THE S^1-EQUIVARIANT YAMABE INVARIANT OF 3-MANIFOLDS

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Abstract. We show that the S^1-equivariant Yamabe invariant of the 3-sphere, endowed with the Hopf action, is equal to the (non-equivariant) Yamabe invariant of the 3-sphere. More generally, we establish a topological upper bound for the S^1-equivariant Yamabe invariant of any closed oriented 3-manifold endowed with an S^1-action. Furthermore, we prove a convergence result for the equivariant Yamabe constants of an accumulating sequence of subgroups of a compact Lie group acting on a closed manifold.

1. Introduction

Before we can describe the main results of the article in Section 1.2 we will give a summary on known results about the non-equivariant (= classical) Yamabe invariant in Section 1.1.

1.1. **Overview of the classical Yamabe invariant.** The Yamabe constant $\mu(M,[g])$ of an n-dimensional conformal compact manifold $(M,[g])$ is the infimum of the restriction to the conformal class $[g]$ of the Einstein–Hilbert functional defined on the set of all Riemannian metrics as

$$h \longmapsto \frac{\int_M \mathrm{Scal}_h \, dv_h}{\mathrm{vol}(M,h)^{\frac{n-2}{n}}}.$$

Aubin [10] proved that the Yamabe constant of $(M,[g])$ is bounded above by the Yamabe constant of the sphere, *i.e.* $\mu(M,[g]) \leq \mu(S^n,[g_{st}])$. The Yamabe invariant $\sigma(M)$ of a compact manifold M is defined as

$$\sigma(M) := \sup_{[g]\in C(M)} \mu(M,[g]),$$

where $C(M)$ is the set of all conformal classes on M. It follows that $\sigma(M) \leq \sigma(S^n) = \mu(S^n,[g_{st}])$. In particular, the Yamabe invariant of any compact manifold is finite. The Yamabe invariant $\sigma(M)$ is positive if and only if a metric of positive scalar curvature exists on M.

In dimension 2, the Yamabe invariant is a multiple of the Euler characteristic. For $n \geq 3$ it is in general a difficult problem to compute the Yamabe invariant, and only in a few cases it can be calculated explicitly. Aubin [10] proved for the n-dimensional sphere $\sigma(S^n) = \mu(S^n,[g_{st}]) = n(n-1)(\mathrm{vol}(S^n,g_{st}))^{2/n}$. Kobayashi [21] and Schoen [33] proved that $\sigma(S^{n-1}\times S^1) = \sigma(S^n)$. For many closed manifolds M, one can show $\sigma(M) = 0$ as the existence of metrics with positive scalar curvature is obstructed, whereas conformal classes $[g_i]$ with $\mu(M,[g_i]) \to 0$ can be constructed explicitly. For example the n-torus T^n does not carry a metric of positive scalar curvature which can be shown with enlargeability type index obstructions by Gromov and Lawson or with the hypersurface obstruction by Schoen and Yau. For the

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standard metric g_0 we have $\mu(T^n,[g_0])=0$, so $\sigma(T^n)=0$. Similarly we know that $\sigma(M)$ is zero for all nilmanifolds and its quotients.

In order to determine non-zero values for σ, several recent developments in mathematics contibuted important techniques: Ricci-flow, Atiyah-Singer index theorem, Seiberg-Witten theory, and the Bray-Huisken inverse mean curvature proof of the Penrose inequality.

In dimension 3, values for the Yamabe invariant of irreducible manifolds were already conjectured and partially studied in [7, 8]. In this dimension it follows from Perelman's work on the Ricci flow that for 3-manifolds with $\sigma(M)\leq 0$, the value of $\sigma(M)$ is determined by the volume of the hyperblic pieces in the Thurston decomposition. We learned this from [20, Prop. 93.10 on page 2832], but ideas for this application go back to [9]. In particular, $\mu(H^3/\Gamma,[g_{\rm hyp}])$ is attained in the hyperbolic metric $g_{\rm hyp}$. More generally, In the case $\sigma(M)>0$, $n=3$, M is the connected sum of copies of quotients $S^2\times S^1$ and of quotients of S^3. For connected sums of copies of $S^2\times S^1$ we have $\sigma(M)=\sigma(S^3)$ but the precise value cannot be determined in most cases. Using inverse mean curvature flow, the Yamabe invariants of $\mathbb{R}P^3$ and some related spaces were determined in [15] and [1], e.g. $\sigma(\mathbb{R}P^3)=2^{-2/3}\sigma(S^3)$. This is indeed a special case of Schoen's conjecture explained below.

The case of positive Yamabe invariant is notoriously difficult in higher dimensions as well. It is already a difficult problem to determine the manifolds with $\sigma(M)>0$, see [35] for a survey.

In dimension $n\geq 5$ it is expected that there are many compact n-dimensional manifold M with $0<\sigma(M)<\sigma(S^n)$, but there is no compact manifold of dimension ≥ 5 for which this inequality can be proven. In dimensions $n=4$ there are some examples for which exact calculations can be carried out, even in the positive case. The values for $\mathbb{C}P^2$ and some related spaces were calculated by LeBrun [24] using Seiberg-Witten theory. The calculation then was simplified considerably by Gursky and LeBrun [17]. This proof no longer uses Seiberg-Witten theory, but only the index theorem by Atiyah and Singer. See also [17, 23, 25] for related results.

Recently, surgery techniques known from the work of Gromov and Lawson and from similar constructions by Schoen and Yau could be refined to obtain explicit positive lower bounds for the Yamabe invariant. Such bounds are easily obtained for *special* manifolds, e. g. for manifolds with Einstein metrics or connected sums of such manifolds. Namely, a theorem by Obata [28] states that the Einstein–Hilbert functional of an Einstein metric g equals $\mu(M,[g])$, thus providing a lower bound for $\sigma(M)$. For instance, if M is S^n, T^n, $\mathbb{R}P^n$ or $\mathbb{C}P^n$, the canonical Einstein metrics provide lower bounds for $\sigma(M)$. However, obtaining a lower bound for $\sigma(M)$ is difficult in general if M carries a metric of positive scalar curvature but no Einstein metric. Using surgery theory, Petean and Yun have proven that $\sigma(M)\geq 0$ for all simply-connected manifolds of dimension at least 5, see [30], [31]. Stronger results can be obtained with the surgery formula developed in [3]. For example, it now can be shown, see [5] and [4, 6], that simply-connected manifolds of dimension 5 resp. 6 satisfy $\sigma(M)\geq 45.1$ resp. $\sigma(M)\geq 49.9$.

The non-simply connected case is much more involved. The results by Petean and Yun were partially extended to the non-simply-connected case by Botvinnik and Rosenberg [14], they proved for example for closed spin manifolds of dimension at least 5 that $\sigma(M)\geq 0$ if the fundamental group is of odd order and has all Sylow subgroups abelian. However, an extension of the work of Ammann, Dahl and Humbert to this case has not yet been achieved.

