The Einstein-Hilbert Action as a Spectral Action

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1 Generalized Laplacians and the heat equation

We start by examining the analysis of so-called generalized Laplacians. A detailed exposition can be found in [2]. Throughout this section let M be a compact Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle over M. Let ∇ be a metric connection on E, i.e. for smooth sections φ and ψ in E and $X \in TM$ we have

$$\partial_X \langle \varphi, \psi \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Riemannian resp. Hermitian metric on E. If φ is a smooth section in E, then $\nabla \varphi$ is a smooth section in $T^*M \otimes E$. Note that the Riemannian metric and the Levi-Civita connection on M together with the metric and ∇ on E induce a metric and a compatible connection on $T^*M \otimes E$, again denoted $\langle \cdot, \cdot \rangle$ and ∇. Similarly, the k^{th} covariant derivative, $\nabla^k \varphi$, is a section in $\underbrace{T^*M \otimes \ldots \otimes T^*M}_{k \text{ times}} \otimes E$ and this bundle carries a natural metric and connection. For φ a smooth section in E, $\varphi \in C^\infty(E)$, we define the *L^2-scalar product*

$$(\varphi, \psi)_{L^2} := \int_M \langle \varphi, \psi \rangle \, dV$$

and the associated *L^2-norm*

$$\|\varphi\|^2_{L^2} := \int_M \langle \varphi, \varphi \rangle \, dV.$$

More generally, for any k we have the *Sobolev-Norms*

$$\|\varphi\|^2_{H^k} := \|\varphi\|^2_{L^2} + \|\nabla\varphi\|^2_{L^2} + \ldots + \|\nabla^k\varphi\|^2_{L^2}.$$

The completions of $C^\infty(E)$ with respect to these norms are denoted $L^2(E)$ and $H^k(E)$, the spaces of square-integrable sections and Sobolev-sections in E.

The *C^k-norm* is defined in a similar manner,

$$\begin{aligned} \|\varphi\|_{C^0} &:= \sup_M |\varphi|, \\ \|\varphi\|_{C^k} &:= \max\{\|\varphi\|_{C^0}, \|\nabla\varphi\|_{C^0}, \ldots, \|\nabla^k\varphi\|_{C^0}\}. \end{aligned}$$

The two families of norms, $\|\cdot\|_{H^k}$ and $\|\cdot\|_{C^k}$, are equivalent in the following sense: It is trivial to see that $\|\cdot\|_{H^k}$ can be estimated against $\|\cdot\|_{C^k}$,

$$\|\varphi\|_{H^k} \leq \mathrm{vol}(M)^{\frac{1}{2}} \cdot (k+1)^{\frac{1}{2}} \cdot \|\varphi\|_{C^k}.$$

Conversely, we have [8, Thm. III.2.5]

Proposition 1.1 (Sobolev Embedding Theorem) *For each k there exists a constant $c = c(k, M, \Delta)$ such that*

$$\|\varphi\|_{C^k} \;\leq\; c \cdot \|\varphi\|_{H^\ell}$$

whenever $\ell > k + \frac{n}{2}$, $n = \dim(M)$.

Now let ∇^* be the L^2-adjoint of ∇, i.e. $(\nabla\varphi, \psi)_{L^2} = (\varphi, \nabla^*\psi)$ for all $\varphi \in C^\infty(E)$, $\psi \in C^\infty(T^*M \otimes E)$, and let $\mathcal{K} \in C^\infty(\mathrm{End}(E))$ be a symmetric endomorphism field. Then the operator

$$\Delta := \nabla^*\nabla + \mathcal{K} : C^\infty(E) \to C^\infty(E)$$

is called a *generalized Laplacian.*

Since Δ^k is a differential operator of order $2k$ we have

$$\|\Delta^k\varphi\|_{L^2} \leq C \cdot \|\varphi\|_{H^{2k}}.$$

But Δ is *elliptic* and this implies the following converse [8, Thm. III.5.2]

Proposition 1.2 (Elliptic Estimates) *For each $k \in \mathbb{N}$ there is a constant $C = C(k, M, \Delta)$ such that*

$$\|\varphi\|_{H^{2k}} \leq C \cdot \left(\|\varphi\|_{L^2} + \|\Delta^k\varphi\|_{L^2}\right).$$

Finally, we need the following fundamental result [8, Thm. III.5.8]

Theorem 1.3 *There exists a Hilbert space orthonormal basis $\varphi_1, \varphi_2, \ldots$ of $L^2(E)$ and real numbers $\lambda_1, \lambda_2, \ldots$ such that*

$$\Delta\varphi_k = \lambda_k \cdot \varphi_k,$$

$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \nearrow +\infty$, and each λ_k is repeated only finitely many times. All φ_k are smooth, $\varphi_k \in C^\infty(E)$.

The theorem says in particular that the eigenvalues tend to $+\infty$. To get started we need some control on how fast they grow. The following proposition will later be improved considerably, c.f. Theorem 2.6.

Proposition 1.4 *There exists a positive constant $c = c(M, \Delta)$ such that*

$$\lambda_k \geq c \cdot k^{\frac{4}{n(n+6)}} + \lambda_1 - 1$$

for all k.

Proof. Replacing $\mathcal{K}$ by $\mathcal{K} - \lambda_1 \cdot \mathrm{id}$ will shift the spectrum of Δ by λ_1. Hence we can assume w.l.o.g. that $\lambda_1 = 0$. Now let $\epsilon > 0$ and let $\{p_1, \ldots, p_N\}$ be a minimal ϵ-dense subset of M, i.e. $M = \bigcup_{i=1}^{N} B(p_i, \epsilon)$ with N minimal. Here $B(p, \epsilon)$ denotes the ball of radius ε about p. It is not hard to see that there is a constant $c_1 = c_1(M)$ such that for all $\epsilon > 0$

$$N = N(\epsilon) \leq c_1 \cdot \epsilon^{-n}.$$

Let $V \subset L^2(E)$ be the subspace spanned by $\varphi_1, \ldots, \varphi_k$. Consider $\varphi = \sum_{i=1}^{k} \alpha_i \varphi_i \in V$ and assume $\varphi(p_i) = 0$, $i = 1, \ldots, N$. Given $x \in M$ choose p_i such that $\mathrm{dist}(x, p_i) < \epsilon$. Differentiation along a shortest geodesic from p_i to x yields

$$|\varphi(x)| \;=\; |\varphi(x)| - |\varphi(p_i)| \;\leq\; \epsilon \cdot \|\nabla\varphi\|_{C^0} \;\leq\; \epsilon \cdot \|\varphi\|_{C^1}.$$

Integration over M gives

$$\|\varphi\|_{L^2} \;\leq\; \epsilon \cdot \|\varphi\|_{C^1} \cdot \;\mathrm{vol}(M)^{\frac{1}{2}}.$$

Let $\ell := \left[\frac{n}{2}\right] + 2$. By the Sobolev embedding theorem we have

$$\|\varphi\|_{C^1} \;\leq\; c_2 \cdot \|\varphi\|_{H^\ell}.$$

By the elliptic estimates

$$\begin{aligned}
\|\varphi\|_{H^\ell} &\leq c_3 \cdot \left(\|\varphi\|_{L^2} + \|\Delta^{\left[\frac{\ell+1}{2}\right]}\varphi\|_{L^2}\right) \\
&\leq c_3 \cdot \left(1 + \lambda_k^{\left[\frac{\ell+1}{2}\right]}\right) \cdot \|\varphi\|_{L^2} \\
&\leq c_3 \cdot (1+\lambda_k)^{\left[\frac{\ell+1}{2}\right]} \cdot \|\varphi\|_{L^2} \\
&\leq c_3 \cdot (1+\lambda_k)^{\frac{n}{4}+\frac{3}{2}} \cdot \|\varphi\|_{L^2}.
\end{aligned}$$

Combining these estimates we obtain

$$\begin{aligned}
\|\varphi\|_{L^2} &\leq \epsilon \cdot \mathrm{vol}(M)^{\frac{1}{2}} \cdot c_2 \cdot c_3 \cdot (1+\lambda_k)^{\frac{n+6}{4}} \cdot \|\varphi\|_{L^2} \\
&= \epsilon \cdot c_4 \cdot (1+\lambda_k)^{\frac{n+6}{4}} \cdot \|\varphi\|_{L^2}.
\end{aligned}$$

For $\epsilon = \frac{1}{2c_4} \cdot (1+\lambda_k)^{-\frac{n+6}{4}}$ we conclude $\|\varphi\|_{L^2} \;\leq\; \frac{1}{2}\|\varphi\|_{L^2}$, hence $\varphi = 0$. Thus for this ε the linear mapping

$$\begin{aligned}
V &\longrightarrow E_{p_1} \oplus \cdots \oplus E_{p_N} \\
\varphi &\longmapsto \big(\varphi(p_1), \ldots, \varphi(p_N)\big)
\end{aligned}$$

is injective. Therefore

$$\begin{aligned}
k &= \dim V \leq \dim(E_{p_1} \oplus \ldots \oplus E_{p_N}) = N \cdot \mathrm{rk}(E) \\
&\leq c_1 \cdot \epsilon^{-n} \cdot \mathrm{rk}(E) = c_5 \cdot (1+\lambda_k)^{\frac{n(n+6)}{4}}.
\end{aligned}$$

Hence $$1 + \lambda_k \geq \left(\frac{k}{c_5}\right)^{\frac{4}{n(n+6)}} = c_6 \cdot k^{\frac{4}{n(n+6)}}.$$ □

The main purpose of this section is to study the *heat equation*

$$\frac{\partial \varphi_t}{\partial t} + \Delta\varphi_t = 0$$

where φ_t is a smooth section in E for each $t \geq 0$ and φ_t depends smoothly on t.

The connection ∇ on E induces a connection, again denoted ∇, on the dual bundle E^*. The endomorphism field $\mathcal{K}$ of E gives the endomorphism field $\mathcal{K}^*$ on E^*. Hence we obtain a generalized Laplacion

$$\Delta = \nabla^*\nabla + \mathcal{K}^*$$

on E^*.

For a section φ in E we define the section φ^* in E^* by

$$\varphi^*(\psi) := \langle \varphi, \psi \rangle \qquad \forall \psi \in E.$$

One easily checks $\nabla_X(\varphi^*) = (\nabla_X \varphi)^*$ for $X \in TM$, $(\mathcal{K}\varphi)^* = \mathcal{K}^* \varphi^*$, and $(\Delta\varphi)^* = \Delta(\varphi^*)$. Hence if $\varphi_1, \varphi_2, \ldots$ is an orthonormal basis of $L^2(E)$ consisting of eigenvectors of Δ, then we get an orthonormal eigenbasis of $L^2(E^*)$ by $\varphi_1^*, \varphi_2^*, \ldots$ for the same eigenvalues.

Now we form the bundle $E \boxtimes E^*$ over $M \times M$ whose fiber over $(x, y) \in M \times M$ is given by

$$(E \boxtimes E^*)_{(x,y)} = E_x \otimes E_y^* = \mathrm{Hom}(E_y, E_x).$$

Again, we get an induced connection $\tilde{\nabla}$ on $E \boxtimes E^*$. We put $\tilde{\mathcal{K}} = \mathcal{K} \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{K}^*$ and obtain the corresponding generalized Laplacian

$$\tilde{\Delta} = \tilde{\nabla}^* \tilde{\nabla} + \tilde{\mathcal{K}}.$$

If φ and ψ are sections in E we get a section $\varphi \boxtimes \psi^*$ in $E \boxtimes E^*$ by

$$(\varphi \boxtimes \psi^*)(x, y) = \varphi(x) \otimes \psi^*(y).$$

One sees that $\tilde{\Delta}(\varphi_j \boxtimes \varphi_k^*) = (\Delta\varphi_j) \boxtimes \varphi_k^* + \varphi_j \boxtimes (\Delta\varphi_k^*) = (\lambda_j + \lambda_k)(\varphi_j \boxtimes \varphi_k^*)$. Hence $\varphi_j \boxtimes \varphi_k^*$, $j, k \geq 1$, form an orthonormal basis of $L^2(E \boxtimes E^*)$ consisting of eigensections for $\tilde{\Delta}$.

Definition. The infinite sum

$$k_t(x, y) := \sum_{j=1}^{\infty} e^{-t\lambda_j} \varphi_j(x) \otimes \varphi_j^*(y),$$

$x, y \in M$, $t > 0$, is called the *heat kernel* of Δ on M.

