THE SPINORIAL τ-INVARIANT AND 0-DIMENSIONAL SURGERY.

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Abstract. Let M be a compact manifold with a metric g and with a fixed spin structure χ. Let $\lambda_1^+(g)$ be the first non-negative eigenvalue of the Dirac operator on (M, g, χ). We set

$$\tau(M, \chi) := \sup \inf \lambda_1^+(g)$$

where the infimum runs over all metrics g of volume 1 in a conformal class $[g_0]$ on M and where the supremum runs over all conformal classes $[g_0]$ on M.

Let $(M^\#, \chi^\#)$ be obtained from (M, χ) by 0-dimensional surgery. We prove that

$$\tau(M^\#, \chi^\#) \geq \tau(M, \chi).$$

As a corollary we can calculate $\tau(M, \chi)$ for any Riemann surface M.

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Contents

1. Introduction

We assume that (M, χ) is a compact spin manifold of dimension $n \geq 2$. We will always use the terminology "spin manifold" in the sense of an oriented manifold together with a chosen spin structure. The open ball around $x \in M$ with radius ε is denoted as $B_x(\varepsilon)$. We choose $p, q \in M$, $p \neq q$, and

$\varepsilon \in (0, \min\{d(p,q), \mathrm{injrad}(M)\}/2)$. Then we define

$$M^{\#} := (M \setminus (B_p(\varepsilon) \cup B_q(\varepsilon))) \cup ([-1,1] \times S^{n-1})/\sim$$

where $\sim$ indicates that we glue $\partial(M \setminus (B_p(\varepsilon) \cup B_q(\varepsilon))) \cong S^{n-1}\dot{\cup}S^{n-1}$ together with $\partial([-1,1] \times S^{n-1}) \cong S^{n-1}\dot{\cup}S^{n-1}$ such that orientations are preserved. $M^{\#}$ carries a differential structure and a spin structure such that $M \setminus (B_p(\varepsilon) \cup B_q(\varepsilon)) \hookrightarrow M^{\#}$ is smooth and preserves the spin structure. The spin structure on $M^{\#}$ is uniquely determined by the spin structure of M in the sense of Lemma 2.1. We say that *$M^{\#}$ is obtained from M by 0-dimensional surgery.* The connected sum construction is a special case of 0-dimensional surgery, namely the case when p and q are in different connected components. In some parts of the literature, 0-dimensional surgery is also called "adding a handle". However, we will only use the term "adding a handle" in the sense of cobordism theory (see subsection 2.3 and [Kos93] for details).

For any metric g on M let $\lambda_+^1(M, g, \chi)$ be the first non-negative eigenvalue of the Dirac operator on (M, g, χ). We set

$$\lambda_{\min}^+(M, [g], \chi) := \inf_{\tilde{g}\in[g]} \lambda_+^1(M, \tilde{g}, \chi)\mathrm{Vol}(M, \tilde{g})^{1/n}$$

and

$$\tau(M, \chi) := \sup_{[g]\in C(M)} \lambda_{\min}^+(M, [g], \chi) \in [0, (n/2)\omega_n^{1/n}]$$

where $[g]$ denotes the conformal class of g, where $C(M)$ is the set of conformal classes on M, and where ω_n denotes the volume of the standard sphere. It follows from [Amm03b] that $\tau(M, \chi) > 0$, if and only if there exists a metric on M with invertible Dirac operator. Recall that the Atiyah-Milnor-Singer-invariant (see e.g. [LM89, II.7] for details) associates to any n-dimensional spin manifold (M, χ) an element $\alpha(M, \chi) \in KO^{-n}(\mathrm{pt})$, where $KO^{-n}(\mathrm{pt}) \cong \mathbb{Z}$ if n is divisible by 4, where $KO^{-n}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$ if $n \equiv 1, 2 \mod 8$ and where $KO^{-n}(\mathrm{pt}) = \{0\}$ in all other dimensions. The map α defines a surjective ring homomorphism from the spin cobordism ring Ω_{spin}^* to $\bigoplus_{n\in\mathbb{N}} KO^{-n}(\mathrm{pt})$. In particular, $\alpha(M, \chi)$ is preserved under k-dimensional surgery on M, $k \in \{0, \ldots, n\}$.

If $\alpha(M, \chi) \neq 0$, then the Dirac operator has a nontrivial kernel for any metric, hence $\tau(M, \chi) = 0$. The converse statement, i.e. that $\alpha(M, \chi) = 0$ implies $\tau(M, \chi) > 0$ for connected M was proved in successive steps by [Hit74, Mai97, BD02, ADH06]. The essential step in [BD02] is that the positivity of $\tau(M, \chi)$ is preserved under k-dimensional surgery for $k \in \{0, 1, \ldots, n-3\}$ and in [ADH06] it is shown that positivity of $\tau(M, \chi)$ is also preserved under $(n-2)$-dimensional surgery.

The goal of the present article is to compare the τ-invariants of M and $M^{\#}$ where $M^{\#}$ is obtained from M by 0-dimensional surgery.

Theorem 1.1. *Let (M, g, χ) be a compact connected Riemannian spin manifold and let $(M^{\#}, \chi^{\#})$ be obtained by 0-dimensional surgery on (M, χ). We assume that the Dirac operator D acting on (M, g, χ) is invertible. Then, there exists a sequence of Riemannian metrics $(g_\varepsilon^{\#})_\varepsilon$ on $(M^{\#}, \chi^{\#})$ such that*

$$\lim_{\varepsilon\to 0} \lambda_{\min}^+(M^{\#}, [g_\varepsilon^{\#}], \chi^{\#}) = \lambda_{\min}^+(M, [g], \chi).$$

As an immediate corollary, we get

Corollary 1.2. *Let (M, χ) be a compact spin manifold and $(M^{\#}, \chi^{\#})$ be obtained by 0-dimensional surgery on (M, χ). Then*

$$\tau(M^{\#}, \chi^{\#}) \geq \tau(M, \chi).$$

Theorem 1.1 and its corollary were already known in the special case $M = S^2$ [AH03].

In the case $n = 2$ the corollary admits to calculate $\tau(M, \chi)$.

Theorem 1.3. *Let M be a compact oriented surface with spin structure χ.*

$$\tau(M, \chi) = \begin{cases} 0 & \text{if } \alpha(M, \chi) = 1 \\ 2\sqrt{\pi} & \text{if } \alpha(M, \chi) = 0 \end{cases}$$

For Riemann surfaces the α-invariant can be easily calculated: one calculates a quadratic form q_χ associated to the spin structure χ, then its Arf invariant $\mathrm{Arf}(q_\chi)$ is related to $\alpha(M,\chi)$ via $\mathrm{Arf}(q_\chi) = (-1)^{\alpha(M,\chi)}$ (see section 6 for details).

If one replaces the first non-negative eigenvalue by the absolute value of the largest non-positive eigenvalue $|\lambda_1^-(g)|$, then one obtains an invariant $\tau_-(M,\chi)$. Our results also hold for $\tau_-(M,\chi)$. In dimensions $n \not\equiv 3 \mod 4$ the spectrum of the Dirac operator is symmetric, and we have $\tau(M,\chi) = \tau_-(M,\chi)$. However, in the case $n \equiv 3 \mod 4$ these invariants are expected to be different.

Let us compare our results to various other results in the literature (see also [AH03]).

The τ-invariant is a spinorial analogue to the σ-constant on compact Riemannian manifolds [Kob87] (also [Sch89]) which is defined on (M,g) by

$$\sigma(M) := \sup \inf \frac{\int \mathrm{Scal}_{\tilde g}\, dv_{\tilde g}}{\mathrm{Vol}(M,\tilde g)^{\frac{n-2}{n}}} \tag{1}$$

where the infimum runs over all metrics in a conformal class $\tilde g \in [g]$, and where the supremum runs over all conformal classes. (In some parts of the literature $\sigma(M)$ is called the Yamabe invariant of M.) When $\sigma(M)$ is positive, the invariant $\sigma(M)$ can be defined also in a way analogous to $\tau(M)$ where we use the smallest eigenvalue of the conformal Laplacian $L_g := 4\,\frac{n-1}{n-2}\,\Delta_g + \mathrm{Scal}_g$ instead of $\lambda_+^1(g)$.

Unfortunately, there are only few examples for which $\sigma(M)$ is known and different from 0, e.g. $\sigma(S^n) = n(n-1)\omega_n^{2/n}$, $\sigma(S^{n-1}\times S^1) = n(n-1)\omega_n^{2/n}$, and $\sigma(\mathbb{R}P^3) = 6\left(\frac{\omega_3}{2}\right)^{2/3}$. The reader might consult [BN04] for a very elegant and amazing calculation of $\sigma(\mathbb{R}P^3)$. In a recent article, Akutagawa and Neves [AN05] could strengthen these methods which lead to a complete classification of compact 3-manifolds M with $\sigma(M) \geq \sigma(\mathbb{R}P^3)$. Gursky and LeBrun [GL98] could calculate the σ-constants of several Kähler surfaces using Seiberg-Witten theory or alternatively the Atiyah-Singer index theorem for twisted Dirac operators; in particular they obtain $\sigma(\mathbb{C}P^2\#m(S^1\times S^3)) = 12\sqrt{2}\pi$ for any $m \in \mathbb{N}$.

Kobayashi proved in [Kob87] that if $M^\#$ is obtained from M by 0-dimensional surgery, then $\sigma(M^\#) \geq \sigma(M)$. A similar monotonicity formula for the σ-invariant was proved by [PY99]. Petean and Yun prove $\sigma(M^\#) \geq \min\{\sigma(M),0\}$ if $M^\#$ is obtained from M by surgery of dimension $1,\ldots,n-3$. See also [AB03] for another approach to this result. Clearly, this surgery result is particularly interesting in the case $\sigma(M) \leq 0$, and it has many fruitful applications. In particular, any simply connected compact manifold of dimension at least 5 has $\sigma(M) \geq 0$ [Pet03]. This result was generalized to some other simple fundamental groups in [BR02]. It also allows to rule out Einstein metrics on many spaces [Pet98]. However, in the case $\sigma(M) > 0$ it is still open under what conditions one has a monotonicity formula $\sigma(M^\#) \geq \sigma(M)$. The article [Joy03] studies 0-dimensional surgery in more details, in particular he shows the non-uniqueness of minimizers of the infimum in (1) in the case $\sigma(M) > 0$. Some of these results have been recently generalized to the G-equivariant σ-invariant σ_G [Sung06].