The following conjecture by Schoen [33] would yield manifolds with $0<\sigma(M)<\sigma(S^n)$. This conjecture states: if Γ is a finite group acting freely on S^n, then $\sigma(S^n/\Gamma)=\sigma(S^n)/(\#\Gamma)^{2/n}$. In particular, it would imply that manifolds with positive Yamabe invariant close to 0 actually exist, for large $k := \#\Gamma$. With [3] this would yield that for any odd $n \geq 5$ and sufficiently large $k := \#\Gamma$, every manifold M representing the bordism class $[S^n/\Gamma] \in \Omega_n^{\mathrm{spin}}(B\Gamma)$ with maps inducing isomorphisms $\pi_1(M) \cong \Gamma \cong \pi_1(S^n/\Gamma)$ has $\sigma(M) = \sigma(S^n)/(\#\Gamma)^{2/n}$.

This is one example of a series of similar applications, explained in [3, Sec. 1.4], see also [2]. A central fact in these applications is that the Yamabe invariant provides a chain of subgroups of the spin bordism groups $\Omega_n^{\mathrm{spin}}(B\Gamma)$, see [3, Thm. 1.7]. Unfortunately these subgroups are very poorly understood. The existence of a manifold M of dimension $n \geq 5$ with positive Yamabe invariant close to 0 would be helpful to understand these subgroups.

Unfortunately, besides the trivial cases $\Gamma = \{\mathrm{id}\}$ or $n = 2$, Schoen's conjecture has only been proven in the particular case, when $n = 3$ and $\#\Gamma = 2$, which is the determination of $\sigma(\mathbb{R}P^3)$ by Bray and Neves in [15] mentioned above.

1.2. **The G-equivariant Yamabe invariant: Overview and Main Results.** In this article, we study the Yamabe invariant in the G-equivariant setting by taking the supremum and the infimum only among G-invariant metrics and conformal classes where G is a compact Lie group acting on M, see Section 2.1 for details. The associated invariants are called the G-equivariant Yamabe constant or simply the G-Yamabe constant $\mu(M, [g]^G)$, and similarly the G-(equivariant) Yamabe invariant $\sigma^G(M)$. To our knowledge the first reference for the G-equivariant Yamabe constant $\mu(M, [g]^G)$ is Bérard Bergery [12]. In particular, he formulated a G-equivariant version of the Yamabe conjecture, which was the main subject of an article by Hebey and Vaugon [19] and by the second author [26, 27]. In general neither $\sigma^G(M) \leq \sigma(M)$ nor $\sigma^G(M) \geq \sigma(M)$ holds, see Example 3.

One motivation for the present article is to shed new light on Schoen's conjecture which is equivalent to saying $\sigma^\Gamma(S^n) = \sigma(S^n)$. A proof of Schoen's conjecture (or even partial results) would be very helpful, as it would provide interesting conclusions about the Yamabe invariant of non-simply connected manifolds. For example, if we were able to obtain an upper bound on $\sigma^\Gamma(S^n)$ which is uniform in Γ, then the Yamabe invariant would define interesting subgroups of the spin bordism and oriented bordism groups, see [3].

The simplest case of Schoen's conjecture is when $\mathbb{Z}_k \subset S^1 \subset \mathbb{C}$ acts by complex multiplication on $S^3 \subset \mathbb{C}^2$, the so-called Hopf action. As it seems currently out of reach to show $\sigma^{\mathbb{Z}_k}(S^n) = \sigma(S^n)$ for $k > 2$, we study the limit $k \to \infty$ instead, and this leads two the following two questions:

(1) Is $\sigma^{S^1}(S^3) = \sigma(S^3)$ for the Hopf action?
(2) Assume that a sequence (H_i) of subgroups of G converges in a suitable sense to G. Can we conclude that $\sigma^{H_i}(M)$ converges to $\sigma^G(M)$?

The first question is answered affirmatively by our main theorem. More generally, we give an upper bound for the S^1-Yamabe invariant of any 3-dimensional closed oriented manifold M, endowed with an S^1-action. This upper bound depends only on the following topological invariants: the first Chern class of the associated circle bundle $M \to M/S^1$ and the Euler–Poincaré characteristic of the quotient space (see Theorem 9 for the precise statement).

Our strategy is to use the quotient space M/S^1. We distinguish the following three cases, since the isotropy group of any point is either $\{\mathrm{id}\}$, $\mathbb{Z}_k$ or S^1. If the S^1-action has at least one fixed point, a result of Hebey and Vaugon [19] implies that $\sigma^{S^1}(M) \leq \sigma(S^3)$. If the S^1-action is free, then M/S^1 is a smooth surface. In order to find an upper bound in this case, we use O'Neill's formula relating the curvatures of the total space and the base space of a Riemannian submersion and the Gauß–Bonnet theorem. In the last case, when the S^1-action is neither free nor has fixed points (*i.e.* there exists at least one point with non-trivial finite isotropy group), the quotient space M/S^1 is a closed 2-dimensional orbifold. We proceed as in the free action case, since the Gauß–Bonnet theorem still holds on orbifolds (see [32]). In the two latter cases, we find a topological upper bound of $\sigma^{S^1}(M)$, which depends only on the Euler–Poincaré characteristic of M/S^1 and the first Chern number of the circle bundle $M \to M/S^1$.

The last part of the article partially answers the second question. More precisely the statement of Corollary 13 is

$$\liminf_{i\to\infty} \sigma^{H_i}(M) \geq \sigma^G(M).$$

Unfortunately, for the corresponding $\leq$-inequality which would allow the interesting application to Schoen's conjecture a proof is not available yet. The natural idea to study limits of unit volume H_i-invariant metrics g_i with $\mathrm{Scal}_{g_i} = \mu(M, [g_i]^{H_i}) > \sigma^{H_i}(M) - o(i)$ fails, as it is unclear how to find uniform for the curvature of g_i.

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2. Preliminaries, definitions and some known results

2.1. **Definition of the G-equivariant Yamabe invariant.** In this section we assume that a compact Lie group G acts on a compact manifold M. All such actions are supposed to be smooth.

We recall that the Einstein-Hilbert functional of M is given by

$$J(\tilde{g}) := \frac{\int_M \mathrm{Scal}_{\tilde{g}}\, dv_{\tilde{g}}}{\mathrm{vol}(M, \tilde{g})^{\frac{n-2}{n}}}. \tag{1}$$

We denote by $[\tilde{g}]^G$ the set of G-invariant metrics in the conformal class of $\tilde{g}$ and by $C^G(M)$ the set of all conformal classes containing at least one G-invariant metric.

Definition 1 (G-Yamabe constant and G-Yamabe invariant)**.** We define the G-equivariant Yamabe constant (or shorter: the G-Yamabe constant) by

$$\mu(M, [\tilde{g}]^G) = \inf_{g' \in [\tilde{g}]^G} J(g') \tag{2}$$

and the G-equivariant Yamabe invariant of M (or shorter: the G-Yamabe invariant) by

$$\sigma^G(M) = \sup_{[\tilde{g}]^G \in C^G(M)} \mu(M, [\tilde{g}]^G) \in (-\infty, \infty].$$

Remark 2. It follows for the solution of the equivariant Yamabe problem [19] that $\mu(M, [g]^G) > 0$ if and only if $[g]$ contains a G-invariant metric of positive scalar curvature. It thus follows that $\sigma^G(M) > 0$ if and only if M carries a G-invariant metric of positive scalar curvature.