Proposition 1.5 *Let $t_0 > 0$. Then the heat kernel and all its t-derivatives converge uniformly in $t \geq t_0$ in all H^k-norms and all C^k-norms. In particular, $k_t(x, y)$ is smooth in t, x, and y, and we can differentiate term by term.*

Proof. In view of the Sobolev embedding theorem it is sufficient to prove the proposition for the H^k-norms. All but finitely many λ_j fulfill $\lambda_j \geq 1$. By the elliptic estimates we then have

$$\begin{aligned}
\|e^{-t\lambda_j} \varphi_j \boxtimes \varphi_j^*\|_{H^{2k}} &\leq c_1 \cdot e^{-t\lambda_j} \cdot \left(\|\varphi_j \boxtimes \varphi_j^*\|_{L^2} + \|\tilde{\Delta}^k(\varphi_j \boxtimes \varphi_j^*)\|_{L^2} \right) \\
&= c_1 \cdot e^{-t\lambda_j} \cdot \left(1 + (2\lambda_j)^k\right) \\
&\leq c_2 \cdot \lambda_j^k \cdot e^{-t\lambda_j} \\
&\leq c_2 \cdot \lambda_j^k \cdot e^{-t_0\lambda_j}.
\end{aligned}$$

Since for large enough x we have $x^k e^{-t_0 x/2} \leq 1$ we have for almost all j:

$$\|e^{-t\lambda_j} \varphi_j \boxtimes \varphi_j^*\|_{H^{2k}} \leq c_2 \cdot e^{-t_0\lambda_j/2}.$$

By Proposition 1.4 we have

$$\lambda_j \geq c_3 \cdot j^\alpha + c_4 \qquad , \alpha = \frac{4}{n(n+6)},$$

and therefore

$$\|e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^*\|_{H^{2k}} \;\leq\; c_5 \cdot e^{-c_6 \cdot j^\alpha}.$$

Convergence of the series $\sum\limits_j e^{-c_6 \cdot j^\alpha}$ follows from finiteness of the integral

$$\int\limits_0^\infty e^{-c_6 \cdot t^\alpha}\, dt = c_7 \cdot \int\limits_0^\infty e^{-s} \cdot s^{\frac{1-\alpha}{\alpha}}\, ds = c_7 \cdot \Gamma\left(\frac{1}{\alpha}\right).$$

We have shown that

$$\sum_{j=1}^\infty e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^*$$

converges in each H^k-norm, uniformly in $t \geq t_0$. The same argument applies to the t-derivatives

$$\sum_{j=1}^\infty \left(\frac{d}{dt}\right)^m \left(e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^*\right) = \sum_{j=1}^\infty (-\lambda_j)^m e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^*.$$

□

Since we are allowed to differentiate term by term we compute for y fixed

$$\begin{aligned}
\frac{\partial}{\partial t} k_t(x,y) &= \frac{\partial}{\partial t}\sum_j e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^* \\
&= \sum_j \frac{\partial}{\partial t} e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^* \\
&= \sum_j (-\lambda_j)\, e^{-t\lambda_j}\varphi_j \boxtimes \varphi_j^* \\
&= -\sum_j e^{-t\lambda_j}\,(\Delta\varphi_j) \boxtimes \varphi_j^* \\
&= -\Delta_x k_t(x,y).
\end{aligned}$$

For $u_0 \in L^2(E)$ we put $u_t(x) := \int\limits_M k_t(x,y) u_0(y)\, dV(y)$ and we see

$$\frac{\partial u_t}{\partial t} + \Delta u_t = 0.$$

Hence u_t solves the heat equation. Moreover,

$$\begin{aligned}
\int\limits_M k_t(x,y)\varphi_k(y)\, dV(y) &= \sum_j e^{-t\lambda_j}\varphi_j \cdot (\varphi_j, \varphi_k)_{L^2} \\
&= e^{-t\lambda_k}\varphi_k.
\end{aligned}$$

Thus $k_t(x,y)$ is the integral kernel of the operator $e^{-t\Delta}$. As $t \searrow 0$ the heat kernel becomes singular. Indeed, since $e^{-0\cdot\Delta} = \mathrm{id}$ we expect the heat kernel to concentrate along the diagonal $\{(y,y) \in M \times M \mid y \in M\}$. We next want to examine the asymptotic behavior of $k_t(x,y)$ for $t \searrow 0$.

2 The formal heat kernel

We start with the *Euclidean heat kernel*

$$q_t : M \times M \to \mathbb{R}, \;\; q_t(x,y) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{\mathrm{dist}(x,y)^2}{4t}\right).$$

A formal series

$$\tilde{k}_t(x,y) = q_t(x,y) \cdot \sum_{j=0}^{\infty} t^j \cdot \Phi_j(x,y),$$

$\Phi_j \in C^\infty(E \boxtimes E^*)$, is called a *formal heat kernel* if for each $N \in \mathbb{N}$ there exists m_0 such that for all $m \geq m_0$

$$\left(\frac{\partial}{\partial t} + \Delta_x\right)\left\{q_t \cdot \sum_{j=0}^{m} t^j \cdot \Phi_j\right\} = q_t \cdot O(t^N).$$

Proposition 2.1 *Let ϵ_0 be the injectivity radius of M. Then there exists a unique formal heat kernel with Φ_j defined and smooth on $(M \times M)_{\epsilon_0} := \{(x,y) \in M \times M \mid \mathrm{dist}(x,y) < \epsilon_0\}$ such that*

$$\Phi_0(x,x) = \mathrm{id}_{E_x} \in \mathrm{Hom}(E_x, E_x) = E_x \otimes E_x^*.$$

Lemma 2.2 *Let Δ_0 denote the standard Laplace-Beltrami operator acting on functions. Then*

$$\left(\frac{\partial}{\partial t} + \Delta_{0,x}\right) q_t(x,y) = \frac{a(x,y)}{t} \cdot q_t(x,y)$$

where a is smooth on $(M \times M)_{\epsilon_0}$ and a vanishes along the diagonal, $a(x,x) = 0$. In geodesic polar coordinates about y we have

$$a(x,y) = \frac{r}{2}\frac{d}{dr}\left(\ln \det(d\exp_y(rX))\right),$$

$x = \exp_y(rX)$, $X \in T_yM$, $\|X\| = 1$. Hence a is essentially given by the radial logarithmic derivative of volume distortion of the exponential map.

Here $\exp_y : T_yM \to M$ denotes the Riemannian exponential map.

Proof of Lemma. Fix $y \in M$. We express Δ_0 in polar coordinates about y:

$$\Delta_0 = \Delta^{S_r} - \frac{\partial^2}{\partial r^2} + (n-1) \cdot H \cdot \frac{\partial}{\partial r}.$$

Here S_r denotes the distance sphere of radius r, $S_r = \{x \in M \mid \mathrm{dist}(x,y) = r\}$, and H is its mean curvature. A direct calculation yields

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \Delta_{0,x}\right) q_t \\ = \;& \left(\frac{\partial}{\partial t} + \Delta^{S_r} - \frac{\partial^2}{\partial r^2} + (n-1) \cdot H \cdot \frac{\partial}{\partial r}\right)\left((4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{4t}\right)\right) \\ = \;& -(n-1)\frac{1+Hr}{2t} \cdot q_t. \end{aligned}$$

Hence $a(x,y) = -\frac{n-1}{2}(1+Hr)$.

In order to identify this term we fix $X \in T_yM$, $\|X\| = 1$, and let $c(r) = \exp_y(rX)$ be the unit speed geodesic emanating from y in direction X. Let $e_1 = X, e_2, \ldots, e_n$ be an orthonormal basis of T_yM. Let V_i be the Jacobi field along c determined by the initial condition $V_i(0) = 0$ and $\frac{\nabla}{dr}V_i(0) = e_i$, $i = 1, \ldots, n$. It is well-known that [6, 1.2.2] the differential of the exponential map at the point rX is given by

$$d\exp_y(rX)(e_i) = \frac{1}{r}V_i(r).$$

Thus $\left(\frac{\nabla}{dr}d\exp_y(rX)\right)(e_i) = -\frac{1}{r^2}V_i(r) + \frac{1}{r}\frac{\nabla}{dr}V_i(r)$. In particular, $V_1(r) = rc'(r)$ and hence $\left(\frac{\nabla}{dr}d\exp_y(rX)\right)(e_1) = 0$. For $i = 2, \ldots, n$ we have $\frac{\nabla}{dr}V_i(r) = -B(V_i(r))$ where B is the Weingarten map (second fundamental form) of S_r [6, 1.2.6]. It follows

$$\begin{aligned} \left(\frac{\nabla}{dr}d\exp_y(rX)\right)(e_i) &= \left(-\frac{1}{r^2}\mathrm{id} - \frac{1}{r}B\right)V_i(r) \\ &= \left(-\frac{1}{r}\mathrm{id} - B\right)d\exp_y(rX)(e_i) \end{aligned}$$

and thus

$$\begin{aligned} \frac{d}{dr}\det(d\exp_y(rX)) &= \det(d\exp_y(rX))\,\mathrm{tr}\left(\left(\frac{\nabla}{dr}d\exp_y(rX)\right)\cdot(d\exp_y(rX))^{-1}\right) \\ &= \det(d\exp_y(rX))\,\mathrm{tr}\left(-\frac{1}{r}\mathrm{id}_{X^\perp} - B\right) \\ &= \det(d\exp_y(rX))\left(-\frac{n-1}{r} - (n-1)H\right) \\ &= \frac{2}{r}\cdot\det(d\exp_y(rX))\cdot a. \end{aligned}$$

Hence

$$\begin{aligned} a &= \frac{r}{2}\det(d\exp_y(rX))^{-1}\cdot\frac{d}{dr}\det(d\exp_y(rX)) \\ &= \frac{r}{2}\frac{d}{dr}\ln\det(d\exp_y(rX)). \end{aligned}$$

□

Proof of Proposition. We first show uniqueness of the Φ_j. To do this we differentiate the formal series $\tilde{k}_t(x,y)$ term by term, order the result by powers of t and equate the resulting coefficients to zero. We use the formula

$$\Delta(f\cdot\varphi) = (\Delta_0 f)\cdot\varphi - 2\nabla_{\mathrm{grad} f}\varphi + f\Delta\varphi$$

where f is a function and φ a section in E. Now

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \Delta_x\right)\tilde{k}_t \\ = &\left(\left(\frac{\partial}{\partial t} + \Delta_{0,x}\right)q_t\right)\cdot\sum_j t^j\Phi_j - 2\nabla_{\mathrm{grad}_x q_t}\sum_j t^j\Phi_j + q_t\left(\frac{\partial}{\partial t} + \Delta_x\right)\sum_j t^j\Phi_j \\ = &\frac{a}{t}q_t\cdot\sum_j t^j\Phi_j + \frac{1}{2t}\cdot q_t\cdot\nabla_{\mathrm{grad}_x(r^2)}\sum_j t^j\Phi_j + q_t\cdot\sum_j t^j\Delta_x\Phi_j + q_t\cdot\sum_j jt^{j-1}\Phi_j \\ = &q_t\cdot\sum_{j=-1}^{\infty} t^j\cdot\{a\cdot\Phi_{j+1} + r\nabla_{\mathrm{grad}_x r}\Phi_{j+1} + \Delta_x\Phi_j + (j+1)\Phi_{j+1}\} \end{aligned}$$

where again $r = \operatorname{dist}(x,y)$, y fixed, and with the convention that $\Phi_{-1} := 0$. Along any unit speed geodesic $c(r) = \exp_y(rX)$ emanating from y we obtain the following singular ordinary differential equations $\big(\Phi_j(r) := \Phi_j(\exp_y(rX), y)\big)$:

$$(j+1+a(r))\,\Phi_{j+1}(r) + r\frac{\nabla}{dr}\Phi_{j+1}(r) + (\Delta_x\Phi_j)(r) = 0. \tag{1}$$

To solve this equation we introduce the *integrating factor*

$$R_j(r) = r^{j+1}\cdot\exp\left(\int_0^r \frac{a(\rho)}{\rho}\,d\rho\right).$$

Then we have

$$\begin{aligned} & \frac{r}{R_j(r)}\cdot\frac{\nabla}{dr}\left(R_j(r)\Phi_{j+1}(r)\right) \\ = \;& \frac{r}{R_j(r)}\cdot\left\{\frac{j+1}{r}R_j(r)\Phi_{j+1}(r) + R_j(r)\cdot\frac{a(r)}{r}\cdot\Phi_{j+1}(r) + R_j(r)\frac{\nabla}{dr}\Phi_{j+1}(r)\right\} \\ = \;& -(\Delta_x\Phi_j)(r). \end{aligned}$$

We denote parallel translation along $c(r)$ from $c(r_1)$ to $c(r_2)$ by π_{r_1,r_2} and we obtain

$$R_j(r)\Phi_{j+1}(r) = -\int_0^r \frac{R_j(\rho)}{\rho}\pi_{\rho,r}\left(\Delta_x\Phi_j\right)(\rho)\,d\rho + \pi_{0,r}C_j.$$

Evaluating this equation for $j=-1$ at $r=0$ yields

$$1\cdot\mathrm{id}_{E_y} = 0 + C_{-1}.$$

Hence $C_{-1} = \mathrm{id}_{E_y}$ and

$$\begin{aligned} \Phi_0(r) &= \frac{1}{R_{-1}(r)}\cdot\pi_{0,r}\cdot C_{-1} = \exp\left(-\int_0^r \frac{a(\rho)}{\rho}\,d\rho\right)\pi_{0,r}\mathrm{id}_{E_y} \\ &= \det\left(d\exp_y(rX)\right)^{-\frac{1}{2}}\cdot\pi_{0,r}. \end{aligned}$$

We have computed Φ_0:

$$\Phi_0(x,y) = \det\left(d(\exp_y^{-1})(x)\right)^{\frac{1}{2}}\cdot\pi_{y,x}$$

where $\pi_{y,x}$ denotes parallel translation from y to x (along the unique shortest geodesic connecting y and x).

For $j \geq 0$ we get at $r=0$:

$$0\cdot\Phi_{j+1}(0) = 0 + C_j.$$

Hence $C_j = 0$ and

$$\Phi_{j+1}(r) = -\frac{1}{R_j(r)}\int_0^r \frac{R_j(\rho)}{\rho}\pi_{\rho,r}\left(\Delta_x\Phi_j\right)(\rho)\,d\rho.$$

This way we can recursively determine the Φ_j and uniqueness is proven.
For the existence part simply use the above equations to define the Φ_j recursively.

