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Dirac eigenvalue estimates on surfaces

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Abstract. We prove lower Dirac eigenvalue bounds for closed surfaces with a spin structure whose Arf invariant equals 1. Besides the area only one geometric quantity enters in these estimates, the spin-cut-diameter $\delta(M)$ which depends on the choice of spin structure. It can be expressed in terms of various distances on the surfaces or, alternatively, by stable norms of certain cohomology classes. In case of the 2-torus we obtain a positive lower bound for all Riemannian metrics and all nontrivial spin structures. For higher genus g the estimate is given by

$$|\lambda| \geq \frac{2\sqrt{\pi}}{(2g+1)\sqrt{\mathrm{area}(M)}} - \frac{1}{\delta(M)}.$$

The corresponding estimate also holds for the L^2-spectrum of the Dirac operator on a noncompact complete surface of finite area. As a corollary we get positive lower bounds on the Willmore integral for all 2-tori embedded in $\mathbb{R}^3$.

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1 Introduction

Relating analytic invariants of the Dirac operator such as the eigenvalues to the geometry of the underlying manifold is in general a difficult problem. Explicit computation of the spectrum is possible only in cases

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of very large symmetry, see [4,6,7,10,12–14,16,18,24,33,34,36,39,40, 42–45,47,48] for examples. In general, the best one can hope for are geometric bounds on the eigenvalues. The first lower eigenvalue bounds [17], [25],[26],[27],[28] for the Dirac spectrum require positivity of the scalar curvature since they are based on variations of the Lichnerowicz formula $D^2 = \nabla^*\nabla + \mathrm{scal}/4$. Refining this technique Hijazi [22,23] could estimate the smallest Dirac eigenvalue against the corresponding eigenvalue of the Yamabe operator. A completely different approach building on Sobolev embedding theorems was used by Lott [31] and the first author [2] to show that for each closed spin manifold M and each conformal class $[g_0]$ on M there exists a constant $C = C(M,[g_0])$ such that all nonzero Dirac eigenvalues λ with respect to all Riemannian metrics $g \in [g_0]$ satisfy

$$\lambda^2 \geq \frac{C}{\mathrm{vol}(M)^{2/n}}.$$

On the 2-sphere $M = S^2$ there is only one conformal class of metrics (up to the action of the diffeomorphism group) and we therefore get a nontrivial lower bound for all metrics. Lott conjectured that in this case the optimal constant should be $C = 4\pi$. Returning to the Bochner technique the second author showed that this is in fact true:

Theorem 1.1 ([5, Theorem 2]). *Let λ be any Dirac eigenvalue of the 2-sphere S^2 equipped with an arbitrary Riemannian metric. Then*

$$\lambda^2 \geq \frac{4\pi}{\mathrm{area}(S^2)}.$$

Equality is attained if and only if S^2 carries a metric of constant Gauss curvature.

In particular, there are no harmonic spinors on S^2. Theorem 1.1 will be the central tool to derive our new estimates in the present paper. Examples [7], [41] show that such an estimate is neither possible for higher dimensional spheres nor for surfaces of higher genus, at least not in this generality. Every closed surface of genus at least 1 has a spin structure and a metric such that 0 is an eigenvalue, i. e. there are nontrivial harmonic spinors [18], [24]. The 2-torus T^2 has four spin structures one of which is called trivial and the others nontrivial. Provided with the trivial spin structure, T^2 has harmonic spinors for all Riemannian metrics. On the other hand, for the three nontrivial spin structures 0 is never an eigenvalue. So it should in principle be possible to give a geometric lower bound in this latter case. The problem is that this estimate must take into account the choice of spin structure but the Bochner technique is based on local computation where the spin structure is invisible. Hence new techniques are needed.

The first estimate using information from the choice of spin structure has been derived by the first author [1, Corollary 2.4]. On a torus with a Riemannian metric and a nontrivial spin structure there is a lower bound for any eigenvalue λ of the Dirac operator. Let K denote Gauss curvature. Recall that the systole is the minimum of the lengths of all noncontractible closed curves. The spinning systole spin-sys(T^2) is the minimum of the lengths of all noncontractible simple closed curves, along which the spin structure is nontrivial. If there exists $p > 1$ with $\|K\|_{L^p} \cdot \mathrm{area}(T^2)^{1-(1/p)} < 4\pi$, then there is a positive number $C > 0$ such that

$$\lambda^2 \geq \frac{C}{\text{spin-sys}(T^2)^2}.$$

Here C is an explicitly given expression in p, $\|K\|_{L^p}$, the area, and the systole.

The Arf invariant associates to each spin structure on a closed surface the number 1 or -1. In case of the 2-torus the Arf invariant of the trivial spin structure is -1 while the three nontrivial spin structures have Arf invariant 1. In the present paper we prove explicit geometric lower bounds for the first eigenvalue of the square of the Dirac operator on closed surfaces M of genus ≥ 1 provided the spin structure has Arf invariant 1. Only two geometric quantities enter, the area of the surface and an invariant we call the spin-cut-diameter $\delta(M)$. The number $\delta(M)$ is defined by looking at distances between loops in the surface along which the spin structure is nontrivial and which are linearly independent in homology. It exists if and only if the Arf invariant of the spin structure equals 1. It can also be defined in terms of stable norms of certain cohomology classes which depend on the choice of spin structure (Proposition 4.1).

In the case of a 2-torus we show:

Theorem 5.1. *Let T^2 be the 2-torus equipped with an arbitrary Riemannian metric and a spin structure whose Arf invariant equals 1. Let λ be an eigenvalue of the Dirac operator and let $\delta(T^2)$ be the spin-cut-diameter. Then for any $k \in \mathbb{N}$,*

$$|\lambda| \geq -\frac{2}{k\,\delta(T^2)} + \sqrt{\frac{\pi}{k\,\mathrm{area}(T^2)} + \frac{2}{k^2\delta(T^2)^2}}.$$

The right hand side of this inequality is positive for sufficiently large k. Hence this theorem gives a nontrivial lower eigenvalue bound for the Dirac operator for all Riemannian metrics and all nontrivial spin structures on the 2-torus.

Similarly, for higher genus we obtain:

Theorem 6.1. *Let M be a closed surface of genus $g \geq 1$ with a Riemannian metric and a spin structure whose Arf invariant equals 1. Let $\delta(M)$ be the spin-cut-diameter of M. Then for all eigenvalues λ of the Dirac operator we have*

$$|\lambda| \geq \frac{2\sqrt{\pi}}{(2g+1)\sqrt{\mathrm{area}(M)}} - \frac{1}{\delta(M)}.$$

In the case $g = 1$ this estimate is simpler but weaker than Theorem 5.1. Every surface of genus $g \geq 2$ admits metrics and spin structures such that this estimate is nontrivial. But in contrast to the first theorem there are also Riemannian metrics and spin structures on surfaces of genus $g \geq 1$ for which the right hand side of this inequality is negative although there are no harmonic spinors.

If one restricts one's attention to surfaces embedded in $\mathbb{R}^3$, then one has the Willmore integral $W(M)$ defined as the integral of the square of the mean curvature. It is well-known that the Willmore integral can be estimated against Dirac eigenvalues. Thus as a corollary to Theorem 5.1 we obtain

Theorem 7.1 *Let $T^2 \subset \mathbb{R}^3$ be an embedded torus. Let $\delta(T^2)$ be its spin-cut-diameter and let $W(T^2)$ be its Willmore integral. Then for any $k \in \mathbb{N}$*

$$\sqrt{W(T^2)} \geq \sqrt{\frac{\pi}{k} + \frac{2\,\mathrm{area}(T^2)}{k^2\,\delta(T^2)^2}} - \frac{2\sqrt{\mathrm{area}(T^2)}}{k\,\delta(T^2)}$$

In the end of the paper we show that our spectral estimates also work for noncompact complete surfaces of finite area. In this case the spectrum need not consist of eigenvalues only. We estimate the fundamental tone of the square of the Dirac operator which gives the length of the spectral gap about 0 in the L^2-spectrum, see Theorem 8.1.

The paper is organized as follows. We start by recalling some basic definitions related to spin structures and Dirac operators on surfaces. We put some emphasis on the case of a surface embedded in $\mathbb{R}^3$. We then recall the Arf invariant and define the spin-cut-diameter $\delta(M)$. In Sect. 4 we show how $\delta(M)$ relates to the stable norm of certain cohomology classes. In Sects. 5 and 6 we prove Theorems 5.1 and 6.1. The central idea of proof consists of constructing a surface of genus 0 out of the given surface by cutting and pasting. Then we apply Theorem 1.1. The estimate for the Willmore integral is proved in Sect. 7 and in Sect. 8 we study the L^2-spectrum of noncompact complete surfaces of finite area.