The σ-invariant and the τ-invariant, are not only related by analogy, but also via Hijazi's inequality [Hij86, Hij91] that implies

$$\tau(M,\chi)^2 \geq \frac{n}{4(n-1)}\,\sigma(M). \tag{2}$$

If $M = S^n$ then equality is attained in this inequality. Upper bounds for $\tau(M,\chi)$ may help to determine the σ-constant.

The structure of the article is as follows. In Section 2 we introduce some notations and recall some preliminaries. The aim of Section 3 is to show that we can assume without loss of generality that g is flat on small neighborhoods of p and q. The metrics $g_\varepsilon^\#$ are constructed in Section 4, and we devote Section 5 to the proof of Theorem 1.1. In the last section, namely in Section 6, the calculation of the τ-invariant for any Riemann surface with spin structure is explained.

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2. Preliminaries

In this section we want to introduce some notations and recall some preliminaries. We refer to [LM89, Fr00, Hij99, BG92] for more details.

2.1. **Notation.** The round metric on S^n, i.e. the metric of sectional curvature 1, will always be denoted by g^n_{round}. We also abbreviate $\mathbb{S}^n$ for the Riemannian manifold $(S^n, g^n_{\mathrm{round}})$ equipped with the spin structure χ^n that arises as the boundary of the $(n+1)$-dimensional disk. The open ball of radius 1 around 0 in $\mathbb{R}^n$ is denoted by $B_{0,\mathbb{R}^n}(1)$.

2.2. **Topological spin structures versus metric spin structures.** The bundle $\mathrm{Gl}_+(M)$ of positively oriented frames over an oriented manifold M of dimension $n \geq 2$ is a $\mathrm{Gl}_+(n,\mathbb{R})$-principal bundle over M. The group $\mathrm{Gl}_+(n,\mathbb{R})$ has fundamental group $\mathbb{Z}$ if $n = 2$ and fundamental group $\mathbb{Z}/2\mathbb{Z}$ if $n \geq 3$. We denote the unique connected double cover of $\mathrm{Gl}_+(n,\mathbb{R})$ by $\widetilde{\mathrm{Gl}}_+(n,\mathbb{R})$. A *topological spin structure on M* consists of a $\widetilde{\mathrm{Gl}}_+(n,\mathbb{R})$-principal bundle $\widetilde{\mathrm{Gl}}_+(M)$ together with a $\widetilde{\mathrm{Gl}}_+(n,\mathbb{R}) \to \mathrm{Gl}_+(n,\mathbb{R})$ equivariant map $\chi : \widetilde{\mathrm{Gl}}_+(M) \to \mathrm{Gl}_+(M)$ over the identity $M \to M$. Two topological spin structures $\chi_1 : \widetilde{\mathrm{Gl}}_+(M)_1 \to \mathrm{Gl}_+(M)$ and $\chi_2 : \widetilde{\mathrm{Gl}}_+(M)_2 \to \mathrm{Gl}_+(M)$ are said to be *equivalent* if there is a $\widetilde{\mathrm{Gl}}_+(n,\mathbb{R})$-equivariant map $H : \widetilde{\mathrm{Gl}}_+(M)_1 \to \widetilde{\mathrm{Gl}}_+(M)_2$ with $\chi_1 = \chi_2 \circ H$. We denote a topological spin structure just by χ.

Note that $\mathrm{Spin}(n)$ is the preimage of $\mathrm{SO}(n)$ under the homomorphism $\widetilde{\mathrm{Gl}}_+(n,\mathbb{R}) \to \mathrm{Gl}_+(n,\mathbb{R})$. A $\mathrm{Spin}(n)$-principal bundle $\mathrm{Spin}(M,g)$ together with a $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ equivariant map $\widehat{\chi} : \mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)$ is called a *metric spin structure on* M. Two metric spin structures $\widehat{\chi}_1$ and $\widehat{\chi}_2$ are *equivalent* if there is a $\mathrm{Spin}(n)$-equivariant map H between the $\mathrm{Spin}(n)$-principal bundles such that $\widehat{\chi}_1 = \widehat{\chi}_2 \circ H$.

If M carries a metric g and if $\mathrm{SO}(M,g)$ denotes the bundle of g-orthonormal positively oriented frames, then the restriction of a topological spin structure χ to $\mathrm{Spin}(M,g) := \chi^{-1}(\mathrm{SO}(M,g))$ defines a metric spin structure on (M,g). This restriction yields a map from equivalence classes of topological spin structures on M to equivalence classes of metric spin structures on M, and one easily sees that this map is bijective. Working with topological spin structures has the advantage that it does not depend on a choice of metric. However, working with metric spin structures allows the definition of the spinor bundle.

Namely, the *spinor bundle* of a Riemannian manifold (M,g) with spin structure $\widehat{\chi}$ is defined as the associated bundle $\Sigma_{g,\widehat{\chi}}M := \mathrm{Spin}(M,g) \times_\sigma \Sigma_n$ where (σ, Σ_n) is the spinor representation of $\mathrm{Spin}(n)$. Sometimes we will just write $\Sigma_g M$ or ΣM when the spin structure or the metric is clear from the context.

As restriction is a bijection from equivalence classes of topological spin structures to equivalence classes of metric spin structures, we will identify topological and metric spin structures from now on and just call them *spin structures*.

2.3. **Surgery, handles and spin structures.** In the introduction we introduced 0-dimensional surgery. In order to understand the behavior of spin structures under surgery it is useful to see it as a bordism.

An $(n+1)$-dimensional (spin) manifold W with boundary $-M_1 \dot\cup M_2$, is called a (spin) bordism from M_1 to M_2 The manifold W carries an orientation and a spin structure, M_2 carries the induced orientation and spin structure, and M_1 carries the induced spin structure and the opposite of the induced orientation. This opposite orientation is indicated by the minus-sign before M_1. For example $W := M \times [0,1]$ is a spin bordism from M to M. If $M^\#$ is obtained from M by a k-dimensional surgery, then there is a bordism from M to $M^\#$. This bordism is obtained by a construction called *adding a k-handle to* $M \times [0,1]$. We will explain this construction in the case $k = 0$, see e.g. [Kos93, VI.6] for details and the general case.

We start with the manifold $W := M \times [0,1]$. Choose two points $p, q \in M$ and two diffeomorphisms $\varphi_p : \overline{B_{0,\mathbb{R}^n}(1)} \to \overline{B_p(\varepsilon)} \subset M$ and $\varphi_q : \overline{B_{0,\mathbb{R}^n}(1)} \to \overline{B_q(\varepsilon)} \subset M$. For each $x \in \overline{B_{0,\mathbb{R}^n}(1)}$ we identify

$$W \ni (\varphi_p(x), 1) \sim (x, -1) \in \overline{B_{0,\mathbb{R}^n}(1)} \times [-1,1] \qquad W \ni (\varphi_q(x), 1) \sim (x, 1) \in \overline{B_{0,\mathbb{R}^n}(1)} \times [-1,1].$$

The topological space

$$W^\# := W \cup (\overline{B_{0,\mathbb{R}^n}(1)} \times [-1,1])/\sim$$

carries naturally the differential structure of a manifold with corners, but by smoothing out the corners one finds a differential structure on W such that the boundary $\partial W^\# = (-M) \dot\cup M^\#$ is smooth. [Kos93, Chapter VI].

We assume that M comes with a fixed orientation and spin structure. Furthermore we assume that φ_p preserves orientation and that φ_q reverses orientation. Then $W^\#$ and $M^\#$ also carry natural orientations.

In order to define the spin structures, we equip W with the product spin structure, denoted by κ. The spin structure on W extends to a spin structure on $W^\#$. We choose the boundary spin structure on $M^\#$. If p and q are in different connected components of M, then the spin structure on $W^\#$ (resp. $M^\#$) is uniquely determined. However, if p and q are in the same connected component, then there are two non-equivalent choices of spin structures on $W^\#$ that extend the spin structure on W. These two spin structures are obtained from each other by a diffeomorphism as explained in the following lemma.

Lemma 2.1. *Let $n = \dim M = \dim W - 1 \geq 2$. If $\kappa_1^\#$ and $\kappa_2^\#$ are two spin structures on $W^\#$ such that the inclusions $(W, \kappa) \hookrightarrow (W^\#, \kappa_j^\#)$, $j = 1, 2$, are spin preserving embeddings, then there is a diffeomorphism $f : W^\# \to W^\#$ which is the identity on W and such that $f^*\kappa_1^\# = \kappa_2^\#$. Similarly, if $\chi_1^\#$ and $\chi_2^\#$ are two spin structures on $M^\#$ such that the inclusions $(M \setminus (B_p(\varepsilon) \cup B_q(\varepsilon)), \chi) \hookrightarrow (M^\#, \chi_j^\#)$, $j = 1, 2$, are spin preserving embeddings, then there is a diffeomorphism $f : M^\# \to M^\#$ which is the identity on $M \setminus (B_p(\varepsilon) \cup B_q(\varepsilon))$ and such that $f^*\chi_1^\# = \chi_2^\#$.*

For proving this lemma, we can assume that p and q are in the same connected component. Let $\kappa_1^\#$ and $\kappa_2^\#$ be the two disjoint extensions of the spin structure κ on W. Let $\gamma : [-1,1] \to \mathrm{SO}(n)$ be a closed loop in $\mathrm{SO}(n)$ generating $\pi_1(\mathrm{SO}(n))$. We can assume that $\gamma(t)$ is the unit in $\mathrm{SO}(n)$ for t close to -1 or 1. We define the diffeomorphism $f_0 : [-1,1] \times \overline{B_{0,\mathbb{R}^n}(1)} \to [-1,1] \times \overline{B_{0,\mathbb{R}^n}(1)}$, $f_0(t,x) := (t, \gamma(t)(x))$. By extending with the identity map on W (and the usual smoothing procedure at the balls where $[-1,1] \times \overline{B_{0,\mathbb{R}^n}(1)}$ is glued to W) one obtains a diffeomorphism $f : W^\# \to W^\#$. One checks that $f^*(\kappa_1^\#) = \kappa_2^\#$.