The following examples show that both $\sigma^G(M) > \sigma(M)$ and $\sigma^G(M) < \sigma(M)$ may occur.

Example 3. For example if S^1 acts on the S^1 factor of $N \times S^1$, $\dim N = n-1$, and if N is a compact manifold carrying a metric of positive scalar curvature, then $\sigma^{S^1}(N \times S^1) = \infty$, whereas $\sigma(N \times S^1) \leq \sigma(S^n) < \infty$. On the other hand, if M is a simply-connected circle bundle over a K3-surface then $\sigma(M) > 0$ but $\sigma^{S^1}(M) = 0$. Here $\sigma(M) > 0$ follows from the surgery construction by Gromov and Lawson and from the fact that every compact simply connected spin 5-manifolds is a spin boundary. For $\sigma^{S^1}(M) \leq 0$ we refer to [37, Theorem 6.2]. The inequality $\sigma^{S^1}(M) \geq 0$ follows from (3) by taking an S^1-invariant metric g_1 on M, we rescale the fibers by a factor $\ell > 0$ and obtain g_ℓ and then $\lim_{\ell\to 0} \mu(M, [g_\ell]^{S^1}) = 0$.

In the non-positive case the situation changes. If $\sigma^G(M) \leq 0$, then we have $\mu(M,[g]^G) = \mu(M,[g])$ for any G-invariant conformal class $[g]$, as the maximum principle implies that minimizers are unique up to a constant. Thus $\sigma(M) \geq \sigma^G(M)$ in this case.

2.2. **Some known results.** In [19], Hebey and Vaugon gave the following upper bound for the G-Yamabe constant:

Theorem 4 (Hebey–Vaugon). *Let M be an n-dimensional compact connected oriented manifold endowed with an action of a compact Lie group G, admitting at least one orbit of finite cardinality. Then the following inequality holds:*

$$\sigma^G(M) \leq \sigma(S^n)\big(\inf_{p\in M} \operatorname{card}(G\cdot p)\big)^{\frac{2}{n}}.$$

Other results in the literature can be rephrased as follows.

Theorem 5 (Lawson–Yau, [22]). *If a compact manifold M admits a smooth, effective action by a compact, connected, non-abelian Lie group G, then $\sigma^G(M) > 0$.*

Theorem 6 (Bérard Bergery [12] $n=3$, Wiemeler [36] all n). *Let an abelian Lie group G act effectively on a closed connected manifold M, and assume that the fixed point set has a component of codimension 2. Then $\sigma^G(M) > 0$.*

More recent progress about the question whether $\sigma^G(M) > 0$ can be found in [18] and [37].

2.3. **Scalar curvature of S^1-bundles.** Let M^n be a compact, oriented, and connected manifold, which is an S^1-bundle over N, let $\pi : M \to N$ be the projection, let $\tilde{g}$ be an S^1-invariant metric on M and g its projection under π on N. Let K denote the tangent vector field induced by the S^1-action and let ℓ be its length (with respect to $\tilde{g}$) and $e_0 := \frac{K}{\ell}$. We define the $(2,1)$-tensor fields A and T on M as in [13, 9.C.], *i.e.* for all vector fields U, V on M:

$$A_U V = \mathscr{H}\nabla_{\mathscr{H}U}\mathscr{V}V + \mathscr{V}\nabla_{\mathscr{H}U}\mathscr{H}V,$$

$$T_U V = \mathscr{H}\nabla_{\mathscr{V}U}\mathscr{V}V + \mathscr{V}\nabla_{\mathscr{V}U}\mathscr{H}V,$$

where $\mathscr{H}$ and $\mathscr{V}$ denote the horizontal, resp. vertical part of a vector field. The tensor A measures the non-integrability of the horizontal distribution, whereas T is essentially the second fundamental form of the S^1-orbits. Thenthe following formula [13, 9.37] relating the scalar curvatures of $(M,\tilde{g})$ and (N,g) holds:

$$\widetilde{\mathrm{Scal}} = \mathrm{Scal} - |A|^2 - |T|^2 - |T_{e_0}e_0|^2 - 2\check{\delta}(T_{e_0}e_0),$$

where $\check{\delta}$ is the codifferential in the horizontal direction. For any vector fields X, Y on N with horizontal lifts $\widetilde{X}, \widetilde{Y}$, the vertical part of $[\widetilde{X}, \widetilde{Y}]$ equals $\Omega(X,Y)K := 2A_{\widetilde{X}}\widetilde{Y}$. We compute:

$$|A|^2 = \frac{\ell^2}{4}|\Omega|^2, \quad T_K\widetilde{X} = \nabla_K\widetilde{X} = \nabla_{\widetilde{X}}K = \frac{\partial_{\widetilde{X}}\ell}{\ell}K, \quad T_{e_0}e_0 = -\frac{\mathrm{grad}\ell}{\ell} = -\mathrm{grad}\log\ell,$$

which yield

$$\mathrm{Scal}_{\tilde{g}} = \mathrm{Scal}_g - \frac{\ell^2}{4}|\Omega|^2 - 2\frac{|d\ell|^2}{\ell^2} + 2\Delta_g(\log\ell) = \mathrm{Scal}_g - \frac{\ell^2}{4}|\Omega|^2 + 2\frac{\Delta_g\ell}{\ell}, \qquad (3)$$

where Δ_g is the Laplacian of the base (N,g).

2.4. **An analytical ingredient.** We recall that the following classical result still holds on orbifolds:

Lemma 7. *Let (Σ, g) be a closed 2-dimensional orbifold. Let $f \in C^k(\Sigma)$ be a function with $\int_\Sigma f dv_g = 0$. Then there exists a solution $u \in C^{k+2}(\Sigma)$ of the equation $\Delta_g u = f$, which is unique up to an additive constant.*

The proof of Lemma 7 is analogous to the case of manifolds, see for instance the construction of the Green function of the Laplacian on orbifolds given by Chiang, [16].

3. The S^1-Yamabe invariant

In this section we always have $G = S^1$, and we use the notation $N = M/S^1$ similar to Section 2.3. Here N may have singular points, i.e. orbifold points or boundary points.

3.1. **Yamabe functional on S^1-bundles.** If the action of S^1 is free, then by (3), we obtain from (1):

$$J(\tilde{g}) = \frac{2\pi \int_N (\mathrm{Scal}_g - \frac{\ell^2}{4}|\Omega|_g^2)\ell \, \mathrm{d}v_g}{\left(\int_N 2\pi\ell \, \mathrm{d}v_g\right)^{\frac{n-2}{n}}}, \tag{4}$$

since the length of any fibre is $2\pi\ell$. The Yamabe functional of $(M, \tilde{g})$ is the restriction of the Einstein-Hilbert functional to the conformal class of $\tilde{g}$. It can be written equivalently as follows:

$$J(u^{\frac{4}{n-2}}\tilde{g}) = \frac{\int_N 2\pi\ell\left(\frac{4(n-1)}{n-2}|du|_g^2 + \mathrm{Scal}_{\tilde{g}} u^2\right) \mathrm{d}v_g}{\left(\int_N 2\pi\ell u^{\frac{2n}{n-2}} \, \mathrm{d}v_g\right)^{\frac{n-2}{n}}}, \tag{5}$$

where $\mathrm{Scal}_{\tilde{g}}$ is given by (3).

