MASS ENDOMORPHISM AND SPINORIAL YAMABE TYPE PROBLEMS ON CONFORMALLY FLAT MANIFOLDS

BERND AMMANN, EMMANUEL HUMBERT AND BERTRAND MOREL

ABSTRACT. Let M be a compact manifold equipped with a Riemannian metric g and a spin structure σ. We let $\lambda^+_{\min}(M,[g],\sigma) = \inf_{\tilde g\in[g]} \lambda_1^+(\tilde g) Vol(M,\tilde g)^{1/n}$ where $\lambda_1^+(\tilde g)$ is the smallest positive eigenvalue of the Dirac operator D in the metric $\tilde g$. A previous result stated that $\lambda^+_{\min}(M,[g],\sigma) \le \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\omega_n^{1/n}$ where ω_n stands for the volume of the standard n-sphere. In this paper, we study this problem for conformally flat manifolds of dimension $n \ge 2$ such that D is invertible. E.g. we show that strict inequality holds in dimension $n \equiv 0,1,2 \mod 4$ if a certain endomorphism does not vanish. Because of its tight relations to the ADM mass in General Relativity, the endomorphism will be called mass endomorphism. We apply the strict inequality to spin-conformal spectral theory and show that the smallest positive Dirac eigenvalue attains its infimum inside the enlarged volume-1-conformal class of g.

MSC: 53A30, 53C27

CONTENTS

1. INTRODUCTION

Let (M,g,σ) be a compact spin manifold of dimension $n \ge 2$. For a metric $\tilde g$ in the conformal class $[g]$ of g, let $\lambda_1^+(\tilde g)$ be the smallest positive eigenvalue of the Dirac operator D. Similarly, let $\lambda_1^-(\tilde g)$ be the largest negative eigenvalue of D. We define

$$\lambda^+_{\min}(M,[g],\sigma) = \inf_{\tilde g\in[g]} \lambda_1^+(\tilde g)\mathrm{Vol}(M,\tilde g)^{1/n}$$

and

$$\lambda^-_{\min}(M,[g],\sigma) = \inf_{\tilde g\in[g]} |\lambda_1^-(\tilde g)|\ \mathrm{Vol}(M,\tilde g)^{1/n}$$

It was proven in [Amm03] that

$$\lambda^+_{\min}(M,[g],\sigma) > 0 \quad \text{and} \quad \lambda^-_{\min}(M,[g],\sigma)) > 0.$$

Several works have been devoted to the study of this conformal invariant. A non-exhaustive list is [Hij86, Hij91, Lot86, Bär92, Amm03a, Amm03b]. In [Amm03], we proved the following result:

Date: Jan 15, 2006.
Ammann was supported by the NSF grant DMS-9810361 of MSRI, Berkeley.

Theorem 1.1. *Let (M,g,σ) be a compact spin manifold of dimension $n \geq 2$. Then*

$$\lambda^+_{\min}(M,[g],\sigma) \leq \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\,\omega_n^{\frac{1}{n}} \qquad \textit{and} \qquad \lambda^-_{\min}(M,[g],\sigma) \leq \lambda^+_{\min}(\mathbb{S}^n),$$

where ω_n stands for the volume of the standard sphere $\mathbb{S}^n$.

The strict inequalities

$$\lambda^+_{\min}(M,[g],\sigma) < \frac{n}{2}\,\omega_n^{\frac{1}{n}} \qquad \text{and} \qquad \lambda^-_{\min}(M,[g],\sigma) < \frac{n}{2}\,\omega_n^{\frac{1}{n}}\ , \tag{1}$$