Similarly, the restriction of f to $M^\#$ yields a diffeomorphism of $M^\#$ to itself that pulls back $\chi_1^\#$ to $\chi_2^\#$.

Also note a particularity in the case $n = 2$: $S^1 \times [-1,1]$ carries 2 non-equivalent spin structures. However, it is clear from the construction above that the restriction of $\chi^\#$ to $S^1 \times [-1,1]$ is the product spin structure of the bounding spin structure on S^1 with the unique spin structure on $[-1,1]$.

Remark 2.2. Let $\chi_1^\#$ and $\chi_2^\#$ be two spin structures on $M^\#$ constructed from a spin structure χ on M. Let also $g^\#$ be a Riemannian metric on $M^\#$. Since $\chi_1^\#$ and $\chi_2^\#$ may be non-equivalent, the spectra of the corresponding Dirac operators (defined for the same metric $g^\#$) do not coincide in general. However, let $f : M^\# \to M^\#$ be the diffeomorphism given by Lemma 2.1 and let D_1 (resp. D_2) be the Dirac operator associated to $\chi_1^\#$ (resp. $\chi_2^\#$) in the metric $g^\#$ (resp. $f^*g^\#$). Lemma 2.1 then implies that D_1 and D_2 have the same spectrum. As a consequence, we get that $\tau(M^\#, \chi_1^\#) = \tau(M^\#, \chi_2^\#)$.

2.4. **Identifying spinors for different metrics.** Throughout the paper, we need to identify spinor fields for a manifold M with a fixed (topological) spin structure χ, but two different metrics g and h. In order to recall this identification we follow [BG92].

Let $x \in M$. Since the metrics g and h are symmetric and positive definite bilinear forms, there is a unique symmetric $b_h^g \in \mathrm{End}(T_xM)$ such that for all $v, w \in T_xM$,

$$h(b_h^g v, b_h^g w) = g(v, w).$$

Note that b^g_h depends smoothly on x. Hence, the map

$$\begin{array}{rcl} (b^g_h)^n : \mathrm{SO}(M,g) & \to & \mathrm{SO}(M,h) \\ (e_1,\cdots,e_n) & \mapsto & (b^g_h(e_1),\cdots,b^g_h(e_n)) \end{array}$$

is an isomorphism of $SO(n)$-principal bundles between the oriented frame bundles of (M,g) and (M,h). This map commutes with the right action of SO_n. Furthermore, $(b^g_h)^n$ can be lifted to

$$\begin{tikzcd} \mathrm{Spin}(M,g) \arrow[r, "\beta^g_h"] \arrow[d] & \mathrm{Spin}(M,h) \arrow[d] \\ \mathrm{SO}(M,g) \arrow[r, "(b^g_h)^n"] \arrow[d] & \mathrm{SO}(M,h) \arrow[d] \\ M \arrow[r, "\mathrm{Id}"] & M \end{tikzcd}$$

Hence, we obtain a map, also denoted by β^g_h, between the spinor bundles $\Sigma_g M$ and $\Sigma_h M$ in the following way:

$$\begin{array}{rcl} \Sigma_g M = \mathrm{Spin}(M,g)\times_\sigma \Sigma_n & \longrightarrow & \Sigma_h M = \mathrm{Spin}(M,h)\times_\sigma \Sigma_n \\ \psi = [s,\varphi] & \longmapsto & \beta^g_h\psi = [\beta^g_h(s),\varphi] \end{array} \tag{3}$$

where (σ,Σ_n) is the complex spinor representation, and $[s,\varphi]$ denotes the equivalence class of (s,φ) under the diagonal action of $\mathrm{Spin}(n)$. The identification β^g_h of spinors preserves the pointwise norm of spinors. Apparently, $\beta^g_h \circ \beta^h_g = \mathrm{id}$. However, in general, for three metrics g, k, h we have $\beta^h_g\circ\beta^k_h\circ\beta^g_k \neq \mathrm{id}$. From the construction of the bundle $\mathrm{Spin}(M,g)$ and the spinor bundle $\Sigma_g M$ it follows directly that there is a section $B^g_h : TM\otimes\Sigma_g M \to \Sigma_g M$

$$\beta^g_h\nabla^g_X\psi - \nabla^h_X\beta^g_h\psi = \beta^g_h(B^g_h(X\otimes\psi)) \tag{4}$$

The expression B^g_h only depends on g, h and their first derivatives. In particular, $\|B^g_h\|\to 0$ if h converges to g in the C^1-topology. The Dirac operators with respect to g and h are locally defined as $D_g = \sum_i e_i\cdot\nabla^g_{e_i}$ and $D_h = \sum_i b^g_h(e_i)\cdot\nabla^h_{b^g_h(e_i)}$ where $(e_1,\dots,e_n)\in\mathrm{SO}(M,g)$. It follows that

$$|\beta^g_h D_g\psi - D_h\beta^g_h\psi| \le C^g_h\,(|\psi| + |\nabla^g\psi|) \tag{5}$$

where $C^g_h\to 0$ if $h\to g$ in C^1.

In the special case when the metric h is conformal to g, i.e. when we have $h = f^2 g$ with f being a smooth positive function, it is shown in [Hit74, Hij86] that

$$D_h(f^{-\frac{n-1}{2}}\beta^g_h\psi) = f^{-\frac{n+1}{2}}\beta^g_h D_g\psi. \tag{6}$$

This equation implies, in particular, that the dimension of the kernel of the Dirac operator is constant on any conformal class.

2.5. **The first eigenvalue of the Dirac operator in a conformal class.** Let (M,g,χ) be a compact spin manifold of dimension n, $\ker D = \{0\}$. For $\psi\in\Gamma(\Sigma M)$ we define

$$J(\psi) = \frac{\left(\int_M |D\psi|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{n}}}{\int_M \langle D\psi,\psi\rangle\,dv_g}.$$

Using techniques from [Lot86], Ammann proved in [Amm03b] that

$$\lambda^+_{\min}(M,[g],\chi) = \inf_\psi J(\psi) \tag{7}$$

where the infimum is taken over the set of smooth spinor fields for which

$$\left(\int_M \langle D\psi,\psi\rangle\,dv_g\right) > 0.$$

We will need the following result:

Proposition 2.3. *Let (M, g, χ) be a compact spin manifolds of dimension $n \geq 2$. Then,*

$$\lambda^+_{\min}(M, [g], \chi) \leq \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\, \omega_n^{\frac{1}{n}} \tag{8}$$

where ω_n stands for the volume of the standard sphere $\mathbb{S}^n$.

The proposition was proven in [Amm03b] using geometric methods if $n \geq 3$. In the case $n = 2$ the article [Amm03b] only provides a proof if $\ker D = \{0\}$. Another method, that yields the proposition in full generality, is to construct for any $p \in M$ and $\varepsilon > 0$ a suitable test spinor field ψ_ε supported in $B_p(\varepsilon)$ that verifies

$$J(\psi_\varepsilon) \leq \lambda^+_{\min}(\mathbb{S}^n) + o(\varepsilon)$$

(see [AHM03A] for details).

If inequality (8) holds even strictly, then one can show that the infimum in equation (7) is attained, say in φ. Then the infimum in the definition of $\lambda^+_{\min}(M, [g], \chi)$ is attained in the generalized conformal metric $\tilde{g} := |\varphi|^{4/(n-1)} g$ (see [Amm03c] for details).

This fact, summarized in the following theorem will be a central ingredient in the proof of our main result Theorem 1.1.

Theorem 2.4 ([Amm03c, Amm03a])**.** *Assume that Inequality (8) is strict. Then there exists a spinor field $\varphi \in C^{2,\alpha}(\Sigma M) \cap C^\infty(\Sigma M \setminus \varphi^{-1}(0))$ on (M, g), $\alpha \in (0,1) \cap (0, 2/(n-1)]$ such that*

$$D(\varphi) = \lambda^+_{\min}(M, [g], \chi)\, |\varphi|^{\frac{2}{n-1}} \varphi, \text{ and } \|\, \varphi \,\|_{\frac{2n}{n-1}} = 1. \tag{9}$$

Furthermore, there is a generalized conformal metric $\tilde{g}$ (see [Amm03a] *for the definition) such that $\lambda^+_1(\tilde{g}) \mathrm{Vol}(M, \tilde{g})^{1/n} = \lambda^+_{\min}(M, [g], \chi)$.*

2.6. **A conformal Sobolev inequality.** In some following estimates we have to compare $\int_M |D\psi|^{\frac{2n}{n+1}}$ and $\int_M |\nabla\psi|^{\frac{2n}{n+1}}$. The pointwise identity

$$|D\psi| \leq \sqrt{n} |\nabla\psi|$$

implies

$$\frac{\int_M |D\psi|^{\frac{2n}{n+1}}\, dv_g}{\int_M |\nabla\psi|^{\frac{2n}{n+1}}} \leq n^{\frac{n}{n+1}}.$$

The fact that this fraction is also bounded from below is a standard result of elliptic regularity theory.