3.2. **Classification of 3-manifolds with $\sigma^{S^1}(M) > 0$.** In dimension 3, it is completely understood, under which condition there is an S^1-invariant metric of positive scalar curvature, in other words, when we have $\sigma^{S^1}(M) > 0$.

Theorem 8 ([12, Theorem 12.1])**.** *Let M be a compact connected 3-dimensional manifold with a smooth S^1-action on M.*

- *a) If the action has a fixed point, then $\sigma^{S^1}(M) > 0$.*
- *b) If the action has no fixed point, then $\sigma^{S^1}(M) > 0$ if and only if M is a finite quotient of S^3 or of $S^2 \times S^1$.*

Note that every finite quotient of S^3 by a freely acting subgroup of $\mathrm{SO}(4)$ admits a non-trivial S^1-action [29, Sec. 6, Theorem 5].

3.3. **Oriented 3-manifolds.** From now on, we assume that M is a 3-dimensional compact oriented connected manifold endowed with an S^1-action. If this S^1-action has at least one fixed point, Theorem 4 implies that the Yamabe invariant of S^3 is an upper bound for the S^1-Yamabe invariant: $\sigma^{S^1}(M) \leq \sigma(S^3)$.

We want to determine an upper bound for the S^1-Yamabe invariant in the complementary case, i.e. we consider S^1-actions without fixed points. This implies that M is an S^1-principal (orbi)bundle over $\Sigma := M/S^1$, which is a 2-dimensional orbifold (a smooth surface, if the action is free). As usual, we use the correspondence between S^1-principal bundles and complex line bundles defined by

$$M \longmapsto L := M \times_{S^1} \mathbb{C}.$$

We write $c_1(L,\Sigma) := \langle c_1(L),[\Sigma]\rangle \in \mathbb{Q}$, where $c_1(L) \in H^2(\Sigma,\mathbb{Q})$ is the first rational Chern class of L in the orbifold sense. Let $\chi(\Sigma) = c_1(T\Sigma,\Sigma)$ be the (orbifold) Euler-Poincaré characteristic of Σ.

We are now ready to state our main result:

Theorem 9. *Let M be a 3-dimensional compact connected oriented manifold endowed with an S^1-action without fixed points. With the above notation, the following assertions hold:*

i) If $\chi(\Sigma) > 0$ and $c_1(L,\Sigma) \neq 0$, then

$$0 < \sigma^{S^1}(M) \leq \sigma(S^3)\left(\frac{\chi(\Sigma)}{2\sqrt{|c_1(L,\Sigma)|}}\right)^{\frac{4}{3}}.$$

ii) If $\chi(\Sigma) > 0$ and $c_1(L,\Sigma) = 0$, then $\sigma^{S^1}(M) = \infty$.

iii) If $\chi(\Sigma) \leq 0$, then $\sigma^{S^1}(M) = 0$.

In particular, $\sigma^{S^1}(M)$ is positive if and only if $\chi(\Sigma)$ is positive. This coincides with the characterization in [12], as explained in Section 3.2.

Proof of Theorem 9. Let $[\tilde g]^{S^1} \in \mathrm{Conf}^{S^1}(M)$ be the class of S^1-invariant metrics conformal to $\tilde g$ on M. Without loss of generality, we assume that the length of the vector field K generating the S^1-action $\ell := |K|_{\tilde g}$ is constant (otherwise we take a different representative of the class $[\tilde g]^{S^1}$). Let g be the projection of the metric $\tilde g$ on Σ, so that $(M,\tilde g) \to (\Sigma, g)$ is a Riemannian submersion. Since ℓ is constant, the O'Neill formula (3) yields that $\mathrm{Scal}_{\tilde g} = \mathrm{Scal}_g - \frac{\ell^2}{4}|\Omega|_g^2$. Strictly speaking in Subsection 2.3 we only discussed the case that Σ is a smooth manifold, but as (3) is a local equation, the orbifold case immediately follows by locally covering with an S^1-bundle over a smooth base manifold. Using the Gauß–Bonnet theorem, we compute the Yamabe functional as follows:

$$\begin{aligned} J(\tilde g) &= \frac{2\pi\int_\Sigma(\mathrm{Scal}_g - \frac{\ell^2}{4}|\Omega|_g^2)\ell\,\mathrm{d}v_g}{(2\pi)^{\frac13}(\int_\Sigma \ell\,\mathrm{d}v_g)^{\frac13}} \\ &= (2\pi)^{\frac23}\frac{\ell(\int_\Sigma \mathrm{Scal}_g\,\mathrm{d}v_g) - \frac{\ell^3}{4}(\int_\Sigma|\Omega|_g^2\,\mathrm{d}v_g)}{\ell^{\frac13}(\int_\Sigma \mathrm{d}v_g)^{\frac13}} \\ &= \left(\frac{\pi^2}{16\mathrm{vol}(\Sigma,g)}\right)^{\frac13}\left(16\pi\chi(\Sigma)\ell^{\frac23} - \|\Omega\|_2^2\ell^{\frac83}\right). \end{aligned} \tag{6}$$

If we have $\chi(\Sigma) \geq 0$ and $\|\Omega\|_2 > 0$, then the maximal value of this expression as a function of ℓ is attained for $\ell = \sqrt{4\pi\chi(\Sigma)}\|\Omega\|_2^{-1}$ and its maximal value equals

$$3\cdot 2^{\frac43}\pi^2(\mathrm{vol}(\Sigma,g))^{-\frac13}\chi(\Sigma)^{\frac43}\|\Omega\|_2^{-\frac23}.$$

We now consider cases i) to iii) in the theorem.

i) Note that in this case $c_1(L,\Sigma) \neq 0$ implies $\|\Omega\|_2 > 0$. By the Cauchy–Schwarz inequality, it further follows that

$$J(\tilde g) \leq 3\cdot 2^{\frac43}\pi^2\chi(\Sigma)^{\frac43}\|\Omega\|_1^{-\frac23}. \tag{7}$$

On the other hand, we claim that $\|\Omega\|_1 \geq 2\sqrt2\pi|c_1(L,\Sigma)|$, since

$$\frac{1}{\sqrt2}\int_\Sigma|\Omega|_g\,\mathrm{d}v_g \geq \left|\int_\Sigma\Omega\right| = 2\pi|c_1(L,\Sigma)|,$$

where the volume form $\mathrm{d}v_g$ has length $\sqrt2$, by convention. Using $\sigma(S^3) = 3\cdot 2^{5/3}\cdot\pi^{4/3}$ it follows that $J(\tilde g) \leq \sigma(S^3)\chi(\Sigma)^{\frac43}|4c_1(L,\Sigma)|^{-\frac23}$, for all S^1-invariant metrics $\tilde{g}$ on M with $\ell = |K|_{\tilde{g}}$ constant. This yields

$$\mu(M, [\tilde{g}]^{S^1}) \le \sigma(S^3)\chi(\Sigma)^{\frac{4}{3}}|4c_1(L,\Sigma)|^{-\frac{2}{3}},$$

for all S^1-invariant conformal classes $[\tilde{g}]^{S^1} \in \mathrm{Conf}^{S^1}(M)$.

