ARE ALL DIRAC-HARMONIC MAPS UNCOUPLED?

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Abstract. Dirac-harmonic maps (f, φ) consist of a map $f : M \to N$ and a twisted spinor $\varphi \in \Gamma(\Sigma M \otimes f^*TN)$ and they are defined as critical points of the super-symmetric energy functional. A Dirac-harmonic map is called uncoupled, if f is a harmonic map. We show that under some minimality assumption Dirac-harmonic maps defined on a closed domain are uncoupled.

We can apply similar arguments to the heat flow (f_t, φ_t) for Dirac-harmonic maps. If M has no boundary or if the boundary condition for the Dirac equation is homogeneous, similar arguments show that f_t is a solution of the classical heat equation. However, for inhomogeneous boundary data, this conclusion no longer holds true in general.

1. Introduction

Let M and N be Riemannian manifolds, and $f : M \to N$ a C^1. If M is compact, we can define the energy of f as

$$\mathcal{E}^{\mathrm{class}}(f) = \frac{1}{2} \int_M |\mathrm{d}f|^2 \,\mathrm{dvol}^g$$

considered as a functional on $C^1(M, N)$. Critical points of this functional are called harmonic maps, and there is an extensive literature about harmonic maps, with many interesting applications, also weak solutions were studied. In the recent years, there is a growing number of publications about a super-symmetric analogue of harmonic maps, called Dirac-harmonic maps. They are defined as critical points of the sypersymmetric energy functional $\mathcal{E}$ defined in (1), see Section 2 for more details. In particular, a Dirac-harmonic map is a pair (f, φ) of a map $f : M \to N$ and a twisted spinor $\varphi \in \Gamma(\Sigma M \otimes f^*TN)$. An early article [6] about such maps was written in 2005 by Chen, Jost, Li and Wang, studying regularity issues for Dirac-harmonic maps, followed by [7, 9] and stimulating many associated questions and results. Between 2005 and September 2022, MathSciNet lists about 45 publications with "Dirac-harmonic" in the title. Many questions answered for harmonic maps can be discussed in the Dirac-harmonic context, often it is hard to include the spinorial part into the estimates, and good progress was achieved.

Obiously, Dirac-harmonic maps with $\varphi \equiv 0$ are uninteresting, as then (f, φ) is Dirac-harmonic if and only if f is harmonic; such solution are called spinor-trivial. Similarly, solutions with f constant, called map-trivial solutions, are not in the focus of interest within this subject either; in this case the problem is equivalent to finding harmonic spinors in the classical sense, see e. g. [12, 4, 1]. We say that a Dirac-harmonic map is uncoupled if f is harmonic, otherwise we say that this

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Dirac-harmonic map is coupled. Being uncoupled is equivalent to the vanishing of the $\mathcal{R}$-term, see Section 2. A major question discussed in this article is whether coupled Dirac-harmonic maps with compact domain exist. Let us discuss examples of Dirac-harmonic maps in the literature first.

Progress in constructing Dirac-harmonic maps was achieved in [14] by using (untwisted) harmonic spinors, twistor spinors and similar solutions of other spinorial equations, although it remained unclear, how one could get solutions of these spinorial equations. This was analyzed later by Ginoux and the author in [3]; we showed – see [3, Theorems 1.1 and 1.3] – that the conditions in [14, Theorems 1 and 3] can be satisfied only in exceptional cases.[1] In particular, for $\dim M \geq 3$ and M complete, one concludes that M has to be simply-connected of constant negative curvature. Other strong obstructions exist for $\dim M = 2$, in particular f has to be necessarily harmonic, thus any Dirac-harmonic map with compact domain obtained this way is uncoupled. The solutions in [14, Theorems 2], reproven as [3, Corollary 2.3] are uncoupled as well.

Many more Dirac-harmonic maps (f, φ) were constructed in [2]. Here, the domain M of f is closed. The method starts with a harmonic map $f : M \to N$ and then one uses index theory to get a non-vanishing harmonic spinor $\varphi \in \Gamma(\Sigma M \otimes f^*T^*N)$. In particular, these solutions are uncoupled.