Proposition 2.5. *Let (M, g, χ) be a compact spin manifold of dimension $n \geq 2$. We assume that the Dirac operator D is invertible. Then for any $p \in (1, \infty)$ there exists a constant $C > 0$ such that for all spinor fields ψ of class C^1, we have :*

$$\int_M |\nabla\psi|^p\, dv_g \leq C \int_M |D\psi|^p\, dv_g. \tag{10}$$

Furthermore assume $p' \in (1, \infty)$, $1 - (n/p) \geq -(n/p')$. Then

$$\left(\int_M |\psi|^{p'}\, dv_g \right)^{1/p'} \leq C \left(\int_M |D\psi|^p\, dv_g \right)^{1/p}. \tag{11}$$

Proof. Inequality (10) is a classical elliptic inequality for invertible elliptic operators on compact manifolds. Recall (e.g. [LM89, Corollary III.5.7]) that any elliptic differential operator P on a compact manifold has a Green operator G_P i.e. a classical pseudo-differential operator G such that

$$G_P P = P G_P = \mathrm{Id} - \pi_{\ker P},$$

where $\pi_{\ker P}$ denotes L^2-orthogonal projection to the kernel of P. Applied to an invertible Dirac operator D this means that D^{-1} is a well-defined classical pseudodifferential operator of order -1.

We will now explain how to show that such an operator is a continuous map

$$D^{-1} : L^p(\Sigma M) \to L_1^p(\Sigma M), \qquad 1 < p < \infty,$$

where $L_1^p(\Sigma M)$ denotes the Sobolev space of spinors to the norm $\|\psi\|_{L_1^p} := \|\psi\|_{L^p} + \|\nabla\psi\|_{L^p}$. At first, we choose a finite system of coordinate charts $(\alpha_i : U_i \to V_i)$ covering M and a corresponding partition of unity (χ_i), i.e. smooth functions $\chi_i : M \to [0,1]$ with $K_i := \mathrm{supp}\chi_i \subset U_i$ and $\sum \chi_i = 1$. We can assume that the spinor bundle is trivialized on U_i via the bundle isomorphisms $A_i : \Sigma M|_{U_i} \to V_i \times \mathbb{C}^N =: \Sigma V_i$, where $N = 2^{[n/2]}$ is the complex rank of the spinor bundle. If φ is a spinor on M (or U_i), then $(A_i)_*\varphi := A_i \circ \varphi \circ \alpha_i^{-1}$ is spinor on V_i. As the symbol $\sigma(\chi_i D^{-1}) = \chi_i \sigma(D^{-1})$ has support in K_i, it is clear from the definition of pseudo-differential operators on compact manifolds (e.g. [LM89, III, Def. 3.15 and following statements]) that we can decompose

$$\chi_i D^{-1} = (A_i)_*^{-1} \circ P_i \circ (A_i)_* + S_i$$

where $P_i : \Gamma(\Sigma V_i) \to \Gamma(\Sigma V_i)$ is a pseudodifferential operator of order -1 and where $S_i : \Gamma(\Sigma M) \to \chi_i \Gamma(\Sigma M)$ is a smoothing operator. Obviously S_i is bounded from L^p to L_1^p. The operator $\nabla \circ P_i : \Gamma(\Sigma V_i) \to \Gamma(\mathbb{R}^n \otimes \Sigma V_i)$ is a pseudodifferential operator of order 0, and such an operator is a continuous operator from L^p to L^p [Tay81, Theorem XI.2.1] for all $p \in (1,\infty)$. It follows that $\chi_i D^{-1}$ and hence $D^{-1} = \sum \chi_i D^{-1}$ is continuous from L^p to L_1^p. This might be equivalently expressed as

$$\|\nabla D^{-1}\varphi\|_{L^p} + \|D^{-1}\varphi\|_{L^p} \le C\|\varphi\|_{L^p} \tag{12}$$

and this obviously implies (10) by substituting $\psi := D^{-1}\varphi$.

Inequality (11) then follows from (12) together with the classical bounded Sobolev empedding $L_1^p \to L^{p'}$.

Corollary 2.6. *Let (M, g, χ) be a compact Riemannian spin manifold with invertible Dirac operator. Let (Ω, h) be conformal to an open subset of (M, g). Then there is a constant $C = C(M, g, \chi)$ such that any compactly supported spinor field $\psi \in \Gamma(\Sigma(\Omega, h, \chi))$ of class C^1 satisfies*

$$\left(\int_\Omega |\psi|^{\frac{2n}{n-1}}\, dv_h\right)^{\frac{n-1}{n}} \le C\left(\int_\Omega |D_h\psi|^{\frac{2n}{n+1}}\, dv_h\right)^{\frac{n+1}{n}}.$$

Proof. Let f be the conformal factor, i.e. $h = f^2 g$. With the notation of subsection 2.4, we let $\varphi := f^{\frac{n-1}{2}} \beta_g^h \psi \in \Gamma(\Sigma(\Omega, g, \chi))$. As φ is compactly supported we can extend it with zero to a spinor of class C^1 on M. Since the map $\beta_g^h : \Sigma_h M \to \Sigma_g M$ is a pointwise isometry, and since $dv_h = f^n\, dv_g$ and by (6), we get that

$$\int_\Omega |\psi|^{\frac{2n}{n-1}}\, dv_h = \int_\Omega |\varphi|^{\frac{2n}{n-1}}\, dv_g$$

and

$$\int_\Omega |D_h\psi|^{\frac{2n}{n+1}}\, dv_h = \int_\Omega |D_g\varphi|^{\frac{2n}{n+1}}\, dv_g.$$

The result is then an immediate consequence of inequality (11) for $p := \frac{2n}{n+1}$ and $p' := \frac{2n}{n-1}$. □

Example 2.7. Let $M = \mathbb{S}^n$ be the sphere. Then the Mercator projection is a conformal embedding $F : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{S}^n$

$$F : \begin{pmatrix} t \\ x_1 \\ \cdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \tanh(t) \\ \frac{x_1}{\cosh(t)} \\ \cdots \\ \frac{x_n}{\cosh(t)} \end{pmatrix}.$$

The image of F is $\mathbb{S}^n$ with the North and South pole removed. Recall from the preliminaries that we always equip $\mathbb{S}^m$ with the spin structure χ^m that arises as restriction of the unique spin structure on $\overline{B_{0,\mathbb{R}^{m+1}}(1)}$ to the boundary. We also remind that on S^m, $m \ge 2$ this is the only spin structure, whereas there is a second spin structures on S^1. One easily verifies that $F^*(\chi^n)$ is the product structure of the unique spin structure on $\mathbb{R}$ and of χ^{n-1}.

Hence, any compactly supported spinor ψ on the cylinder $\mathbb{R}\times\mathbb{S}^{n-1}$ satisfies

$$\left(\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}}|\psi|^{\frac{2n}{n-1}}\,dv_{g^{n-1}_{\mathrm{round}}}\,dt\right)^{\frac{n-1}{n}}\le C\left(\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}}|D\psi|^{\frac{2n}{n+1}}\,dv_{g^{n-1}_{\mathrm{round}}}\,dt\right)^{\frac{n+1}{n}}. \tag{13}$$

3. Approximation by locally flat manifolds

The first step in the proof of Theorem 1.1 is to approximate (M,g) by metrics that are flat in a neighborhood of p and q, and then show that $\lambda^+_{\min}$ does not change much under this approximation. The results of this section will allow us to assume that g is flat in a neighborhood of p and q.

Lemma 3.1. *Let (M,g,χ) be a compact spin manifold of dimension $n\ge 2$. We assume that the Dirac operator D is invertible. Let $p,q\in M$. There is a family of Riemannian metrics $(g_\varepsilon)_{\varepsilon\in(0,\alpha)}$ on M, such that each g_ε is flat in a neighborhood of p and q and such that $g_\varepsilon\to g$ in C^1 when $\varepsilon\to 0$.*

Proof. Let $\eta:[0,\infty)\to[0,1]$ be a smooth function that equals to 1 on $[0,1]$ and whose support is contained in $[0,2)$. Choose a small number $\varepsilon>0$. On $M\setminus(B_p(2\varepsilon)\cup B_q(2\varepsilon))$ we define $g_\varepsilon=g$. In normal coordinates centered in p and defined on $B_p(2\varepsilon)$ we write $g(x)=\sum_{ij}g_{ij}(x)\,dx^i\,dx^j$. We define on $B_p(2\varepsilon)$

$$g_\varepsilon(x):=\sum_{ij}\Big(\eta(\varepsilon^{-1}|x|)\,\delta_{ij}+(1-\eta(\varepsilon^{-1}|x|))\,g_{ij}(x)\Big)\,dx^i\,dx^j,$$

and similarly on $B_q(2\varepsilon)$. Then one calculates that $g_\varepsilon\to g$ in C^1 when $\varepsilon\to 0$. □

Proposition 3.2. *Let (M,g,χ) be a compact spin manifold of dimension $n\ge 2$ with invertible Dirac operator D. If g_ε is a sequence of Riemannian metrics converging to g in C^1, then*

$$\lim_{\varepsilon\to 0}\lambda^+_{\min}(M,[g_\varepsilon],\chi)=\lambda^+_{\min}(M,[g],\chi).$$

Proof. Since D is invertible, and as the spectrum of D seen as of function of g depends continuously on g, it follows that the Dirac operator on (M,g_ε,χ), denoted by D_ε, is invertible as well. Let J (resp. J_ε) be the functional (see Paragraph 2.5) associated to $(M,[g],\chi)$ (resp. $(M,[g_\varepsilon],\chi)$). Let $\psi\in\Gamma(\Sigma_g M)$ be a smooth spinor field on (M,g,χ). We denote by D_ε the Dirac operator acting on (M,g_ε,χ). We can choose ψ such that $J(\psi)\le\lambda^+_{\min}(M,[g],\chi)+\delta$ where $\delta>0$ is small. By (5), (10) and(11) for $p'=2n/(n-1)$ and $p=2n/(n+1)$ one easily gets that

$$\limsup_{\varepsilon\to 0}\lambda^+_{\min}(M,[g_\varepsilon],\chi)\le\limsup_{\varepsilon\to 0}J_\varepsilon(\beta^g_{g_\varepsilon}\psi)=J(\psi)\le\lambda^+_{\min}(M,[g],\chi)+\delta.$$