have several applications. At first, when $n > 2$, together with Hijazi inequality, each one of the two inequalities (1) implies the existence of a solution of the Yamabe problem. This problem is a famous problem of conformal geometry which has been solved by Aubin [Aub76] and Schoen [Sch84].
Another application of the inequalities (1) is the solution of a conformally invariant PDE which can be read as a nonlinear eigenvalue equation for the Dirac operator. The nonlinearity involves a critical exponent from the point of view of Sobolev embeddings and hence, this PDE cannot be solved by standard methods. Moreover, from this solution, one can construct a generalized metric $\tilde{g}$ (see [Amm03b, Amm03a]) such that $Vol_{\tilde{g}}(M) = 1$ and such that $\lambda^+_1(\tilde{g}) = \lambda^+_{\min}(M,[g],\sigma)$ (resp. $\lambda^-_1(\tilde{g}) = \lambda^-_{\min}(M,[g],\sigma)$). In other words, this proves that $\lambda^+_{\min}(M,[g],\sigma)$ (resp. $\lambda^-_{\min}(M,[g],\sigma)$) is attained by the generalized metric $\tilde{g}$.
In this paper we are interested in obtaining the inequalities (1) when M is a conformally flat manifold of dimension $n \geq 2$ such that D is invertible. In this goal, we introduce in section 2 the notion of *mass endomorphism*. This endomorphism corresponds to the constant term in the development of the Green function for D near the diagonal with respect to a conformal chart. In Remark 2.12 we will show that the pointwise eigenvalues of the mass endomorphism are all real. The mass endomorphism plays the same role as the constant term of the Green function $\gamma(\,\cdot\,,p)$ of the Yamabe operator. In the Yamabe problem, the constant term of γ can be interpreted as the mass [ADM62] of the asymptotically flat manifold $(M \setminus \{p\}, \gamma(\,\cdot\,,p)^{4/(n-2)}g)$ (see also [LP87]). This is why we use the name *mass endomorphism*. Schoen shows in [Sch84] that the positivity of the mass implies the solution of the Yamabe problem. In this paper, the eigenvalues of the mass endomorphism play the same role as the mass in Yamabe problem. Namely, we obtain the following result:

Theorem 1.2. *Let (M,g,σ) be a conformally flat compact spin manifold of dimension $n \geq 2$ with* $\ker(D) = \{0\}$. *Assume that the mass endomorphism (see next section) possesses a positive (resp. negative) eigenvalue. Then*

$$\lambda^+_{\min}(M,[g],\sigma) \quad (\textit{resp. } \lambda^-_{\min}(M,[g],\sigma)) < \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\,\omega_n^{\frac{1}{n}}.$$

Assume that $n \not\equiv 3 \mod 4$, then the spectrum of the Dirac operator and the pointwise spectrum of the mass endomorphism are symmetric (see Subsection 2.5). In particular,

$$\lambda^+_{\min}(M,[g],\sigma) = \lambda^-_{\min}(M,[g],\sigma).$$

This implies

Theorem 1.3. *Let (M,g,σ) be a conformally flat compact spin manifold of dimension $n \geq 2$ with* $\ker(D) = \{0\}$. *Assume that $n \not\equiv 3 \mod 4$ and that the mass endomorphism α is not identically zero. Then*

$$\lambda^+_{\min}(M,[g],\sigma) < \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\,\omega_n^{\frac{1}{n}} \qquad \textit{and} \qquad \lambda^-_{\min}(M,[g],\sigma) < \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\,\omega_n^{\frac{1}{n}}.$$

This is no longer true if $n \equiv 3 \mod 4$. In Example 2.15 we study the real projective spaces $\mathbb{R}P^{4k+3}$. Here the mass endomorphism is a non-vanishing multiple of the identity section, hence has constant sign which depends on the spin structure. Furthermore, the two spin structures σ_+ and σ_- on $\mathbb{R}P^{4k+3}$ satisfy

$$\lambda^-_{\min}(\mathbb{R}P^{4k+3}, g_0, \sigma_+) > \lambda^+_{\min}(\mathbb{R}P^{4k+3}, g_0, \sigma_+) = \frac{n}{2}\left(\frac{\omega_n}{2}\right)^{\frac{1}{n}},$$

$$\lambda^+_{\min}(\mathbb{R}P^{4k+3}, g_0, \sigma_-) > \lambda^-_{\min}(\mathbb{R}P^{4k+3}, g_0, \sigma_-) = \frac{n}{2}\left(\frac{\omega_n}{2}\right)^{\frac{1}{n}}.$$