Now, we show that $\sigma^{S^1}(M)$ is positive. The function $f := \frac{2\pi}{vol(\Sigma,g)}\chi(\Sigma) - \frac{1}{2}\mathrm{Scal}_g$ has zero average over Σ. By Lemma 7, there exists a solution u of the equation $\Delta_g u = f$. Therefore the scalar curvature of $g_u := e^{2u}g$ is given by

$$\mathrm{Scal}_{g_u} = 2e^{-2u}(\Delta_g u + \frac{1}{2}\mathrm{Scal}_g) = \frac{4\pi}{\mathrm{vol}(\Sigma, g)}\chi(\Sigma)e^{-2u}. \tag{8}$$

Hence, the scalar curvature of g_u is positive. Using the identity (3) and choosing the length of the S^1-fibre constant and sufficiently small, we construct an S^1-invariant metric $\tilde{g}_u$ (which is not necessarily conformal to $\tilde{g}$) with positive scalar curvature. Therefore, the Yamabe constant $\mu(M, [\tilde{g}_u]^{S^1})$ is positive, so $\sigma^{S^1}(M) > 0$.

ii) If $c_1(L,\Sigma) = 0$, then there exists an S^1-equivariant finite covering $S^1 \times \widetilde{\Sigma}$ of M of degree d, where $\widetilde{\Sigma}$ is a smooth compact surface finitely covering Σ (for more details, see *e.g.* [34, Lemma 3.7]). Since $\chi(\Sigma) > 0$, we see that $\widetilde{\Sigma}$ is diffeomorphic to S^2. As in the previous case, we know that a metric of positive Gauß curvature exists on Σ. The product metric $\tilde{g}_\ell$ of its lift to $\widetilde{\Sigma}$ with a rescaled standard metric on S^1 of length $2\pi\ell$ is invariant under the deck transformation group of $S^1 \times \widetilde{\Sigma} \to M$. As this deck transformation group commutes with the S^1-action, $\tilde{g}_\ell$ descends to an S^1-invariant metric g_ℓ on M. From (5), we get $\mu(S^1 \times \widetilde{\Sigma}, [\tilde{g}_\ell]^{S^1}) = \mu(S^1 \times \widetilde{\Sigma}, [\tilde{g}_1]^{S^1})\ell^{2/3}$. Obviously we have $\mu(S^1 \times \widetilde{\Sigma}, [\tilde{g}_\ell]^{S^1}) \le d^{2/3}\mu(M, [g_\ell]^{S^1})$. Then $\mu(M, [g_\ell]^{S^1})$ converges to ∞ for $\ell \to \infty$, which implies the statement.

iii) Assume that the Euler-Poincaré characteristic of Σ is non-positive. By (6), we have

$$\mu(M, [\hat{g}]^{S^1}) \le J(\hat{\ell}^{-2}\hat{g}) \le 2(2\pi)^{\frac{5}{3}}\chi(\Sigma)\mathrm{vol}(\Sigma, \hat{g}_\Sigma)^{-\frac{1}{3}} \le 0,$$

for any S^1-invariant Riemannian metric $\hat{g}$ on M, where $\hat{\ell} := |K|_{\hat{g}}$. This yields $\sigma^{S^1}(M) \le 0$. Moreover, if we fix a Riemannian metric $\hat{g}_\Sigma$ on Σ, we define $(\hat{g}_j)$ to be a sequence of metrics on M with constant functions $\hat{\ell}_j := |K|_{\hat{g}_j} \le 1$ converging to 0 and $\pi^*\hat{g}_\Sigma = \hat{g}_j$. From (5) and using the Hölder inequality, we obtain

$$\mu(M, [\hat{g}_j]^{S^1}) \ge -(2\pi\hat{\ell}_j)^{\frac{2}{3}}(\|\mathrm{Scal}_{\hat{g}_\Sigma}\|_{\frac{3}{2}} + \frac{1}{4}\|\Omega\|_3^2).$$

Hence, when j goes to $+\infty$, it follows that $\sigma^{S^1}(M) \ge 0$. We conclude that $\sigma^{S^1}(M) = 0$.

□

3.4. **The case of S^3.** We now consider the special case of S^1-actions on $S^3 \subset \mathbb{C}^2$.

For $m_1, m_2 \in \mathbb{N}$ assumed to be relatively prime as long as $m_1 m_2 \neq 0$, we define

$$\phi_{m_1,m_2} : S^1 \to \mathrm{Diff}(S^3), \quad \phi_{m_1,m_2}(x)(z_1, z_2) := (x^{m_1}z_1, x^{m_2}z_2). \tag{9}$$

With this notation, the Hopf action of S^1 on S^3 corresponds to $\phi_{1,1}$. Up to diffeomorphism these are the only possible smooth S^1-actions on S^3 (see *e.g.* [29]). Note that such an action has fixed points if and only if $m_1 m_2 = 0$.

Theorem 10. *For the Hopf action of S^1 on S^3 it holds:*

$$\sigma^{S^1}(S^3) = \sigma(S^3).$$

Moreover, the S^1-equivariant Yamabe invariant of any S^1-action ϕ_{m_1,m_2} on S^3 satisfies the following:

i) If $m_1 m_2 = 0$, then $\sigma^{S^1}(S^3) = \sigma(S^3) = 6 \cdot 2^{\frac{2}{3}} \cdot \pi^{\frac{4}{3}}$.

ii) If $m_1 m_2 \neq 0$, then

$$\sigma(S^3) \leq \sigma^{S^1}(S^3) \leq \sigma(S^3) \left(\frac{m_1 + m_2}{2\sqrt{m_1 m_2}} \right)^{\frac{4}{3}}.$$

Proof. Let us first remark that, since the standard metric g_{st} on S^3 is S^1-invariant for any S^1-action ϕ_{m_1,m_2}, it follows that $\mu(S^3, [g_{\mathrm{st}}]^{S^1}) \geq \mu(S^3, [g_{\mathrm{st}}]) = \sigma(S^3)$. Hence, we obtain the inequality: $\sigma^{S^1}(S^3) \geq \sigma(S^3)$.

i) If $m_1 m_2 = 0$, then the S^1-action has fixed points and by Theorem 4 we also obtain the reverse inequality: $\sigma^{S^1}(S^3) \leq \sigma(S^3)$.

ii) If $m_1 m_2 \neq 0$, then the quotient orbifold is the so-called 1-dimensional weighted projective space denoted by $\mathbb{C}P^1(m_1, m_2)$. In order to use the upper bound provided by Theorem 9, we need to compute $\chi(\mathbb{C}P^1(m_1, m_2))$ and $c_1(L, \mathbb{C}P^1(m_1, m_2))$. Using the Seifert invariants of S^1-bundles (see *e.g.* [29], [34]), one obtains: $\chi(\mathbb{C}P^1(m_1, m_2)) = \frac{1}{m_1} + \frac{1}{m_2}$ and $|c_1(L, \mathbb{C}P^1(m_1, m_2))| = \frac{1}{m_1 m_2}$. Alternatively, we give in the Appendix an explicit geometric computation of these topological invariants. Substituting these values in Theorem 9, i), we obtain the desired inequality.