2 Dirac operators on surfaces

Let M be an oriented surface with a Riemannian metric. Rotation by 90 degrees in the positive direction defines a complex multiplication J on TM. The bundle $\mathrm{SO}(M)$ of oriented orthonormal frames is an S^1-principal bundle over M. Let SM be the bundle of unit tangent vectors on M. Then $v \mapsto (v, Jv)$ is a fiber preserving diffeomorphism from SM to $\mathrm{SO}(M)$ with inverse given by projection to the first vector.

Let $\Theta : S^1 \to S^1$ be the nontrivial double covering of S^1. A spin structure on M is an S^1-principal bundle $\mathrm{Spin}(M)$ over M together with a twofold covering map $\theta : \mathrm{Spin}(M) \to \mathrm{SO}(M)$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Spin}(M) \times S^1 & \to & \mathrm{Spin}(M) & & \\ & & & \searrow & \\ \downarrow \theta \times \Theta & & \downarrow \theta & & M \\ & & & \nearrow & \\ \mathrm{SO}(M) \times S^1 & \to & \mathrm{SO}(M) & & \end{array} \tag{1}$$

commutes.

Every orientable surface admits a spin structure, but it is in general not unique. The number of possible spin structures on M equals the number of elements in $H^1(M, \mathbb{Z}_2)$.

Example. Let $i : M \hookrightarrow \mathbb{R}^3$ be an immersion of an oriented surface (not necessarily compact, and possibly with boundary) into $\mathbb{R}^3$. We define a map $i_* : \mathrm{SO}(M) \to \mathrm{SO}(3)$ as follows: $(v, Jv) \in \mathrm{SO}(M)$ over a basepoint $m \in M$ is mapped to $(v, Jv, v \times Jv) \in \mathrm{SO}(3)$. Here $\times$ denotes the vector cross product in $\mathbb{R}^3$. Let $\mathrm{Spin}(M)$ be the pullback of the double covering $\Theta_3 : \mathrm{Spin}(3) \to \mathrm{SO}(3)$, i. e.

$$\begin{aligned} &\mathrm{Spin}(M) \\ &:= \left\{ ((v, Jv), A) \in \mathrm{SO}(M) \times \mathrm{Spin}(3) \,\middle|\, i_*(\mathrm{SO}(M)) = \Theta_3(A) \right\}. \end{aligned}$$

Then $\mathrm{Spin}(M) \to \mathrm{SO}(M)$ is a fiberwise nontrivial double covering. Let $\pi : \mathrm{SO}(M) \times \mathrm{Spin}(3) \to \mathrm{SO}(M)$ be the projection onto the first component. Then $(\mathrm{Spin}(M), \pi|_{\mathrm{Spin}(M)})$ is a spin structure on M, the *spin structure induced by the immersion.*

Let $\gamma : S^1 \to M$ be an immersion or, in other words, a regular closed curve. Then the vector field $\frac{\dot\gamma}{|\dot\gamma|}$ is a section of SM along γ, which, by the above diffeomorphism from SM to $\mathrm{SO}(M)$, yields the section $(\frac{\dot\gamma}{|\dot\gamma|}, J\frac{\dot\gamma}{|\dot\gamma|})$ of $\mathrm{SO}(M)$ along γ.

Definition. The spin structure $(\mathrm{Spin}(M), \theta)$ is said to be *trivial along* γ if this section lifts to a closed curve in $\mathrm{Spin}(M)$ via θ.

This notion is invariant under homotopic deformation of γ within the class of immersions.

Example. The unique spin structure on $\mathbb{R}^2$ is nontrivial along any simple closed curve. More generally, any spin structure on a surface M is nontrivial along any contractible simple closed curve.

Proposition 2.1. *Let $i : M \hookrightarrow \mathbb{R}^3$ be an immersion. Let $\gamma : S^1 \to M$ be a simple closed curve. If γ is a parametrization of the boundary of an immersed two-dimensional disk $j : D \hookrightarrow \mathbb{R}^3$ intersecting $i(M)$ transversally, then the spin structure on M induced by i is nontrivial along γ.*

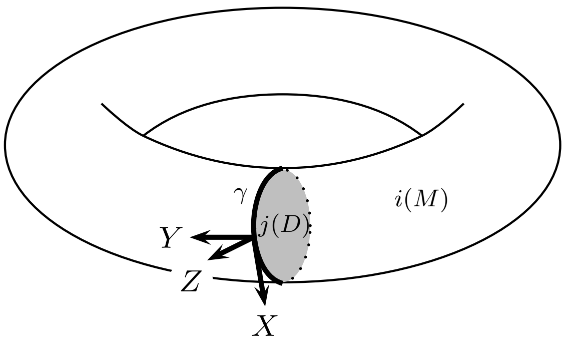

Fig. 1. A criterion for nontriviality of the spin structure along γ

Proof. We can assume that $j(D)$ and $i(M)$ intersect orthogonally along γ. We set $X(t) := \frac{\dot\gamma(t)}{|\dot\gamma(t)|}$, $Y(t) := J_M X(t)$ and $Z(t) := X(t) \times Y(t)$. The induced spin structure on M is trivial along γ if and only if $S^1 \to \mathrm{SO}(3), t \mapsto (X(t), Y(t), Z(t))$, lifts to a closed loop in $\mathrm{Spin}(3)$. Analogously, we view γ as a curve on $j(D)$, we define the vector fields $\hat{Y}(t) := J_D X(t)$ and $\hat{Z}(t) := X(t) \times \hat{Y}(t)$. Because of the orthogonality of $j(D)$ and $i(M)$ we have $\hat{Y}(t) = \pm Z(t)$ and $\hat{Z}(t) = \mp Y(t)$.

Hence $t \mapsto (X(t), \hat{Y}(t), \hat{Z}(t))$ lifts to $\mathrm{Spin}(3)$ if and only if $t \mapsto (X(t), Y(t), Z(t))$ lifts. The induced spin structure on M is nontrivial along γ if and only if the spin structure on D is nontrivial along γ. This is always true according to the previous example. □

Example. Let $Z := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$ be the cylinder with the induced spin structure. Let γ be any simple closed curve in Z. We show that the spin structure is nontrivial along γ: If γ is contractible then the spin structure is nontrivial because of the preceeding example. If γ is noncontractible, then $[\gamma]$ generates $\pi_1(Z)$ (Lemma A.1). Hence it bounds a disk transversal to Z.

Let $\Sigma^+M := \mathrm{Spin}(M) \times_\iota \mathbb{C}$ be the complex line bundle over M associated to the S^1-principal bundle $\mathrm{Spin}(M)$ and to the standard representation $\iota : S^1 \to U(1)$. This line bundle is called the *bundle of positive half-spinors,* its complex conjugate $\Sigma^-M := \overline{\Sigma^+M}$ is the *bundle of negative half-spinors* and their sum $\Sigma M := \Sigma^+M \oplus \Sigma^-M$ is the *spinor bundle*.

Clifford multiplication consists of complex linear maps

$$TM \otimes_{\mathbb{C}} \Sigma^+M \to \Sigma^-M$$
$$\overline{TM} \otimes_{\mathbb{C}} \Sigma^-M \to \Sigma^+M$$

denoted by $v \otimes \sigma \mapsto v \cdot \sigma$. It satisfies the Clifford relations

$$v \cdot w \cdot \sigma + w \cdot v \cdot \sigma + 2\langle v, w \rangle\, \sigma = 0$$

for all $v, w \in TM$ and $\sigma \in \Sigma M$ over a common base point.

The Levi-Civita connection on TM gives rise to a connection-1-form on $\mathrm{Spin}(M)$ and this in turn defines a Hermitian connection ∇ on ΣM.

Definition. The *Dirac operator* D is a map from smooth sections of ΣM to smooth sections of ΣM which is locally given by the formula

$$D\Psi := e_1 \cdot \nabla_{e_1}\Psi + e_2 \cdot \nabla_{e_2}\Psi$$

for a local orthonormal frame (e_1, e_2) of TM.

It is easily checked that the definition does not depend on the choice of the local frame and that D is a formally self-adjoint elliptic operator. Hence, if M is closed, the spectrum of D is real and discrete with finite multiplicities.

For any smooth function f and smooth spinor Ψ the equation

$$D(f\Psi) = \nabla f \cdot \psi + fD\Psi$$

holds. Here ∇f denotes the gradient of f.

For more background material on Dirac operators and spin structures see e. g. [30], [19], or [38].

To simplify notation a *closed* surface will always mean a surface which is compact, without boundary, and *connected*.

3 Arf invariant and spin-cuts

In this section we review some properties of the Arf invariant which is an invariant of a spin structure on a surface (see [37] for more details). For closed oriented surfaces with spin structures whose Arf invariant equals 1 we define a geometric quantity, the spin-cut-diameter, which will play an important role in our estimate.

