ricci-flat metrics such that $g_0 = g$, then $\mathrm{Hol}(M,g_t)$ is conjugate to $\mathrm{Hol}(M,g)$ in $\mathrm{GL}(n,\mathbb{R})$.*

3. Rigidity of products

An important ingredient in the proof of Theorem 1 is the following product formula by Kröncke [13]. It essentially tells us that product metrics $g \times h$ on $N \times P$, with $g \in \mathcal{M}_{\|}(N)$, $h \in \mathcal{M}_{\|}(P)$ and N, P compact, are rigid in the sense that deformations within $\mathcal{M}_{\|}(N \times P)$ are again product metrics. This cannot hold in the strict sense, as flat tori and pull-backs of $g \times h$ by non-product diffeomorphisms provide counterexamples to rigidity in the strict sense. However, rigidity holds modulo diffeomorphisms, provided that one of the factors does not carry parallel vector fields.

To describe this in detail, let g be a Riemannian metric on a compact manifold N. An infinitesimal deformation of g given by a symmetric 2-tensor h is orthogonal to the conformal class if h is trace-free (i.e. $\mathrm{tr}_g h = 0$), and h is orthogonal to the diffeomorphism orbit of g if h is divergence-free. On trace-free, divergence-free symmetric 2-tensors the linearization of the Ricci curvature functional $g \mapsto \mathrm{Ric}^g$ is

$\mathrm{Hol}(M^n,g)$	$\dim \mathcal{P}(M,g)$
$\mathrm{SO}(n)$	–
$\mathrm{SU}(m)$, $n = 2m$	2
$\mathrm{Sp}(k)$, $n = 4k$	$k+1$
$\mathrm{Spin}(7)$, $n = 8$	1
G_2, $n = 7$	1

Table 1. Special holonomy groups and the dimension of the space of parallel spinors.

given by the Einstein operator $\Delta_E^N = \frac{1}{2}\nabla^*\nabla - \mathring{R}$, where $\mathring{R}h(X,Y) := \sum h(R_{e_i,X}Y, e_i)$ for a frame (e_i) for g. For any smooth family g_t of Ricci-flat metrics of some fixed volume such that $g_0 = g$ and such that $h = \frac{d}{dt}\big|_{t=0} g_t$ is divergence-free, h is in addition trace-free and $\Delta_E^N h = 0$, see [3, Chapter 12].
If a trace-free, divergence-free symmetric 2-tensor h is in $\ker(\Delta_E^N)$, then it is called an *infinitesimal Ricci-flat deformation*. An infinitesmal Ricci-flat deformation h is called *integrable* if there exists a smooth family g_t of Ricci-flat metrics such that $g_0 = g$ and $\frac{d}{dt}\big|_{t=0} g_t = h$. We say that g is a *stable* Ricci-flat metric if Δ_E^N is positive-semidefinite on trace-free, divergence-free symmetric 2-tensors. All metrics in $\mathcal{M}_\parallel(N)$ are stable, see [20] and also [7]. We want to remark that [7] even provides a local version of stability which is stronger than the infinitesimal version of stability that we are using here.

Theorem B (Ricci-flat deformations of products, [13, Prop. 4.5 and 4.6])**.** *If (N^n, g) and (P^p, h) are two stable Ricci-flat manifolds, then $(N \times P, g + h)$ is also stable.*
Furthermore, on trace-free, divergence-free symmetric 2-tensors we have

$$\ker(\Delta_E^{N\times P}) = \mathbb{R}(p\cdot g - n\cdot h) \oplus \big(\Gamma_\parallel(TN) \odot \Gamma_\parallel(TP)\big) \oplus \ker(\Delta_E^N) \oplus \ker(\Delta_E^P),$$

where $\Gamma_\parallel$ is the space of parallel sections. Thus, if all infinitesimal Ricci-flat deformations of (N, g) and (P, h) are integrable, then all infinitesimal Ricci-flat deformations of $(N \times P, g + h)$ are integrable.

In particular, if $\Gamma_\parallel(TN) = 0$ or if $\Gamma_\parallel(TP) = 0$, then Ricci-flat deformations of $g \times h$ are up to diffeomorphisms again of product form.

4. Rigidity of the restricted holonomy group

In order to prove the theorem we first prove the analogous statement for the restricted holonomy.