The first statement of the theorem follows from *ii)* for $m_1 = m_2 = 1$. □

4. Convergence result

Definition 11. Let G be a Lie group, and let $(H_i)_{i\in\mathbb{N}}$ be a sequence of (closed) Lie subgroups. We say that $h \in G$ is an accumulation point for $(H_i)_{i\in\mathbb{N}}$ if there is a sequence $(h_i)_{i\in\mathbb{N}}$ with $h_i \in H_i$ and $h_i \to h$. The set of accumulation points is a closed subgroup of G. We say that $(H_i)_{i\in\mathbb{N}}$ is accumulating if every element of G is an accumulation point.

Proposition 12. *Assume that a compact Lie group G acts on a closed manifold M. Let $(H_i)_{i\in\mathbb{N}}$ be an accumulating sequence of Lie subgroups of G. Then for any G-equivariant conformal class $[g]$ we get*

$$\lim_{i\to\infty} \mu(M, [g]^{H_i}) = \mu(M, [g]^G).$$

Proof. We distinguish the following two cases:

- If the (non-equivariant) Yamabe constant satisfies $\mu(M, [g]) \leq 0$, then there is, up to a multiplicative constant, a unique metric $u_\infty^{\frac{4}{n-2}} g$ of constant scalar curvature and u_∞ is G-invariant. This implies $\mu(M, [g]) = \mu(M, [g]^G) = \mu(M, [g]^{H_i})$.
- Now we assume that the Yamabe constant satisfies $\mu(M, [g]) > 0$. Set $\mu_i := \mu(M, [g]^{H_i})$, $\mu := \mu(M, [g]^G)$. Obviously $\mu_i \leq \mu$. After passing to a subsequence we can assume that μ_i converges to a number $\bar{\mu} \leq \mu$ and it remains to show that $\bar{\mu} < \mu$ leads to a contradiction. For an orbit O we will use the convention that $\#O$ takes values in $\mathbb{N} \cup \{\infty\}$, in particular $\#O = \infty$ if and only if O is of any infinite cardinality. We claim that $\lim_{i\to\infty} \#(H_i \cdot p) = \#(G \cdot p)$, for any $p \in M$. The inequality $\#(H_i \cdot p) \leq \#(G \cdot p)$ is obvious as $H_i \subset G$.

  To prove the claim in the case $\#(G \cdot p) < \infty$, we choose pairwise disjoint neighborhoods of all the G-orbit points of p and for i sufficiently large, we find in each such neighborhood an element of the H_i-orbit of p, showing that $\#(H_i \cdot p) \geq \#(G \cdot p)$. If $\#(G \cdot p) = \infty$, then we apply the previous argument to a finite subset of the G-orbit of p and then let its cardinality converge to ∞. This shows that $\lim_{i\to\infty} \#(H_i \cdot p) = \infty$.

Without loss of generality, we assume that $\mu_i \leq \tilde{\mu} := (\mu + \bar{\mu})/2 < \mu$. Let k be the cardinality of the smallest G-orbit, again sloppily written as ∞ in the case that k is infinite. Then by Theorem 4, we have $\mu \leq \sigma(S^n)k^{2/n}$. This implies $\mu_i \leq \tilde{\mu} < \sigma(S^n)k^{2/n}$. Hence, by the above claim, we obtain the inequality $\mu_i \leq \tilde{\mu} < \sigma(S^n)(\min_{p\in M} \#(H_i{\cdot}p))^{2/n}$, for $i \geq i_0$, where i_0 is sufficiently large. By Hebey and Vaugon [19], it follows that, for $i \geq i_0$, there exists a sequence $(u_i^{\frac{4}{n-2}} g)_{i\in\mathbb{N}}$ of H_i-invariant metrics, which minimizes the functional J among all H_i-invariant metrics in $[g]$. Furthermore u_i is a positive smooth H_i-invariant solution of the Yamabe equation, and we may assume $\|u_i\|_{\frac{2n}{n-2}} = 1$. The sequence $(u_i)_{i\in\mathbb{N}}$ is uniformly bounded in $H^1(M)$. Hence there exists a nonnegative function $u_\infty \in H^1(M)$, such that $(u_i)_{i\in\mathbb{N}}$ converges strongly in $L^q(M)$, for $1 \leq q < \frac{2n}{n-2}$, and weakly in $H^1(M)$ to u_∞. We now claim, that u_i is bounded in the L^∞-norm. Suppose that it is not bounded. Then we find a sequence of $x_i \in M$ such that $u_i(x_i) \to \infty$, and after taking a subsequence x_i converges to a point $\bar{x}$. For any point $g\bar{x}$ in its orbit, there is a sequence of $h_i \in H_i$ with $h_i x_i \to g\bar{x}$, $u_i(h_i x_i) \to \infty$. If the orbit $G{\cdot}\bar{x}$ contains at least $\tilde{k}$ points, then we can do classical blowup-analysis in $\tilde{k}$ points, which would yield $\bar{\mu} \geq \sigma(S^n)\tilde{k}^{2/n}$ (see for example [11, Chapter 6.5.]). This implies $\bar{\mu} \geq \sigma(S^n)k^{2/n}$ which contradicts $\bar{\mu} < \sigma(S^n)k^{2/n}$. We obtain the claim, i.e. the boundedness of u_i in L^∞. By a standard bootstrap argument this yields the boundedness of u_i in $C^{2,\alpha}$ for $0 < \alpha < 1$, and thus u_i converges to u_∞ in C^2. It follows that u_∞ is a smooth, positive G-invariant solution of the Yamabe equation, with $\|u_\infty\|_{\frac{2n}{n-2}} = 1$ and $J(u_\infty^{\frac{4}{n-2}} g) = \bar{\mu}$. Thus $\mu \leq \bar{\mu}$.