$\square$

Remark. By assumption we have

$$\Phi_0(y,y) = \mathrm{id}_{E_y}.$$

Plugging $r=0$ into (1) for $j=0$ we obtain

$$\Phi_1(0) = -\left(\Delta_x \Phi_0\right)(0).$$

Let us compute this term. We use the Taylor expansion of the metric in normal coordinates about $y(\widehat{=}0)$:

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\sum_{kl} R_{ikjl}(0)x^k x^l + O(\|x\|^3). \tag{2}$$

Hence

$$\begin{aligned}
\det\left(d\exp_y\right) &= \det\left((g_{ij})_{i,j=1,\ldots,n}\right)^{\frac{1}{2}} \\
&= \left[1 + \mathrm{tr}\left(\frac{1}{3}\sum_{kl} R_{ikjl}(0)x^k x^l + O(\|x\|^3)\right) + O(\|x\|^4)\right]^{\frac{1}{2}} \\
&= 1 - \frac{1}{6}\sum_{kl} \mathrm{ric}_{kl}(0)x^k x^l + O(\|x\|^3)
\end{aligned}$$

Here $\mathrm{ric}_{kl} = \sum_{ij} g^{ij} R_{iklj} = -\sum_{ij} g^{ij} R_{ikjl}$ denotes *Ricci curvature*. Thus $\det(d\exp_y)^{-\frac{1}{2}} = 1 + \frac{1}{12}\sum_{kl} \mathrm{ric}_{kl}(0)x^k x^l + O(\|x\|^3)$ and therefore

$$\begin{aligned}
\Delta_{0,x}\left(\det(d\exp_y)^{-\frac{1}{2}}\right) &= -\frac{1}{6}\sum_{k} \mathrm{ric}_{kk}(0) + O(\|x\|) \\
&= -\frac{1}{6}\mathrm{scal}(0) + O(\|x\|).
\end{aligned}$$

Here $\mathrm{scal} = \sum_k \mathrm{ric}_{kk}$ denotes the *scalar curvature*.

Now $\left(\Delta_x \Phi_0\right)(x,y) = \left(\Delta_{0,x}\left(\det(d\exp_y)^{-\frac{1}{2}}\right)\right)\cdot \pi_{y,x} + \det(d\exp_y)^{-\frac{1}{2}}\cdot \mathcal{K}_x \circ \pi_{y,x}$ and therefore $\Delta_x \Phi_0(y,y) = -\frac{1}{6}\mathrm{scal}(y) + \mathcal{K}_y$.

We have shown

$$\Phi_1(y,y) = \frac{1}{6}\mathrm{scal}(y)\cdot \mathrm{id}_{E_y} - \mathcal{K}_y.$$

This is of greatest importance to us because this function will give us the Einstein-Hilbert action.

It remains to see what the formal heat kernel and the true heat kernel have to do with each other. Pick a smooth cut-off function $\chi : \mathbb{R} \to \mathbb{R}$, such that $\chi(r) = 1$ for $r \leq \frac{\epsilon_0}{3}$, $\chi(r) = 0$ for $r \geq \frac{2\epsilon_0}{3}$, and $0 \leq \chi \leq 1$ everywhere. We define

$$\widehat{k}_t(x,y) := \widetilde{k}_t(x,y)\cdot \chi\left(\mathrm{dist}(x,y)\right).$$

Hence $\widehat{k}_t$ coincides with the formal heat kernel $\widetilde{k}_t$ on a neighborhood of the diagonal but $\widehat{k}_t$ is defined and smooth on all of $M \times M$ (or, more precisely, its finite partial sums $\widehat{k}_t^{(m)}(x,y) := \chi(\mathrm{dist}(x,y))\cdot q_t(x,y)\cdot \sum_{j=0}^{m} t^j \Phi_j \cdot (x,y))$.

Proposition 2.3 *$\widehat{k}_t$ is asymptotic to k_t, in symbols*

$$k_t \overset{t\searrow 0}{\sim} \widehat{k}_t,$$

in the following sense: For each $N \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$ and $t_0 > 0$ such that for all $m \geq m_0$ there is a constant $C_{N,m} > 0$ with

$$|k_t(x,y) - \widehat{k}_t^{(m)}(x,y)| \leq C_{N,m} \cdot t^N$$

for all $t \in (0, t_0)$, $x, y \in M$.

Proof. Let $\varphi \in C^0(E)$ such that the support of φ is contained in a ball of radius $\frac{\epsilon_0}{2}$. Recall that ε_0 is the injectivity radius of M. Since q_t is the Euclidean heat kernel we see

$$\lim_{t\searrow 0} \int_M q_t(x,y)\Phi_0(x,y)\varphi(y)\, dV(y) = \Phi_0(x,x)\varphi(x) = \varphi(x).$$

A partition of unity argument yields for arbitrary $\varphi \in C^0(E)$

$$\lim_{t\searrow 0} \int_M \widehat{k}_t^{(0)}(x,y)\varphi(y)\, dV(y) = \varphi(x).$$

Since higher powers of t do not contribute to the limit for $t \searrow 0$ we have

$$\lim_{t\searrow 0} \int_M \widehat{k}_t^{(m)}(x,y)\varphi(y)\, dV(y) = \varphi(x)$$

for all $m \in \mathbb{N}$ and $\varphi \in C^0(E)$. On the other hand, since $e^{-t\Delta}$ tends to $e^{-0\cdot\Delta} = \mathrm{id}$, we also have

$$\lim_{t\searrow 0} \int_M k_t(x,y)\varphi(y)\, dV(y) = \varphi(x).$$

Thus for $\delta_t^{(m)} := k_t - \widehat{k}_t^{(m)}$ we get

$$\lim_{t\searrow 0} \int_M \delta_t^{(m)}(x,y)\varphi(y)\, dV(y) = 0.$$

Now put $\left(\frac{\partial}{\partial t} + \Delta_x\right)\delta_t^{(m)} =: \eta_t^{(m)}$ and $\widetilde{\delta}_t^{(m)} := \int_0^t e^{-(t-\tau)\Delta_x}\eta_\tau^{(m)} d\tau$. We know that

$$\eta_t^{(m)} = -\left(\frac{\partial}{\partial t} + \Delta_x\right)\widehat{k}_t^{(m)} = -\left(\frac{\partial}{\partial t} + \Delta_x\right)\left(\chi \cdot \widetilde{k}_t^{(m)}\right)$$

where $\chi(x,y) = \chi(\mathrm{dist}(x,y))$. Hence

$$\begin{aligned} \eta_t^{(m)} &= -\chi \cdot \left(\frac{\partial}{\partial t} + \Delta_x\right)\widetilde{k}_t^{(m)} + \underbrace{(\Delta_{0,x}\chi)\cdot \widetilde{k}_t^{(m)} - 2\nabla_{\mathrm{grad}_x \chi}\widetilde{k}_t^{(m)}}_{=: R_t^{(m)}} \\ &= q_t \cdot O(t^N) + R_t^{(m)}. \end{aligned}$$

Now $R_t^{(m)}$ is of the form $q_t \times$ smooth section vanishing for $\mathrm{dist}(x,y) < \frac{\epsilon_0}{3}$. For $\mathrm{dist}(x,y) \geq \frac{\epsilon_0}{3}$ we have

$$q_t(x,y) \leq c_1 \cdot \exp\left(-\frac{c_2}{t}\right) \cdot q_{2t}(x,y)$$

for suitable constants $c_1, c_2 > 0$. Therefore

$$\eta_t^{(m)} = q_{2t} \cdot O(t^N).$$

From the definition of $\widetilde{\delta}_t^{(m)}$ we have

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{\delta}_t^{(m)} &= e^{-(t-t)\Delta_x}\eta_t^{(m)} + \int_0^t -\Delta_x e^{-(t-\tau)\Delta_x}\eta_\tau^{(m)} d\tau \\ &= \eta_t^{(m)} - \Delta_x \widetilde{\delta}_t^{(m)}. \end{aligned}$$

Therefore $\left(\frac{\partial}{\partial t} + \Delta_x\right)\widetilde{\delta}_t^{(m)} = \eta_t^{(m)}$ and $\left(\frac{\partial}{\partial t} + \Delta_x\right)\left(\widetilde{\delta}_t^{(m)} - \delta_t^{(m)}\right) = 0$. Since $\widetilde{\delta}_t^{(m)} - \delta_t^{(m)} \xrightarrow{t\searrow 0} 0$ it follows $\widetilde{\delta}_t^{(m)} - \delta_t^{(m)} = e^{-t\Delta}0 = 0$, thus

$$\delta_t^{(m)} = \widetilde{\delta}_t^{(m)} = \int_0^t e^{-(t-\tau)\Delta_x}\eta_\tau^{(m)} d\tau$$

and hence

$$\|\delta_t^{(m)}\|_{H^k} \leq t \cdot \sup_{\tau\in[0,t]} \|e^{-(t-\tau)\Delta_x}\|_{H^k,H^k} \cdot \sup_{\tau\in[0,t]} \|\eta_\tau^{(m)}\|_{H^k} = O\left(t^{N+1}\right).$$

The Sobolev embedding theorem implies for $k > \frac{n}{2}$

$$\|k_t - \widehat{k}_t^{(m)}\|_{C^0} = \|\delta_t^{(m)}\|_{C^0} = O\left(t^{N+1}\right).$$

□

Corollary 2.4

$$\begin{aligned} k_t(x,x) &\overset{t\searrow 0}{\sim} \widehat{k}_t(x,x) = \widetilde{k}_t(x,x) \\ &= (4\pi t)^{-\frac{n}{2}} \cdot \left\{ \mathrm{id}_{E_x} + t \cdot \left(\frac{1}{6}\mathrm{scal}(x)\cdot \mathrm{id}_{E_x} - \mathcal{K}_x\right) + O(t^2)\right\}. \end{aligned}$$

□

Corollary 2.5

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} = \mathrm{Tr}\left(e^{-t\Delta}\right) = \int_M \mathrm{tr}\left(k_t(x,x)\right)\, dV(x) \overset{t\searrow 0}{\sim}$$

$$(4\pi t)^{-\frac{n}{2}} \cdot \left\{ \mathrm{rk}(E)\cdot \mathrm{vol}(M) + t\cdot \left(\frac{\mathrm{rk}(E)}{6}\int_M \mathrm{scal}(x)\, dV(x) - \int_M \mathrm{tr}(\mathcal{K}_x)\, dV(x)\right) + O(t^2)\right\}.$$

□

Theorem 2.6 (Weyl) *Let $\Delta : C^\infty(E) \to C^\infty(E)$ be a generalized Laplace operator over an n-dimensional compact Riemannian manifold. For each $\lambda \in \mathbb{R}$ let $N(\lambda)$ be the number of eigenvalues of Δ less than λ. Then*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\mathrm{rk}(E)\cdot \mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}}\cdot \Gamma\left(\frac{n}{2}+1\right)}.$$

For the proof we need the following tool:

Lemma 2.7 (Karamata) *Let $d\mu$ be a positive measure on $(0,\infty)$, let $\alpha > 0$ and $C > 0$. We assume*

$$\int_0^\infty e^{-t\lambda} d\mu(\lambda) < \infty$$

for all $t > 0$ and

$$\lim_{t \searrow 0} t^\alpha \int_0^\infty e^{-t\lambda} d\mu(\lambda) = C.$$

Then for all continuous functions f on $[0,1]$ the following holds:

$$\lim_{t \searrow 0} t^\alpha \int_0^\infty f\left(e^{-t\lambda}\right) e^{-t\lambda} d\mu(\lambda) = \frac{C}{\Gamma(\alpha)} \int_0^\infty f\left(e^{-t}\right) t^{\alpha-1} e^{-t} dt.$$

Proof of Theorem 2.6. Since a shift of the spectrum by a constant will not alter the limit $\lim\limits_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}}$ we may w.l.o.g. assume that all eigenvalues λ_i are positive. We apply Karamata's lemma with $\alpha = \frac{n}{2}$, $C = (4\pi)^{-\frac{n}{2}} \mathrm{rk}(E) \mathrm{vol}(M)$, and the spectral measure $d\mu = \sum\limits_{i=1}^{\infty} \delta_{\lambda_i}$. Since $\int\limits_0^\infty e^{-t\lambda} d\mu(\lambda) = \sum\limits_{i=1}^{\infty} e^{-t\lambda_i} = \mathrm{Tr}\left(e^{-t\Delta}\right) < \infty$ and $\lim\limits_{t\searrow 0} t^\alpha \cdot \int\limits_0^\infty e^{-t\lambda} d\mu(\lambda) = \lim\limits_{t\searrow 0} t^{\frac{n}{2}} \cdot \mathrm{Tr}\left(e^{-t\Delta}\right) = C$ by Corollary 2.5 the assumptions in Karamata's lemma are satisfied.