Since $\delta>0$ is arbitrary, we obtain that

$$\lim_{\varepsilon\to 0}\lambda^+_{\min}(M,[g_\varepsilon],\chi)\le\lambda^+_{\min}(M,[g],\chi). \tag{14}$$

Now, let $\psi_\varepsilon\in\Gamma(\Sigma_g M)$ be a smooth spinor field on (M,g,χ) such that

$$J_\varepsilon(\beta^g_{g_\varepsilon}\psi_\varepsilon)\le\lambda^+_{\min}(M,[g_\varepsilon],\chi)+\delta.$$

In order to abbreviate, we set $\overline{\psi}_\varepsilon:=\beta^g_{g_\varepsilon}\psi_\varepsilon$. Without loss of generality, we can assume that

$$\int_M\langle D_\varepsilon\overline{\psi}_\varepsilon,\overline{\psi}_\varepsilon\rangle\,dv_{g_\varepsilon}=1.$$

Then, since $J_\varepsilon(\overline{\psi}_\varepsilon)$ is bounded (by Relation (8)), then $\int_M|D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_{g_\varepsilon}$ is also bounded. We have

$$\begin{aligned}\Big|\,\|D_\varepsilon\overline{\psi}_\varepsilon\|_{L^{\frac{2n}{n+1}}(M,g_\varepsilon)}-\|D\psi_\varepsilon\|_{L^{\frac{2n}{n+1}}(M,g)}\Big|&\le\Big|\,\|D_\varepsilon\overline{\psi}_\varepsilon\|_{L^{\frac{2n}{n+1}}(M,g_\varepsilon)}-\|D_\varepsilon\overline{\psi}_\varepsilon\|_{L^{\frac{2n}{n+1}}(M,g)}\Big|\\&\quad+\Big|\,\|D_\varepsilon\overline{\psi}_\varepsilon\|_{L^{\frac{2n}{n+1}}(M,g)}-\|D\psi_\varepsilon\|_{L^{\frac{2n}{n+1}}(M,g)}\Big|.\end{aligned} \tag{15}$$

By definition of g_ε, it is clear that $(1-o(\varepsilon))\,dv_g \le dv_{g_\varepsilon} \le (1+o(\varepsilon))\,dv_g$, and hence for $\varepsilon \to 0$

$$\left|\left(\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_{g_\varepsilon}\right)^{\frac{n+1}{2n}} - \left(\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}}\right| \le o(\varepsilon)\underbrace{\left(\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_{g_\varepsilon}\right)^{\frac{n+1}{2n}}}_{J_\varepsilon(\overline{\psi}_\varepsilon)} \to 0.$$

In addition, we get from relation (5) that

$$\left|\left(\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}} - \left(\int_M |D\psi_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}}\right| \le a_\varepsilon\left(\int (|\psi_\varepsilon| + |\nabla_g\psi_\varepsilon|)|^{\frac{2n}{n+1}}dv_g\right)^{\frac{n+1}{2n}}.$$

where $\lim_{\varepsilon\to 0} a_\varepsilon = 0$. Using the triangle inequality and Hölder's inequality, we see the existence of a constant $C > 0$ independent of ε such that

$$\left|\left(\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}} - \left(\int_M |D\psi_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}}\right| \le Ca_\varepsilon\left(\left(\int |\psi_\varepsilon|^{\frac{2n}{n-1}}dv_g\right)^{\frac{n-1}{2n}} + \left(\int |\nabla_g\psi_\varepsilon|^{\frac{2n}{n+1}}dv_g\right)^{\frac{n+1}{2n}}\right).$$

By inequalities (10) and (11), we obtain that

$$\left|\left(\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}} - \left(\int_M |D\psi_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{2n}}\right| \le Ca_\varepsilon\left(\int |\nabla_g\psi_\varepsilon|^{\frac{2n}{n+1}}dv_g\right)^{\frac{n+1}{2n}}.$$

Coming back to (15), this clearly implies that $\int_M |D\psi_\varepsilon|^{\frac{2n}{n+1}}\,dv_g$ is bounded and that $\int_M |\psi_\varepsilon|^{\frac{2n}{n-1}}\,dv_g$ is bounded, too. Since $\lim_{\varepsilon\to 0} a_\varepsilon = 0$, this also implies that

$$\lim_{\varepsilon\to 0}\left|\int_M |D_\varepsilon\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}\,dv_{g_\varepsilon} - \int_M |D\psi_\varepsilon|^{\frac{2n}{n+1}}\,dv_g\right| = 0. \tag{16}$$

Similarly, using relation (5) we have

$$\lim_{\varepsilon\to 0}\left|\int_M \langle D_\varepsilon\overline{\psi}_\varepsilon, \overline{\psi}_\varepsilon\rangle\,dv_{g_\varepsilon} - \int_M \langle D\psi_\varepsilon, \psi_\varepsilon\rangle\,dv_g\right| = 0. \tag{17}$$

Relations (16) and (17) imply that

$$\lambda^+_{\min}(M,[g_\varepsilon],\chi) + \delta \ge J_\varepsilon(\overline{\psi}_\varepsilon) \ge J(\psi_\varepsilon) + o(1) \ge \lambda^+_{\min}(M,[g],\chi) + o(1).$$

Since δ is arbitrary, we get that

$$\liminf_{\varepsilon\to 0}\lambda^+_{\min}(M,[g_\varepsilon],\chi) \ge \lambda^+_{\min}(M,[g],\chi). \tag{18}$$

Together with (14), this proves Theorem 3.2.

4. Construction of the metrics on $M^\#$

The aim of this section is to construct the sequence of metrics $g^\#_\varepsilon$ of Theorem 1.1. Using the result of the previous section, *we can assume from now on and in the rest of the article that the metric g is flat on $B_p(\rho)$ and $B_q(\rho)$ for a small $\rho > 0$.* For $0 < \alpha < \beta < \rho$ we introduce the notation

$$B_{p,q}(\alpha) = B_p(\alpha) \cup B_q(\alpha) \text{ and } C_{p,q}(\alpha,\beta) = B_{p,q}(\beta) \setminus B_{p,q}(\alpha).$$

Let $\varepsilon > 0$ be small. We explained in the introduction that $M^\#$ is obtained (as a topological space) by gluing a cylinder $[-1,1]\times S^{n-1}$ with $M \setminus B_{p,q}(\varepsilon)$ along $\partial B_p(\varepsilon)$ on one side and along $\partial B_q(\varepsilon)$ on the other side. Evidently this can also be expressed by saying

$$M^\# = M \setminus B_{p,q}(\varepsilon)/\sim$$

where $\sim$ indicates that we glue $\partial B_p(\varepsilon)$ with $\partial B_q(\varepsilon)$ via an orientation preserving diffeomorphism. $M^\#$ is equipped with a differential structure and a spin structure such that $I_\varepsilon : M \setminus B_{p,q}(\varepsilon) \to M^\#$ is smooth and compatible with the spin structures. We also introduce for all $a \in (\varepsilon, \rho)$ the notation

$$H(a,\varepsilon) := I_\varepsilon(C_{p,q}(\varepsilon,a)) = C_p(\varepsilon,a) \cup C_q(\varepsilon,a)/\sim.$$

Hence $M^\#$ is the disjoint union of $M \setminus B_{p,q}(a)$ and $H(a,\varepsilon)$.

Let us denote by d_p (resp. d_q) the distance in (M,g) to the point p (resp. q). We define a function f_ε on $(M \setminus B_{p,q}(3\varepsilon)) \cup C_{p,q}(\varepsilon, 2\varepsilon)$ by setting

$$f_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in M \setminus B_{p,q}(3\varepsilon) \\ d_p^{-1} & \text{if} \quad x \in C_p(\varepsilon, 2\varepsilon) \\ d_q^{-1} & \text{if} \quad x \in C_q(\varepsilon, 2\varepsilon). \end{cases}$$

We can extend f_ε smoothly and positively to $M \setminus B_{p,q}(\varepsilon)$ such that f_ε satisfies on $C_{p,q}(2\varepsilon, 3\varepsilon)$:

$$|df_\varepsilon| \leq \frac{2}{\varepsilon}.$$

As $(C_p(\varepsilon,2\varepsilon), f_\varepsilon^2 g)$ and $(C_q(\varepsilon,2\varepsilon), f_\varepsilon^2 g)$ are isometric to $([0,\log 2) \times S^{n-1}, dt^2 + g_{\mathrm{round}}^{n-1})$, $f_\varepsilon^2 g$ defines a metric $g_\varepsilon^\#$ on $M^\#$, or more precisely: there is a unique metric $g_\varepsilon^\#$ on $M^\#$ such that

$$f_\varepsilon^2 g = I_\varepsilon^* g_\varepsilon^\# \qquad \text{holds on } M \setminus B_{p,q}(\varepsilon).$$

Note that $(H(a,\varepsilon), g_\varepsilon^\#)$ is then conformal (but in general not isometric) to $((-\log(a/\varepsilon), \log(a/\varepsilon)) \times S^{n-1}, dt^2 + g_{\mathrm{round}}^{n-1})$.

5. Proof of Theorem 1.1

In this section we will prove that the metrics $(g_\varepsilon^\#)_\varepsilon$ are the desired metrics. We denote by D_ε the Dirac operator acting on $(M^\#, g_\varepsilon^\#, \chi^\#)$. We set $\lambda = \lambda_{\min}^+(M,[g],\chi)$ and $\lambda_\varepsilon = \lambda_{\min}^+(M^\#, [g_\varepsilon^\#], \chi^\#)$. We denote by J (resp. J_ε) the functional associated to λ (resp. λ_ε) (see Paragraph 2.5).