In the special case $M = N = S^2$ with the round metric, it was shown in [23] that every Dirac-harmonic map is uncoupled, and this was recently reproven in [15, Proposition 1.1].

Summarizing this, the author is not aware of any publication in which the existence of any coupled Dirac-harmonic map with compact domain was proven.

In the current article, we will show, that "generically" every Dirac-harmonic map with closed domain is uncoupled, see Corollary 3 for details. The argument may be adapted to compact manifolds M with boundary for many suitable homogeneous boundary conditions, an extension of the results that we will not work out.•1.1

•1.1: *more details*

It thus remains questionable whether coupled Dirac-harmonic maps with closed domain actually exist. Suppose such solutions did not exist, then the main results in several recent publications would not provide new result, compared to what is known already for harmonic maps. However, spectral flow methods might possibly be used to construct coupled Dirac-harmonic maps with closed domain. Thus it is hard to predict whether an overseen reason implies $\mathcal{R} = 0$ for M closed, or whether we did not yet find the right tools to find coupled solutions.

Our method also applies to the Dirac-harmonic map heat flow as discussed in [8] and [20]. In this situation the genericity condition is trivially satisfied. As a consequence it turns out that several results in [8] and [20] may be obtained as corollaries from well-known statements about the heat flow for harmonic maps.

What we do not discuss here. In order to avoid missunderstandings, we want to add here some issues that we *do not* prove in this article. First, there is a variant of the super-symmetric energy functional that includes a quartic spinor term, depending on the curvature of the target manifold N, see e. g. [13, (2.4)]. Orally, I was told, that this variant is even more important from the perspective of applications to physics. I am unable to adapt the arguments of the current article to this modified functional. On the other hand I expect that many publications about

[1] Note that the proof of [14, Theorems 1] uses that the immersion is isometric, although this is not stated in this theorem [14, Theorems 1] explicitly.

Dirac-harmonic maps are sufficiently robust in order to be applicable also to critical points of this modified functional. Similar arguments apply to other perturbations of the supersymmetric energy functional. As a consequence, the techniques in the articles listed above would provide valuable contributions even when a non-existence result for coupled solution on closed domains were finally proven.

Second, we did not yet discuss the case of non-compact domains. Our argument relies essentially on the fact that the Euler-Lagrange equations (2) describe the stationary points of a well-defined functional, that $\not{D}^f$ is self-adjoint and that we do not get boundary terms by partial integration. In [14] and [3] an example of a Dirac-harmonic map with non-vanishing $\mathcal{R}$-term is (implicitly) given: here M^m is a simply connected complete Riemannian manifold of constant negative sectional curvature $-4/(m+2)$ (i. e. a space form), and f is an isometric embedding into $(m+1)$-dimensional hyperbolic space N, for which $f(M)$ is totally umbillic with parallel shape tensor in N. Due to the conformal covariance of the Dirac operator and of the Penrose operator, M carries many twistor spinors and harmonic spinors, thus the Jost-Mo-Zhu method [14] may be applied, and one easily checks that $\tau(f)$ is a non-zero constant. However, in this case $|\mathrm{d}f|^2$ is not integrable, and thus the super-symmetric energy functional $\mathcal{E}$ does not converge. The Dirac-harmonic map (f,φ) solves the system of partial differential equations (2) without being the stationarity equation of a functional defined on all pairs of maps with spinors.

Third, we only considered solutions to the Dirac-harmonic map equation in the strong sense. We do not know to which extend our methods also generalize to weak settings. For the harmonic maps, weak solutions gave rise to involved research, see e.g. Bethuel's work [5].