Let $\epsilon > 0$ and pick a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq e^{-(1+\epsilon)}$, $f(x) = x^{-1}$ for $x \geq e^{-1}$ and $0 \leq f(x) \leq x^{-1}$ everywhere. For the left hand side in Karamata's lemma we get

$$\begin{aligned}
\lim_{t\searrow 0} t^{\frac{n}{2}} \int_0^\infty f\left(e^{-t\lambda}\right) e^{-t\lambda} d\mu(\lambda) &= \lim_{t\searrow 0} t^{\frac{n}{2}} \int_0^{(1+\epsilon)t^{-1}} f\left(e^{-t\lambda}\right) e^{-t\lambda} d\mu(\lambda) \\
&\geq \limsup_{t\searrow 0} t^{\frac{n}{2}} \int_0^{t^{-1}} d\mu(\lambda) \\
&= \limsup_{t\searrow 0} t^{\frac{n}{2}} N(t^{-1}) \\
&= \limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}}.
\end{aligned}$$

For the right hand side we obtain

$$\begin{aligned}
\frac{C}{\Gamma(\alpha)} \int_0^\infty f\left(e^{-t}\right) t^{\alpha-1} e^{-t} dt &= \frac{C}{\Gamma(\alpha)} \int_0^{1+\epsilon} f\left(e^{-t}\right) t^{\alpha-1} e^{-t} dt \\
&\leq \frac{C}{\Gamma(\alpha)} \int_0^{1+\epsilon} t^{\alpha-1} dt \\
&= \frac{C \cdot (1+\epsilon)^\alpha}{\Gamma(\alpha) \cdot \alpha} = \frac{C \cdot (1+\epsilon)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} \leq \frac{C\cdot(1+\epsilon)^\alpha}{\Gamma(\alpha+1)}$$

and $\epsilon \searrow 0$ yields

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} \leq \frac{C}{\Gamma(\alpha+1)} = \frac{\mathrm{rk}(E)\cdot \mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}.$$

The proof of $\liminf\limits_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} \geq \frac{C}{\Gamma(\alpha+1)}$ is completely analogous. One uses continuous functions $f:[0,1]\to\mathbb{R}$ satisfying $f(x)=0$ for $x\leq e^{-1}$, $f(x)=x^{-1}$ for $x\geq e^{-1+\epsilon}$ and $0\leq f(x)\leq x^{-1}$ everywhere. $\square$

Proof of Lemma. By Weierstrass' theorem the polynomials lie dense in $C^0([0,1])$ (w.r.t. the C^0-norm). Hence it is sufficient to prove the lemma for f a polynomial. Then we can assume w.l.o.g. that $f(x)=x^k$. For the left hand side we get

$$\begin{aligned}
\lim_{t\searrow 0} t^\alpha \int_0^\infty f\left(e^{-t\lambda}\right)e^{-t\lambda}d\mu(\lambda) &= \lim_{t\searrow 0} t^\alpha \int_0^\infty e^{-(k+1)t\lambda}d\mu(\lambda)\\
&= \lim_{s\searrow 0}\left(\frac{s}{k+1}\right)^\alpha \int_0^\infty e^{-s\lambda}d\mu(\lambda)\\
&= \frac{C}{(k+1)^\alpha}.
\end{aligned}$$

The right hand side turns out to be the same

$$\begin{aligned}
\frac{C}{\Gamma(\alpha)}\int_0^\infty f\left(e^{-t}\right)t^{\alpha-1}e^{-t}dt &= \frac{C}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(k+1)t}dt\\
&= \frac{C}{\Gamma(\alpha)}\int_0^\infty \left(\frac{s}{k+1}\right)^{\alpha-1}\cdot e^{-s}\cdot\frac{ds}{k+1}\\
&= \frac{C}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)}{(k+1)^\alpha}.
\end{aligned}$$

$\square$

3 Dirac operators and Weitzenböck formulas

Again, let M be a compact Riemannian manifold. Let $\mathrm{Cl}(M)$ denote the *Clifford bundle* of M, i.e. at each point $p\in M$ the fiber $\mathrm{Cl}(M)_p$ is the Clifford algebra of T_pM. There is a canonical vector bundle isomorphism $\mathrm{Cl}(M)\xrightarrow{\cong}\bigoplus_{k=0}^{n}\Lambda^n\mathrm{TM}$ which we use to define the Levi-Civita connection ∇ on $\mathrm{Cl}(M)$. For an orthonormal basis $e_1,\ldots,e_n$ of T_pM this isomorphism is given by $e_{i_1}\cdot\ldots\cdot e_{i_k}\mapsto e_{i_1}\wedge\ldots\wedge e_{i_k}$, $i_1<i_2<\ldots<i_k$. Note that this is **not** an algebra homomorphism.

Now let $E\to M$ be a $\mathrm{Cl}(M)$-module bundle, i.e. for each $p\in M$ there is an action of $\mathrm{Cl}(M)_p$ on E_p. We will assume that this action depends smoothly on p. Suppose furthermore that E carries a Hermitian or Riemannian metric with respect to which the action of vectors $X \in T_pM \subset \mathrm{Cl}(M)_p$ is skew-adjoint,

$$\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle, \qquad \varphi, \psi \in E_p,$$

and a metric connection ∇^E which is compatible with the Levi-Civita connection in the following sense:

$$\nabla^E_X(\omega \cdot \varphi) = (\nabla_X \omega) \cdot \varphi + \omega \cdot \nabla^E_X \varphi$$

for all $X \in \mathrm{TM}$, $\omega \in C^\infty(\mathrm{Cl}(M))$, $\varphi \in C^\infty(E)$.

Now the *Dirac operator* $D : C^\infty(E) \to C^\infty(E)$ is defined by

$$D\varphi := \sum_{k=1}^{n} e_k \cdot \nabla^E_{e_k} \varphi.$$

This definition is independent of the choice of local orthonormal frame $e_1, \ldots, e_n$.

The Dirac operator is an elliptic differential operator of first order. It is self-adjoint in $L^2(E)$ with domain $H^1(E)$.

Example. If M is a Riemannian **spin** manifold, then we can take $E := \Sigma M$, the *spinor bundle*. The resulting operator D is the *classical Dirac operator*, sometimes also called *Atiyah-Singer operator*.

Example. If E is a $\mathrm{Cl}(M)$-module bundle as above and V is another Hermitian or Riemannian vector bundle over M with a metric connection, then $E \otimes V$ is again a $\mathrm{Cl}(M)$-module bundle. Here the $\mathrm{Cl}(M)$-action is on the first factor,

$$\omega \cdot (\varphi \otimes v) = (\omega \cdot \varphi) \otimes v, \qquad \omega \in \mathrm{Cl}(M)_p,\ \varphi \in E_p,\ v \in V_p,$$

and $E \otimes V$ carries the induced metric and connection, $\nabla^{E\otimes V} = \nabla^E \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^V$. The resulting Dirac operator is called a *twisted Dirac operator* with coefficients in V.

For any Dirac operator direct computation yields

$$D(f \cdot \varphi) = \mathrm{grad} f \cdot \varphi + f \cdot D\varphi \tag{3}$$

for $\varphi \in C^\infty(E)$ and a smooth function f on M. This can also be expressed by saying that the principal symbol of D is given by Clifford multiplication.

The link to the previous section is now established by

Proposition 3.1 (Bochner-Weitzenböck formula) *Let E be a $\mathrm{Cl}(M)$-module bundle over M. Then the square of its Dirac operator is a generalized Laplace operator*

$$D^2 = (\nabla^E)^* \nabla^E + \mathcal{K}$$

where $\mathcal{K} = \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot R^E(e_i, e_j)$.

Proof. Fix $p \in M$ and choose an orthonormal frame $e_1, \ldots, e_n$ near p *synchronous* at p, i.e. $(\nabla e_k)(p) = 0$ for all k. Then at p

$$D^2\varphi = \sum_{ij} e_i \nabla^E_{e_i} \left(e_j \nabla^E_{e_j} \varphi \right)$$

$$
\begin{aligned}
&= \sum_{ij} e_i e_j \nabla^E_{e_i} \nabla^E_{e_j} \varphi \\
&= \sum_i e_i^2 \nabla^E_{e_i} \nabla^{E\varphi}_{e_i} + \sum_{i<j} \left(e_i e_j \nabla^E_{e_i} \nabla^E_{e_j} + e_j e_i \nabla^E_{e_j} \nabla^E_{e_i} \right) \varphi \\
&= -\sum_i \nabla^E_{e_i} \nabla^E_{e_i} \varphi + \sum_{i<j} e_i e_j \left(\nabla^E_{e_i} \nabla^E_{e_j} - \nabla^E_{e_j} \nabla^E_{e_i} \right) \varphi \\
&= (\nabla^E)^* \nabla^E \varphi + \sum_{i<j} e_i e_j R^E(e_i, e_j) \varphi.
\end{aligned}
$$

□

Example. In the case of the classical Dirac operator acting on spinors the *curvature endomorphism* $\mathcal{K}$ takes a very simple form [9, 10]

$$\mathcal{K}_p = \frac{1}{4} \mathrm{scal}(p) \cdot \mathrm{id}_{\Sigma_p M}.$$

Example. In the case of a twisted Dirac operator we have

$$R^{E \otimes V}(X, Y) = R^E(X, Y) \otimes \mathrm{id} + \mathrm{id} \otimes R^V(X, Y)$$

and hence

$$
\begin{aligned}
\mathcal{K}^{E \otimes V}(\varphi \otimes v) &= \frac{1}{2} \sum_{ij} e_i e_j R^{E \otimes V}(e_i, e_j)(\varphi \otimes v) \\
&= \frac{1}{2} \sum_{ij} e_i e_j R^E(e_i, e_j) \varphi \otimes v + \frac{1}{2} \sum_{ij} e_i e_j \varphi \otimes R^V(e_i, e_j) v \\
&= \mathcal{K}^E \varphi \otimes v + \mathcal{F}^V(\varphi \otimes v),
\end{aligned}
$$

i.e. $\mathcal{K}^{E \otimes V} = \mathcal{K}^E \otimes \mathrm{id} + \mathcal{F}^V$.

Here $\mathcal{F}^V$ is the so-called *twisting curvature.* If E is the spinor bundle as in the previous example, then the twisted classical Dirac operator has

$$\mathcal{K} = \frac{1}{4} \mathrm{scal} \cdot \mathrm{id} + \mathcal{F}^V$$

as its curvature endomorphism. In particular, we can write down the heat asymptotics. By Corollary 2.4 we have for the heat kernel of D^2

$$
\begin{aligned}
k_t(x, x) &\overset{t \searrow 0}{\sim} (4\pi t)^{-\frac{n}{2}} \cdot \left\{ \mathrm{id}_{\Sigma_x M} + t \cdot \left(\frac{1}{6} \mathrm{scal}(x) \cdot \mathrm{id}_{\Sigma_x M} - \mathcal{K}_x \right) + O(t^2) \right\} \\
&= (4\pi t)^{-\frac{n}{2}} \cdot \left\{ \mathrm{id}_{\Sigma_x M} - t \cdot \left(\frac{1}{12} \mathrm{scal}(x) \cdot \mathrm{id}_{\Sigma_x M} + \mathcal{F}^V_x \right) + O(t^2) \right\}.
\end{aligned}
$$

Since the rank of the spinor bundle is $2^{[n/2]}$ integration yields

$$
\begin{gathered}
\mathrm{Tr}\left(e^{-tD^2} \right) \overset{t \searrow 0}{\sim} \\
2^{[n/2]} (4\pi t)^{-\frac{n}{2}} \cdot \left\{ \mathrm{vol}(M) - t \cdot \int_M \left(\frac{1}{12} \mathrm{scal}(x) + 2^{-[n/2]} \mathrm{tr}(\mathcal{F}^V_x) \right) dV(x) + O(t^2) \right\}
\end{gathered}
\tag{4}
$$

4 Integration and Dixmier trace

In noncommutative geometry one replaces a "classical" compact Riemannian spin manifold M by the tripel $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A} = C^\infty(M)$ is the pre-C^*-algebra of smooth functions on M (with respect to the C^0-norm), $\mathcal{H} = L^2(\Sigma M)$ is the Hilbert space of square-integrable spinors, and D is the classical Dirac operator. The algebra $\mathcal{A}$ acts on $\mathcal{H}$ by pointwise multiplication. For any $f \in \mathcal{A}$ the commutator of f and D is given by Clifford multiplication with the gradient of f, cf. (3). Hence the condition $\|[D, f]\|_{C^0} \leq 1$ means that the gradient of f is bounded by 1. This observation is important since it implies that we can reconstruct the distance function and hence the metric on M from the triple $(\mathcal{A}, \mathcal{H}, D)$:

$$\mathrm{dist}(x, y) = \sup \left\{ |f(x) - f(y)| \;\middle|\; f \in \mathcal{A},\ \|[D, f]\|_{C^0} \leq 1 \right\}.$$

In order to get noncommutative generalizations we have to express classical geometric operations in terms of the triple $(\mathcal{A}, \mathcal{H}, D)$. We will do this now for integration of functions over M.

Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of a generalized Dirac operator, ordered by increasing absolute values, $|\lambda_1| \leq |\lambda_2| \leq \ldots \nearrow \infty$. We assume that 0 is not an eigenvalue of D. The square D^2 is a generalized Laplacian with eigenvalues $0 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots \nearrow \infty$. By Weyl's theorem $\lim_{k\to\infty} k/|\lambda_k|^n = C$ with $C = (\mathrm{rk}(E) \cdot \mathrm{vol}(M))/((4\pi)^{n/2} \cdot \Gamma(n/2 + 1))$. In particular, there exists a constant $C' > 0$ such that

$$|\lambda_k| \geq C' \cdot k^{1/n}$$

for all but finitely many k. Therefore

$$\mathrm{Tr}_\omega(|D|^{-n}) := \lim_{N\to\infty} \frac{1}{\log(N)} \sum_{k=1}^{N} |\lambda_k|^{-n} \leq C'' \cdot \lim_{N\to\infty} \frac{1}{\log(N)} \sum_{k=1}^{N} \frac{1}{k} < \infty.$$

The number $\mathrm{Tr}_\omega(|D|^{-n})$ is called the *Dixmier trace* of $|D|^{-n}$. Let $\Psi \in C^\infty(\mathrm{End}(E))$ be an endomorphism field, for example $\Psi = f \cdot \mathrm{id}$ where $f \in C^\infty(M)$. Connes' trace theorem tells us that the Dixmier trace is a residue, more precisely

$$\mathrm{Tr}_\omega(\Psi \circ |D|^{-n}) = \frac{1}{n} \lim_{p \searrow n} (p - n) \mathrm{Tr}(\Psi \circ |D|^{-p}). \tag{5}$$

In order to apply this we have to control the integral kernel of $|D|^{-p}$ with $p > n$. Let k_t be the heat kernel of the generalized Laplacian D^2. We perform the following *Mellin transformation*: After restriction to the λ-eigenspace of D we have

$$\begin{aligned}
\Gamma\left(\frac{p}{2}\right) |D|^{-p} &= |\lambda|^{-p} \int_0^\infty e^{-t} t^{p/2-1} dt \\
&= |\lambda|^{-p} \int_0^\infty e^{-s\lambda^2} \left(s\lambda^2\right)^{p/2-1} \lambda^2 ds \\
&= \int_0^\infty e^{-s\lambda^2} s^{p/2-1} ds,
\end{aligned}$$

hence

$$|D|^{-p} = \frac{1}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty t^{p/2-1} e^{-tD^2} dt.$$

Therefore $|D|^{-p}$ has the integral kernel

$$k(x, y; |D|^{-p}) = \frac{1}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty t^{p/2-1} k_t(x, y) dt.$$

Then $\Psi \circ |D|^{-p}$ has integral kernel

$$k(x,y;\Psi \circ |D|^{-p}) = \frac{1}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty t^{p/2-1}\Psi(x) \circ k_t(x,y)dt.$$

Therefore

$$\begin{aligned}\mathrm{Tr}(\Psi \circ |D|^{-p}) &= \int_M \mathrm{tr}\left(k(x,x;\Psi \circ |D|^{-p})\right)\, dV(x) \\ &= \frac{1}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty t^{p/2-1} \int_M \mathrm{tr}\left(\Psi(x)k_t(x,x)\right)\, dV(x)dt.\end{aligned}$$

For any $t_0 > 0$ the integral

$$\int_{t_0}^\infty t^{p/2-1} \int_M \mathrm{tr}\left(\Psi(x)k_t(x,x)\right)\, dV(x)dt$$

remains bounded for $p \searrow n$ (remember that $e^{-tD^2} \le e^{-t\lambda_1^2}$ tends to zero exponentially fast for $t \to \infty$) and hence does not contribute to the residue. For $0 < t < t_0$, t_0 sufficiently small, we have by Corollary 2.4 that

$$k_t(x,x) = (4\pi t)^{-\frac{n}{2}}\mathrm{id} + O(t^{-\frac{n}{2}+1}).$$

Thus

$$\begin{aligned}&\int_0^{t_0} t^{p/2-1} \int_M \mathrm{tr}\left(\Psi(x)k_t(x,x)\right)\, dV(x)dt \\ &= (4\pi)^{-\frac{n}{2}} \int_0^{t_0} \left(\int_M t^{\frac{p-n}{2}-1}\mathrm{tr}\left(\Psi(x)\right)\, dV(x) + O(t^{\frac{p-n}{2}})\right) dt \\ &= (4\pi)^{-\frac{n}{2}} \frac{2}{p-n} t_0^{\frac{p-n}{2}} \int_M \mathrm{tr}\left(\Psi(x)\right)\, dV(x) + O(1)\end{aligned}$$

and therefore by (5)

$$\mathrm{Tr}_\omega(\Psi \circ |D|^{-n}) = (4\pi)^{-\frac{n}{2}} \frac{2}{n\Gamma(n/2)} \int_M \mathrm{tr}\left(\Psi(x)\right)\, dV(x).$$

We have shown

Proposition 4.1 *Let $\Psi \in C^\infty(\mathrm{End}(E))$. Then*

$$\mathrm{Tr}_\omega(\Psi \circ |D|^{-n}) = (4\pi)^{-\frac{n}{2}} \frac{2}{n\Gamma(n/2)} \int_M \mathrm{tr}\left(\Psi(x)\right)\, dV(x).$$

In particular, for $\Psi = f \cdot \mathrm{id}$

$$\mathrm{Tr}_\omega(f \cdot |D|^{-n}) = (4\pi)^{-\frac{n}{2}} \frac{2\mathrm{rk}(E)}{n\Gamma(n/2)} \int_M f(x)\, dV(x).$$

and for $f = 1$

$$\mathrm{Tr}_\omega(|D|^{-n}) = (4\pi)^{-\frac{n}{2}} \frac{2\mathrm{rk}(E)}{n\Gamma(n/2)} \mathrm{vol}(M).$$

This justifies to call $|D|^{-n}$ the operator theoretic volume element and to interprete Tr_ω as integration, c.f. [4, 5].

5 Variation formulas and the Einstein-Hilbert action

In this section we want to calculate the variation of the *gravity action*

$$\int_M (\mathrm{scal}_g + \lambda)\, dV_g$$

under changes of the Riemannian metric g. Here λ is a real constant closely related to the cosmological constant as we will see at the end of this section. The Euler-Lagrange equations of this functional will turn out to be the Einstein equations of General Relativity. In this section we follow [3, Ch. 1.K].

At first we fix some notation.

Let $\mathcal{M}$ be the space of smooth semi-Riemannian metrics on a manifold M. In contrast to all other sections of this article the manifold M need not be compact and g need not be Riemannian. We view the Riemannian curvature tensor R as a functional $\mathcal{M} \to C^\infty(T^{3,1}M)$, $g \mapsto R_g$, where $T^{i,j}M$ denotes the bundle of (i,j)-tensors on M.

The corresponding differential R'_g at g is defined as

$$R'_g h(X,Y)Z = \left.\frac{d}{dt}\right|_{t=0} R_{g+th}(X,Y)Z$$

where h is an arbitrary smooth symmetric $(2,0)$-tensor on M. Similarly we consider the Ricci curvature ric, the scalar curvature scal and the Levi-Civita connection ∇, and we denote their differentials by ric'_g, scal'_g and ∇'_g.

Connections are not tensorial in the second slot, but differences of two connections are. Therefore $\nabla'_g h$ is a $(2,1)$-tensor.

If v and w are symmetric $(2,0)$-tensors, we define the *composition* $v \circ w$ to be the $(2,0)$-tensor given by

$$(v \circ w)(X,Y) = \sum_{i=1}^n \varepsilon_i v(X,e_i) w(e_i,Y) \qquad X,Y \in T_pM,$$

where $e_1,\dots,e_n$ is an orthonormal basis of T_pM, i. e. $g(e_i,e_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_i = \pm 1$. In the Riemannian case all $\varepsilon_i = +1$.

The Riemannian curvature tensor acts on symmetric $(2,0)$-tensors via

$$\mathring{R}_g h(X,Y) := \sum_{i=1}^n \varepsilon_i h(R(e_i,X)Y,e_i).$$

In Corollary 5.3 it will be proven that $\mathring{R}_g h$ is actually a symmetric $(2,0)$-tensor.

The *Lichnerowicz Laplacian* Δ_L on symmetric $(2,0)$-tensors is defined by

$$\Delta_L h := \nabla^*\nabla h + \mathrm{ric}_g \circ h + h \circ \mathrm{ric}_g - 2\mathring{R}_g h.$$

The semi-Riemannian metric g on M defines a scalar product on the bundle $T^*M \otimes T^*M$ given locally by

$$\langle h_1, h_2 \rangle_g := \sum_{i,j=1}^n \varepsilon_i \varepsilon_j h_1(e_i,e_j) h_2(e_i,e_j)$$

where $e_1, \dots, e_n$ is an orthonormal frame, i.e. $g(e_i, e_j) = \varepsilon_i \delta_{ij}$.

Now we can formulate the variation formulas for the curvature.

Proposition 5.1 *Let (M, g) be a semi-Riemannian manifold and let h be a symmetric $(2,0)$-tensor on M. The differentials of R,* ric, scal *and ∇ at g, in the direction of h, are given by the formulas:*

(a) Levi-Civita connection

$$g\left(\nabla'_g h(X,Y), Z\right) = \frac{1}{2}\left\{(\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) - (\nabla_Z h)(X,Y)\right\},$$

(b) Riemannian curvature tensor

$$R'_g h(X,Y)Z = \left(\nabla_X \nabla'_g h\right)(Y,Z) - \left(\nabla_Y \nabla'_g h\right)(X,Z),$$

(c) Ricci tensor

$$\mathrm{ric}'_g h = \frac{1}{2}\Delta_L h - \delta_g^*\left(\delta_g h\right) - \frac{1}{2}\nabla_g d(\mathrm{tr}_g h),$$

(d) scalar curvature

$$\mathrm{scal}'_g h = \Delta_g(\mathrm{tr}_g h) + \delta_g(\delta_g h) - \left\langle \mathrm{ric}_g, h\right\rangle_g.$$

Proof of (a).

We set $g_t := g + th$. Then

$$T_t(X,Y,Z) := g(\nabla^t_X Y, Z) - g(\nabla^0_X Y, Z)$$

is a $(3,0)$-tensor field on M for any t near 0. We want to compute $\frac{\partial}{\partial t} T_t\big|_{t=0}$. We can assume that X, Y and Z are vectorfields on M that are synchronous for g at a fixed point $p \in M$. That is, $\nabla^0_W X = \nabla^0_W Y = \nabla^0_W Z = 0$ for any $W \in T_pM$. This implies that the commutators of X, Y and Z vanish at p, too.

By the Koszul formula we get at p

$$\begin{aligned} 2g_t(\nabla^t_X Y, Z) &= 2g(\nabla^0_X Y, Z) + t\left\{\partial_X(h(Y,Z)) + \partial_Y(h(X,Z)) - \partial_Z(h(X,Y))\right\} \\ &= 2g(\nabla^0_X Y, Z) + t\left\{(\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) - (\nabla_Z h)(X,Y)\right\}. \end{aligned}$$

On the other hand, the left hand side is equal to

$$2g(\nabla^t_X Y, Z) + 2th(\underbrace{\nabla^0_X Y}_{=0 \text{ at } p}, Z) + O(t^2).$$

Therefore

$$\left.\frac{\partial}{\partial t}\right|_{t=0} T_t(X,Y,Z) = \frac{1}{2}\left\{(\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) - (\nabla_Z h)(X,Y)\right\}$$

which proves part (a). $\square$

Proof of (b). The Riemannian curvature tensor is defined as

$$R_g(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

We will calculate its differential $(R'_g h)$ at the point $p \in M$. For the calculation of $(R'_g h)(X,Y,Z)$ we can assume that X, Y and Z are synchronous vector fields at p.

$$\begin{aligned}
(R'_g h)(X,Y,Z) &= (\nabla'_g h)(X, \nabla_Y Z) + \nabla_X((\nabla'_g h)(Y,Z)) \\
&\quad -(\nabla'_g h)(Y, \nabla_X Z) - \nabla_Y((\nabla'_g h)(X,Z)) - (\nabla'_g h)([X,Y],Z) \\
&= (\nabla_X(\nabla'_g h))(Y,Z) - (\nabla_Y(\nabla'_g h))(X,Z)
\end{aligned}$$

□

Before we go on proving the proposition, we will prove a lemma and a corollary.

Lemma 5.2 *Let h be a symmetric $(2,0)$-tensor, let $e_1, \ldots, e_n$ be a locally defined orthonormal frame, i. e. $g(e_i, e_j) = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$.*

Then for any $X \in TM$

$$\sum_{i=1}^{n} \varepsilon_i h(\nabla_X e_i, e_i) = 0.$$

Proof of Lemma 5.2. We write $\nabla_X e_i = \sum_{j=1}^n \alpha_{ji} e_j$. Differentiation of the orthogonality relation yields

$$\begin{aligned}
0 &= \partial_X(g(e_i,e_j)) = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) \\
&= \sum_{k=1}^{n} (\alpha_{ki}\varepsilon_k\delta_{kj} + \alpha_{kj}\varepsilon_k\delta_{ki}) \\
&= \alpha_{ji}\varepsilon_j + \alpha_{ij}\varepsilon_i.
\end{aligned}$$

Using this we calculate

$$\begin{aligned}
\sum_{i=1}^{n} \varepsilon_i h(\nabla_X e_i, e_i) &= \sum_{i,k=1}^{n} \varepsilon_i h(\alpha_{ki} e_k, e_i) \\
&= \frac{1}{2} \sum_{i,k=1}^{n} (\alpha_{ki}\varepsilon_i + \alpha_{ik}\varepsilon_k) h(e_k, e_i) \\
&= \frac{1}{2} \sum_{i,k=1}^{n} \varepsilon_i\varepsilon_k \underbrace{(\alpha_{ki}\varepsilon_k + \alpha_{ik}\varepsilon_i)}_{=0} h(e_k, e_i) = 0.
\end{aligned}$$

□

Corollary 5.3 *Let h be a symmetric $(2,0)$-tensor. Then $\mathring{R}_g h$, defined as above, is a symmetric $(2,0)$-tensor.*

Proof of the corollary. We have to show that

$$\left(\mathring{R}_g h\right)(X,Y) = \left(\mathring{R}_g h\right)(Y,X).$$

By definition

$$\left(\overset{\circ}{R}_g h\right)(X,Y) = \sum_{i=1}^{n} \varepsilon_i h(R(e_i,X)Y,e_i)$$

for any orthonormal frame $e_1,\dots,e_n$. Using the Bianchi identity this is equal to

$$-\sum_{i=1}^{n} \varepsilon_i h(R(X,Y)e_i,e_i) - \sum_{i=1}^{n} \varepsilon_i h(R(Y,e_i)X,e_i)).$$

The second term is just $(\overset{\circ}{R}_g h)(Y,X)$, so we have to show that the first term vanishes.

$$\begin{aligned} h(R(X,Y)e_i,e_i) &= h(\nabla_X\nabla_Y e_i,e_i) - h(\nabla_Y\nabla_X e_i,e_i) - h(\nabla_{[X,Y]}e_i,e_i) \\ &= \partial_X(h(\nabla_Y e_i,e_i)) - (\nabla_X h)(\nabla_Y e_i,e_i) - h(\nabla_Y e_i,\nabla_X e_i) \\ &\quad -\partial_Y(h(\nabla_X e_i,e_i)) + (\nabla_Y h)(\nabla_X e_i,e_i) + h(\nabla_X e_i,\nabla_Y e_i) \\ &\quad -h(\nabla_{[X,Y]}e_i,e_i). \end{aligned}$$

If we apply Lemma 5.2 to the symmetric $(2,0)$-tensors h, $\nabla_X h$ and $\nabla_Y h$ we get

$$\sum_{i=1}^{n} \varepsilon_i h(R(X,Y)e_i,e_i) = 0. \qquad \square$$

We return to the proof of Proposition 5.1.

