Proof of a well-known theorem about finitely presented groups

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In the proof of Proposition, in [Ammann, Große, *Relations Between Threshold Constants for Yamabe Type Bordism Invariants*, J. Geom. Analysis (2016), 26:2842–2882, see DOI 10.1007/s12220-015-9651-1 and the articles web page] we implicitly used the following theorem which is standard in surgery theory for positive scalar curvature metrics. As we did not find a suitable reference, we give a proof of this easy and probably well-known statement here.

Theorem 1. *Let G be a finitely generated group and H be a finitely presented group. Let $\phi : G \to H$ be a surjective homorphism. Then $\ker \phi$ is finitely generated as a normal subgroup of G.*

Proof. In the following, for a subset T of a group G_a we write $\langle T\rangle$ for the subgroup generated by T, and $\langle T\rangle_{G_a}$ for the normal subgroup generated by T.

We fix a finite presentation (S, R) for H. This consists of a finite subset $S = \{s_1, \dots, s_k\} \subset H$ of generators for H, and of a finite set R of relations. In order to describe the relations, let $F_k := \mathbb{Z}^{*k}$ be the free group in k generators, denoted as $e_1, \dots, e_k$, and we write f_k for the unique homomorphism $G \to H$ with $e_i \mapsto s_i$. This is obviously surjective. We formalize R as a subset of F_k. Then the statement, that R is a set of relations for H is (essentially) equivalent to

$$\ker f_k = \langle R\rangle_{F_k}.$$

Note that f_k factors to a group isomorphism $F_k/\langle R\rangle_{F_k} \to H$. As assumed above R is finite.

Now we choose for any s_i a $g_i \in G$ with $\phi(g_i) = s_i$. The set $\{g_1, \dots, g_k\}$, may be completed to a set of generators $\{g_1, \dots, g_\ell\}$, $\ell \geq k$ of G, and for $k < j \leq \ell$ one may choose g_i in the kernel of ϕ. Let $\gamma_k : F_k \to G$ be the unique homomorphism with $\gamma_k(e_i) = g_i$ for $i = \{1, \dots, k\}$. Obviously $f_k = \phi \circ \gamma_k$.

We claim

$$\langle \gamma_k(R) \cup \{g_{k+1}, \dots, g_\ell\}\rangle_G = \ker \phi, \tag{1}$$

and as the set $\gamma_k(R) \cup \{g_{k+1}, \dots, g_\ell\}$ is obviously finite, the theorem follows from this claim.

In the claim the inclusion $\subset$ follows from $f_k(R) = \phi(\gamma_k(R)) = \{1\}$ and $\{g_{k+1}, \dots, g_\ell\} \subset \ker \phi$.

To prove the converse inclusion, let g be in $\ker \phi$. Let π be the quotient map

$$\pi : G \to G/\langle \gamma_k(R) \cup \{g_{k+1}, \dots, g_\ell\}\rangle_G.$$

We will show $\pi(g) = 1$.

We write g as a word in the g_i's, which can be written as

$$g = \gamma_k(a_1)g_{j_1}\gamma_k(a_2)g_{j_2} \cdots \gamma_k(a_s)g_{j_s}$$

where for each $i \in \{1, \dots, s\}$ we have $a_i \in F_k$ and $k < j_i \leq \ell$.

We calculate

$$\pi(g) = \pi(\gamma_k(a_1))\underbrace{\pi(g_{j_1})}_{=1} \cdots \pi(\gamma_k(a_s))\underbrace{\pi(g_{j_s})}_{=1} = \pi(\gamma_k(a_1 \cdots a_k)). \tag{2}$$

Furthermore, we conlude

$$1 = \phi(g) = \phi(\gamma_k(a_1)) \cdots \phi(\gamma_k(a_s)) = f_k(a_1) \cdots f_k(a_s) = f_k(a_1 \cdots a_s),$$

and we get $a_1 \cdots a_s \in \ker f_k = \langle R \rangle_{F_k}$. Thus

$$\gamma_k(a_1 \cdots a_s) \in \gamma_k(\langle R \rangle_{F_k}) = \langle \gamma_k(R) \rangle_{\gamma_k(F_k)} \subset \langle \gamma_k(R) \rangle_G \subset \langle \gamma_k(R) \cup \{g_{k+1}, \ldots, g_\ell\} \rangle_G .$$

This implies $\pi(\gamma_k(a_1 \cdots a_k)) = 1$ and together with (2) we get $\pi(g) = 1$ which completes the proof of the claim and of the theorem. □

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