$\square$

Corollary 13. *Assume that a compact Lie group G acts on a closed manifold M. Let $(H_i)_{i\in\mathbb{N}}$ be an accumulating sequence of subgroups of G. Then*

$$\liminf_{i\to\infty} \sigma^{H_i}(M) \geq \sigma^G(M).$$

$\square$

Appendix

Computation of $c_1(L, \mathbb{C}P^1(m_1, m_2))$. We consider the action of S^1 on $S^3 \subset \mathbb{C}^2$ given by

$$\phi_{m_1,m_2} : e^{i\theta} \mapsto \big((z_1, z_2) \mapsto (e^{im_1\theta} z_1, e^{im_2\theta} z_2)\big),$$

where m_1 and m_2 are two positive relatively prime integers. Let $\pi : S^3 \to S^3/S^1$ denote the projection, where the quotient $S^3/S^1 =: \mathbb{C}P^1(m_1, m_2)$ is the one dimensional weighted projective space. We consider the round metric of S^3 induced by the standard metric on $\mathbb{R}^4 \simeq \mathbb{C}^2$: $\langle (z_1, z_2), (w_1, w_2)\rangle = \mathrm{Re}(z_1\bar{w}_1 + z_2\bar{w}_2)$. The vector field induced by the S^1-action is given by:

$$K_p = i(m_1 z_1, m_2 z_2) \in T_p S^3 = p^\perp, \text{ where } p = (z_1, z_2) \in S^3.$$

The vector field K vanishes nowhere, since $|K_p|^2 = m_1^2|z_1|^2 + m_2^2|z_2|^2 > 0$, for all $p \in S^3$. For $p \in S^3 \setminus (\{0\} \times S^1 \cup S^1 \times \{0\})$, the orthogonal complement of K_p in $T_p S^3$ (w.r.t. the round metric) is spanned by the horizontal vector fields

$$\widetilde{X}_1(p) := i(m_2|z_2|^2 z_1, -m_1|z_1|^2 z_2), \quad \widetilde{X}_2(p) := (|z_2|^2 z_1, -|z_1|^2 z_2),$$

which are also S^1-invariant. Hence they project to the vector fields X_1, resp. X_2 on $\mathbb{C}P^1(m_1, m_2)$.

We define the connection 1-form ω on S^3 whose kernel is given by the orthogonal complement of K and normalized such that $\omega(K) = 1$, $\omega := \frac{\langle K, \cdot\rangle}{|K|^2}$. The 2-form

$\Omega := \mathrm{d}\omega$ is S^1-invariant and thus projects onto a 2-form on $\mathbb{C}P^1(m_1, m_2)$, which we denote by the same symbol. It follows that

$$\Omega_{\pi(p)}(X_1, X_2) = -\omega_p([\widetilde{X}_1, \widetilde{X}_2]) = \frac{2m_1m_2|z_1|^2|z_2|^2}{m_1^2|z_1|^2 + m_2^2|z_2|^2},$$

since we have $\mathrm{d}\widetilde{X}_1(\widetilde{X}_2) - \mathrm{d}\widetilde{X}_2(\widetilde{X}_1) = -2i|z_1|^2|z_2|^2(m_2z_1, m_1z_2)$.

We now introduce the following complex coordinates on $\mathbb{C}P^1(m_1, m_2) \setminus \{[0:1]\}$,

$$\begin{aligned} \varphi : \mathbb{C}P^1(m_1, m_2) \setminus \{[0:1]\} &\longrightarrow \mathbb{C} \\ [z_1 : z_2] &\longmapsto z := \frac{z_2^{m_1}}{z_1^{m_2}}. \end{aligned}$$

It follows that for any $p \in S^3 \setminus (\{0\} \times S^1)$, the tangent linear map of the projection is given by

$$\pi_*(p) = \left(-m_2 \frac{z_2^{m_1}}{z_1^{m_2+1}}, m_1 \frac{z_2^{m_1-1}}{z_1^{m_2}} \right)$$

and the vector fields X_1 and X_2 are

$$X_1(z) = -(m_2^2|z_2|^2 + m_1^2|z_1|^2)iz, \quad X_2(z) = -(m_2|z_2|^2 + m_1|z_1|^2)z.$$

These together imply the following:

$$\Omega_z = \frac{-m_1m_2|z_1|^2|z_2|^2}{(m_2^2|z_2|^2 + m_1^2|z_1|^2)^2(m_2|z_2|^2 + m_1|z_1|^2)|z|^2} i\, \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

$$\begin{aligned} c_1(L, \mathbb{C}P^1(m_1, m_2)) &= \frac{1}{2\pi} \int_{\mathbb{C}P^1(m_1,m_2)} \Omega \\ &= -\int_0^\infty \frac{2m_1m_2r(1-r)}{(m_2^2 + (m_1^2 - m_2^2)r)^2(m_2 + (m_1 - m_2)r)} \frac{\mathrm{d}\rho}{\rho} \\ &= \int_0^1 \frac{m_1m_2}{(m_2^2 + (m_1^2 - m_2^2)r)^2} \mathrm{d}r = \frac{1}{m_1m_2}, \end{aligned}$$

where $r = |z_1|^2$, $\rho = |z|$, and $\rho = \frac{(1-r)^{\frac{m_1}{2}}}{r^{\frac{m_2}{2}}}$.

Computation of $\chi(\mathbb{C}P^1(m_1, m_2))$. The quotient metric g induced on $\mathbb{C}P^1(m_1, m_2)$ by the standard metric of S^3 is uniquely determined by the requirement that the following two vector fields of the tangent space of $\mathbb{C}P^1(m_1, m_2)$ at $z \in \mathbb{C} \setminus \{0\}$ constitute an orthonormal base:

$$e_1(z) := \frac{X_1(z)}{|\widetilde{X}_1(z)|} = \lambda_1(z)iz, \quad e_2(z) := \frac{X_2(z)}{|\widetilde{X}_2(z)|} = \lambda_2(z)z,$$

where $\lambda_j(z) := \widetilde{\lambda}_j \circ \gamma^{-1}(|z|^2)$, $\widetilde{\lambda}_1(t) := -\frac{\sqrt{(m_1^2-m_2^2)t+m_2^2}}{\sqrt{t(1-t)}}$, $\widetilde{\lambda}_2(t) := -\frac{(m_1-m_2)t+m_2}{\sqrt{t(1-t)}}$ and γ is the diffeomorphism $\gamma(r) := \frac{(1-r)^{m_1}}{r^{m_2}}$, for $r \in (0,1)$ and $|z|^2 = \frac{(1-|z_1|^2)^{m_1}}{|z_1|^{2m_2}} = \gamma(|z_1|^2)$. Let Θ be the Levi-Civita connection 1-form of the frame bundle pulled back via the trivialization given by e_1 to a 1-form on M, i.e.

$$\Theta(v) := g(\nabla_v e_2, e_1) = g([e_1, e_2], v).$$

We first compute the Lie bracket:

$$[e_1, e_2] = \lambda_1\lambda_2[iz, z] + \lambda_1 \,\mathrm{d}\lambda_2(iz)z - \lambda_2 \,\mathrm{d}\lambda_1(z)iz = -\frac{\mathrm{d}\lambda_1(e_2)}{\lambda_1} e_1,$$

since $[iz, z] = 0$ and $\mathrm{d}\lambda_j = 2(\widetilde{\lambda}_j \circ \gamma^{-1})'(|\cdot|^2)z$, for $j = 1, 2$, which implies $\mathrm{d}\lambda_2(iz) = 0$. Secondly, we compute the Gaussian curvature of $\mathbb{C}P^1(m_1, m_2)$:

$$\kappa = \mathrm{d}\Theta(e_1, e_2) = -\mathrm{d}(g([e_1, e_2], e_1))(e_2) - \Theta([e_1, e_2]) = \mathrm{d}\left(\frac{\mathrm{d}\lambda_1(e_2)}{\lambda_1}\right)(e_2) - \left(\frac{\mathrm{d}\lambda_1(e_2)}{\lambda_1}\right)^2$$