Let V be a $2g$-dimensional vector space over the field $\mathbb{Z}_2$, $g \in \mathbb{N}$, together with a symplectic 2-form $\omega : V \to \mathbb{Z}_2 = \{0, 1\}$. A *quadratic form* on (V, ω) is a map $q : V \to \mathbb{Z}_2$, such that

$$q(a+b) = q(a) + q(b) + \omega(a,b) \qquad a, b \in V.$$

The difference of two quadratic forms on (V, ω) is a linear map from V to $\mathbb{Z}_2$ and vice versa the sum of a linear map $V \to \mathbb{Z}_2$ and a quadratic form is again a quadratic form. Hence the space of quadratic forms on V is an affine space over $\mathrm{Hom}(V, \mathbb{Z}_2)$.

Example. Let M be a closed oriented surface. Let $V := H_1(M, \mathbb{Z}_2)$ and let ω be the intersection form $\cap$. Fix a spin structure on M. We associate to each spin structure a quadratic form q_{spin} on (V, ω) as follows. Each homology class $a \in H_1(M, \mathbb{Z}_2)$ is represented by an embedding $\gamma : S^1 \to M$. We set $q_{\mathrm{spin}}(a) := 1$, if $(\dot\gamma, J(\dot\gamma)) : S^1 \to \mathrm{SO}(M)$ lifts to $\mathrm{Spin}(M)$, otherwise we set $q_{\mathrm{spin}}(a) := 0$.

According to Theorem 1 of [29] the map q_{spin} is a well-defined quadratic form on $(H_1(M, \mathbb{Z}_2), \cap)$.

The set of all spin structures on M is an affine space over $H^1(M, \mathbb{Z}_2) = \mathrm{Hom}(H_1(M, \mathbb{Z}_2), \mathbb{Z}_2)$ and it is a well known fact that the map which associates to any spin structure the corresponding quadratic form q_{spin} is an isomorphism of affine $H^1(M, \mathbb{Z}_2)$-spaces from the space of spin structures on M to the space of quadratic forms on $(V, \omega) = (H_1(M, \mathbb{Z}_2), \cap)$.

Definition. For any quadratic form q on (V, ω) the *Arf invariant* is defined by

$$\mathrm{Arf}(q) := \frac{1}{\sqrt{\#V}} \sum_{a \in V} (-1)^{q(a)}.$$

The Arf invariant of a quadratic form corresponding to a spin structure will be called the *Arf invariant* of that spin structure.

Lemma 3.1. *Let q_i be a quadratic form on (V_i, ω_i) for $i = 1, 2$. Then $q_1 \oplus q_2$, given by*

$$(q_1 \oplus q_2)(v_1 + v_2) = q_1(v_1) + q(v_2),$$

is a quadratic form on $(V_1 \oplus V_2, \omega_1 \oplus \omega_2)$. Moreover,

$$\mathrm{Arf}(q_1 \oplus q_2) = \mathrm{Arf}(q_1)\mathrm{Arf}(q_2).$$

The proof is a simple counting argument. □

Any $2g$-dimensional symplectic vector space V with a symplectic form ω is isomorphic to the g-fold sum $V_2 \oplus \cdots \oplus V_2$ where V_2 is the standard 2-dimensional symplectic vector space. Since the Arf invariants of the four possible choices of quadratic forms on V_2 are either 1 or -1 the above lemma implies

$$\mathrm{Arf}(q) \in \{-1, +1\}$$

for any quadratic form q on any symplectic $\mathbb{Z}_2$-vector space.

Proposition 3.2. *Let q be a quadratic form on (V, ω), $\dim V = 2g$. Then the following statements are equivalent:*

(1) $\mathrm{Arf}(q) = 1$.
(2) *There is a basis $e_1, f_1, \ldots, e_g, f_g$ of V such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, $\omega(e_i, f_j) = \delta_{ij}$, and $q(e_i) = q(f_j) = 0$ for all i, j.*
(3) *There are linearly independent vectors $e_1, \ldots, e_g$ in V such that $\omega(e_i, e_j) = 0$ and $q(e_i) = 0$ for all i, j.*

Proof. (2)⇒(1) follows directly from Lemma 3.1.

To show (3)⇒(2) let $e_1, \ldots, e_g$ be linearly independent vectors with $\omega(e_i, e_j) = 0$ and $q(e_i) = 0$ for all i, j. Since ω is symplectic, we can find $\tilde{f}_1, \ldots, \tilde{f}_g$ satisfying $\omega(e_i, \tilde{f}_j) = \delta_{ij}$ and $\omega(\tilde{f}_i, \tilde{f}_j) = 0$ for all i, j. If $q(\tilde{f}_i) = 0$, we set $f_i := \tilde{f}_i$, otherwise we put $f_i := \tilde{f}_i + e_i$.

To see (1)⇒(3), we take a basis $e_1, f_1, \ldots, e_g, f_g$ of V satisfying $\omega(e_i, f_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. For every i exactly one of the following holds:

(a) $q(e_i) = q(f_i) = q(e_i + f_i) = 1$, or
(b) q takes the value 0 at exactly two of the vectors e_i, f_i and $e_i + f_i$.

In the second case, we can assume without loss of generality that $q(e_i) = q(f_i) = 0$. Let I be the set of all i for which (a) holds. Then by Lemma 3.1 $\mathrm{Arf}(q) = (-1)^{\#I}$. If (1) holds, then $\#I$ is even, hence we may assume $I := \{1, \ldots, 2k\}$. For $j = 1, \ldots, k$ we replace e_{2j-1} by $e_{2j-1} + f_{2j}$ and e_{2j} by $e_{2j} + f_{2j-1}$. Then (3) holds. □

Example. Let $M \hookrightarrow \mathbb{R}^3$ be an embedded closed surface with the induced spin structure. Then because of Propositions 2.1 and 3.2 (3) the Arf invariant of the spin structure is 1. As a consequence any immersion $M \hookrightarrow \mathbb{R}^3$ whose induced spin structure has Arf invariant -1 is not regularly homotopic to an embedding.

Remark. In the literatur the 3 spin structures on the 2-torus T^2 with Arf invariant 1 are called *nontrivial* spin structures and the unique spin structure with Arf invariant -1 is called the *trivial* spin structure.

Definition. Let M be a closed oriented surface of genus g. A *cut* of M is a family of pairwise disjoint simple closed curves $\gamma_i : S^1 \to M, i = 1, \dots, g$, such that $[\gamma_1], \dots, [\gamma_g]$ are linearly independent in $H_1(M, \mathbb{Z})$. If, in addition, M carries a spin structure, and if the spin structure is nontrivial along each of the γ_i, then we call $\gamma_1, \dots, \gamma_g$ a *spin-cut* of M.

Corollary 3.3. *Let M be a closed oriented surface equipped with a spin structure. Then M admits a spin-cut if and only if the Arf invariant of the spin structure equals* 1.

Proof. If the Arf invariant is 1, we can find vectors $e_1, \dots, e_g \in H_1(M, \mathbb{Z}_2)$ for which (3) of Proposition 3.2 holds. For each e_i we choose a preimage $\tilde{e}_i \in H_1(M, \mathbb{Z})$ under the natural map $H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z}_2)$. We choose $\tilde{e}_i$ such that $\tilde{e}_i$ is primitive, i. e. there are no $a_i \in H_1(M, \mathbb{Z})$, $n \geq 2$ with $e_i = n \cdot a_i$. This choice can be made such that $\tilde{e}_i \cap \tilde{e}_j = 0$ for all i, j. We choose a hyperbolic metric g_{hyp} on M and represent $\tilde{e}_i$ by closed curves γ_i of minimal length. Then the γ_i are closed geodesics. They are simple closed curves because the $\tilde{e}_i$ are primitive. Since $\tilde{e}_i \cap \tilde{e}_j = 0$ and g_{hyp} is hyperbolic, γ_i and γ_j are disjoint for $i \neq j$. The spin structure is nontrivial along each γ_i because of $q_{\mathrm{spin}}(e_i) = 0$. Hence $\gamma_1, \dots, \gamma_g$ form a spin-cut of M.

Conversely, if $\gamma_1, \dots, \gamma_g$ form a spin-cut of M, then $[\gamma_1], \dots, [\gamma_g] \in H_1(M, \mathbb{Z})$ form a linearly independent set of primitive elements in $H_1(M, \mathbb{Z})$. Hence their images e_i in $H_1(M, \mathbb{Z}_2)$ are also linearly independent. The e_i satisfy (3) of Proposition 3.2 and thus the Arf invariant is 1. □

Definition. Let M be a closed surface. Let $\gamma_1, \dots, \gamma_g$ be a cut. The *cut-open* $\widetilde{M}$ of M is a surface with boundary, such that there is a smooth map $\widetilde{M} \to M$ which is a diffeomorphism from the interior of $\widetilde{M}$ onto $M \setminus \bigcup_{j=1}^{g} \gamma_j$ and a twofold covering from the boundary $\partial\widetilde{M}$ onto $\bigcup_{j=1}^{g} \gamma_j$.