Proposition 2 (Rigidity of the restricted holonomy group)**.** *Let (M, g) be a compact Riemannian manifold whose universal covering is spin and carries a parallel spinor. If g_t, $t \in I := [0, T]$ is a smooth family of Ricci-flat metrics such that $g_0 = g$, then $\mathrm{Hol}_0(M, g_t)$ is conjugate to $\mathrm{Hol}_0(M, g)$ in $\mathrm{GL}(n, \mathbb{R})$, i.e. there are $Q_t \in \mathrm{GL}(n, \mathbb{R})$ with $\mathrm{Hol}_0(M, g_t) = Q_t \mathrm{Hol}_0(M, g) Q_t^{-1}$. Moreover, the map $t \mapsto Q_t$ can be chosen continuously.*

In the proposition we have fixed a base point $x \in M$ and an identification $T_xM \cong \mathbb{R}^n$. Thus $\mathrm{Hol}_0(M, g_t) \in \mathrm{GL}(n, \mathbb{R})$.

Remark.

(i) The proposition (and our main theorem) apply for example to Riemannian spin manifolds carrying a parallel spinor. Under these assumptions, the universal covering $(\widetilde{M}, \tilde{g})$ is spin as well, and the parallel spinor on (M, g) lifts to a parallel spinor on $(\widetilde{M}, \tilde{g})$.

(ii) The proposition (and our main theorem) are false for non-compact manifolds. Indeed, let g be a (possibly non-complete) Ricci-flat metric on $\mathbb{R}^n$ with a parallel spinor and non-trivial holonomy group H. Let $\mu_s : \mathbb{R}^n \to \mathbb{R}^n$ be multiplication with $1 - s$ for $1 > s \geq 0$. Then $g_s := (1 - s)^{-2}\mu_s^* g$ also has holonomy H for $1 > s \geq 0$. However, for $s \to 1$ the metric g_s converges in the compact-open C^∞ topology to the flat metric g_1 with trivial holonomy.

Proof of the proposition. Consider the universal Riemannian covering $(\widetilde{M}, \tilde{g})$ of (M, g), and let $\Gamma \cong \pi_1(M)$ be the group of deck transformations. Using the structure theorem for Ricci-flat metrics [6, 8] recalled in Section 2, we know that

(M,g) has a finite normal Riemannian covering isometric to $(\overline{M},\bar{g})\times(T^q,g_{\mathrm{fl}})$, where $(\overline{M},\bar{g})$ is compact, simply-connected and Ricci-flat, and (T^q,g_{fl}) a flat torus. By the de Rham decomposition theorem, $(\overline{M},\bar{g})$ is globally isometric to a Riemannian product of compact, simply-connected and *irreducible* Ricci-flat manifolds $(\overline{M}_i,\bar{g}_i)$, $i=1,\dots,r$. Thus we obtain a finite Riemannian covering

$$(\overline{M}\times T^q=\overline{M}_1\times\ldots\times\overline{M}_r\times T^q,\bar{g}_1\times\ldots\times\bar{g}_r\times g_{\mathrm{fl}})\to(M,g)$$

whose holonomy group equals $H_1\times\ldots\times H_r=\mathrm{Hol}(\widetilde{M},\tilde{g})=\mathrm{Hol}_0(M,g)$.
Since $\widetilde{M}$ is spin, so is $\overline{M}$, and thus all $\overline{M}_i$ are spin. The spin structure on each $\overline{M}_i$ is unique as $\overline{M}_i$ is simply-connected. On T^q there are several distinct spin structures; we choose the unique spin structure that admits parallel spinors. A parallel spinor on a product corresponds to a product of parallel spinors (this well-known fact follows from simple representation theoretic considerations as in [1, Prop. 4.5], see for instance [14, Theorem 2.5] for a detailed proof). Thus the given parallel spinor on $\widetilde{M}$ implies the existence of parallel spinors on each $\overline{M}_i$. The holonomy group $H_i:=\mathrm{Hol}(\overline{M}_i,\bar{g}_i)$ is therefore conjugate to one of the groups of Table 1.
Then by taking products, we obtain a parallel spinor on $(\overline{M}\times T^q,\bar{g}\times g_{\mathrm{fl}})$, which lifts to a parallel spinor on $\widetilde{M}$, and we can assume without loss of generality that this is the given parallel spinor on $\widetilde{M}$ discussed above.