Proof of (c). The Ricci curvature is defined as

$$\mathrm{ric}_g(X,Y) := \mathrm{tr} R_g(\,\cdot\,,X)Y.$$

Since here tr denotes the trace of a linear map it does not depend on the metric. Therefore tr commutes with differentiation in direction h. Using (b) we get

$$\mathrm{ric}'_g h(X,Y) = \sum_{i=1}^{n} \varepsilon_i \left\{ (\nabla_{e_i}(C_g h))(X,Y,e_i) - (\nabla_X(C_g h))(e_i,Y,e_i) \right\}, \tag{6}$$

with $C_g h(X,Y,Z) := g(\nabla'_g h(X,Y),Z)$.

The second term can easily be computed using (a). We will suppose that X and Y and the orthonormal frame $e_1,\dots,e_n$ are synchronous at p. Then we get at p:

$$\begin{aligned} &\sum_{i=1}^{n} \varepsilon_i (\nabla_X(C_g h))(e_i,Y,e_i) \\ &\quad = \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \partial_X \left\{ (\nabla_{e_i}h)(Y,e_i) + (\nabla_Y h)(e_i,e_i) - (\nabla_{e_i}h)(e_i,Y) \right\} \\ &\quad = \frac{1}{2}\partial_X\partial_Y(\mathrm{tr}_g h) = \frac{1}{2}(\nabla d(\mathrm{tr}_g h))(X,Y). \end{aligned}$$

Now we turn to the first term of (6). Applying (a) shows that the first term is equal to

$$\frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \left\{ \left(\nabla^2_{e_i,X} h\right)(Y,e_i) + \left(\nabla^2_{e_i,Y} h\right)(X,e_i) - \left(\nabla^2_{e_i,e_i} h\right)(X,Y) \right\}. \tag{7}$$

The last term hereof is one half the connection Laplacian

$$\nabla^*\nabla h = -\sum_{i=1}^{n} \varepsilon_i \nabla^2_{e_i,e_i} h.$$

The first term of (7) can be rewritten using the curvature tensor on the bundle of (2,0)-tensors:

$$R_{X,Y} = \nabla^2_{X,Y} - \nabla^2_{Y,X},$$
$$\left(\nabla^2_{e_i,X} h\right)(Y,e_i) = \left(\nabla^2_{X,e_i} h\right)(Y,e_i) + \left(R_{e_i,X} h\right)(Y,e_i). \tag{8}$$

The curvature of a $(2,0)$-tensor can be expressed in terms of the Riemannian curvature tensor:

$$\left(R_{A,B} h\right)(V,W) = -h(R(A,B)V,W) - h(V,R(A,B)W).$$

On the other hand note that

$$\begin{aligned}(h \circ \mathrm{ric}_g)(X,Y) &= \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j h(X,e_i) g(R_g(e_j,e_i)Y,e_j) \\ &= \sum_{j=1}^{n} \varepsilon_j h(X, R_g(Y,e_j)e_j)\end{aligned}$$

and similarly

$$(\mathrm{ric}_g \circ h)(X,Y) = \sum_{j=1}^{n} \varepsilon_j h(R_g(X,e_j)e_j, Y).$$

Altogether we obtain

$$\begin{aligned}\mathrm{ric}'_g h(X,Y) &= -\frac{1}{2}\left(\nabla d(\mathrm{tr}_g h)\right)(X,Y) \\ &\quad + \frac{1}{2}\left[\nabla^*\nabla h + \mathrm{ric}_g \circ h + h \circ \mathrm{ric}_g - 2\mathring{R}_g h\right](X,Y) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \left[\left(\nabla^2_{X,e_i} h\right)(Y,e_i) + \left(\nabla^2_{Y,e_i} h\right)(X,e_i)\right].\end{aligned} \tag{9}$$

Now we calculate the divergence of h

$$\delta_g h = -\sum_{i=1}^{n} \varepsilon_i \left(\nabla_{e_i} h\right)(e_i, \cdot\,)$$

and applying the formal adjoint, δ_g^*, we get

$$\left(\delta_g^* \delta_g h\right)(X,Y) = -\frac{1}{2}\sum_{i=1}^{n} \varepsilon_i \left\{\left(\nabla^2_{X,e_i} h\right)(e_i,Y) + \left(\nabla^2_{Y,e_i} h\right)(e_i,X)\right\}$$

which is up to a sign the last term of (9). As the second term of (9) is one half the Lichnerowicz Laplacian, we have

$$\mathrm{ric}'_g h = \frac{1}{2}\Delta_L h - \delta_g^* \delta_g h - \frac{1}{2}\nabla d(\mathrm{tr}_g h)$$

which proves (c). □

In the following we will generalize our previous definition of the composition: If A and B are tensors, then $A \circ B$ means contraction of $A \otimes B$ in the last slot of A with the first slot of B.

For the semi-Riemannian metric g which is a $(2,0)$-tensor there is a unique $(0,2)$-tensor $L(g)$ such that $L(g) \circ g = \mathrm{id}|_{TM}$. If $e_1, \ldots, e_n$ are orthonormal with respect to g, i. e. $g(e_i,e_j) = \varepsilon_i \delta_{ij}$, then $L(g) = \sum_{i=1}^{n} \varepsilon_i \, e_i \otimes e_i$.

Now the metric trace $\mathrm{tr}_g(h)$ of a $(2,0)$-tensor h can be expressed as a metric-independent trace via

$$\mathrm{tr}_g(h) = \mathrm{tr}\big(L(g) \circ h\big)$$

and the metric on symmetric $(2,0)$-tensors h_1, h_2 fulfills

$$\langle h_1, h_2 \rangle_g = \mathrm{tr}\Big(L(g) \circ h_1 \circ L(g) \circ h_2\Big).$$

Lemma 5.4 *For symmetric* $(2,0)$*-tensors* h *and* r *we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{tr}_{g+th}(r) = -\langle h, r \rangle_g .$$

Proof. Because of

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \Big(L(g+th) \circ (g+th)\Big) \\ &= L'_g h \circ g + L(g) \circ h \end{aligned}$$

we get

$$L'_g h = -L(g) \circ h \circ L(g)$$

and therefore

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathrm{tr}_{g+th}(r) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{tr}(L(g+th) \circ r) \\ &= \mathrm{tr}\left(L'_g h \circ r\right) \\ &= -\mathrm{tr}\Big(L(g) \circ h \circ L(g) \circ r\Big) \\ &= -\langle h, r \rangle_g . \end{aligned}$$

□

Now we are ready to calculate the variation of the scalar curvature.

Proof of (d).

$$\begin{aligned} \mathrm{scal}'_g h &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{tr}_{g+th}(\mathrm{ric}_{g+th}) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \mathrm{tr}_{g+th}(\mathrm{ric}_g) \right) + \mathrm{tr}_g\left(\mathrm{ric}'_g h\right) \\ &= -\langle h, \mathrm{ric}_g \rangle_g + \mathrm{tr}_g\left(\frac{1}{2}\Delta_L h - \delta_g^* \delta_g h - \frac{1}{2}\nabla d(\mathrm{tr}_g h) \right) . \end{aligned}$$

Note that $\Delta_g f = -\mathrm{tr}_g(\nabla df)$. Furthermore for any 1-form ω we have

$$\delta_g^* \omega(X,Y) = \frac{1}{2}\Big((\nabla\omega)(X,Y) - (\nabla\omega)(Y,X)\Big)$$

$$\Rightarrow \quad \mathrm{tr}_g(\delta_g^*\omega) = \sum_{i=1}^{n} \varepsilon_i (\nabla\omega)(e_i, e_i) = -\delta_g \omega.$$

Now we want to compute $\mathrm{tr}_g \Delta_L h$. It is straightforward to show that

$$\mathrm{tr}_g(h \circ \mathrm{ric}_g) = \mathrm{tr}_g(\mathrm{ric}_g \circ h) = \mathrm{tr}_g(\mathring{R}_g h).$$

So we have $\mathrm{tr}_g \Delta_L h = \mathrm{tr}_g(\nabla^* \nabla h)$. On the other hand, since $L(g)$ is parallel we get

$$\begin{aligned}
\Delta_g(\mathrm{tr}_g h) &= \Delta_g(\mathrm{tr}(L(g) \circ h)) \\
&= -\sum_{i=1}^{n} \varepsilon_i \nabla^2_{e_i, e_i} \mathrm{tr}\,(L(g) \circ h) \\
&= -\mathrm{tr}\left(L(g) \circ \big(\sum_{i=1}^{n} \varepsilon_i \nabla^2_{e_i, e_i} h\big)\right) \\
&= \mathrm{tr}_g(\nabla^* \nabla h).
\end{aligned}$$

Hence

$$\Delta_g(\mathrm{tr}_g h) = \mathrm{tr}_g(\Delta_L h).$$

Putting everything together we obtain

$$\mathrm{scal}'_g h = \Delta_g(\mathrm{tr}_g h) + \delta_g \delta_g h - \langle \mathrm{ric}_g, h \rangle_g$$

and therefore the proposition is proven. □

As a next step we want to calculate the variation of the volume element.

Proposition 5.5 *Let (M, g) be a semi-Riemannian manifold. Then the differential of the volume element dV_g is given by*

$$dV'_g h = \frac{1}{2}(\mathrm{tr}_g h) dV_g.$$

Proof. We consider dV as a map from symmetric $(2,0)$-tensors to volume densities, locally given by

$$\sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j \mapsto \sqrt{|\det(g_{ij})|}\; dx^1 dx^2 \cdots dx^n.$$

For $A \in \mathrm{End}(TM)$ we write $g\&A(X,Y) := g(AX, AY)$. Then

$$dV_{g\&A} = dV_g \cdot |\det A|.$$

If h is a symmetric $(2,0)$-tensor, then $H := L(g) \circ h \in \mathrm{End}(TM)$ satisfies

$$h(X,Y) = g(HX, Y) = g(X, HY)$$

and

$$\begin{aligned}
h \circ h(X,Y) &= g(HX, HY) = g\&H(X,Y), \\
g\&(\mathrm{id} + tH) &= g + 2th + t^2 h \circ h, \\
dV_{g+2th+t^2 h\circ h} &= dV_g\, |\det(\mathrm{id} + tH)|.
\end{aligned}$$

We differentiate w.r.t. t at $t = 0$ and get

$$2dV'_g h = dV_g \,\mathrm{tr}(H) = (\mathrm{tr}_g h)\, dV_g.$$ □

Now we calculate the Euler-Lagrange equations for the gravity action $\int(\mathrm{scal}_g + \lambda)\, dV_g$.

In order to have a finite integral, we suppose that the variation h of the metric has compact support contained in an open and relatively compact subset $U \subset M$.

The variation of $S_M := \int_M (\mathrm{scal}_g + \lambda)\, dV_g$ is given by

$$\begin{aligned} S'_U h = S'_M h &= \int_M \mathrm{scal}'_g h \, dV_g + \int_M (\mathrm{scal}_g + \lambda)\, dV'_g h \\ &= \int_M \Big\{ \Delta_g(\mathrm{tr}_g h) + \delta_g(\delta_g h) - \langle \mathrm{ric}_g, h \rangle_g \Big\}\, dV_g \\ &\quad + \frac{1}{2} \int_M (\mathrm{scal}_g + \lambda)\,(\mathrm{tr}_g h)\, dV_g . \end{aligned}$$

The first two summands of the first integral vanish since they are divergences. We rewrite $(\mathrm{tr}_g h)$ as $\langle g, h \rangle_g$.

$$S'_M h = - \int_M \left\langle \mathrm{ric}_g - \frac{1}{2} \mathrm{scal}_g \cdot g - \frac{1}{2} \lambda \cdot g \, , \, h \right\rangle_g dV_g .$$

We have shown

Proposition 5.6 *Stationarity of the functional S_M at g is equivalent to the Einstein equations*

$$\mathrm{ric}_g - \frac{1}{2} \mathrm{scal}_g \cdot g - \Lambda \cdot g = 0$$

of the vacuum with cosmological constant $\Lambda = \lambda/2$.

6 Einstein-Hilbert action and Wodzicki residue

In the fourth section we have seen how to characterize integration of functions over a closed Riemannian manifold using the Dirac operator and the Dixmier trace. This was based on the first coefficient Φ_0 in the heat asymptotics. In the previous section we have shown that the total scalar curvature functional gives rise to the field equations of General Relativity. But this is exactly the second term Φ_1 in the heat asymptotics. Therefore the question arises if we can extract the second heat coefficient using some kind of a trace. This is what we do in this section. Here we follow closely the work of Kalau and Walze [5].