Riemannian metrics and spin structures on M can be pulled back to $\widetilde{M}$.

Lemma 3.4. *Let $\gamma_1, \dots, \gamma_g$ be a cut of M. Then the cut-open $\widetilde{M}$ is diffeomorphic to a sphere S^2 with $2g$ disks removed. Moreover, if it is a spin-cut, $\widetilde{M}$ carries the spin structure inherited from S^2.*

Proof. At first we prove that $\widetilde{M}$ is connected. Assume that $\widetilde{M}$ is not connected. This would imply that the boundary of one of the connected components of $\widetilde{M}$ is homologous to zero. Hence a nontrivial linear combination of the $[\gamma_i]$ vanishes which is impossible by the definition of a cut.

Since the Euler characteristic of $\widetilde{M}$ satisfies $\chi(\widetilde{M}) = \chi(M) = 2 - 2g$ and $\widetilde{M}$ has $2g$ boundary circles, it must be diffeomorphic to a sphere S^2 with $2g$ disks removed.

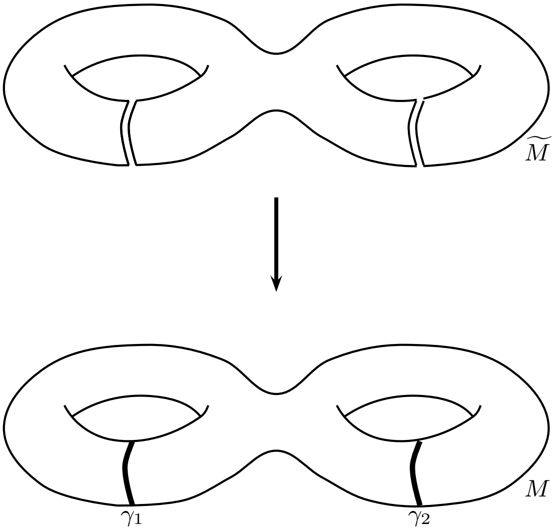

Fig. 2. The cut-open $\widetilde{M}$ and its projection onto M

In the case of a spin-cut, the spin structure is nontrivial along each of the boundary components. Therefore the spin structure extends to the disk which has been removed. Hence $\widetilde{M}$ carries the spin structure which is the pullback of the unique spin structure on S^2 under any injective immersion $\widetilde{M} \hookrightarrow S^2$. $\square$

Definition. Let M be a closed surface with a fixed Riemannian metric and a fixed spin structure with Arf invariant 1. Let $\gamma_1, \dots, \gamma_g$ be a spin-cut. Denote by $\partial_1 \widetilde{M}, \dots, \partial_{2g} \widetilde{M}$ the boundary components of the cut-open $\widetilde{M}$. We define the *cut-diameter* of the spin-cut by

$$\delta(\gamma_1, \dots, \gamma_g) := \min_{1 \le i < j \le 2g} d(\partial_i \widetilde{M}, \partial_j \widetilde{M}),$$

where $d(A, B)$ denotes the length of a shortest path joining A and B. The *spin-cut-diameter* of M is defined as

$$\delta(M) := \sup \delta(\gamma_1, \dots, \gamma_g)$$

with the supremum running over all spin-cuts. The spin-cut-diameter $\delta(M)$ is a finite positive number depending on the surface M, the Riemannian metric and the spin structure.

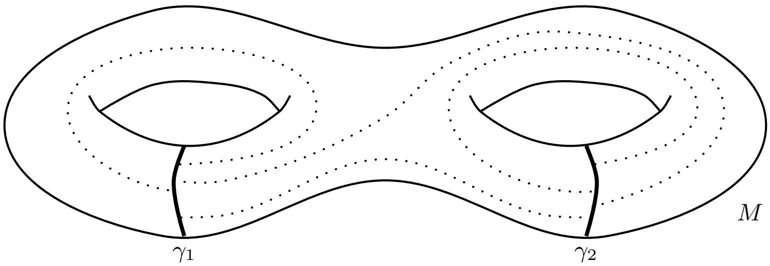

Fig. 3. The cut-diameter is the length of the shortest dotted line (only representatives of 4 of the 6 homotopy classes of lines are shown)

4 Stable norms and the spin-cut-diameter

Let M be a closed Riemannian manifold. In this section we define norms on $H_1(M,\mathbb{R})$ and $H^1(M,\mathbb{R})$, the stable norms, and we recall some of their properties. We will be able to express the spin-cut-diameter defined in the previous section in terms of stable norms of certain cohomology classes which depend on the spin structure. A good reference for stable norms is [21], Chapter 4C. A more detailed exposition of stable norms can be found in [15].

For any $v \in H_1(M,\mathbb{R})$ the *stable norm* is defined as

$$\|v\|_{\mathrm{st}} := \inf \left\{ \sum_{i=1}^{k} |a_i| \cdot \mathrm{length}(c_i) \right\}$$

where the infimum runs over all 1-cycles $\sum_{i=1}^{k} a_i c_i$ representing v with $a_i \in \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$ and $c_i : S^1 \to M$ smooth.

For cohomology classes $\alpha \in H^1(M,\mathbb{R})$ we define the *stable norm* by

$$\|\alpha\|_{\mathrm{st}} := \inf \|\omega\|_{L^\infty},$$

where the infimum runs over all closed smooth 1-forms ω representing α.

These norms are dual to each other in the following sense:

$$\|\alpha\|_{\mathrm{st}} = \sup \{\alpha(v) \,|\, v \in H_1(M,\mathbb{R}), \|v\|_{\mathrm{st}} = 1\},$$

$$\|v\|_{\mathrm{st}} = \sup \left\{\alpha(v) \,|\, \alpha \in H^1(M,\mathbb{R}), \|\alpha\|_{\mathrm{st}} = 1\right\}.$$

We can also characterize the stable norm on $H_1(M,\mathbb{R})$ in terms of lengths of closed curves. For any 1-cycle $v \in H^1(M,\mathbb{R})$ which lies in the image of the map $H^1(M,\mathbb{Z}) \to H^1(M,\mathbb{R})$ the relation

$$\|v\|_{\mathrm{st}} = \inf \left\{ \frac{1}{n}\,\mathrm{length}(\gamma) \,\middle|\, \gamma \text{ is a closed curve representing } nv, n \in \mathbb{N} \right\}$$

holds.

If $M = T^n$, the n-dimensional torus with an arbitrary Riemannian metric, then one can identify $H_1(T^n, \mathbb{R})$ with the universal covering of T^n. Let d be the distance function on $H_1(T^n, \mathbb{R})$ induced by the pullback of the Riemannian metric on T^n. Burago [11] proved that there is a constant C, such that for any $x, y \in H_1(T^n, \mathbb{R})$

$$|d(x, y) - \|x - y\|_{\mathrm{st}}| \leq C.$$

Roughly speaking, this result says that the stable norm is a good approximation for the distance d.

The stable norm also plays a central role in Bangert's criterion [3] for the existence of globally minimizing geodesics on the universal covering $\widetilde{M}$ of a closed Riemannian manifold M. E. g. if $b_1(M) \geq 2$, and if the stable norm on $H_1(M, \mathbb{R})$ is strongly convex, then there are infinitely many geodesics on M whose lifts are globally minimizing geodesics on $\widetilde{M}$.

In the special case that M is a closed orientable surface of positive genus, any closed curve γ representing a nontrivial $[\gamma] = [\alpha]^n \in \pi_1(M)$ with $n \geq 2$ has a self-intersection. To see this, let $\overline{M}$ be the universal covering. We lift γ to $\overline{M}/\langle[\alpha]\rangle$ where $[\alpha]$ acts via deck transformations and apply Lemma A.1 for $S \setminus \{N, S\} \cong \overline{M}/\langle[\alpha]\rangle$. A standard curve shortening argument shows that in this case we can characterize the stable norm of an integral class v as follows:

$$\|v\|_{\mathrm{st}} = \inf \{\mathrm{length}(\gamma) \,|\, \gamma \text{ is a closed curve in } M \text{ representing } v\}.$$

Remark. An intersection argument implies that $\|\cdot\|_{\mathrm{st}}$ is a strictly convex norm on $H_1(T^2, \mathbb{R})$ [32]. In contrast to this, on any surface of genus ≥ 2 the stable norm is not strictly convex [32].

In the remaining part of this section we specialize to the case $M = T^2$, and we will show how the stable norm can be used to express the spin-cut-diameter of a spin structure.