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2. Dirac-harmonic maps with closed domain are generically uncoupled

Let (M,g) be a compact Riemannian spin manifold, and (N,h) a Riemannian manifold. For a map $f : M \to N$ and a twisted spinor $\varphi \in \Gamma(\Sigma M \otimes f^*TN)$ we define

$$\begin{aligned} \mathcal{E}^{\mathrm{class}}(f) &:= \frac{1}{2}\int_M |\mathrm{d}f|^2 \,\mathrm{dvol}^g \\ \mathcal{E}^{\mathrm{odd}}(f,\varphi) &:= \frac{1}{2}\int_M \langle \varphi, \not{D}^f \varphi\rangle \,\mathrm{dvol}^g \\ \mathcal{E}(f,\varphi) &:= \mathcal{E}^{\mathrm{class}}(f) + \mathcal{E}^{\mathrm{odd}}(f,\varphi)\,. \end{aligned} \tag{1}$$

The functional $\mathcal{E}$ is called the super-symmetric energy functional. We define

$$\begin{aligned}\tau(f) &:= -\frac{\partial\mathcal{E}^{\mathrm{class}}}{\partial f} = \mathrm{tr}\nabla df \in \Gamma(f^*TN)\\ \mathcal{R}(f,\varphi) &:= \frac{\partial\mathcal{E}^{\mathrm{odd}}}{\partial f}\\ \Psi(f,\varphi) &:= \frac{\partial\mathcal{E}^{\mathrm{odd}}}{\partial\varphi} = D\!\!\!\!/^{f}\varphi\,.\end{aligned}$$

In these equations, $D\!\!\!\!/^{f}$ denotes the twisted Dirac operator acting on $\Gamma(\Sigma M \otimes f^*TN)$. Further, we used the canonical L^2-scalar product on $\Gamma(f^*TN)$ and on $\Gamma(\Sigma \otimes f^*TN)$, in order to identify the partial derivatives of $\mathcal{E}^{\mathrm{class}}$ and $\mathcal{E}^{\mathrm{odd}}$, given above, with the corresponding gradients.

Stationary points of $\mathcal{E}$ are called Dirac-harmonic maps. Thus this condition is equivalent to

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$$D\!\!\!\!/^{f}\varphi = 0 \text{ and } \tau(f) = \mathcal{R}(f,\varphi). \tag{2}$$

We say that (f,φ) is uncoupled, if $\mathcal{R}(f,\varphi) = 0$. In this publication we only consider *smooth* Dirac-harmonic maps.

Remark. Note that $\Sigma M \otimes f^*TN$ will always denote the real tensor product, even when f^*TN and ΣM carry natural complex structures. Note that ΣM may be defined as a real, a complex, a quaternionic or a Cl_m-linear Dirac operator, this does not matter. For simplicity we only restrict to the complex version in this article, as it is classically the most studied one.

Remark. The terminology for the $\mathcal{R}$-term is standard, see e.g. [jost.mo.zhu:09 14, Page 1514, after (9)], [chen.jost.sun.zhu:19 8, (1.3)].

R.null

Theorem 1. *Let (f_0,φ_0) be given. We assume that there is an open neighborhood U of f_0, and a C^1-map $U \ni f \mapsto \hat\varphi(f)$, such that $D\!\!\!\!/^{f}(\hat\varphi(f)) = 0$ and such that $\hat\varphi(f_0) = \varphi_0$. Then (f_0,φ_0) is uncoupled.*

Proof. Obviously we have $\mathcal{E}^{\mathrm{odd}}(f,\hat\varphi(f)) = 0$. In the following we write d/df for the total[2] derivative with respect to f and – as before – $\partial/\partial f$ for the partial derivative with respect to f. We thus get