Hence, as predicted by Theorem 1.2 either $\lambda^+_{\min}$ or $\lambda^-_{\min}$ is smaller than $(n/2)\omega_n^{1/n}$. As a remark, if $\min(\lambda^+_{\min}(\mathbb{R}P^{4k+3}, g_0, \sigma_-), \lambda^-_{\min}(\mathbb{R}P^{4k+3}, g_0, \sigma_-)) < (n/2)\omega_n^{1/n}$ then the infimum in the definition of $\lambda^\pm_{\min}$ would be attained by a metric of non-constant curvature. It is then natural to think that we cannot obtain the strict inequality (1) for $\lambda^+_{\min}$ and $\lambda^-_{\min}$ for all manifolds of dimension $n \equiv 3 \mod 4$.
In order to prove inequalities (1) for arbitrary conformally flat manifolds of dimension $n \geq 2$ such that D is invertible, then one has to find some nonzero eigenvalues of the mass endomorphism. Some questions arise naturally. At first, on the sphere $\mathbb{S}^n$, the mass endomorphism is null. Otherwise, we could apply Theorem 1.2. One may wonder if the mass endormorphism is not always zero. The answer is no since as mentioned above, the projective spaces give examples for which the mass endomorphism is a nonzero multiple of the identity.
Recall once again that the constant term of the Green function of the Yamabe operator can be interpreted as the mass of an asymptotically flat manifold; according to the positive mass theorem it is positive unless is M is conformally diffeomorphic to $\mathbb{S}^n$. Then, one could hope to find a result of the same type for the mass endomorphism. However, we show that this is false in general. Namely, we show in Section 4 that the mass endomorphism of flat tori always vanishes.

2. The mass endomorphism

In the following, we assume that (M, g, σ) is a conformally flat manifold of dimension $n \geq 2$. We also assume that D is invertible, i.e. that $\ker(D) = \{0\}$. The mass endomorphism is defined as the constant term in the Green function for D. In this section, we give a precise definition and some properties of the mass endomorphism.

2.1. **Trivialization of the spinor bundle.** Let $\rho : U \subset (M, g) \to V \subset (N, g_N)$ be a conformal map preserving orientation and spin structure. We write $g = f^2 \rho^* g_N$. Then according to [Hit74, Hij86] there is a fiberwise isomorphism

$$\begin{array}{cccc} \rho_* : & \Sigma M|_U & \to & \Sigma N|_V \\ & \downarrow & & \downarrow \\ & U & \to & V, \end{array}$$

such that $D_N \rho_* \varphi = f \rho_* D\varphi$ for all $\varphi \in \Gamma(\Sigma M)$ and $f^{-\frac{n-1}{2}} \rho_*$ is an isometry, where D and D_N denote the Dirac operators on M and N. The most important case we will use is that N is Euclidean space, i.e. ρ is a conformal chart of (M, g) preserving orientation and the spin structure. In this case the above map yields a well chosen trivialization of the spinor bundle.
The definition of the mass endomorphism will be done by working with a conformal charts. For simplicity we will first define it in the special case that g is flat in a neighborhood of a given base point, and then extend it to the general case.

2.2. **Green function for the Dirac operator.** Let $\pi_1, \pi_2 : M \times M \to M$ be the projection to the first and second component. Then we define

$$\Sigma M \boxtimes \Sigma M^* := \pi_1^*(\Sigma M) \otimes (\pi_2^*(\Sigma M))^*,$$

i.e. it is the bundle whose fiber over (x, y) is given by $\mathrm{Hom}(\Sigma_y M, \Sigma_x M)$. Let $\Delta := \{(p, p) \,|\, p \in M\}$ be the diagonal.

Definition 2.1. A smooth section $G_D : M \times M \setminus \Delta \to \Sigma M \boxtimes \Sigma M^*$ that is locally integrable on $M \times M$ is called *the Green function* for the Dirac operator D if in the sense of distributions

$$D_x(G_D(x, y)) = \delta_y Id_{\Sigma_y M}$$

In other words, we have for any $y \in M$, $\psi_0 \in \Sigma_y M$, and $\varphi \in \Gamma(\Sigma M)$

$$\int \langle G_D(x, y)\psi_0, D^*\varphi(x)\rangle \, dx = \langle \psi_0, \varphi(y)\rangle.$$

For $k : M \times M \setminus \Delta \to \Sigma M \boxtimes \Sigma M^*$ we denote the corresponding integral operator by P_k, i.e. we define for $\varphi \in \Gamma(\Sigma M)$

$$P_k(\varphi)(x) := \int k(x,y)\varphi(y)\,dy.$$

If k is smooth on $M\times M\setminus\Delta$ and locally integrable on $M\times M$, then the integral exists in the principal value sense of distributions. P_k uniquely determines k. In this notation P_{G_D} is the inverse of $D : L^2_1(\Sigma M) \to L^2(\Sigma M)$. Hence, the Green function is unique.