Let $\gamma : S^1 \to T^2$ be a noncontractible simple closed curve along which the spin structure is nontrivial. Then $[\gamma] \in H_1(T^2, \mathbb{Z}) \setminus \{0\}$. We define $\alpha_\gamma \in H^1(T^2, \mathbb{Z})$ via the relation

$$\langle \alpha_\gamma, \beta \rangle = [\gamma] \cap \beta, \qquad \forall \beta \in H_1(T^2, \mathbb{Z}).$$

Proposition 4.1. *Let $\delta(M)$ be the spin-cut-diameter of a 2-torus with spin structure whose Arf invariant equals 1. Let $\gamma_0 : S^1 \to T^2$ be a noncontractible simple closed curve along which the spin structure is nontrivial, i. e. γ_0 is a spin-cut of M. Then for*

$$\delta_0 := \sup\{\delta(\gamma) \,|\, \gamma \text{ is a simple closed curve homotopic to } \gamma_0\}$$

we have

$$\delta_0 = \frac{1}{\|\alpha_{\gamma_0}\|_{\mathrm{st}}}.$$

Proof.

(a) We show $\delta_0 \le 1/\|\alpha_{\gamma_0}\|_{\mathrm{st}}$.
Let $\varepsilon > 0$. Choose a simple closed curve γ homotopic to γ_0 such that $\delta(\gamma) \ge (1+\varepsilon)^{-1}\delta_0$. We cut T^2 along γ. Then the cut-open $\widetilde{M}$ thus obtained is a topological cylinder. Let $\tilde{c} : [a,b] \to \widetilde{M}$ be a curve of minimal length joining the two boundary components $\partial_1\widetilde{M}$ and $\partial_2\widetilde{M}$ of $\widetilde{M}$. Let c be the image of $\tilde{c}$ under the map $\widetilde{M} \to T^2$. Clearly $\mathrm{length}(c) = \mathrm{length}(\tilde{c}) = \delta(\gamma) \ge (1+\varepsilon)^{-1}\delta_0$. Let $f : \widetilde{M} \to [0,\delta_0]$ be a smooth function with the following properties:

$$\begin{aligned} &|df| \le 1 + 2\varepsilon, \\ &f \equiv 0 \quad \text{on a neighborhood of } \partial_1\widetilde{M}, \\ &f \equiv \delta_0 \quad \text{on a neighborhood of } \partial_2\widetilde{M}. \end{aligned}$$

Such an f can be obtained for example by a smooth approximation of the Lipschitz function

$$\begin{aligned} \bar{f} : \widetilde{M} &\to [0,\delta_0], \\ x &\mapsto \frac{\delta_0}{\delta(\gamma)} \min\Big\{d(x,\partial_1\widetilde{M}), \delta(\gamma)\Big\}. \end{aligned}$$

Let ω be the 1-form on T^2 such that df equals the pullback of ω.
We now prove $\delta_0 \cdot \alpha_\gamma = \pm[\omega]$.
Observe that $\omega(\dot{\gamma}(t)) = \frac{d}{dt}\,(f \circ \gamma) \equiv 0$, since f is constant along $\partial_1\widetilde{M}$.
Hence $\int_{\gamma|_I} \omega = 0$ for any $I \subset S^1$. In particular,

$$\langle [\omega], [\gamma] \rangle = \int_\gamma \omega = 0.$$

There are $t_1, t_2 \in S^1$ such that $\gamma(t_1) = c(a)$, $\gamma(t_2) = c(b)$. Let β be the product path $\beta := \gamma|_{[t_2,t_1]} * c$.
Then $[\gamma] \cap [\beta] = \pm 1$. Moreover,

$$\begin{aligned} \langle [\omega], [\beta] \rangle = \underbrace{\int_{\gamma|_{[t_2,t_1]}} \omega}_{=0} + \int_c \omega \;&=\; \int_c df \;=\; f(c(b)) - f(c(a)) \\ = \delta_0 \;&=\; \pm\delta_0\, [\gamma] \cap [\beta] \;=\; \pm\delta_0\, \langle \alpha_\gamma, [\beta] \rangle. \end{aligned}$$

Therefore $[\omega] \mp \delta_0 \cdot \alpha_\gamma$ vanishes on $[\gamma]$ and on $[\beta]$. Since $[\gamma]$ and $[\beta]$ form a basis of $H_1(T^2, \mathbb{Z})$ we obtain $\delta_0 \cdot \alpha_\gamma = \pm[\omega]$.
From

$$\delta_0 \cdot \|\alpha_\gamma\|_{\mathrm{st}} = \|[\omega]\|_{\mathrm{st}} \le \|\omega\|_{L^\infty} \le 1 + 2\varepsilon$$

we get the $\le$-part of the equation by taking the limit $\varepsilon \to 0$.

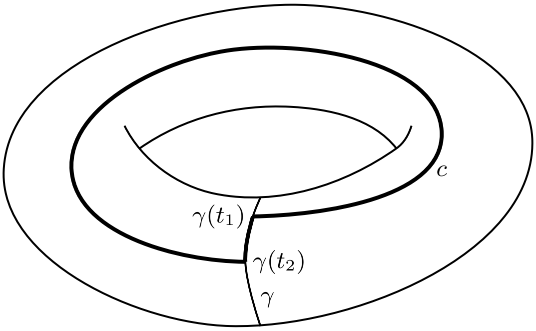

Fig. 4. The curve β in the proof of Proposition 4.1 (thick line)

(b) Now we prove $\delta_0 \geq 1/\|\alpha_{\gamma_0}\|_{\mathrm{st}}$.
We choose a smooth closed 1-form ω on T^2 such that $[\omega] = \alpha_{\gamma_0}$ and $\|\omega\|_{L^\infty} \leq \|\alpha_{\gamma_0}\|_{\mathrm{st}} + \varepsilon$ for small $\varepsilon > 0$. The cyclic subgroup $\langle[\gamma_0]\rangle$ of $\pi_1(T^2)$ generated by $[\gamma_0]$ acts via deck transformations on the universal covering $\mathbb{R}^2$ of T^2. Define the cylinder $Z := \mathbb{R}^2/\langle[\gamma_0]\rangle$. Since $[\gamma_0]$ generates the first cohomology of Z and α_{γ_0} vanishes on $[\gamma_0]$ the pullback of the cohomology class $[\omega] = \alpha_{\gamma_0}$ is trivial on Z. Hence we can find a smooth function $f : Z \to \mathbb{R}$ such that df is the pullback of ω under the covering $Z \to T^2$.
The function f is proper. Without loss of generality we can assume that 0 is a regular value of f. Then $f^{-1}(0)$ is a union of simple closed curves. According to Lemma A.2 there is a simple closed curve γ in $f^{-1}(0)$ whose homotopy class generates $\pi_1(Z)$. Choose the orientation of γ such that γ is homotopic to γ_0. The spin struture is nontrivial along γ, hence γ defines a spin-cut $\widetilde{M} \to M$, i. e. a map which is a diffeomorphism from the interior of $\widetilde{M}$ onto $M \setminus \gamma(S^1)$ and a trivial double covering from $\partial\widetilde{M}$ onto $\gamma(S^1)$.
We can identify $\widetilde{M}$ with a closed subset of Z, and we can assume that $f|_{\partial_1\widetilde{M}} \equiv 0$, $f|_{\partial_2\widetilde{M}} \equiv 1$, where $\partial_1\widetilde{M}$ and $\partial_2\widetilde{M}$ denote the two boundary components of $\widetilde{M}$.
Let $c : [a,b] \to \widetilde{M}$ be a curve of minimal length joining the two boundary components $\partial_1\widetilde{M}$ and $\partial_2\widetilde{M}$. By definition we have $\delta(\gamma) = \mathrm{length}(c)$. It follows

$$\begin{aligned} 1 = f(c(b)) - f(c(a)) &= \int_c df \\ &\leq \mathrm{length}(c)\,\|df\|_{L^\infty} \\ &= \delta(\gamma)\,\|\omega\|_{L^\infty} \end{aligned}$$

$$\leq \delta_0 \Big(\|\alpha\|_{\mathrm{st}} + \varepsilon \Big).$$

The limit $\varepsilon \to 0$ yields $\delta_0 \geq /\|\alpha_{\gamma_0}\|_{\mathrm{st}}$. □

Corollary 4.2. *The spin-cut-diameter satisfies*

$$\delta(M) = \sup \left\{ \frac{1}{\|\alpha_\gamma\|_{\mathrm{st}}} \;\middle|\; \gamma \text{ is a noncontractible simple closed curve along which the spin structure is nontrivial.} \right\}$$

□

5 An estimate for the 2-torus

We now come to the first main result of this paper. We give a geometric lower bound for the eigenvalues of the Dirac operator on a 2-torus which is nontrivial for all metrics and for all spin structures.