$$\begin{aligned}-\tau(f) &= \frac{d}{df}\mathcal{E}^{\mathrm{class}}(f)\\ &= \frac{d}{df}\Big(\mathcal{E}\big(f,\hat\varphi(f)\big)\Big)\\ &= \frac{\partial\mathcal{E}}{\partial f}\Big|_f + \frac{\partial\mathcal{E}}{\partial\varphi}\Big|_{(f,\hat\varphi(f))}\frac{\partial\hat\varphi}{\partial f}\Big|_f\\ &= \frac{\partial\mathcal{E}^{\mathrm{class}}}{\partial f}\Big|_f + \frac{\partial\mathcal{E}^{\mathrm{odd}}}{\partial f}\Big|_{(f,\hat\varphi(f))} + \underbrace{\frac{\partial\mathcal{E}^{\mathrm{odd}}}{\partial\varphi}\Big|_{(f,\hat\varphi(f))}}_{=\Psi(f,\hat\varphi(f))=0}\frac{\partial\hat\varphi}{\partial f}\Big|_f\\ &= -\tau(f) + \mathcal{R}(f,\hat\varphi(f)).\end{aligned}$$

and this obviously implies $\mathcal{R}(f_0,\varphi_0) = 0$. □

[2]In the sense used in Lagrangian mechanics: we derive the expression $f \mapsto \mathcal{E}(f,\hat\varphi(f))$

Corollary 2. *Suppose f_0 has a neighborhood U such that for all $f \in U$ we have $\dim \not{D}^f \geq \dim \not{D}^{f_0}$. Then for every $\varphi_0 \in \ker \not{D}^{f_0}$ the pair (f_0, φ_0) is uncoupled.*

Or as a special case we get.

dirac-harm.trivial

Corollary 3. *Suppose (f_0, φ_0) is a Dirac-harmonic map with M closed. Then*

(1) (f_0, φ_0) is uncoupled, or

(2) any neighborhood U of f_0 contains an $f \in U$ with $\dim \not{D}^f < \dim \not{D}^{f_0}$.

This corollary shows that the construction of coupled Dirac-harmonic maps with compact domain is difficult. This fits to the fact that to the author's knowledge no coupled Dirac-harmonic map with closed domain has been found so far.

Note that these results may be easily generalized to compact manifolds with boundary with suitable homogeneous boundary conditions.[2.1] However, our methods are not robust enough to generalize to more general type of functionals $\mathcal{E}$, e. g. to the one with quartic spinor term discussed in [13, (2.4)].

•2.1: More here

Remark 4. For studying harmonic maps, Sacks and Uhlenbeck [19] applied a perturbation of the energy functional successfully, namely they defined for $\alpha \in (1,2)$

$$\mathcal{E}_\alpha(f) = \frac{1}{2}\int_M (1 + |\mathrm{d}f|^2)^\alpha \,\mathrm{dvol}^g.$$

Recently, a similar perturbation was also used for the super-symmetric energy functional, i. e. one defines $\mathcal{E}(f,\varphi) = \mathcal{E}_\alpha(f) + \mathcal{E}^{\mathrm{odd}}(f,\varphi)$, see e. g. [16, Eq. (1.2)], [17] and [18]. Obviously Theorem 1 remains true for this modification. The condition of Theorem 1 are satisfied as index theoretical methods and minimal kernel methods are used. As a consequence all α-Dirac-harmonic maps in [17] are uncoupled and most of the results for (α)-Dirac-harmonic maps in [17] directly follow from the corresponding statements for (α)-harmonic maps of the same article. In the results in [16, Eq. (1.1)] an additional perturbation F is allowed, which makes our trick non-applicable, while it applies to stationary points of (1.2) in the same article. Again, it seems unanswered whether coupled α-Dirac-harmonic maps with closed domain exist.

3. Application to the heat flow for Dirac-harmonic maps and similar problems

We now want to apply our argument to the heat flow for Dirac-harmonic maps, provided that no inhomogeneous boundary data are assumed. We try do carry out or procedure generalization as large as possible in order to allow a large class of situation. Unfortunately, the larger setup requires a bit more formalism, but the idea is simple again. Some of the formalism we work out here, in fact is used in many articles on Dirac-harmonic maps already implicitely without mentioning it, e. g. if one studies the gradient flow of a function defined the Banach manifold

$$\left\{(f,\varphi) \mid f \in C^1(M,N), \quad \varphi \in \Gamma(\Sigma M \times f^*TN)\right\}$$

then the Banach manifold has to equipped with a Riemannian metric in order to define a gradient, and this requires a connection as explained below (See: Connection of $\Sigma\mathcal{C}$).