Remark 2.2. Analogously, one can define the Green function of Euclidean space as a section $G_{\mathrm{eucl}} : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \Sigma M \boxtimes \Sigma M^*$ such that $P_{G_{\mathrm{eucl}}}$ is the inverse of $D_{\mathrm{eucl}} : L^2_1(\Sigma M) \to L^2(\Sigma M)$. Again we have unicity. One easily checks that

$$G_{\mathrm{eucl}}(x,y) = -\frac{1}{\omega_{n-1}}\frac{x-y}{|x-y|^n}\cdot .$$

In our construction of the test spinor, we will need the asymptotics of Green functions close to the diagonal. This is provided by the following proposition.

Proposition 2.3. *Assume that the metric is flat near $y \in M$, and let ρ be an* isometric *chart. Then, the Green function G_D for D exists, and in the above trivialization has the following expansion when x tends to y*

$$\omega_{n-1}\, G_D(x,y)\psi_0 = -\frac{x-y}{|x-y|^n}\cdot\psi_0 + v(x,y)\psi_0$$

where $D_x v(x,y)(\psi_0) = 0$ on a neighborhood of y.

Proof: Let $G_{D_{\mathrm{eucl}}}$ be the Green function for D_{eucl} on $\mathbb{R}^n$ given by Remark 2.2. We take a cut-off function η with support in $B_y(\delta)$ which is equal to 1 on $B_y(\delta/2)$ where $\delta > 0$ is a small number. We set $\Phi(x) = \eta(x)G_{D_{\mathrm{eucl}}}(x,y)\cdot\psi_0$ where ψ_0 is constant. The spinor Φ is harmonic on $B_y(\delta/2)\setminus\{y\}$. We extend Φ by zero, and obtain a smooth spinor on $M\setminus\{y\}$. As $D\Phi|_{B_y(\delta/2)} \equiv 0$, we see that $D\Phi$ extends to a smooth spinor on M, denoted by Ψ. Since D is assumed to be invertible, there exists a smooth spinor field ζ such that $D\zeta = -\Psi$. We then define $G_D(x,y)\psi_0 := \Phi(x) - \zeta(x)$. It is easy to see that G_D is the Green function for D and has the development described above. As Ψ and hence ζ depend smoothly on y, it is clear that $G_D(x,y)$ depends smoothly on y outside of the diagonal. □

Lemma 2.4 (Conformal change and Green functions)**.** *Let $\rho : (M, g_M, \sigma) \to (N, g_N, \sigma_N)$ be a conformal diffeomorphism preserving the orientation and the spin structure and write $g_M = f^2\rho^* g_N$. Let G_M resp. G_N be the Green function on M resp. N, then*

$$G_N(\rho(x),\rho(y)) = f^{n-1}(y)\rho_{*,x}\circ G_M(x,y)\circ\rho_{*,y}^{-1}.$$

Proof. We already know that $D_N = f\rho_* D\rho_*^{-1}$. This implies

$$P_{G_N} = D_N^{-1} = \rho_* D^{-1}\rho_*^{-1} f^{-1} = \rho_* P_{G_M}\rho_*^{-1} f^{-1},$$

Hence,

$$G_N(\rho(x),\rho(y))\,d\mu_N(\rho(y)) = f^{-1}(y)\rho_{*,x}\circ G_M(x,y)\circ\rho_{*,y}^{-1}\,d\mu(y).$$

With $d\mu = f^n d\mu_N$ this implies the lemma. □

In particular, the previous proposition and the lemma imply that the Green function G_D for D exists on any conformally flat manifold for which D is invertible.