Theorem 5.1. *Let T^2 be the 2-torus equipped with an arbitrary Riemannian metric and a spin structure whose Arf invariant equals 1. Let λ be an eigenvalue of the Dirac operator and let $\delta(T^2)$ be the spin-cut-diameter. Then for any $k \in \mathbb{N}$,*

$$|\lambda| \geq -\frac{2}{k\,\delta(T^2)} + \sqrt{\frac{\pi}{k\,\mathrm{area}(T^2)} + \frac{2}{k^2 \delta(T^2)^2}}.$$

Note that the right hand side of this inequality is positive for sufficiently large k, but tends to 0 for $k \to \infty$. The best bound is obtained by choosing

$$k = \left\lfloor 4\,(1+\sqrt{2})\,\frac{\mathrm{area}(T^2)}{\pi\,\delta(T^2)^2} \right\rfloor$$

or

$$k = \left\lfloor 4\,(1+\sqrt{2})\,\frac{\mathrm{area}(T^2)}{\pi\,\delta(T^2)^2} \right\rfloor + 1.$$

Proof. Let γ be a spin-cut, i. e. γ is a simple closed curve in T^2 along which the spin structure is nontrivial. Assume $\delta(\gamma) \geq (1+\varepsilon)^{-1}\,\delta(T^2)$ for small $\varepsilon > 0$.

We now proceed as in part (a) of the proof of Proposition 4.1. On the cut-open $\widetilde{T}^2$ we obtain a function $f : \widetilde{T}^2 \to [0, \delta(T^2)]$ satisfying

$$\begin{aligned} &|df| \leq 1 + 2\varepsilon, \\ &f \equiv 0 \qquad\quad \text{on a neighborhood of } \partial_1 \widetilde{T}^2, \\ &f \equiv \delta(T^2) \quad \text{on a neighborhood of } \partial_2 \widetilde{T}^2. \end{aligned}$$

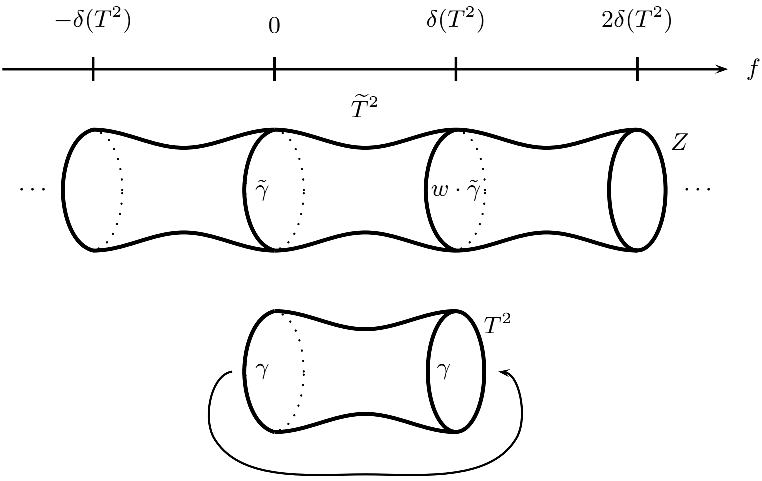

Fig. 5. The cylinder Z and a fundamental domain

Let ω be the 1-form on T^2 such that df equals the pullback of ω.

The homotopy class $[\gamma] \in \pi_1(T^2)$ acts on the universal covering $\mathbb{R}^2$ of T^2, and

$$Z := \mathbb{R}^2/\langle[\gamma]\rangle$$

is a cylinder covering T^2. We pull the metric and the spin structure on T^2 back to a metric and a spin structure on Z.

We fix a $w \in \pi_1(T^2)$ with $[\gamma] \cap w = 1$. Then w generates the deck transformation group of the covering $Z \to T^2$. Let $\tilde{\gamma} : S^1 \to Z$ be a lift of γ. Then $Z \setminus (\tilde{\gamma}(S^1) \cup w \cdot \tilde{\gamma}(S^1))$ consists of three connected components. Two of them are unbounded and one is bounded. The closure of the bounded component can be identified with the cut-open $\widetilde{T}^2$. The function f can then be extended "pseudo-periodically" to Z, more precisely,

$$f(w + p) = \delta(T^2) + f(p) \tag{2}$$

for all $p \in Z$, where w acts as a deck transformation on Z. Note that

$$\operatorname{area}\Big(f^{-1}((t, t + \delta(T^2)])\Big) = \operatorname{area}(\widetilde{T}^2) = \operatorname{area}(T^2).$$

We set

$$T_{-k} := f^{-1}\Big([-k\delta(T^2), 0]\Big),$$

$$T_k := f^{-1}\Big([0, k\delta(T^2)]\Big).$$

Both T_{-k} and T_k are isometric to k copies of $\widetilde{T}^2$ glued together to a cylinder. Similarly, we consider $T_{-k} \cup T_k$ as a cylinder consisting of $2k$ copies of $\widetilde{T}^2$.

We glue two disks to the remaining two boundary components of $T_{-k} \cup T_k$ and obtain a surface N of genus 0. We extend the metric on $T_{-k} \cup T_k$ to one on N such that the total area of the two disk glued in is smaller than ε. Hence

$$\mathrm{area}(N) \leq 2k\,\mathrm{area}(T^2) + \varepsilon.$$

By Proposition 2.1 the spin structure on $T_{-k} \cup T_k$ extends to the unique spin structure on N.

For fixed $k \in \mathbb{N}$ let $X_1 : \mathbb{R} \to [0,1]$ be a smooth function with

$$X_1(t) = 1 \text{ for } t \leq 0,$$
$$X_1(t) = 0 \text{ for } t \geq k,$$
$$|X_1'(t)| \leq \frac{1+\varepsilon}{k} \text{ for all } t.$$

We set $X(t) := X_1(t) - X_1(t+k)$. Then

$$\chi(p) := X\left(\frac{f(p)}{\delta(T^2)}\right)$$

is a compactly supported smooth function on Z with

$$k \cdot \|\nabla\chi\|_{L^\infty} \leq k \cdot \|X'\|_{L^\infty} \cdot \frac{\|df\|_{L^\infty}}{\delta(T^2)} \leq \frac{(1+\varepsilon)(1+2\varepsilon)}{\delta(T^2)} =: a_\varepsilon.$$

We denote the L^2-norm of a spinor φ on a subset A of the manifold on which φ is defined by

$$\|\varphi\|_A := \sqrt{\int_A |\varphi|^2 \,\mathrm{d\,area}}.$$

If A equals the whole manifold we simply write

$$\|\varphi\|_A =: \|\varphi\|.$$

Now let φ be an eigenspinor on T^2 corresponding to an eigenvalue λ of the Dirac operator. By the preceeding lemma, the spin structure pulled back via π extends to the unique spin structure on N. Thus $\chi \cdot \pi^*\varphi$ is a well-defined spinor on N, and we obtain the following estimate

$$\begin{aligned}
\|D(\chi \cdot \pi^*\varphi)\|^2_{T_{-k}} &= \|\nabla\chi \cdot \pi^*\varphi + \chi \cdot D(\pi^*\varphi)\|^2_{T_{-k}} \\
&\leq \left(\frac{a_\varepsilon}{k} \cdot \|\pi^*\varphi\|_{T_{-k}} + |\lambda|\, \|\chi \cdot \pi^*\varphi\|_{T_{-k}}\right)^2 \\
&\leq \frac{a_\varepsilon^2}{k^2} \cdot \|\pi^*\varphi\|^2_{T_{-k}} + \frac{2|\lambda| a_\varepsilon}{k} \|\pi^*\varphi\|^2_{T_{-k}} \\
&\quad + \lambda^2\, \|\chi \cdot \pi^*\varphi\|^2_{T_{-k}} \\
&= \left(\frac{a_\varepsilon^2}{k} + 2|\lambda|\, a_\varepsilon\right) \|\varphi\|^2_{T^2} + \lambda^2\, \|\chi \cdot \pi^*\varphi\|^2_{T_{-k}}. \qquad (3)
\end{aligned}$$

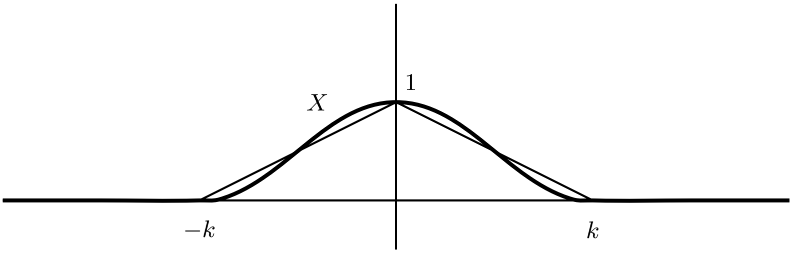

Fig. 6. The graph of $t \mapsto X(t)$

In a similar manner we obtain

$$\|D(\chi \cdot \pi^*\varphi)\|^2_{T_k} \leq \left(\frac{a_\varepsilon^2}{k} + 2|\lambda|\, a_\varepsilon \right) \|\varphi\|^2_{T^2} + \lambda^2 \, \|\chi \cdot \pi^*\varphi\|^2_{T_k}. \qquad (4)$$