Let M be a (finite-dimensional) smooth compact Riemannian manifold, possibly with boundary.[3.1] Let $\mathcal{S}$ be the set[3.2]of generalized spinor bundles, in the sense of Gromov and Lawson, [].[3.3], possibly by allowing additional parallel structures (as

•3.1: oriented?

•3.2: Not even a set, but need manifold!

•3.3: Details!

e. g. Cl_n-linear structures, quaternionic or Real structures), also real bundles are allowed.

Example 5. *In case M carries a spin structure, then we have a well-defined spinor bundle ΣM, which is already an element of $\mathcal{C}$. Now, let $\mathcal{T}$ be the space of vector bundles V*[3.4] *over M, with a scalar product on the fibers, smoothly depeding on the base point, and a compatible connection. We will say $\mathcal{T}$ is the space of twist bundles. Then $V \mapsto \Sigma M \otimes V$ defines a map $\mathcal{T} \to \mathcal{S}$.*

•3.4: which kind

Remark. With a suitable specification of additional parallel structures on ΣM and the elements of $\mathcal{S}$, etc...[3.5], one can modify these definitions such that the map $\mathcal{T} \to \mathcal{S}$ is a bijection up to equivalence of bundles. We do elaborate on this, as this is not needed here.

•3.5: Details!

Furthermore we assume that $\mathcal{C}$ is a Banach manifold, modeled on a reflexive Banach space (the model Banach space). The Banach manifold $\mathcal{C}$ will be called the configuration space. Further we assume that it is equipped with a Finsler metric in the sense, that any tangent space of $\mathcal{C}$ is equipped with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{C}}$, continuous with respect to the norm on the model Banach space, and that this scalar product depends continuously on the base point. We have a differentiable function $\mathcal{E}^{\mathrm{class}}: \mathcal{C} \to \mathbb{R}$, denoted as the classical energy. We assume that there is a continuous vector fields on $\mathcal{C}$, called the gradient of $\mathcal{E}^{\mathrm{class}}$, written as $\operatorname{grad} \mathcal{E}^{\mathrm{class}}$, such that the differential $\mathrm{d}_f \mathcal{E}^{\mathrm{class}}$ of $\mathcal{E}^{\mathrm{class}}$ at $f \in \mathcal{C}$ satisfies

$$\mathrm{d}_f \mathcal{E}^{\mathrm{class}} = \left\langle \operatorname{grad} \mathcal{E}^{\mathrm{class}}, \cdot \right\rangle .$$

For expressing that the gradient is a continuous vector field on $\mathcal{C}$ we write $\operatorname{grad} \mathcal{E}^{\mathrm{class}} \in C^0(\mathcal{C}; T\mathcal{C})$.

Furthermore we assume that a C^1-map $t: \mathcal{C} \to \mathcal{S}$ is given.

Examples 6.

(1) *We assume that (N, h) is a Riemannian manifold. Let $F: \partial M \to N$ be a given smooth*[3.6] *map. We set*

•3.6: time-dependent?

$$\mathcal{C} := \left\{ f \in C^1(M, N) \mid f\big|_{\partial M} = F \right\} .$$

Then $T_f \mathcal{C}$ can (and will) be naturally identified with C^1-sections of $f^* TN$ that vanish at the boundary of M. In particular, the value of this section at $x \in M$ is an element of $T_{f(x)} N$. On two such sections $V, W \in \Gamma(f^* TN)$ we may defined the scalar product $(V, W) := \int_M h_{f(x)}(V(x), W(x)) \,\mathrm{dvol}^g(x)$.