Example 2.5. If φ_λ is an eigenspinor for the eigenvalue λ of D, and if G_D is the unique Green function for D, then

$$\int_M G_D(x,y)\varphi_\lambda(y)\,dy = \frac{1}{\lambda}\varphi_\lambda(x)$$

Example 2.6. Let $\rho : S^n\setminus\{N\} \to \mathbb{R}^n$ be the stereographic projection. Then

$$g_{S^n} = \left(\frac{2}{|\rho(x)|^2+1}\right)^2 \rho^* g_{\mathrm{eucl}}.$$

Let G_{S^n} be the Green function on S^n. An obvious modification of Lemma 2.4 tells us that

$$\rho_* G_{S^n}(x,y)\rho_*^{-1} = \left(\frac{2}{|\rho(y)|^2+1}\right)^{1-n} G_{\mathrm{eucl}}(\rho(x),\rho(y)).$$

Example 2.7. Let $\rho : S^n \to S^n$ be a Moebius transformation of S^n with $g_{S^n} = f^2\rho^*(g_{S^n})$. Then

$$G_{S^n}(x,y) = f^{n-1}(y)\rho_{*,x} \circ G_{S^n}(x,y) \circ \rho_{*,y}^{-1}.$$

In the sequel the following self-adjointness result for G_D will be important.

Proposition 2.8. *Let V be an open set of M in which g is flat. Then, for all $x \neq y \in V$, $G_D(x,y)^* = G_D(y,x)$. In other words,*

$$\langle G_D(x,y)\psi_y, \varphi_x\rangle = \langle \psi_y, G_D(y,x)\varphi_x\rangle$$

for all $\psi_y \in \Sigma_y M$ and $\varphi_x \in \Sigma_x M$.

Proof: We have

$$P_{G_D}^* = \left(D^{-1}\right)^* = (D^*)^{-1} = D^{-1} = P_{G_D}.$$

As the operator uniquely determines the kernel, this implies the proposition. □

Proposition 2.9. *Assume that we have two metrics g_1 and g_2 on M, $g_1 = f^2 g_2$. We assume that both metrics are flat in a neighborhood U of y. We define v_1 and v_2 as above. Then*

$$v_1(x,x) = f^{1-n} v_2(x,x)$$

Proof. Let G^1 (resp. G^2) be the Green function for g_1 (resp. g_2). We identify spinors via ρ_*. By Lemma 2.4, G^1 and G^2 are related by the following formula: for all $x, y \in M$,

$$G^1(x,y) = f^{1-n}(y) G^2(x,y) \tag{2}$$

Let $h_1 : (U, g_1) \to \mathbb{R}^n$ and $h_2 : (U, g_2) \to \mathbb{R}^n$ be isometric embeddings. According to Liouville's theorem the conformal diffeomorphism $h_2 \circ h_1^{-1} : h_1(U) \to h_2(U)$ extends to a Möbius transformation of the Alexandrov compactification S^n of $\mathbb{R}^n$. Because of Examples 2.6 and 2.7 this implies that

$$G^1_{\mathrm{eucl}}(x,y) = f^{1-n}(y) G^2_{\mathrm{eucl}}(x,y).$$

Subtracting this from (2), and taking the limit $x \to y$ one obtains the desired formula. □

2.3. **Definition and first properties of the mass endomorphism.** The mass endomorphism is defined as the constant term of the Green function for D with respect to a conformal chart. Let us make this precise.

Definition 2.10. Let (M, g) be a compact manifold which is conformally flat on a neighborhood of $y \in M$. Choose a metric $\tilde{g} \in [g]$ that is flat on a neighborhood of y and such that $\tilde{g}_y = g_y$. Let G_D be the Green function for D. Then we define the *mass endomorphism* as

$$\alpha_y : \left| \begin{array}{ccc} \Sigma_y(M) & \to & \Sigma_y(M) \\ \psi_0 & \mapsto & v(y,y)(\psi_0) \end{array} \right.$$

where v is as in the previous paragraph with respect to $\tilde{g}$.

Because of Proposition 2.9 this definition does not depend on the choice of $\tilde{g}$.

Proposition 2.11. *For each $y \in M$, the mass endomorphism α_y is linear and self-adjoint.*

Proof: Let $y \in M$ and $\psi_0, \varphi_0 \in \Sigma_y(M)$. We have

$$\alpha_y(\psi_0) = \lim_{x \to y} \omega_{n-1} G_D(x,y)\psi_0 + \frac{x-y}{|x-y|^n} \cdot \psi_0$$

It follows immediately that α_y is linear. Taking the limit when $x \to y$ one gets that

$$\langle \alpha_y(\psi_0), \varphi_0\rangle = \langle \psi_0, \alpha_y(\varphi_0)\rangle$$

□

Remark 2.12. Proposition 2.11 immediately implies that the mass endomorphism has only real eigenvalues.