From

$$X(t)^2 + X(t-k)^2 = X(t)^2 + (1 - X(t))^2 \in [1/2, 1]$$

for $0 \leq t \leq k$ we obtain

$$\frac{k}{2} \, \|\varphi\|^2_{T^2} \leq \|\chi \cdot \pi^*\varphi\|^2_{T_{-k} \cup T_k} \leq k \|\varphi\|^2_{T^2}$$

which together with (3) and (4) gives

$$\|D(\chi \cdot \pi^*\varphi)\|^2_{T_{-k} \cup T_k} \leq \left\{ 2 \left(\frac{a_\varepsilon^2}{k} + 2|\lambda|\, a_\varepsilon \right) + k \cdot \lambda^2 \right\} \|\varphi\|^2_{T^2}.$$

We plug $\chi\varphi$ into the Rayleigh quotient and use Theorem 1.1 to get

$$\begin{aligned} \frac{4\pi}{2k \,\mathrm{area}(T^2) + \varepsilon} &\leq \frac{4\pi}{\mathrm{area}(N)} \\ &\leq \frac{\|D(\chi \cdot \pi^*\varphi)\|^2_{T_{-k} \cup T_k}}{\|\chi \cdot \pi^*\varphi\|^2_{T_{-k} \cup T_k}} \\ &\leq \frac{2a_\varepsilon^2/k + 4|\lambda|\, a_\varepsilon + k \cdot \lambda^2}{k/2}. \end{aligned}$$

Thus

$$\frac{\pi}{2k \,\mathrm{area}(T^2) + \varepsilon} \leq \frac{a_\varepsilon^2}{k^2} + \frac{2|\lambda|\, a_\varepsilon}{k} + \frac{\lambda^2}{2}.$$

In the limit as $\varepsilon \to 0$ we obtain

$$\frac{\pi}{k \,\mathrm{area}(T^2)} \leq \frac{2}{k^2 \delta(T^2)^2} + \frac{4|\lambda|}{k \,\delta(T^2)} + \lambda^2.$$

Solving this inequality proves the theorem. □

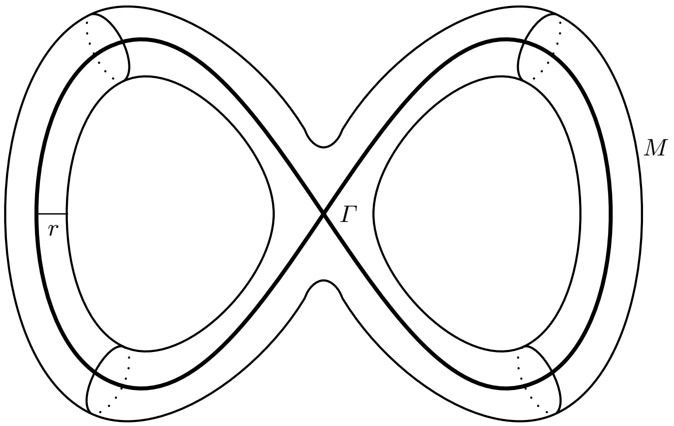

Fig. 7. The boundary M of a small neighborhood of a graph Γ in $\mathbb{R}^3$ has a large maximal spin-cut-diameter compared to the area

6 Compact surfaces of higher genus

Using a similar technique we can also obtain a lower bound for the Dirac spectrum on closed surfaces M of higher genus.

Theorem 6.1. *Let M be a closed surface of genus $g \geq 1$ with a Riemannian metric and a spin structure whose Arf invariant equals 1. Let $\delta(M)$ be the spin-cut-diameter of M. Then for all eigenvalues λ of the Dirac operator we have*

$$|\lambda| \geq \frac{2\sqrt{\pi}}{(2g+1)\sqrt{\mathrm{area}(M)}} - \frac{1}{\delta(M)}.$$

Note that on any closed oriented surface of genus $g \geq 1$ there is a Riemannian metric and a spin structure such that $\delta(M)^2/\mathrm{area}(M)$ is arbitrarily large. To see this take a suitable finite graph Γ embedded in $\mathbb{R}^3$ and let M be the boundary (smoothed out appropriately) of a tubular neighborhood of Γ of small tubular radius $r > 0$. Provide M with the Riemannian metric and the spin structure induced from $\mathbb{R}^3$. Then for $r \to 0$ the spin-cut-diameter stays bounded while the area tends to 0. By Theorem 6.1 the smallest eigenvalue of D^2 must then tend to ∞. Hence any closed oriented surface carries a Riemannian metric and a spin structure such that the above estimate is not trivial.

Theorem 6.1 also holds for $g = 1$ but in this case Theorem 5.1 with $k = 2$ gives a better estimate.

Proof. Let $\gamma_1, \dots, \gamma_g$ be a spin-cut of M. We cut M along the γ_i and obtain the cut-open $\widetilde{M}$. According to Lemma 3.4, $\widetilde{M}$ is a compact orientable surface of genus 0 with $2g$ boundary components. The two boundary components of $\widetilde{M}$ that arise from cutting along γ_i we denote by $\partial_i^1 \widetilde{M}$ and $\partial_i^2 \widetilde{M}$.

We assume that the spin-cut has been chosen such that $\delta(\gamma_1, \ldots, \gamma_g) \geq \delta(M) - \varepsilon$ with $\varepsilon > 0$ small.

We take $2g+1$ copies of $\widetilde{M}$, denoted by $\widetilde{M}_0, \ldots, \widetilde{M}_{2g}$. For $t = 1, \ldots, g$ we glue $\partial_t^1 \widetilde{M}_t$ to $\partial_t^2 \widetilde{M}_0$ and $\partial_t^2 \widetilde{M}_{g+t}$ to $\partial_t^1 \widetilde{M}_0$. The resulting surface S_0 is of genus 0 with $2g(2g-1)$ boundary components. We glue disks to these boundaries and obtain a surface S diffeomorphic to S^2.

The Riemannian metric on M pulls back to a Riemannian metric on $\widetilde{M}$ and gives rise to a smooth metric on S_0. We extend this metric to a metric on S such that

$$\operatorname{area}(S) \leq \operatorname{area}(S_0) + \varepsilon = (2g+1)\operatorname{area}(M) + \varepsilon. \tag{5}$$

Since the spin structure of M is nontrivial along each γ_i, the induced spin structures on $\widetilde{M}_i$ fit together to the unique spin structure on S.

There is a smooth function $\chi\colon S \to [0,1]$ with the following properties:

(1) $\chi|_{\widetilde{M}_0} \equiv 1$,
(2) $\chi|_{S \setminus S_0} \equiv 0$,
(3) $\|\nabla \chi\|_{L^\infty} \leq \dfrac{1}{\delta(M) - 2\varepsilon}$.

Let φ be an eigenspinor of the Dirac operator on M to the eigenvalue λ. This spinor lifts to an eigenspinor φ_0 of the Dirac operator on S_0. Thus $\chi \cdot \varphi_0$ is a well-defined spinor on S. We use it as a test spinor for the Rayleigh quotient. Theorem 1.1 yields

$$\frac{4\pi}{\operatorname{area}(S)} \leq \frac{\|D(\chi \cdot \varphi_0)\|_S^2}{\|\chi \cdot \varphi_0\|_S^2.} \tag{6}$$

We compute

$$\|D(\chi \cdot \varphi_0)\|_{\widetilde{M}_i}^2 \leq \left(\frac{1}{(\delta(M) - 2\varepsilon)^2} + |\lambda| \right)^2 \|\varphi\|_M^2.$$

Summing over i yields

$$\|D(\chi \cdot \varphi_0)\|_S^2 \leq (2g+1) \left(\frac{1}{(\delta(M) - 2\varepsilon)^2} + |\lambda| \right)^2 \|\varphi\|_M^2. \tag{7}$$

The denominator of the Rayleigh quotient is estimated by

$$\|\chi \cdot \varphi_0\|_S^2 \geq \|\varphi_0\|_{\widetilde{M}_0}^2 = \|\varphi\|_M^2. \tag{8}$$

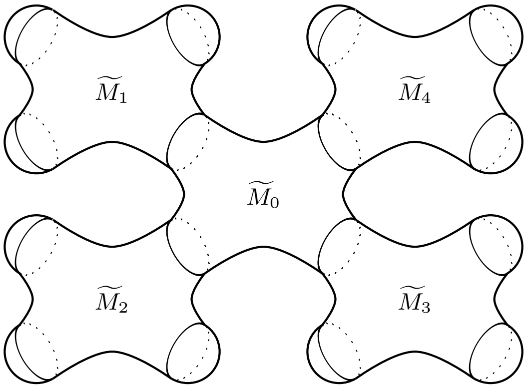

Fig. 8. The surface S for $g = 2$

Combining (5),(6),(7), and (8) we obtain

$$\frac{4\pi}{(2g+1)\,\mathrm{area}(M)+\varepsilon} \le (2g+1)\,\left(\frac{1}{(\delta(M)-2\varepsilon)^2}+|\lambda|\right)^2$$

which yields in the limit $\varepsilon \to 0$

$$\frac{2\sqrt{\pi}}{(2g+1)\,\sqrt{\mathrm{area}(M)}} - \frac{1}{\delta(M)} \le |\lambda|.$$

□

7 An application to the Willmore integral

The *Willmore integral* of an embedded closed surface $M \subset \mathbb{R}^3$ is defined by

$$W(M) = \int_M H^2 \mathrm{dvol} = \|H\|^2$$

where H denotes the mean curvature of M. The famous *Willmore conjecture* states that for an embedded 2-torus the Willmore integral is bounded by

$$W(M) \ge 2\pi^2.$$

This conjecture has been proven for various classes of embedded 2-tori (see [46] for a good overview), but in full generality it is still open. We will not resolve this problem here but our estimates on Dirac eigenvalues imply lower bounds on the Willmore integral as well.