$df \in \Gamma(T^* M \otimes f^* TN)$ Classical harmonic energy

$$\mathcal{E}^{\mathrm{class}}(f) := \frac{1}{2} \int_M |\mathrm{d} f|^2 \,\mathrm{dvol}^g$$

(2) *Let ... be as above. For $\alpha \geq 1$ we define the α-energy $\mathcal{E}^{\mathrm{class}}_\alpha$ as*

$$\mathcal{E}^{\mathrm{class}}_\alpha(f) := \frac{1}{2} \int_M (1 + |\mathrm{d} f|^2)^\alpha \,\mathrm{dvol}^g.$$

Obviously, we have $\mathcal{E}^{\mathrm{class}}_1(f) = \mathcal{E}^{\mathrm{class}}(f) + \frac{1}{2} \mathrm{vol}(M, g)$.

(3) *Other powers?*

Definition of $\Sigma\mathcal{C}$. Connection on $\Sigma\mathcal{C}$. Defines Riemannian metric on $\Sigma\mathcal{C}$.

For $f \in \mathcal{C}$ let $\not{D}^f$ the associated Dirac operator, we define the odd part of the energy as

$$\mathcal{E}^{\text{odd}}(f,\varphi) := \int_M \langle \varphi, \not{D}^f \varphi \rangle \, \mathrm{dvol}^g \, .$$

So, finally we have the (total) supersymmetric energy

$$\mathcal{E}^{\text{super}}(f,\varphi) := \mathcal{E}^{\text{class}}(f) + \mathcal{E}^{\text{odd}}(f,\varphi) \, .$$

We now fix a connected component $\mathcal{C}_0$ of $\mathcal{C}$. We define

$$d_{\min} := d_{\min}(\mathcal{C}_0) := \min_{f \in \mathcal{C}_0} \dim \ker \not{D}^f \, .$$

Furthermore we define

$$\mathcal{C}_{\min} := \left\{ f \in \mathcal{C}_0 \mid \dim \ker \not{D}^f = d_{\min} \right\} \, .$$

Obviously this is an open and non-empty subset of $\mathcal{C}_0$ and thus of $\mathcal{C}$.

Definition 7 (homogeneous and inhomogeneous boundary conditions)**.** *The boundary condtion $\mathcal{B}$ is called homogeneous, if $\mathcal{B}$ is a linear subspace of ..., otherwise it is called inhomogeneous.*

For homogeneous boundary conditions $\mathcal{B}$, we now consider the gradient $G_{\text{super}}(f,\varphi)$ of the supersymmetric energy functional at (f,φ) and compare it to the gradient of the classical energy functional $G_{\text{class}}(f)$.

Proposition 8. *Suppose that $f \in$[3.7] $\mathcal{C}_{\min}$ and that $\varphi \in \ker \not{D}^f_{\mathcal{B}}$ for a homogeneous boundary condition Then we have*

•3.7: wth respect to the identification given in ...

$$G_{\text{super}}(f,\varphi) = (G_{\text{class}}(f), 0) \, .$$

The proof is a minor modification of the proof of Theorem R.null 1 and its corollaries.

Assumption $(GF_T(f_0))$: for some given initial condition f_0, the gradient flow exists on $[0,T)$ with initial condition f_0.

Theorem 9. *Let $f_0 \in \mathcal{C}_{\min}$ and $\varphi_0 \in \ker \not{D}^f_{\mathcal{B}}$, and we assume $(GF_T(f_0))$. Let $\mathcal{B}$ be a homogeneous boundary condition. Let f_t be the solution of the classical gradient flow (??). We define*

$$T_1 := \inf \left(\left\{ t \in [0,T) \mid \dim \ker \not{D}^{f_t}_{\mathcal{B}} > d_{\min} \right\} \cup \{T\} \right) \, .$$

As $\mathcal{C}_{\min}$ is open in $\mathcal{C}$ it is obvious that $0 < T_{\le} T$.

We define φ_t as the parallel transport along f_t with initial condition φ_0 and with respect to the connection given in !!![3.8]. Then (f_t, φ_t) is a solution of the supersymmetric gradient flow (??) on the interval $[0,T_1)$. Every solution to the supersymmetric gradient flow is obtained this way, thus if the classical theory yields uniqueness, we also get it in the supersymmetric context.