Lemma 5.1.

$$\limsup_{\varepsilon \to 0} \lambda_\varepsilon \leq \lambda. \tag{19}$$

Proof. Let $\gamma > 0$ be small and let ψ be a smooth spinor field such that $J(\psi) \leq \lambda + \gamma$. Clearly, for each small number $\alpha > 0$, one can construct a cut-off function $\eta_\alpha \in C^\infty(M)$, such that $0 \leq \eta_\alpha \leq 1$, equal to 1 on $M \setminus B_{p,q}(2\alpha)$, equal to 0 on $B_{p,q}(\alpha)$ and which satisfies $|d\eta_\alpha| \leq \frac{2}{\alpha}$. As easily seen, we have $\lim_{\alpha \to 0} J(\eta_\alpha \psi) = J(\psi)$. We choose α small enough such that $J(\eta_\alpha\psi) \leq \lambda + 2\gamma$. Now, if $3\varepsilon < \alpha$, then $\eta_\alpha\psi$ is supported on the common part of (M,g,χ) and $(M^\#, g_\varepsilon^\#, \chi^\#)$ and hence can be seen as a spinor field on $(M^\#, g_\varepsilon^\#, \chi^\#)$. We have

$$\lambda_\varepsilon \leq J_\varepsilon(\eta_\alpha\psi) \leq \lambda + 2\gamma$$

for all small $\varepsilon > 0$. Since γ is arbitrary, we proved the lemma. □

It remains to prove that

$$\liminf_{\varepsilon \to 0} \lambda_\varepsilon \geq \lambda. \tag{20}$$

This inequality is more involved than Equation (19) and will occupy the rest of this section.

We set $\lambda_0 := \liminf_{\varepsilon \to 0} \lambda_\varepsilon$ and pass to a sequence of $\varepsilon_i \to 0$ with $\lim_{i \to \infty} \lambda_{\varepsilon_i} = \lambda_0$. To simplify notation we write $\lambda_i := \lambda_{\varepsilon_i}$, $D_i := D_{\varepsilon_i}$, $g_i^\# := g_{\varepsilon_i}^\#$, $dv_i := dv_{g_{\varepsilon_i}^\#}$, and so on. In the following arguments we will frequently pass to subsequences. Slightly abusing the notation, we will continue with the same index notation λ_i, D_i, $g_i^\#$, and so on.

In the case $\lambda_0 = \lambda_{\min}^+(\mathbb{S}^n)$ Equation (20) follows directly from (8). Hence, after possibly passing to a subsequence, we can assume that $\lambda_i < \lambda_{\min}^+(\mathbb{S}^n)$ for all $i \in \mathbb{N}$. By Theorem 2.4, there exists a spinor field ψ_i of class C^1 defined on $(M^\#, g_i^\#, \chi^\#)$ that satisfies

$$D_i\psi_i = \lambda_i |\psi_i|^{\frac{2}{n-1}} \psi_i \tag{21}$$

with the normalization

$$\int_{M^\#} |\psi_i|^{\frac{2n}{n-1}} \, dv_i = 1. \tag{22}$$

Lemma 5.2. *For any $\delta > 0$ and $a_0 > 0$ there is an $a \in (0, a_0)$ such that after possibly passing to a subsequence*

$$\liminf_{i\to\infty} \int_{C_{p,q}(a,2a)} |\psi_i|^{\frac{2n}{n-1}}\, dv_g < \delta \tag{23}$$

and such that g is flat on $C_{p,q}(a, 2a)$.

Note that the integral above has a meaning for $\varepsilon_i \le a$, since $C_{p,q}(a, 2a) \subset M^{\#}$. If in addition, $3\varepsilon_i < a$, we have $g_i = g$ on $C_{p,q}(a, 2a)$. For the proof of Lemma 5.2 we proceed by contradiction. Let N_0 be such that g is flat on $B_{p,q}(2^{-N_0})$. If (23) is false, then for any $N > N_0$ we can find a sequence $i_j \to 0$ such that for all $k \in \{N_0, \cdots, N\}$ we have

$$\lim_{i\to\infty} \int_{C_{p,q}(2^{-(k+1)},2^{-k})} |\psi_{i_j}|^{\frac{2n}{n-1}}\, dv_g \ge \delta.$$

Since

$$\int_{M^{\#}} |\psi_i|^{\frac{2n}{n-1}}\, dv_i = 1$$

and since the cylinders $C_{p,q}(2^{-(k+1)}, 2^{-k})$, $k \in \{N_0, \ldots, N\}$ are disjoint, we obtain a contradiction if $N - N_0 > \frac{1}{\delta}$ for $a = 2^{-(N+1)}$. This proves Relation (23).

We fix $a \in (0, a_0)$ for which Relation (23) is verified. After possibly extracting a subsequence of (ψ_i), we can assume that the limit infimum in (23) is a limit. In other words, we have found a number a such that

$$\int_{C_{p,q}(a,2a)} |\psi_i|^{\frac{2n}{n-1}}\, dv_g < \delta \tag{24}$$

for all i in a suitable subsequence.

Since

$$\int_{M\setminus B_{p,q}(2a)} |\psi_i|^{\frac{2n}{n-1}}\, dv_g + \int_{C_{p,q}(a,2a)} |\psi_i|^{\frac{2n}{n-1}}\, dv_g + \int_{H(a,\varepsilon)} |\psi_i|^{\frac{2n}{n-1}}\, dv_i = 1$$

since for ε small

$$\int_{C_{p,q}(a,2a)} |\psi_i|^{\frac{2n}{n-1}}\, dv_g \le \delta$$

and since δ can be chosen lower than $\frac{1}{3}$, we have:

$$\limsup_{i\to\infty} \int_{M\setminus B_{p,q}(2a)} |\psi_i|^{\frac{2n}{n-1}}\, dv_i \ge \frac{1}{3} \tag{25}$$

or

$$\limsup_{i\to\infty} \int_{H(a,\varepsilon_i)} |\psi_i|^{\frac{2n}{n-1}}\, dv_i \ge \frac{1}{3}. \tag{26}$$

where $H(a, \varepsilon_i)$ is defined as in section 4.

The theorem now follows from the following two lemmata.

Lemma 5.3. *Relation (25) implies $\lambda_0 \ge \lambda$.*

Lemma 5.4. *Relation (26) implies $\lambda_0 \ge \lambda^{+}_{\min}(\mathbb{S}^n)$.*

Proof of Lemma 5.4.
We suppose that Inequality (26) holds.

Step 1. $\{i \in \mathbb{N} \,|\, \lambda_i = 0\}$ **is finite.**

We prove this step by contradiction and assume that after passing to a subsequence $\lambda_i = 0$ for all i.

This means that the spinors ψ_i are harmonic on $(M^{\#}, g_i^{\#})$.

By Example 2.7, the manifold $(H(2a,\varepsilon), g_i^{\#})$ is conformal to a subdomain of the sphere $\mathbb{S}^n$. By Proposition 2.5 there exists a constant C which does not depend on i and δ such that for all spinor field θ of class C^1 and whose support is included in $H(2a,\varepsilon_i)$, we have

$$\left(\int_{M^{\#}} |\theta|^{\frac{2n}{n-1}}\, dv_i\right)^{\frac{n-1}{n}} \leq C\left(\int_{M^{\#}} |D_i\theta|^{\frac{2n}{n+1}}\, dv_i\right)^{\frac{n+1}{n}}. \tag{27}$$

Let $\eta \in C^\infty(M^{\#})$, $0 \leq \eta \leq 1$ be a cut-off function equal to 1 on $H(a,\varepsilon_i)$ supported in $H(2a,\varepsilon_i)$, and which satisfies $|d\eta| \leq \frac{2}{a}$. Then we apply Inequality (27) with $\theta = \eta\psi_i$. Since ψ_i is harmonic and by (26), we get that

$$3^{-\frac{n-1}{n}} \leq C\left(\int_{M^{\#}} |(\mathrm{grad}\ \eta)\cdot\psi_i|^{\frac{2n}{n+1}}\, dv_i\right)^{\frac{n+1}{n}}.$$

Using the Hölder inequality, we get that

$$3^{-\frac{n-1}{n}} \leq C\left(\int_{M^{\#}} |d\eta|^n\, dv_i\right)^{\frac{2}{n}}\left(\int_{Supp(d\eta)} |\psi_i|^{\frac{2n}{n-1}}\, dv_i\right)^{\frac{n-1}{n}}.$$

Since $|d\eta| \leq \frac{2}{a}$, since $d\eta$ is supported in $C_{p,q}(a,2a)$ whose volume for g is bounded by Ca^n and since g_i is constant equal to g on $C_{p,q}(a,2a)$, there exists a constant C' independent of i and δ such that

$$\int_M |d\eta|^n\, dv_i \leq C'. \tag{28}$$

Using (24), we get that

$$\int_{Supp(d\eta)} |\psi_i|^{\frac{2n}{n-1}}\, dv_i \leq \delta.$$

We obtain

$$\frac{1}{3^{\frac{n-1}{n}}} \leq CC'^{\frac{2}{n}}\delta^{\frac{n-1}{n}}.$$

If δ is small enough, we get a contradiction. This proves Relation step 1.