Next consider the family of Ricci-flat metrics g_t on M. We choose the covering $\overline{M}\times T^q\to M$ as described above for the metric $g=g_0$. The metrics $\bar{g}_1,\dots,\bar{g}_r$ and g_{fl} carry a parallel spinor and are thus stable. The factors $(\overline{M}_i,\bar{g}_i)$ are irreducible, hence defined by a torsion-free G-structure for one of the groups $G\neq\mathrm{SO}(n)$ taken from Table 1. It then follows from [18] (building on work of [11]) that infinitesimal Ricci-flat deformations of $\bar{g}_i$ and g_{fl} are integrable. Finally, the manifolds $\overline{M}_i$ admit no harmonic 1-forms as their simply-connectedness implies vanishing of their first Betti number b_1. Therefore Theorem B in Section 3 (Ricci-flat deformations of products) implies that any Ricci-flat deformation of the pull-back of g_0 to $\overline{M}\times T^q$ is — up to pull-back by diffeomorphisms in $\mathit{Diff}_0(\overline{M}\times T^q)$ — of the form $\bar{g}_{t,1}\times\ldots\times\bar{g}_{t,r}\times g_{\mathrm{fl},t}$, where $\bar{g}_{t,i}$ is a smooth Ricci-flat deformation of $\bar{g}_i$ and where $g_{\mathrm{fl},t}$ is a Ricci-flat and thus flat deformation of g_{fl}. By the known rigidity results in the irreducible case [18, 20], the holonomy group of nearby Ricci-flat deformations of $\bar{g}_i$ will be conjugate to H_i, and the conjugating element can be chosen continuously in t. □

The proof of Proposition 2 also extends the validity of the following corollary to compact manifolds with reducible holonomy and arbitrary fundamental group.

Corollary 3 (Local stability, see [7] for a special case)**.** *Let (M,g) be a compact Riemannian manifold whose universal covering is spin and carries a parallel spinor. Then there is a neighborhood $\mathcal{U}$ of g in the space of all metrics on M with respect to the C^2-topology which contains no metric of positive scalar curvature. Furthermore any metric of non-negative scalar curvature in $\mathcal{U}$ must admit a parallel spinor.*

The corollary is proven in [7, Theorem 3.1] for simply connected manifolds with irreducible holonomy. Note that in the general case a slightly weaker version with a more involved proof is provided by [7, Theorem 4.2].
We now explain how to extend the proof of [7, Theorem 3.1] to a proof of the above Corollary. A key argument in the proof of [7, Theorem 3.1] is to show that any infinitesimal Ricci-flat deformation is integrable. This is done by a case-by-case argument, see [7, Section 3]. However the arguments in [7, Section 3] do not provide a proof in the case of reducible holonomy as it is not discussed whether an infinitesimal Ricci-flat deformation infinitesimally preserves the product structure.

This missing piece of the proof for manifolds of reducible holonomy is provided by the following argument:
In the proof of Proposition 2 we have applied Theorem B to the decompostion $\overline{M} \times T^q \to M$ which finitely covers M. This proves that any infinitesimal Ricci-flat deformation of (M, g) is integrable. The arguments in [7] can then be applied to this situation, and one obtains Corollary 3 for manifolds with reducible holonomy and arbitrary fundamental group.

5. Applications

Corollary 4. *Let M be a compact spin manifold. If g_t, $t \in I := [0, T]$ is a smooth family of Ricci-flat metrics on M such that $g_0 \in \mathcal{M}_\parallel(M)$, then $g_t \in \mathcal{M}_\parallel(M)$ for all t and $\dim \mathcal{P}(M, g_t)$ is constant in t.*

Remark. A parallel spinor is a special case of a real Killing spinors. If (M, g) carries a (non-zero) real Killing spinor, then g is Einstein. However, it was shown in [4], that our results no longer hold if we replace parallel spinors by real Killing spinors and Ricci-flat metrics by Einstein metrics.