Let $P : C^\infty(E) \to C^\infty(E)$ be a classical pseudo-differential operator of order m over the closed Riemannian manifold M. After choosing a system of local coordinates and a trivialization of the bundle E we can look at the total sysmbol σ^P of P and develop it into a formal series

$$\sigma^P(x, \xi) \sim \sum_{k=0}^{\infty} \sigma^P_{m-k}(x, \xi), \tag{10}$$

where each σ^P_j is a matrix valued function homogeneous of degree j in ξ (for $\xi \geq \varepsilon > 0$) and satisfies an estimate

$$|\partial_x^\alpha \partial_\xi^\beta \sigma^P_j(x, \xi)| \leq c_{\alpha\beta}(1 + \|\xi\|)^{j - |\beta|}$$

for all multiindices α and β. Conversely, given a formal series as in (10) there exists a classical pseudodifferential operator with this development. The pseudodifferential operator is unique up to smoothing operators.

The "Leibnitz rule" gives us a multiplication in the space of formal developments of symbols which corresponds to the composition of operators [8, Ch. III]

$$\sigma^{P_1 \circ P_2}(x,\xi) \sim \sigma^{P_1} \circ \sigma^{P_2} = \sum_{|\alpha|=0}^{\infty} (-i)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma^{P_1} \partial_x^\alpha \sigma^{P_2}. \tag{11}$$

Except for the leading part σ_m, the *principal symbol*, the total symbol does depend on the choice of local coordinates and trivialization. However, for $p \in M$ the quantity

$$\int_{S_p^{n-1}} \mathrm{tr}\left(\sigma_{-n}^P(p,\xi)\right) \, d\xi$$

is invariantly defined and independent of the choices [11]. Here integration is over the unit sphere S_p^{n-1} in the cotangent bundle T_p^*M. One further integration over the manifold gives us the *Wodzicki residue*,

$$\mathrm{Res}(P) = \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \int_M \int_{S_p^{n-1}} \mathrm{tr}\left(\sigma_{-n}^P(p,\xi)\right) \, d\xi \, dV(p).$$

Now the main result is

Theorem 6.1 (Kalau-Walze[5],Kastler[7]) *Let M be a compact Riemannian manifold of dimension n, n even, $n \geq 4$. Let*

$$\Delta = \nabla^* \nabla + \mathcal{K}$$

be an invertible generalized Laplacian over M. Then for each $p \in M$

$$\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \int_{S_p^{n-1}} \mathrm{tr}\left(\sigma_{-n}^{\Delta^{-(n/2)+1}}(p,\xi)\right) \, d\xi = \frac{n-2}{2} \mathrm{tr}\left(\Phi_1(p,p)\right).$$

In particular,

$$\mathrm{Res}\left(\Delta^{-(n/2)+1}\right) = \frac{n-2}{2} \int_M \mathrm{tr}\left(\Phi_1(p,p)\right) dV(p).$$

Before we can prove the theorem we need to show a lemma

Lemma 6.2 *In Riemannian normal coordinates x^ρ based at the point p we have for the Christoffel symbols $\Gamma_{\mu\nu}^\kappa$,*

$$\sum_{\mu,\nu=1}^{n} \delta^{\mu\nu} \partial_{x^\rho} \Gamma_{\mu\nu}^\kappa = \frac{2}{3} \, \mathrm{ric}_\rho^\kappa$$

where $\delta^{\mu\nu}$ is the Kronecker symbol.

Proof of Lemma 6.2. We use the Einstein summation convention in order to keep notation at a reasonable size. The Koszul formula for the Levi-Civita connection reads in coordinates

$$2\,\Gamma_{\mu\nu}^\kappa = g^{\kappa\lambda}(\partial_{x^\mu} g_{\nu\lambda} + \partial_{x^\nu} g_{\mu\lambda} - \partial_{x^\lambda} g_{\mu\nu})$$

which together with (2) implies

$$
\begin{aligned}
\Gamma^{\kappa}_{\mu\nu} &= \frac{1}{6}\, g^{\kappa\lambda}\Big(R_{\nu\mu\lambda\delta} + R_{\nu\delta\lambda\mu} + R_{\mu\nu\lambda\delta} \\
&\qquad + R_{\mu\delta\lambda\nu} - R_{\mu\lambda\nu\delta} - R_{\mu\delta\nu\lambda}\Big)x^{\delta} \;+\; O(\|x\|^2).
\end{aligned}
$$

So we get

$$
\begin{aligned}
\partial_{x^\rho}\Gamma^{\kappa}_{\mu\nu} &= \frac{1}{6}\, g^{\kappa\lambda}\Big(R_{\nu\mu\lambda\rho} + R_{\nu\rho\lambda\mu} + R_{\mu\nu\lambda\rho} \\
&\qquad + R_{\mu\rho\lambda\nu} - R_{\mu\lambda\nu\rho} - R_{\mu\rho\nu\lambda}\Big) \;+\; O(\|x\|)
\end{aligned}
$$

and therefore at the base point p (corresponding to $x^\mu = 0$)

$$
\delta^{\mu\nu}\partial_{x^\rho}\Gamma^{\kappa}_{\mu\nu} = \frac{2}{3}\, g^{\kappa\lambda}\mathrm{ric}_{\rho\lambda} = \frac{2}{3}\,\mathrm{ric}^{\kappa}_{\rho}
$$

$\square$

Proof of Theorem 6.1.

(i) With respect to any system of local coordinates and to any local trivialization of the bundle E we write down the total symbol of Δ, $\sigma^{\Delta}(x,\xi) := \sigma_2 + \sigma_1 + \sigma_0$. In particular, σ_2 is proportional to $\mathrm{id}_{\mathrm{End}(E)} =: \mathbf{1}$. We introduce a new pseudo-differential operator P by inverting the principal symbol of Δ, $\sigma^P(x,\xi) = \sigma^P_{-2} := (\sigma_2)^{-1}$. By (11) we have

$$
\begin{aligned}
\sigma^{\Delta\circ P - 1} &\sim \sum_{|\alpha|=0}^{\infty} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial^{\alpha}_{\xi}\sigma^{\Delta}\; \partial^{\alpha}_{x}\sigma_2^{-1} - \mathbf{1} \\
&\sim \sum_{k=1}^{2}\sum_{|\alpha|=0}^{k} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial^{\alpha}_{\xi}\sigma_{|\alpha|+2-k}\; \partial^{\alpha}_{x}\sigma_2^{-1} \\
&=: \; -\, r(x,\xi)
\end{aligned}
$$

In other words $\sigma^{\Delta}\circ(\sigma^P\circ(\mathbf{1}-r)^{-1}) \sim \mathbf{1}$. Using the geometric series in symbol-space (this can be done because r is of order -1) we obtain

$$
\sigma^{\Delta^{-1}}(x,\xi) \sim\; \sigma_2^{-1}\circ\sum_{k=0}^{\infty} r^{\circ k}\ .
$$

We begin to compute

$$
\begin{aligned}
r_{-k}(x,\xi) &= -\sum_{|\alpha|=0}^{k} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial^{\alpha}_{\xi}\sigma_{|\alpha|+2-k}\; \partial^{\alpha}_{x}\sigma_2^{-1}\ , \\
r_{-1}(x,\xi) &= -\,\sigma_2^{-1}\sigma_1 - i\,\sigma_2^{-2}\;\partial_{\xi_\mu}\sigma_2\;\partial_{x^\mu}\sigma_2\ , \\
r_{-2}(x,\xi) &= -\,\sigma_2^{-1}\sigma_0 - \sigma_2^{-2}(i\,\partial_{\xi_\mu}\sigma_1\;\partial_{x^\mu}\sigma_2 + \tfrac{1}{2}\,\partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2\;\partial_{x^\mu}\partial_{x^\nu}\sigma_2) \\
&\qquad + \sigma_2^{-3}\;\partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2\;\partial_{x^\mu}\sigma_2\;\partial_{x^\nu}\sigma_2\ , \\
r_{-k}(x,\xi) &= 0 \qquad\qquad \forall\, k > 2\ .
\end{aligned}
\tag{12}
$$

Furthermore we write

$$\sum_{k=0}^{\infty} r^{\circ k} = \sum_{j=0}^{\infty} s_{-j} \qquad \text{with} \qquad s_0 = \mathbf{1}\,, \quad s_{-1} = r_{-1}\,, \quad s_{-2} = r_{-1}^2 + r_{-2}\,, \quad \ldots\,.$$

From this we can read off the symbol of Δ^{-1} :

$$\sigma^{\Delta^{-1}}(x,\xi) \sim \sum_{l=2}^{\infty} \sigma_{-l}^{\Delta^{-1}} \quad \text{with} \quad \sigma_{-l}^{\Delta^{-1}}(x,\xi) = \sum_{|\alpha|=0}^{l-2} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial_\xi^\alpha \sigma_2^{-1}\, \partial_x^\alpha s_{|\alpha|+2-l}\,.$$

We will only need the first three non-vanishing terms :

$$\begin{aligned}
\sigma_{-2}^{\Delta^{-1}}(x,\xi) &= \sigma_2^{-1}\,, \qquad \sigma_{-3}^{\Delta^{-1}}(x,\xi) = \sigma_2^{-1}\, r_{-1}\,, \\
\sigma_{-4}^{\Delta^{-1}}(x,\xi) &= \sigma_2^{-1}\,(r_{-1}^2 + r_{-2}) + i\,\sigma_2^{-2}\,\partial_{\xi_\mu}\sigma_2\,\partial_{x^\mu} r_{-1}\,.
\end{aligned} \tag{13}$$

More generally we get

$$\sigma^{\Delta^{-m}}(x,\xi) \sim \sigma^{\Delta^{-m+1}} \circ \sigma^{\Delta^{-1}} \sim \sum_{|\alpha|=0}^{\infty} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial_\xi^\alpha \sigma^{\Delta^{-m+1}}\, \partial_x^\alpha \sigma^{\Delta^{-1}} = \sum_{l=2m}^{\infty} \sigma_{-l}^{\Delta^{-m}}\,,$$

$$\text{with} \qquad \sigma_{-l}^{\Delta^{-m}}(x,\xi) = \sum_{|\alpha|=0}^{l-2m} \sum_{k=2}^{2+l-|\alpha|-2m} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial_\xi^\alpha \sigma_{|\alpha|+k-l}^{\Delta^{-m+1}}\, \partial_x^\alpha \sigma_{-k}^{\Delta^{-1}}\,.$$

Using this and $\sigma_{-2m}^{\Delta^{-m}} = \sigma_2^{-m}$ we get the recursion relations

$$\sigma_{3-2k}^{\Delta^{-k+2}}(x,\xi) = \sigma_{5-2k}^{\Delta^{-k+3}}\, \sigma_2^{-1} + \sigma_2^{-k+3}\, \sigma_{-3}^{\Delta^{-1}} - i\,\partial_{\xi_\mu} \sigma_2^{-k+3}\, \partial_{x^\mu} \sigma_2^{-1} \tag{14}$$

and

$$\begin{aligned}
\sigma_{-2k}^{\Delta^{-k+1}}(x,\xi) &= \sum_{|\alpha|=0}^{2} \sum_{j=2}^{4-|\alpha|} (-i)^{|\alpha|}\, \frac{1}{\alpha!}\, \partial_\xi^\alpha \sigma_{|\alpha|+j-2k}^{\Delta^{-k+2}}\, \partial_x^\alpha \sigma_{-j}^{\Delta^{-1}} \\
&= \sigma_{2-2k}^{\Delta^{-k+2}}\, \sigma_2^{-1} + \sigma_{3-2k}^{\Delta^{-k+2}}\, \sigma_{-3}^{\Delta^{-1}} + \sigma_2^{-k+2}\, \sigma_{-4}^{\Delta^{-1}} - i\,\partial_{\xi_\mu} \sigma_{3-2k}^{\Delta^{-k+2}}\, \partial_{x^\mu} \sigma_2^{-1} \\
&\quad - i\,\partial_{\xi_\mu} \sigma_2^{-k+2}\, \partial_{x^\mu} \sigma_{-3}^{\Delta^{-1}} - \tfrac{1}{2}\, \partial_{\xi_\mu} \partial_{\xi_\nu} \sigma_2^{-k+2}\, \partial_{x^\mu} \partial_{x^\nu} \sigma_2^{-1}.
\end{aligned} \tag{15}$$

(ii) Since the formula for $Res(\Delta^{-(n/2)+1})$ in an arbitrary coordinate system contains a lot of terms it is more convenient to specialize our formulas to Riemannian normal coordinates x^μ about the base point p for which $x^\mu = 0$. We will also use the Einstein summation convention for all Greek indices. By (2) the $(0,2)$-tensor corresponding to the metric has the Taylor expansion

$$g^{\mu\nu} = \delta^{\mu\nu} - \tfrac{1}{3}\, R^{\mu\ \ \nu}_{\ \ \gamma\ \ \delta}(p)\, x^\gamma x^\delta + O(\|x\|^3)\,.$$

We also have to trivialize the bundle E. Then the connection is given by $\nabla_{x^\mu} = \partial_{x^\mu} + A_\mu$ with A_μ matrix-valued functions. We choose the trivialization such that it simplifies the calculations. Parallel translation of a basis of E_p along the radial geodesics emanating from p yields a trivialization such that $A_\mu(p) = 0$.