Step 2. Proof of the inequality

$$\lim_{i\to\infty} \lambda_i \geq \lambda^+_{\mathrm{min}}(\mathbb{S}^n). \tag{29}$$

Let η be the same function as above. Since the manifold $(H(2a,\varepsilon), g_i^{\#})$ is conformal to a subdomain of the sphere $\mathbb{S}^n$ and since $\eta\psi_i$ is supported in $H(2a,\varepsilon_i)$, we have

$$\lambda^+_{\mathrm{min}}(\mathbb{S}^n) \leq J(\eta\psi_i). \tag{30}$$

By Equation (21) we have

$$\int_{M^{\#}} \langle \eta D_i\psi_i, \eta\psi_i\rangle\, dv_i = \lambda_i \int_{M^{\#}} \eta^2|\psi_i|^{\frac{2n}{n-1}}\, dv_i \in \mathbb{R}. \tag{31}$$

On the other hand

$$\underbrace{\int_{M^{\#}} \langle D_i(\eta\psi_i), \eta\psi_i\rangle\, dv_i}_{\in\mathbb{R}} = \underbrace{\int_{M^{\#}} \langle \mathrm{grad}\ \eta\cdot\psi_i, \eta\psi_i\rangle\, dv_i}_{\in i\mathbb{R}} + \underbrace{\int_{M^{\#}} \eta^2\langle D_i\psi_i, \psi_i\rangle\, dv_i}_{\in\mathbb{R}},$$

implies that we can replace the left hand side of (31) by $\int_{M^{\#}} \langle D_i(\eta\psi_i), \eta\psi_i\rangle$. Since $\eta \equiv 1$ on $H(a,\varepsilon_i)$, we obtain that

$$\int_{M^{\#}} \langle D_i(\eta\psi_i), \eta\psi_i\rangle\, dv_i \geq \lambda_i \int_{H(a,\varepsilon_i)} |\psi_i|^{\frac{2n}{n-1}}\, dv_i. \tag{32}$$

We also have by Equation (21)

$$\left(\int_{M^\#}|D(\eta\psi_i)|^{\frac{2n}{n+1}}\,dv_i\right)^{\frac{n+1}{n}}=\left(\int_{M^\#}\left|\mathrm{grad}\,\eta\cdot\psi_i+\eta\,\lambda_i|\psi_i|^{\frac{2}{n-1}}\psi_i\right|^{\frac{2n}{n+1}}\,dv_i\right)^{\frac{n+1}{n}}.$$

Again since $\eta\equiv 1$ on $H(a,\varepsilon_i)$, we can write that

$$\int_{M^\#}\left|\mathrm{grad}\,\eta\cdot\psi_i+\eta\,\lambda_i|\psi_i|^{\frac{2}{n-1}}\psi_i\right|^{\frac{2n}{n+1}}\,dv_i\le$$
$$\int_{C_{p,q}(a,2a)}\left|\mathrm{grad}\,\eta\cdot\psi_i+\eta\,\lambda_i|\psi_i|^{\frac{2}{n-1}}\psi_i\right|^{\frac{2n}{n+1}}\,dv_i+\lambda_i^{\frac{2n}{n+1}}\int_{H(a,\varepsilon_i)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i.$$

Using the fact that for $s,t\in\mathbb{R}$, we have $|s+t|^{\frac{2n}{n+1}}\le|2s|^{\frac{2n}{n+1}}+|2t|^{\frac{2n}{n+1}}$ and using Hölder inequality, we get that

$$\begin{aligned}&\int_{C_{p,q}(a,2a)}\left|\mathrm{grad}\,\eta\cdot\psi_i+\eta\,\lambda_i|\psi_i|^{\frac{2}{n-1}}\psi_i\right|^{\frac{2n}{n+1}}\,dv_i\\
&\le 2^{\frac{2n}{n+1}}\int_{C_{p,q}(a,2a)}|d\eta|^{\frac{2n}{n+1}}|\psi_i|^{\frac{2n}{n+1}}\,dv_i+2^{\frac{2n}{n+1}}\lambda_i^{\frac{2n}{n+1}}\int_{C_{p,q}(a,2a)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i\\
&\le 2^{\frac{2n}{n+1}}\left(\int_{C_{p,q}(a,2a)}|d\eta|^n\,dv_i\right)^{\frac{2}{n+1}}\left(\int_{C_{p,q}(a,2a)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i\right)^{\frac{n+1}{n-1}}+2^{\frac{2n}{n+1}}\lambda_i^{\frac{2n}{n+1}}\delta.\end{aligned}$$

Using again relation (28), we get that

$$\int_{C_{p,q}(a,2a)}\left|\mathrm{grad}\,\eta\cdot\psi_i+\eta\,\lambda_i|\psi_i|^{\frac{2}{n-1}}\psi_i\right|^{\frac{2n}{n+1}}\,dv_i\le C(\delta^{\frac{n+1}{n-1}}+\delta)$$

where C is a constant independent of i and δ. Finally, we obtain that

$$\left(\int_{M^\#}|D\eta\psi_i|^{\frac{2n}{n+1}}\,dv_i\right)^{\frac{n+1}{n}}\le\left(C(\delta^{\frac{n+1}{n-1}}+\delta)+\lambda_i^{\frac{2n}{n+1}}\int_{H(a,\varepsilon_i)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i\right)^{\frac{n+1}{n}}. \tag{33}$$

Plugging (33) and (32) into (30), we get that

$$\lambda^+_{\min}(\mathbb{S}^n)\le\frac{\left(C(\delta^{\frac{n+1}{n-1}}+\delta)+\lambda_i^{\frac{2n}{n+1}}\int_{H(a,\varepsilon_i)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i\right)^{\frac{n+1}{n}}}{\lambda_i\int_{H(a,\varepsilon_i)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i}.$$

Since

$$\int_{H(a,\varepsilon_i)}|\psi_i|^{\frac{2n}{n-1}}\,dv_i\ge\frac{1}{3}$$

and letting δ go to 0, we obtain Relation (29). Since by (8), we have $\lambda\le\lambda^+_{\min}(\mathbb{S}^n)$, we get (20). This proves the lemma.

Proof of Lemma 5.3. We just sketch the proof of Lemma 5.3 because the method is the same as the proof of Lemma 5.4, At first, we prove step 1. We proceed by contradiction and assume that $\lambda_i=0$ for infinitely many $i\in\mathbb{N}$. This means that the spinors ψ_i are harmonic on $(M^\#,g_i^\#)$. By Proposition 2.5, there exists $C>0$ such that for all spinor θ of class C^1 defined on M, we have

$$\left(\int_M|\theta|^{\frac{2n}{n-1}}\,dv_{g_i^\#}\right)^{\frac{n-1}{n}}\le C\left(\int_M|D\theta|^{\frac{2n}{n+1}}\,dv_g\right)^{\frac{n+1}{n}}.$$

We apply this inequality with $\theta=(1-\eta)\psi_i$ where η is defined as in the proof of Lemma 5.4 and we obtain a contradiction if δ is small enough. This proves step 1. Then, we say that

$$\lambda\le J((1-\eta)\psi_i).$$

As in the proof of Lemma 5.4, we obtain (20) in the limit $\delta \to 0$. This proves Theorem 1.1.

6. The Dirac operator on Riemann surfaces

In this section we want to prove Theorem 1.3. In fact, we use Theorem 1.1 to calculate the τ-invariant for all compact connected Riemann surfaces equipped with spin structures.

As already mentioned in the introduction, the Atiyah-Milnor-Singer invariant α is a ring homomorphisms from the ring of spin-cobordism classes into $KO^{-*}(\mathrm{pt})$. It allows to define an index theorem for the Dirac operator that is non-trivial in dimensions $0, 1, 2, 4 \mod 8$ (see e.g. [LM89, II.7]).

Let us to recall an equivalent definition of the α-invariant in the special case $n = 2$. In this case $\alpha(M) \in KO^{-2}\{\mathrm{pt}\} = \mathbb{Z}_2 := \mathbb{Z}/(2\mathbb{Z}) = \{0, 1\}$. Some more details can also be found in [KS96] and [AB02, Section 2 and 3]. We will also recall the index theorem in dimension 2, Theorem 6.3.

A quadratic form is a map $q : H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ such that

$$q(a + b) = q(a) + q(b) + a \cap b$$

holds for all $a, b \in H_1(M, \mathbb{Z}_2)$. Here $\cap$ denotes the intersection form $H_1(M, \mathbb{Z}_2) \times H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ which is a non-degenerate (anti-)symmetric bilinear form on $H_1(M, \mathbb{Z}_2)$.

The difference of two such quadratic forms is a linear map $H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$, and vice versa if one adds a linear map $H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ to a quadratic form, one easily sees that one obtains a quadratic form again. The space of quadratic forms on $H_1(M, \mathbb{Z}_2)$ is an affine spaces modelled on the space $H^1(M, \mathbb{Z}_2) := \mathrm{Hom}(H_1(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z}_2)$.

We will now associate to any spin structure χ on a Riemann surface M a quadratic form $q_\chi : H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$. This association will define a bijection from the set of equivalence classes of spin structures to the set of quadratic forms.

For simplicity of notation, we fix a Riemannian metric g on M. Let $\mathrm{SO}(M, g)$ denote the S^1-principal bundle of positively oriented orthonormal frames on M. If one specializes the description of a spin structure (Subsection 2.2) to dimension 2, then a spin structure χ consists of a principal S^1-bundle $\mathrm{Spin}(M, g, \chi) \to M$ and a double covering $\mu_\chi : \mathrm{Spin}(M, g) \to \mathrm{SO}(M, g)$ with $\mu_\chi(\varphi \cdot z) = \mu_\chi(\varphi) \cdot z^2$ for all $\varphi \in \mathrm{Spin}(M, g)$ and all $z \in S^1$.

Any homology class in $H_1(M, \mathbb{Z}_2)$ can be represented by a closed embedded loop $\gamma : \mathbb{R}/(L\mathbb{Z}) \to M$, $L > 0$, parameterized by arclength. If $\dot\gamma : \mathbb{R}/(L\mathbb{Z}) \to \mathrm{SO}(M, g)$ lifts to a map $\mathbb{R}/(L\mathbb{Z}) \to \mathrm{Spin}(M, g)$, then we define $q_\chi(\gamma) := 1$, otherwise we define $q_\chi(\gamma) := 0$. One checks that if γ_1 and γ_2 represent the same homology class in $H_1(M, \mathbb{Z}_2)$, then $q_\chi(\gamma_1) = q_\chi(\gamma_2)$, hence q_χ defines a map $q_\chi : H_1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$. One checks that this map is in fact a quadratic map.