Proof. In the proof of Proposition 2 we have chosen a spin structure on $\overline{M} \times T^q$. The pullback of the spin structure on M also yields a spin structure on $\overline{M} \times T^q$, which a priori might be different from the one in the proof of Proposition 2. As $\overline{M}$ is simply connected (and spin), it carries a unique spin structure. If the two spin structures on $\overline{M} \times T^q$ differ, then they come from different spin structures on T^q. However the existence of a parallel spinor on $\overline{M} \times T^q$ implies the existence of a parallel spinor both on $\overline{M}$ and T^q. The only spin structure on T^q which admits parallel spinors is the one taken in Proposition 2. Thus the two spin structures may only be different in the case $\dim \mathcal{P}(M, g_{t_0}) = 0$, for which the corollary is trivially satisfied for t close to t_0 due to continuity of the eigenvalues of the Dirac operator. Hence we assume from now on, that the two spin structure coincide.
Let $\overline{\Gamma}$ be the (finite) group of deck transformations of the covering space $\overline{M} \times T^q \to M$. By normality of this covering space, $\overline{\Gamma}$ acts transitively on each fiber, and this action lifts to the spin structure, and thus to the spinor bundle. The dimension of $\mathcal{P}(\overline{M} \times T^q, \bar{g}_t \times g_{\mathrm{fl},t})$ is determined by the holonomy group $\mathrm{Hol}(\overline{M}, \bar{g}_t) = \mathrm{Hol}(\overline{M} \times T^q, \bar{g}_t \times g_{\mathrm{fl}}) = \mathrm{Hol}_0(M, g_t)$ which therefore does not depend on t. Further, $\mathcal{P}(M, g_t)$ can be identified with the subspace of $\mathcal{P}(\overline{M} \times T^q, \bar{g}_t \times g_{\mathrm{fl},t})$ which is invariant for the representation $\rho_t : \overline{\Gamma} \to \mathrm{GL}(\mathcal{P}(\overline{M} \times T^q, \bar{g}_t \times g_{\mathrm{fl},t}))$. Representation theory of finite groups implies

$$\dim \mathcal{P}(M, g_t) = \frac{1}{|\overline{\Gamma}|} \sum_{\gamma \in \overline{\Gamma}} \mathrm{Tr}\, \rho_t(\gamma)$$

which is continuous in t, hence locally constant. □

This corollary extends Wang's result from [20] which states that the dimension is constant if M is simply-connected and g_0 irreducible. An immediate consequence is that the premoduli space $\mathcal{M}_\parallel(M)/\mathit{Diff}_0(M)$ is an open subset of the premoduli space of Ricci-flat metrics. In general, the premoduli space of Ricci-flat metrics is merely a real analytic subset of a finite-dimensional real analytic manifold, and the tangent space of this manifold can be identified with the space of infinitesimal Ricci-flat deformations and rescalings of the metric (cf. for instance the discussion in [3, Chapter 12.F] building on Koiso's work [12]).

Corollary 5. *Assume that M is a compact spin manifold. Then $\mathcal{M}_\parallel(M)/\mathit{Diff}_0(M)$ is a smooth manifold. Furthermore, $\mathcal{M}_\parallel(M)$ is a smooth submanifold in the space of all metrics on M.*

Proof. We have to check that every infinitesimal Ricci-flat deformation of $g_0 \in \mathcal{M}_\parallel(M)$ is in fact integrable. Consider again the finite cover $\overline{M} \times T^q \to M$. The space of Ricci-flat metrics on $\overline{M} \times T^q$ is smooth by the proof of Proposition 2, and so is the space of Ricci-flat metrics on M as the fixed point set of the finite group $\overline{\Gamma}$ acting on the former space by pull-back. This implies the result. □

This generalises Nordström's result [18] for metrics defined by a torsion-free G-structure for one of the groups $G \neq \mathrm{SO}(n)$ taken from Table 1 to general metrics in $\mathcal{M}_\parallel(M)$. Nordström makes strong use of Goto's earlier result [11] about the unobstructedness of the deformation theory of such G-structures.

A further application involves the spinorial energy functional introduced in [2], to which we refer for details. To define it, consider the space of sections $\mathcal{N}$ of the universal bundle of unit spinors over M. A section $\Phi \in \mathcal{N}$ can be thought of as a pair (g, ϕ) where g is a Riemannian metric and $\phi \in \Gamma(\Sigma_g M)$ is a g-spinor of constant length one. The relevant functional is defined by

$$\mathcal{E} : \mathcal{N} \to \mathbb{R}_{\geq 0}, \quad \Phi \mapsto \frac{1}{2} \int_M |\nabla^g \phi|_g^2 \, dv^g,$$

where ∇^g denotes the Levi-Civita connection on the g-spinor bundle $\Sigma_g M$, $|\cdot|_g$ the pointwise norm on $T^*M \otimes \Sigma_g M$ and dv^g the Riemann-Lebesgue measure given by the volume form of g. By [2, Corollary 4.10] the set of critical points $\mathrm{Crit}(\mathcal{E})$ consists precisely of g-parallel unit spinors (g, ϕ), provided that $\dim M > 2$. Since by Corollary 4 the dimension of the space of parallel spinors is constant under Ricci-flat deformations, the proof of [2, Theorem 4.17] immediately implies the subsequent