Let $M \subset \mathbb{R}^3$ be an embedded surface of genus $g \ge 1$. The discussion from Sects. 2 and 3 shows that the induced spin structure on M admits spin-cuts and hence its spin-cut-diameter $\delta(M)$ is well-defined. A spin-cut can be obtained by choosing disjoint simple closed curves $\gamma_1, \dots, \gamma_g$ on M which bound transversal disks in $\mathbb{R}^3$ and whose homology classes $[\gamma_1], \dots, [\gamma_g]$ in $H_1(M, \mathbb{Z})$ are linearly independent.

Theorem 7.1. *Let $T^2 \subset \mathbb{R}^3$ be an embedded torus. Let $\delta(T^2)$ be its spin-cut-diameter and let $W(T^2)$ be its Willmore integral. Then for any $k \in \mathbb{N}$*

$$\sqrt{W(T^2)} \geq \sqrt{\frac{\pi}{k} + \frac{2\,\mathrm{area}(T^2)}{k^2\,\delta(T^2)^2}} - \frac{2\sqrt{\mathrm{area}(T^2)}}{k\,\delta(T^2)}$$

Proof. In [9] it was shown that a closed surface possesses Dirac eigenvalues λ satisfying

$$\lambda^2 \leq \frac{W(M)}{\mathrm{area}(M)}.$$

Combining this with Theorem 5.1 yields the result. □

This theorem yields a positive lower bound on $W(T^2)$ for all embedded 2-tori.

Remark. From Theorem 6.1 we can obtain a similar bound, but it turns out to be weaker than the well-known bound $W(M) \geq 4\pi$.

8 Noncompact surfaces of finite area

Now we extend the bounds on Dirac eigenvalues to the L^2-spectrum of the Dirac operator on a complete noncompact spin surface of finite area. The *fundamental tone* of the square of the Dirac operator on a noncompact spin manifold is given by

$$\lambda_*^2 = \inf_{\varphi} \frac{\|D\varphi\|^2}{\|\varphi\|^2}$$

where the infimum runs over all smooth spinors φ with compact support. If $\lambda_*^2 > 0$, then the L^2-spectrum of D has a gap about 0, more precisely,

$$\mathrm{spec}_{L^2}(D) \cap (-\lambda_*, \lambda_*) = \emptyset.$$

Any complete surface M of finite area is diffeomorphic to a closed surface $\overline{M}$ with finitely many points removed. The genus g of $\overline{M}$ is then also called the genus of M. By a *cut* of M we mean a collection of simple closed curves $\gamma_1, \dots, \gamma_g$ on M which are mapped under the diffeomorphism to a cut on $\overline{M}$. If M carries a spin structure, then we call the cut a *spin-cut* if the spin structure is nontrivial along all γ_i just as we did for closed surfaces. If the spin structure on M extends to one on $\overline{M}$, then we say the spin structure is *nontrivial along the ends.*

Given a spin-cut on M one can define the *cut-open* as before. It is now a noncompact complete surface of finite area with compact boundary. The *spin-cut-diameter* is again defined as the minimal distance of the various boundary components of the spin-cut. Taking the supremum over all spin-cuts yields the *spin-cut-diameter* $\delta(M)$ depending on the surface, its Riemannian metric and its spin structure.

Let us show that the results for closed surfaces carry over to the complete noncompact case without any essential changes.

Theorem 8.1. *Let M be a complete surface of genus $g \geq 1$ with a Riemannian metric of finite area. Let M be equipped with a spin structure which is nontrivial along the ends and which admits a spin-cut. Let $\delta(M)$ be the spin-cut-diameter of M. Then*

$$\lambda_* \geq \frac{2\sqrt{\pi}}{(2g+1)\sqrt{\operatorname{area}(M)}} - \frac{1}{\delta(M)}.$$

If $g = 1$, then for any $k \in \mathbb{N}$

$$\lambda_* \geq -\frac{2}{k\,\delta(M)} + \sqrt{\frac{\pi}{k\operatorname{area}(T^2)} + \frac{2}{k^2\delta(M)^2}}.$$

Proof. Let $\varepsilon > 0$ and let $\gamma_1, \ldots, \gamma_g$ be a spin-cut such that its spin-cut-diameter satisfies

$$\delta(\gamma_1, \ldots, \gamma_g) \geq \delta(M) - \varepsilon.$$

Pick a smooth spinor φ on M with compact support such that

$$\frac{\|D\varphi\|^2}{\|\varphi\|^2} \leq \lambda_* + \varepsilon.$$

Now we change the metric on M outside the support of φ and away from the γ_i such that it extends to $\overline{M}$ and such that

$$\operatorname{area}(\overline{M}) \leq \operatorname{area}(M) + \varepsilon.$$

Since the spin structure of M is nontrivial along the ends it extends to one on $\overline{M}$. Theorem 6.1 applied to $\overline{M}$ now yields

$$\begin{aligned}
\lambda_* + \varepsilon &\geq \frac{\|D\varphi\|^2}{\|\varphi\|^2} \\
&\geq \frac{2\sqrt{\pi}}{(2g+1)\sqrt{\operatorname{area}(\overline{M})}} - \frac{1}{\delta(\gamma_1, \ldots, \gamma_g)} \\
&\geq \frac{2\sqrt{\pi}}{(2g+1)\sqrt{\operatorname{area}(M) + \varepsilon}} - \frac{1}{\delta(M) - \varepsilon}.
\end{aligned}$$

Taking $\varepsilon \to 0$ finishes the proof of the first assertion. The second part for $g = 1$ is shown similarly. □

The assumption that the spin structure be nontrivial along the ends is crucial. It has been shown by the second author [8] that the L^2-spectrum of the Dirac operator on a complete hyperbolic surface of finite area whose spin structure is not nontrivial along the ends is given by

$$\mathrm{spec}_{L^2}(D) = \mathbb{R}.$$

A Two lemmata about cylinders

Lemma A.1. *Let $\gamma : S^1 \to S^2 \setminus \{N, S\}$ be a simple closed curve in the 2-sphere without North Pole N and South Pole S. Then either γ is contractible in $S^2 \setminus \{N, S\}$ or the homotopy class of γ generates $\pi_1(S^2 \setminus \{N, S\}) \cong \mathbb{Z}$.*

Proof. According to the theorem of Jordan-Schoenfliess there is a diffeomorphism $\varphi : S^2 \to S^2$ mapping γ to the equator. If $\phi(S)$ and $\phi(N)$ lie in the same hemisphere, then γ bounds a disk in $Z = S^2 \setminus \{N, S\}$. In this case γ is contractible in $S^2 \setminus \{N, S\}$. Otherwise $[\gamma]$ generates the fundamental group of $S^2 \setminus \{N, S\}$. □

Lemma A.2. *Let $Z := \left\{(x, y, z) \,|\, x^2 + y^2 = 1\right\} \subset \mathbb{R}^3$ be the cylinder. Let $f : Z \to \mathbb{R}$ be smooth and assume that $f(x, y, z) \to \infty$ for $z \to \infty$ and $f(x, y, z) \to -\infty$ for $z \to -\infty$ uniformly in x, y. This is equivalent to assuming that f is proper and onto. Then for any regular value $t \in \mathbb{R}$ the set $f^{-1}(t)$ has a connected component which is a simple closed curve whose homotopy class generates $\pi_1(Z)$.*

Proof. Since f is proper and t is regular $N := f^{-1}(t)$ is a closed 1-dimensional manifold, i. e. a finite union of simple closed curves. Not every connected component of N is contractible in Z, as otherwise for large K it would be possible to connect $(1, 0, -K)$ and $(1, 0, K)$ by a curve in $Z \setminus N$. This is impossible by the mean value theorem.

Let γ by a parametrization of a noncontractible component of N. According to the previous lemma $[\gamma]$ generates $\pi_1(Z)$. □

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