For any quadratic map $q : V \to \mathbb{Z}_2$ associated to a non-degenerate symmetric bilinear form on a finite-dimensional $\mathbb{Z}_2$-vector space V one defines one defines the Arf-invariant

$$\mathrm{Arf}(q) := \frac{1}{\sqrt{\#V}} \sum_{a \in V} (-1)^{q_\chi(a)}.$$

One can check (e.g. [AB02, Section 3]) that the sum is either $+1$ or -1. We now define $\alpha(M, \chi)$ via $(-1)^{\alpha(M,\chi)} = \mathrm{Arf}(q_\chi)$.

Example 6.1. Let $M = S^2$ with the spin structure χ^2. Then $q_{\chi^2} : \{0\} \to \mathbb{Z}_2$, $q_{\chi^2}(0) = 0$, $\alpha(S^2, \chi^2) = 0$.

Example 6.2. Let M be of genus 1, i.e. diffeomorphic to T^2, with a metric g. After performing a conformal change (which does not affect neither the quadratic form nor the dimension of the kernel) we can assume that g is flat, i.e. (M, g) is isometric to $\mathbb{R}^2/\Gamma$ equipped with the euclidean metric for a lattice $\Gamma \subset \mathbb{R}^2$, $\Gamma = \pi_1(M) = H_1(M, \mathbb{Z})$. Then $SO(M, g)$ is a trivial S^1 bundle, where a trivialization is given by a parallel frame (e_1, e_2). We can also write $SO(M, g) = (\mathbb{R}^2/\Gamma) \times \mathrm{SO}(2)$. Any group homomorphism

$\gamma : \Gamma \to \{-1,+1\} \subset \ker(\mathrm{Spin}(2) \to \mathrm{SO}(2)) \subset \mathrm{Spin}(2) = S^1$ defines a diagonal action of Γ on $\mathbb{R}^2 \times \mathrm{Spin}(2)$, and we obtain a Spin(2)-principal bundle by factoring out this action

$$\mathrm{Spin}(M,g)_\Gamma := \mathbb{R}^2 \times_\gamma \mathrm{Spin}(2).$$

This principal bundle together with the natural map

$$\begin{array}{rcl} \chi_\gamma : \mathrm{Spin}(M,g) = \mathbb{R}^2 \times_\Gamma \mathrm{Spin}(2) & \to & \mathrm{SO}(M,g) = (\mathbb{R}^2/\Gamma) \times \mathrm{SO}(2) \\ {[(x,z)]_\gamma} & \mapsto & (x+\Gamma, z^2) \end{array}$$

defines a spin structure χ_γ on M. Note that in the sense of Spin(2)-principal bundles, the possible $\mathrm{Spin}(M,g)$ are all equivalent. However, χ_{γ_1} and χ_{γ_2} are not equivalent in the sense of spin structures if $\gamma_1 \neq \gamma_2$. Furthermore any spin structure on M is obtained in this way. Denote the image of $v \in \Gamma$ in $H_1(M,\mathbb{Z}_2) = \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_2$ by $\overline{v}$. The quadratic form q of the spin structure associated to γ then fulfills

$$\begin{array}{ll} q(\overline{v}) = \gamma(v) + 1 & \text{for } \overline{v} \neq 0, \\ q(\overline{v}) = \gamma(v) = 0 & \text{for } \overline{v} = 0. \end{array}$$

This implies $\alpha(M,\chi) = 1$ if χ is the spin structure associated to the trivial map γ, and $\alpha(M,\chi) = 0$ in all other cases. The boundary of a solid torus has a map γ which is non-trivial, hence $\alpha(M,\chi) = 0$ in this case.

On the other hand we can also calculate the dimension of the kernel of D. As g is flat, $D\varphi = 0$ is equivalent to $\nabla\varphi = 0$.

If γ is the trivial map, then the spinor bundle is trivialized by parallel spinors, i.e. $\dim\ker D_g = 2$. If γ is non-trivial, then $\dim\ker D_g = 0$.

This terminates the example, and we return to the general case.

We now recall the index theorem for Dirac operators on compact Riemann surfaces. Note that the spinor bundle $\Sigma M \to M$ carries the structure of a quaternionic vector bundle over M, and the quaternionic multiplication commutes with the Dirac operator. Hence, the complex dimension of any eigenspace of the Dirac operator is divisible by 2.

Theorem 6.3 (Index theorem). *For any compact surface with Riemannian metric g and spin structure χ we have*

$$(\dim_{\mathbb{C}} \ker D)/2 \equiv \alpha(M,\chi) \mod 2.$$

In the examples above, we have verified this relation if M is diffeomorphic to S^2 or T^2. We will sketch a short proof of the theorem in Remark 6.7.

Remark 6.4. According to [Hit74] the (complex) dimension k_g of the kernel of the Dirac operator on a compact Riemann surface (M,g) of genus γ is at most $\gamma+1$. Hence, k_g is already determined by $\alpha(M,\chi)$ and γ if $\gamma \leq 2$, or if $\gamma = 3$, $\alpha(M,\chi) = 1$. However, in all other cases, k_g depends on the conformal class of g (see [Hit74], [BS92]). The spectrum of the Dirac operator depends continuously on g in the C^1-topology. Hence, $k_g \geq \limsup_{h\to g} k_h$. If $g_i \to g$ with $k_{g_i} < k_g$ then due to the symmetric of the spectrum of D, some positive and some negative eigenvalue converges to 0, both having the same, even multiplicity. Hence, if k_g jumps then by a multiple of 4, and $k_g \mod 4$ is therefore independent of g.

Proof of Theorem 1.3. In the case $\alpha(M,\chi) = 0$, the index theorem implies that the Dirac operator has a kernel for any given metric on M. This immediately implies $\tau(M,\chi) = 0$ which yields the first case in Theorem 1.3.

In order to derive the second case, we have to study the behavior of the Arf-invariant on sums of quadratic forms and the α-invariant under connected sum.

If $q_i : V_i \to \mathbb{Z}_2$ are quadratic maps for $i = 1,2$, then on $V_1 \oplus V_2$ we have a product quadratic map defined via $q_1 \oplus q_2 : V_1 \oplus V_2 \to \mathbb{Z}_2$, $(q_1 \oplus q_2)(v_1,v_2) = q_1(v_1) + q_2(v_2)$. One checks that $\mathrm{Arf}(q_1 \oplus q_2) = \mathrm{Arf}(q_1)\mathrm{Arf}(q_2)$.

The quadratic form of a disjoint union of (M_1,χ) and (M_2,χ_2) is just the product quadratic map $q_{\chi_1}\otimes q_{\chi_2}$. Furthermore, one easily sees that the quadratic forms of $(M_1\#M_2,\chi_1\#\chi_2)$ and $(M_1,\chi_1)\dot\cup(M_2,\chi_2)$ can be identified.

It follows that

$$\alpha(M_1\#M_2,\chi_1\#\chi_2)\equiv\alpha((M_1,\chi_1)\dot\cup(M_2,\chi_2))\equiv\alpha(M_1,\chi_1)+\alpha(M_2,\chi_2)\mod 2$$

As $\alpha(S^2,\chi_2)=0$, it follows that $\alpha(M,\chi)$ is a spin-cobordism invariant. Hence, α defines a map from the 2-dimensional spin-cobordism group to $\mathbb{Z}_2$.

Inversely, it can be shown (e.g. with the statements of [AB02, section 3]):

Lemma 6.5. *Let M be an compact oriented surface of genus γ with spin structure χ, and $\alpha(M,\chi)=0$. Then (M,χ) is spin-diffeomorphic to the connected sum of γ 2-tori*

$$T^2\#T^2\#\cdots\#T^2$$

where each 2-torus carries a spin structure that is associated to a non-trivial homomorphism.

Corollary 6.6. *Let M be an compact oriented surface of genus γ with spin structure χ, and $\alpha(M,\chi)=0$. Then (M,χ) is obtained from (S^2,χ^2) by a sequence of 0-dimensional surgeries.*

Hence, Theorem 1.1 implies that $\tau(M,\chi)\geq\tau(S^2,\chi^2)=2\sqrt{\pi}$, but as $\tau(M,\chi)\leq 2\sqrt{\pi}$ (Proposition 8), we obtain the conclusion in the second case of Theorem 1.3. □

Remark 6.7. With the methods provided in this section Theorem 6.3 can be proved in a simple and geometric way. From the construction of $\alpha(M,\chi)$ out of the quadratic form it is clear that $\alpha(M,\chi)$ is preserved under 0-dimensional surgery, in particular it is additive under connected sum. Together with $\alpha(S^2,\chi^2)=0$ it follows that it is a spin-cobordism invariant. We have already seen that $\dim\ker D$ mod 4 is invariant on the metric. The following Proposition implies that $\dim\ker D$ might only jump by multiples of 4 when a 0-dimensional surgery is performed that introduces a (long and thin) cylinder. Hence, $\dim\ker D \mod 4$ is spin-cobordant as well. As we have already checked the index theorem for tori, the index theorem Theorem 6.3 follows in general.

Proposition 6.8. *Let (M,g,χ) be a compact spin manifold with Dirac operator D, and let $(M^\#,\chi^\#)$ be obtained from M by 0-dimensional surgery. Then there is a sequence of metrics $g_i^\#$, $i\to\infty$, on $M^\#$ such that the Dirac operator D_i on $(M^\#,g_i^\#,\chi^\#)$ satisfies*

(a) $\dim\ker D_i$ *is independent on i,*
(b) $\dim\ker D_i\leq\dim\ker D$,
(c) *If $\dim\ker D_i<\dim\ker D$, then there are positive and negative eigenvalues converging to 0. Their combined multiplicity is equal to $\dim\ker D-\dim\ker D_i$.*

Part (a) and (b) of the Proposition are proved in [ADH06]. Part (c) is not proved in there explicitly, but the arguments and constructions in [ADH06] can be adapted such that we obtain (c).

The metrics g_i are unchanged outside the attached cylinders, and the cylinders equipped with the metrics g_i get longer and thinner when i tends to ∞.

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