•3.8: !!!

Remark 10. The statements above require homogeneity of the boundary conditions. For inhomogeneous boundary conditions a further terms arises which is a non-local boundary-to-interior term which is in some sense of lower order. However, in the literature, most regularity theorems or results about short time existence in the classical context are essentially local. Thus one expects that for inhomogeneous boundary conditions, such local estimates can be taken over almost literally from the classical context.

4. Application to the heat flow for Dirac-harmonic maps

We now want to apply our argument to the heat flow for Dirac-harmonic maps, as introduced in [8]. This is the following geometric-elliptic problem. One considers now maps $f_t : M \to N$ and spinors $\varphi_t \in \Gamma(\Sigma M \otimes f_t^* TN)$ depending on a parameter $t \in [0,T]$, with some suitable regularity, whose precise definition will be omitted in this short note, see [8, (1.5)] for details. Note that M is non-empty, connected, oriented and spin, and furthermore M might have non-empty boundary ∂M. The heat flow for Dirac-harmonic maps is given by the following equation

eq.1
$$\left.\begin{aligned} \partial_t f_t &= \tau(f_t) - \mathcal{R}(f_t,\varphi_t) \\ D\!\!\!/^{f_t}\varphi_t &= 0 \end{aligned}\right\} \text{ on } M \text{ for all } t \in [0,T]. \tag{3}$$

In the literature, two separate cases are discussed, the case $\partial M \neq \emptyset$ and the case $\partial M = \emptyset$. In both cases we need additional assumptions for obtaining a well-posed problem.

The case $\partial M \neq \emptyset$ was discussed in [8] and then one has to specify boundary conditions $\mathcal{B}$. For simplicity we restrict to the case of the chirality operator $\mathcal{B} = \mathcal{B}^\pm := \frac{1}{2}(\mathrm{Id} \pm \mathbf{n}\cdot G)$ where a sign in $\pm$ is chosen, where $\mathbf{n}$ is the unit outward normal field, and where G is a chiral operator (e. g. the usual the grading operator of the spinor bundle, in case M is even-dimensional). Another example is the MIT bag boundary conditions, see [8, Subsection 2.2] for further details.

Now, let us assume $\dim M \geq 2$. In order to define boundary conditions, following [8], we fix a t-dependent map $F_t : M \to N$ and a t-dependent spinor $\Phi_t \in \Gamma(\Sigma M \otimes F_t^* TN)$. For simplicity of presentation we assume that $F_t(x)$ and $\Phi_t(x)$ depend smoothly on t and x.

eq.2
$$f_t(x) = F_t(x) \qquad \text{for all } (x,t) \in M \times \{0\} \cup \partial M \times [0,T] \tag{4}$$

eq.3
$$\mathcal{B}\varphi_t = \mathcal{B}\Phi_t \qquad \text{on } \partial M \times [0,T] \tag{5}$$

A short-time existence and uniqueness result was derived, see [8, Theorem 1.3].

The case $\partial M = \emptyset$ was discussed in [20]. The investigations were a part of Wittmann's PhD thesis [21] under the supervision of the author of this short note. In this case (4) reduces to

eq.init.dat
$$f_0 = F_0 \tag{6}$$

and (5) is trivially satisfied. However in this situation the spinorial part of the equation is too free to expect a well-posed problem, thus one adds the normalization

eq.normed
$$\int_M \langle \varphi_t, \varphi_t \rangle = 1\,. \tag{7}$$

In the sequel we write (3)&(6)&(7) for the system of equations given by (3), (6), and (7).

In the case $\partial M = \emptyset$, a short-time existence result was obtained for (3)&(6)&(7) assuming that the kernel of the involved Dirac operators remains 1-dimensional in the sense of Dahl [11].[3] In order to explain, what we mean by "1-dimensional" here, one uses the fact that $\Sigma M \otimes f^T N$ carries a quaternionic structure if $m := \dim M \equiv$

[3] In fact [20] was written after [8]. It was heavily inspired and motivated by [8], although it uses different analytic arguments, namely a fixed point argument derived from a contraction.