Corollary 6. *The functional $\mathcal{E}$ is Morse-Bott, i.e. the critical set $\mathrm{Crit}(\mathcal{E})$ is smooth and $\mathcal{E}$ is non-degenerate transverse to $\mathrm{Crit}(\mathcal{E})$.*

Now the universal covering group of $\mathit{Diff}_0(M)$, the diffeomorphisms homotopic to the identity, naturally acts on $\mathrm{Crit}(\mathcal{E})$. It also follows with the arguments above that the premoduli space of critical points, that is, $\mathrm{Crit}(\mathcal{E})$ divided by this action, is smooth as well.

6. Rigidity of the full holonomy group

We now prove Theorem 1, using the following fact from Lie group theory.

Theorem (Montgomery-Zippin [16]). *Let H_0 be a compact subgroup of a Lie group G. Then there exists an open neighbourhood U of H_0 such that if H is a compact subgroup of G contained in U, then there exists $g \in G$ with $g^{-1}Hg \subset H_0$. Moreover, upon sufficiently shrinking U, g can be chosen in any neighbourhood of the identity of G.*

Proof of Theorem 1. The group epimorphisms

$$\alpha_t : \Gamma = \pi_1(M) \to \mathrm{Hol}(M, g_t)/\mathrm{Hol}_0(M, g_t)$$

factor through $\Gamma \to \overline{\Gamma} = \Gamma/\mathbb{Z}^q$ to epimorphisms $\bar{\alpha}_t : \overline{\Gamma} \to \mathrm{Hol}(M, g_t)/\mathrm{Hol}_0(M, g_t)$. Choose loops $\gamma_1, \dots, \gamma_\ell : [0,1] \to M$ where $\gamma_i(0) = \gamma_i(1)$ is the base point of M and such that $\overline{\Gamma} = \{[\gamma_1], \dots, [\gamma_\ell]\}$, $\ell = \#\overline{\Gamma}$. Let $A_{t,i}$ be the parallel transport along γ_i for the metric g_t.
Thus

$$\mathrm{Hol}(M, g_t) = A_{t,1} \cdot \mathrm{Hol}_0(M, g_t) \cup \ldots \cup A_{t,\ell} \cdot \mathrm{Hol}_0(M, g_t).$$

Let $k(t) := \#\{i \in \{1, 2, \dots, \ell\} \mid A_{t,i} \in \mathrm{Hol}_0(M, g_t)\} = \#\ker \overline{\alpha}_t$. Then the index of $\mathrm{Hol}_0(M, g_t)$ in $\mathrm{Hol}(M, g_t)$ is $\ell/k(t)$ and this index is the number of connected components of $\mathrm{Hol}(M, g_t)$. We fix $t_0 \in [0, T]$. Since the $A_{t,i}$ depend continuously on t, the rigidity of the restricted holonomy group as proven in Proposition 2 yields that $k(t)$ is upper semi-continuous. Therefore the index of $\mathrm{Hol}_0(M, g_t)$ in $\mathrm{Hol}(M, g_t)$ is lower semi-continuous in t. Furthermore, the theorem of Montgomery-Zippin implies that $\mathrm{Hol}(M, g_t)$ is conjugate to a subgroup H_t of $\mathrm{Hol}(M, g_{t_0})$ if t is sufficiently close to t_0. Under this conjugation the identity component of $\mathrm{Hol}(M, g_t)$ is mapped to the identity component of H_t, thus $\mathrm{Hol}_0(M, g_{t_0}) \subset H_t \subset \mathrm{Hol}(M, g_{t_0})$ with $(H_t)_0 = \mathrm{Hol}_0(M, g_{t_0})$. As we have shown above that the number of connected components of H_t is lower semi-continuous, we obtain $H_t = \mathrm{Hol}(M, g_{t_0})$ for t close to t_0. This implies the result. □

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