In these coordinates the generalized Laplacian takes the form

$$\begin{aligned}
\Delta &= \nabla^*\nabla + \mathcal{K} \\
&= -g^{\mu\nu}\left\{\nabla_\mu\nabla_\nu - \nabla_{\nabla_\mu \partial_{x^\nu}}\right\} + \mathcal{K} \\
&= -g^{\mu\nu}\left\{(\partial_{x^\mu} + A_\mu)(\partial_{x^\nu} + A_\nu) - \Gamma^\beta_{\mu\nu}(\partial_{x^\beta} + A_\beta)\right\} + \mathcal{K} \\
&= g^{\mu\nu}\left\{-\partial_{x^\mu}\partial_{x^\nu} - 2A_\mu\partial_{x^\nu} + \Gamma^\beta_{\mu\nu}\partial_{x^\beta} - (\partial_{x^\mu}A_\nu) - A_\mu A_\nu + \Gamma^\beta_{\mu\nu}A_\beta\right\} + \mathcal{K}.
\end{aligned}$$

Therefore Δ has the symbols

$$\begin{aligned}
\sigma_2 &= g^{\mu\nu}\xi_\mu\xi_\nu \\
\sigma_1 &= -2ig^{\mu\nu}A_\mu\xi_\nu + ig^{\mu\nu}\Gamma^\beta_{\mu\nu}\xi_\beta \\
\sigma_0 &= -g^{\mu\nu}(\partial_{x^\mu}A_\nu) - g^{\mu\nu}A_\mu A_\nu + g^{\mu\nu}\Gamma^\beta_{\mu\nu}A_\beta + \mathcal{K}.
\end{aligned}$$

At p we get

$$\begin{aligned}
\sigma_2(p,\xi) &= \delta^{\mu\nu}\xi_\mu\xi_\nu \\
\sigma_1(p,\xi) &= 0 \\
\sigma_0(p,\xi) &= -\delta^{\mu\nu}\partial_{x^\mu}A_\nu + \mathcal{K} \\
\partial_{x^\mu}\sigma_1(p,\xi) &= -2i\,(\partial_{x^\mu}A_\nu)\xi^\nu + i\,\underbrace{\delta^{\rho\nu}\partial_{x^\mu}\Gamma^\beta_{\rho\nu}}_{=(2/3)\mathrm{ric}^\beta_\mu}\xi_\beta \\
\partial_{x^\mu}\sigma_2(p,\xi) &= 0 \\
\partial_{x^\gamma}\partial_{x^\delta}\sigma_2(p,\xi) &= -\frac{2}{3}R^\mu{}_\gamma{}^\nu{}_\delta\,\xi_\mu\xi_\nu \\
\partial_{\xi_\mu}\sigma_2(p,\xi) &= 2\,\xi^\mu.
\end{aligned} \tag{16}$$

So we obtain

$$\partial_{\xi_\delta}\sigma_2(p,\xi)\,\partial_{x^\gamma}\partial_{x^\delta}\sigma_2(p,\xi) = -\frac{4}{3}R^\mu{}_\gamma{}^{\nu\delta}\,\xi_\mu\xi_\nu\xi_\delta = 0. \tag{17}$$

With these quantities we can calculate

$$\begin{aligned}
r_{-1}(p,\xi) &= 0, \qquad r_{-2}(p,\xi) = -\,\sigma_2^{-1}\,\sigma_0 + \tfrac{2}{3}\,\sigma_2^{-2}\delta^{\rho\sigma}\,R^\mu{}_\rho{}^\nu{}_\sigma\,\xi_\mu\xi_\nu\ , \\
\partial_{x^\mu}r_{-1}(p,\xi) &= -\sigma_2^{-1}\,\partial_{x^\mu}\sigma_1\,-\,i\,\sigma_2^{-2}\,2\xi^\nu\,\partial_{x^\mu}\partial_{x^\nu}\sigma_2 \\
&= -\,\sigma_2^{-1}\,\partial_{x^\mu}\sigma_1\,+\,\tfrac{4}{3}\,i\,\sigma_2^{-2}\,R^\gamma{}_\mu{}^\delta{}_\nu\,\xi^\nu\xi_\gamma\xi_\delta \\
&\overset{(17)}{=} -\,\sigma_2^{-1}\,\partial_{x^\mu}\sigma_1 \\
\sigma^{\Delta^{-1}}_{-2}(p,\xi) &= \sigma_2^{-1}\ , \qquad \sigma^{\Delta^{-1}}_{-3}(p,\xi) = 0, \\
\sigma^{\Delta^{-1}}_{-4}(p,\xi) &= -\,\sigma_2^{-2}\,\sigma_0 + \sigma_2^{-3}\left(-\,2\,i\,\partial_{x^\mu}\sigma_1\,\xi^\mu + \tfrac{2}{3}\,\delta^{\rho\sigma}\,R^\mu{}_\rho{}^\nu{}_\sigma\,\xi_\mu\xi_\nu\right)\ . \\
&= -\,\sigma_2^{-2}\,\sigma_0 + \sigma_2^{-3}\left(-\,4\,(\partial_{x^\mu}A_\nu)\,\xi^\mu\xi^\nu\,+\,2\cdot\tfrac{2}{3}\,\mathrm{ric}_{\mu\beta}\,\xi^\mu\xi^\beta - \tfrac{2}{3}\,\mathrm{ric}_{\mu\nu}\,\xi^\mu\xi^\nu\right) \\
&= -\,\sigma_2^{-2}\,\sigma_0 + \sigma_2^{-3}\left(-\,4\,\partial_{x^\mu}A_\nu + \tfrac{2}{3}\,\mathrm{ric}_{\mu\nu}\right)\xi^\mu\xi^\nu.
\end{aligned} \tag{18}$$

We define

$$a_k := \sigma^{\Delta^{-k+2}}_{3-2k}(p,\xi).$$

It is easy to check that $a_2 = \sigma^{\mathrm{id}}_{-1}(p,\xi) = 0$. The recursion formula (14) reads as

$$a_k = a_{k-1}\sigma_2^{-1},$$

and therefore

$$a_k = \sigma_2^{-k+2} a_2 = 0.$$

Now we set

$$b_k := \sigma^{\Delta^{-k+1}}_{-2k}(p,\xi)\,\sigma_2^k(p,\xi).$$

Obviously, we have $b_1 = 0$ and the recursion formula (15) yields for b_k

$$\begin{aligned} b_k &= b_{k-1} + \sigma_2^2\sigma^{\Delta^{-1}}_{-4} - i\sigma_2^k\partial_{\xi_\mu}\sigma_2^{-k+2}\partial_{x^\mu}\sigma^{\Delta^{-1}}_{-3} \\ &\quad -\frac{1}{2}\sigma_2^k\partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2^{-k+2}\partial_{x^\mu}\partial_{x^\nu}\sigma_2^{-1} \end{aligned} \tag{19}$$

The term $\sigma^{\Delta^{-1}}_{-3}$ can be expressed in quantities we know already

$$\begin{aligned} \sigma^{\Delta^{-1}}_{-3} &\overset{(13)}{=} \sigma_2^{-1} r_{-1} \\ &\overset{(12)}{=} -\sigma_2^{-2}\sigma_1 - i\sigma_2^{-3}\partial_{\xi_\nu}\sigma_2\partial_{x^\nu}\sigma_2, \end{aligned}$$

so its x^μ-derivative at p is

$$\begin{aligned} \left(\partial_{x^\mu}\sigma^{\Delta^{-1}}_{-3}\right)(p,\xi) &= -\sigma_2^{-2}\partial_{x^\mu}\sigma_1 - i\sigma_2^{-3}\partial_{\xi_\nu}\sigma_2\partial_{x^\mu}\partial_{x^\nu}\sigma_2 \\ &\overset{(17)}{=} -\sigma_2^{-2}\partial_{x^\mu}\sigma_1 \\ &= 2i\,\sigma_2^{-2}\,(\partial_{x^\mu}A_\nu)\,\xi^\nu - \frac{2}{3}\,i\,\sigma_2^{-2}\,\mathrm{ric}^\beta_\mu\,\xi_\beta. \end{aligned}$$

Now we are ready to calculate the summands of the recursion formula (19). We already know the second summand. The third one yields

$$\begin{aligned} &-i\,\sigma_2^k\,\partial_{\xi_\mu}\sigma_2^{-k+2}\,\partial_{x^\mu}\sigma^{\Delta^{-1}}_{-3} \\ &\quad= -2i\,\sigma_2^k\,(-k+2)\,\sigma_2^{-k+1}\,\xi^\mu\,\left(\partial_{x^\mu}\sigma^{\Delta^{-1}}_{-3}\right) \\ &\quad= 4\,(-k+2)\,\sigma_2^{-1}\,\xi^\mu\,(\partial_{x^\mu}A_\nu)\,\xi^\nu - 2\cdot\frac{2}{3}\,(-k+2)\,\sigma_2^{-1}\,\xi^\mu\,\mathrm{ric}_{\mu\beta}\,\xi^\beta. \end{aligned}$$

It is straightforward to transform the last summand of (19).

$$-\frac{1}{2}\,\sigma_2^k\,\partial_{\xi_\mu}\,\partial_{\xi_\nu}\,\sigma_2^{-k+2}\,\partial_{x^\mu}\,\partial_{x^\nu}\,\sigma_2^{-1} \quad = \quad \frac{2}{3}\,(-k+2)\,\sigma_2^{-1}\mathrm{ric}_{\mu\nu}\,\xi^\mu\xi^\nu.$$

The above formulas yield

$$\begin{aligned} b_k &= b_{k-1} + \sigma_2^2\,\sigma^{\Delta^{-1}}_{-4} + \sigma_2^{-1}\,(-k+2)\,\xi^\mu\xi^\nu\,\left\{4\,\partial_{x^\mu}A_\nu - \frac{4}{3}\,\mathrm{ric}_{\mu\nu}\right\} \\ &\quad + \frac{2}{3}\,(-k+2)\,\sigma_2^{-1}\mathrm{ric}_{\mu\nu}\,\xi^\mu\xi^\nu \\ &= b_{k-1} + \sigma_2^2\,\sigma^{\Delta^{-1}}_{-4} + \sigma_2^{-1}\,(-k+2)\,\xi^\mu\xi^\nu\,\left\{4\,\partial_{x^\mu}A_\nu - \frac{2}{3}\,\mathrm{ric}_{\mu\nu}\right\}. \end{aligned}$$

Using $b_1 = 0$ and an induction over k we get

$$\begin{aligned} b_k &= (k-1)\,\sigma_2^2\,\sigma_{-4}^{\Delta^{-1}} - \sigma_2^{-1}\,(k-1)(k-2)\,\left\{2\,\partial_{x^\mu}A_\nu - \frac{1}{3}\,\mathrm{ric}_{\mu\nu}\right\} \\ &\overset{(18)}{=} (k-1)\left\{-\sigma_o + \sigma_2^{-1}\,\left(\left(-4\,(\partial_{x^\mu}A_\nu) + \frac{2}{3}\,\mathrm{ric}_{\mu\nu}\right)\xi^\mu\xi^\nu\right)\right. \\ &\quad \left. + \sigma_2^{-1}\left((-2k+4)\,(\partial_{x^\mu}A_\nu) + \frac{k-2}{3}\,\mathrm{ric}_{\mu\nu}\right)\,\xi^\mu\xi^\nu\right\} \\ &= (k-1)\,\left\{-\sigma_o + \sigma_2^{-1}\,\left(-2\,k\,(\partial_{x^\mu}A_\nu) + \frac{k}{3}\,\mathrm{ric}_{\mu\nu}\right)\,\xi^\mu\xi^\nu\right\}. \end{aligned}$$

Now we want to integrate $\mathrm{End}(E)$-valued $(0,2)$-tensors over the unit sphere in T^*M. For this we have the formula

$$\int_{S^{n-1}} d\xi\, T^{\mu\nu}\,\xi_\mu\xi_\nu = \frac{2\pi^{n/2}}{n\,\Gamma(n/2)}\,g_{\mu\nu}\,T^{\mu\nu}.$$

Note that $\frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the volume of $S^{n-1} \subset \mathbb{R}^n$. We get for every fixed p

$$\begin{aligned} \int_{S^{n-1}} d\xi\,\sigma_{-n}^{\Delta^{-(n/2)+1}} &= \int_{S^{n-1}} d\xi\, b_{n/2}(\xi) \\ &= \frac{2\,\pi^{n/2}\,\left(\frac{n}{2}-1\right)}{n\,\Gamma(n/2)}\,\left\{-n\,\sigma_0 - n\,(\delta^{\mu\nu}\,\partial_{x^\mu}A_\nu) + \frac{n}{6}\,\mathrm{scal}\right\}. \end{aligned}$$

From (16) we know that $\mathcal{K} = \sigma_0 + (\delta^{\mu\nu}\partial_\mu A_\nu)$, hence

$$\int_{S^{n-1}} d\xi\, b_{n/2}(\xi) = \frac{2\,\pi^{n/2}\,\left(\frac{n}{2}-1\right)}{\Gamma(n/2)}\,\left\{-\mathcal{K} + \frac{1}{6}\,\mathrm{scal}\right\}.$$

□

Remark 1 *It was noted by Ackermann [1] that Theorem 6.1 is a special case of a more general relationship between the Wodzicki residue of certain powers of an elliptic operator and the asymptotic expansion of the trace of the corresponding heat operator.*

References

[1] T. Ackermann, *A note on the Wodzicki residue*, J. Geom. Phys. **20** (1996), 404–406.

[2] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin Heidelberg, 1991.

[3] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg, 1987.

[4] A. Connes, *Noncommutative geometry*, Academic Press, San Diego-New York, 1994.

[5] W. Kalau and M. Walze, *Gravity, non-commutative geometry and the Wodzicki residue*, J. Geom. Phys. **16** (1995), 327–344.

[6] H. Karcher, *Riemannian comparison constructions*, Global Differential Geometry (Washington, DC) (S.S. Chern, ed.), Math. Assoc. America, 1989, pp. 170–222.

[7] D. Kastler, *The Dirac operator and gravitation*, Commun. Math. Phys. **166** (1995), 633–643.

[8] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, 1989.

[9] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963), 7–9.

[10] E. Schrödinger, *Diracsches Elektron im Schwerefeld I.*, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. (1932), 105–128.

[11] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), 143–178.