Hence $\mathrm{d}\Theta = \kappa e_1^* \wedge e_2^* = -\frac{\kappa}{|\cdot|^2\lambda_1\lambda_2}\,\mathrm{d}x \wedge \mathrm{d}y$. By the orbifold Gauß–Bonnet theorem it follows that

$$\chi(\mathbb{C}P^1(m_1, m_2)) = \frac{1}{2\pi}\int_{\mathbb{C}P^1(m_1,m_2)} \mathrm{d}\Theta = \frac{-1}{2}\int_0^\infty \frac{\kappa}{|\cdot|^2\lambda_1\lambda_2}\,\mathrm{d}|z|^2 = \frac{1}{2}\int_0^1 \frac{\kappa(r)\gamma'(r)}{\widetilde{\lambda}_1(r)\widetilde{\lambda}_2(r)\gamma(r)}\,\mathrm{d}r,$$

since the functions λ_j are radial and thus κ is also radial. Substituting κ in the last integral, we get

$$\begin{aligned}\chi(\mathbb{C}P^1(m_1, m_2)) =& 2\int_0^1 \left(\left(\frac{\widetilde{\lambda}_2\widetilde{\lambda}_1'\gamma}{\widetilde{\lambda}_1\gamma'}\right)' \frac{\gamma\widetilde{\lambda}_2}{\gamma'} - \left(\frac{\widetilde{\lambda}_2\widetilde{\lambda}_1'\gamma}{\widetilde{\lambda}_1\gamma'}\right)^2 \right) \frac{\gamma'}{\widetilde{\lambda}_1\widetilde{\lambda}_2\gamma}\,\mathrm{d}r \\ =& 2\int_0^1 \left(\frac{\widetilde{\lambda}_2\widetilde{\lambda}_1'\gamma}{\widetilde{\lambda}_1\gamma'}\right)' \frac{1}{\widetilde{\lambda}_1} - \frac{\widetilde{\lambda}_2(\widetilde{\lambda}_1')^2\gamma}{\widetilde{\lambda}_1^3\gamma'}\,\mathrm{d}r = 2\left[\frac{\widetilde{\lambda}_2\widetilde{\lambda}_1'\gamma}{\widetilde{\lambda}_1^2\gamma'}\right]_0^1 = \frac{1}{m_1} + \frac{1}{m_2}.\end{aligned}$$

References

1. K. Akutagawa and A. Neves, *3-manifolds with Yamabe invariant greater than that of* $\mathbb{RP}^3$, J. Differential Geom. **75** (2007), no. 3, 359–386.
2. B. Ammann, M. Dahl, and E. Humbert, *A surgery formula for the smooth Yamabe invariant*, Oberwolfach Report no. 41/2007, 2007.
3. B. Ammann, M. Dahl, and E. Humbert, *Smooth Yamabe invariant and surgery*, J. Diff. Geom. **94** (2013), 1–58.
4. ———, *Square-integrability of solutions of the Yamabe equation*, Comm. Anal. Geom. **21** (2013), 891–916.
5. ———, *The conformal Yamabe constant of product manifolds*, Proc. AMS **141** (2013), 295–307.
6. ———, *Low-dimensional surgery and the Yamabe invariant*, J. Math. Soc. Japan **67** (2015), 159–182.
7. M. T. Anderson, *Scalar curvature and geometrization conjectures for* 3*-manifolds*, Comparison geometry (Berkeley, CA, 1993–94), Math. Sci. Res. Inst. Publ., vol. 30, Cambridge Univ. Press, Cambridge, 1997, pp. 49–82.
8. ———, *Scalar curvature and the existence of geometric structures on 3-manifolds I*, J. Reine Angew. Math. **553** (2002), 125–182.
9. ———, *Canonical metrics on 3-manifolds and 4-manifolds*, Asian J. Math. **10** (2006), 127–163.
10. T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.*, J. Math. Pur. Appl., IX. Ser. **55** (1976), 269–296.
11. ———, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
12. L. Bérard Bergery, *Scalar curvature and isometry group*, Kaigai Publications (1983), 9–28.
13. A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, no. 10, Springer-Verlag, 1987.
14. B. Botvinnik and J. Rosenberg, *The Yamabe invariant for non-simply connected manifolds*, J. Differential Geom. **62** (2002), 175–208.
15. H. L. Bray and A. Neves, *Classification of prime 3-manifolds with Yamabe invariant greater than* $\mathbb{RP}^3$, Ann. of Math. (2) **159** (2004), no. 1, 407–424.
16. Y.-J. Chiang, *Harmonic maps of V-manifolds*, Ann. Global Anal. Geom. **8** (1990), no. 3, 315–344.
17. M. J. Gursky and C. LeBrun, *Yamabe invariants and* Spin^c *structures*, Geom. Funct. Anal. **8** (1998), no. 6, 965–977.
18. B. Hanke, *Positive scalar curvature with symmetry*, J. Reine Angew. Math. **614** (2008), 73–115.
19. E. Hebey and M. Vaugon, *Le problème de Yamabe équivariant*, Bull. Sci. Math. **117** (1993), 241–286.
20. B. Kleiner and J. Lott, *Notes on Perelman's papers*, Geom. Topol. **12** (2008), no. 5, 2587–2855.

21. O. Kobayashi, *Scalar curvature of a metric with unit volume*, Math. Ann. **279** (1987), 253–265.
22. H. B. Lawson and S. T. Yau, *Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres*, Comment. Math. Helv. **49** (1974), 232–244.
23. C. LeBrun, *Four-manifolds without Einstein metrics*, Math. Res. Lett. **3** (1996), no. 2, 133–147.
24. ———, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom. **5** (1997), no. 3, 535–553.
25. ———, *Kodaira dimension and the Yamabe problem*, Comm. Anal. Geom. **7** (1999), no. 1, 133–156.
26. F. Madani, *Equivariant Yamabe problem and Hebey–Vaugon conjecture*, J. Func. Anal. **258** (2010), 241–254.
27. ———, *Hebey–Vaugon conjecture II*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 17-18, 849–852.
28. M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Diff. Geom. **6** (1971/72), 247–258.
29. P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics, Vol. 291, Springer-Verlag, Berlin-New York, 1972.
30. J. Petean, *The Yamabe invariant of simply connected manifolds*, J. Reine Angew. Math. **523** (2000), 225–231.
31. J. Petean and G. Yun, *Surgery and the Yamabe invariant*, Geom. Funct. Anal. **9** (1999), no. 6, 1189–1199.
32. I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492.
33. R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in calculus of variations, Lect. 2nd Sess., Montecatini/Italy 1987, Lect. Notes Math. 1365, 120-154, 1989.
34. P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487.
35. S. Stolz, *Manifolds of positive scalar curvature*, Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), ICTP Lect. Notes, vol. 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 661–709.
36. M. Wiemeler, *Circle actions and scalar curvature*, Arxiv preprint, 2013, `http://arxiv.org/abs/1305.2288`, to appear in Trans. Am. Math. Soc.
37. ———, *Invariant metrics of positive scalar curvature on S^1-manifolds*, Arxiv preprint, 2015, `http://arxiv.org/abs/1506.04073`,.

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