$2, 3, 4 \mod 8$. This implies that $\ker \not{D}^f$ is a (free)[4] $\mathbb{K}$-vector space with $\mathbb{K} = \mathbb{C}$ for $m \equiv 0, 1, 5, 6, 7 \mod 8$ and $\mathbb{K} = \mathbb{H}$ for $m \equiv 2, 3, 4 \mod 8$. Following Dahl we consider the dimension with respect to $\mathbb{K}$, i. e. we claim $\dim_{\mathbb{K}} \ker \not{D}^{f_t} = 1$ for all $t \in [0, T]$.

Furthermore, Wittmann showed that this solution is unique "up to gauge", provided the 1-dimensionalilty condition holds for all $t \in [0, T]$. More precisely, Wittmann obtained uniqueness up to a t-dependend factor in $\mathbb{S}_{\mathbb{K}}$, where $\mathbb{S}_{\mathbb{K}}$ is the unit sphere in $\mathbb{K}$. Multiplication by a function $[0, T] \to \mathbb{S}_{\mathbb{K}}$ will be considered as a gauge transformation. It preserves solutions of (3)&(6)&(7), and for $\dim_{\mathbb{K}} \ker \not{D}^f = 1$ the solution is unique up to such a transformation.

Examples satisfying the 1-dimensionality condition were derived in [22].

Because of the smoothness assumption for F_t and Φ_t, one obtains smooth solutions f_t and φ_t in both cases (see [8] for $\partial M \neq \emptyset$ and [20] for $\partial M = \emptyset$).

In both cases the spinor φ_f satisfying $\not{D}^f \varphi_f = 0$ and $\int_M \langle \varphi_f, \varphi_f \rangle = 1$ is unique – for $\partial M = \emptyset$: up to gauge. In the boundary case this is discussed in [8], see Cor. 3.7 for $m = 2$ and the considerations with the weak unique continuation property (WUCP, Thm 3.9). In the boundary free case this is an immediate consequence of the assumption $\dim_{\mathbb{K}} \ker \not{D}^f = 1$. This uniqueness together with standard implicit function theorem arguments implies that one can choose φ_f to depend smoothly on f (at least locallly close to some given $f_0 : M \to N$). Theorem 1 thus says that the $\mathcal{R}$-term of the solution φ_t vanishes.

We thus have proven:

Theorem 11. *For $\partial M \neq \emptyset$: If (f_t, φ_t) is a solution of (3)–(5), then f_t is a solution of the heat flow for harmonic maps (in the classical sense) with boundary*

$$\partial_t f_t = \tau(f_t) \text{ on } M$$
$$f_t(x) = F_t(x) \text{ for all } (x, t) \in M \times \{0\} \cup \partial M \times [0, T]$$

For $\partial M = \emptyset$: If (f_t, φ_t) is a solution of (3)&(6)&(7) with $\dim_{\mathbb{K}} \ker \not{D}^{f_t} = 1 \quad \forall t$, then f_t is a solution of the heat flow for harmonic maps, i. e. we have

$$\partial_t f_t = \tau(f_t)$$
$$f_0 = F_0$$

on all of M.

Obviously if f_t is a solution to the classical harmonic map heat flow (with boundary), the uniqueness – possibly up to gauge – implies that we obtain a solution of (3)–(5) resp. (3)&(6)&(7). Thus the main statement in [8] and [20] may also be deduced from classical results as described above.

5. Comments to literature

- [10] Qun Chen, Jrgen Jost, Guofang Wang, Miaomiao Zhu *The boundary value problem for Dirac-harmonic maps*: General theory about boundary values of spinors coupled to maps.

[4]In fact, any module over a skew field is free, thus we follow the standard convention to say "$\mathbb{K}$-vector space" instead of "$\mathbb{K}$-module", and we will use the word "dimension" instead of "rank" in the following.

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