Remark 2.13. If $\dim M \equiv 2,3,4 \mod 8$, then the spinor bundle carries a quaternionic structure, i.e. a basepoint-preserving, parallel, complex anti-linear map $Q : \Sigma M \to \Sigma M$ with Q^{-1} commuting with the Clifford multiplication. As a consequence Q commutes with D, G_D and with α.

2.4. **Examples.**

Example 2.14 (Flat tori)**.** Let (M, g) be an n-dimensional flat torus. It carries 2^n spin structures. For one spin structure, the so-called *trivial* spin structure, we have a $\mathrm{rank}(\Sigma M) = 2^{[n/2]}$-dimensional space of parallel sections. All other spin structures admit no non-trivial parallel spinors. Because M is scalar-flat, the kernel of D consists exactly of the parallel spinors, in particular D is invertible for all non-trivial spin structures χ. Because translations act spin-isometric on (M, g, χ), the Green function G_D satisfies $G_D(x, y) = G_D(x - y, 0)$. Also $D_x(G_D(y, x)) = -\delta(x - y)$, hence $G_D(x, y) = -G_D(y, x)$. Therefore, all terms of even order in the development of G_D have to vanish. In particular, the mass endomorphism vanishes.

Example 2.15 (Real Projective Spaces)**.** Besides $\mathbb{R}P^1 = S^1$, the only real projective spaces that are orientable and spin are $\mathbb{R}P^{4k+3}$ with $k \in \mathbb{N}$. The space $M = \mathbb{R}P^{4k+3}$ carries exactly two spin structures. The universal covering $\pi : S^{4k+3} \to \mathbb{R}P^{4k+3}$ induces a push-forward of the spinor bundles, which is a fiberwise isomorphism $\pi_* : \Sigma_p S^{4k+3} \to \Sigma_{\pi(p)}\mathbb{R}P^{4k+3}$. One calculates

$$G_D^{\mathbb{R}P^{4k+3}}(\pi x, \pi y) \circ \pi_* = \pi_* G_D^{S^{4k+3}}(x, y) + \pi_* G_D^{S^{4k+3}}(x, -y), \tag{3}$$

where $-y$ denotes the antipodal point of y. Stereographic projection based in $y \notin \{x, -x\}$ defines a conformal chart containing x and $-x$. Example 2.6 implies that $G_D^{S^{4k+3}}(x, -x) \neq 0$. Hence, the mass endomorphism $\alpha_{\pi(x)}$ of $\mathbb{R}P^{4k+3}$ does not vanish anywhere on $\mathbb{R}P^{4k+3}$.
The group of orientation preserving isometries fixing $\pi(x)$ is $\mathrm{SO}(4k + 3)$. After passing to the double cover $\mathrm{Spin}(4k+3)$, we obtain a $\mathrm{Spin}(4k+3)$-action on ΣS^{4k+3}, that pushes down to a $\mathrm{Spin}(4k+3)$-action on ΣS^{4k+3} which commutes with the Dirac operator. Hence, this action also commutes with the mass endomorphism, and as the $\mathrm{Spin}(4k + 3)$-action on $\Sigma_{\pi(x)}\mathbb{R}P^{4k+3}$ is irreducible, the mass endomorphism is a constant multiple of the identity [LM89, Prop. I.5.15], [Fri00, section 1.5]. If one changes the spin structure, then the second summand in (3) changes its sign. Hence, the sign of the mass endomorphism depends on the choice of the spin structure, which are denoted by σ_+ and σ_-.

2.5. **Endomorphisms generating symmetries.** The aim of this section is to show that if $n \not\equiv 3 \mod 4$, then there is an automorphism $\Gamma(\mathrm{Aut}_{\mathbb{R}}(\Sigma M))$ that anticommutes with the Dirac operator. This result is well-known, see for example [Fri00, 1.7], [Dah03, Prop. 5]. As a consequence, one sees that it also anticommutes with the mass endomorphism.
Let (W, γ) be an irreducible complex representation of the Clifford algebra of an Euclidean vector space V of dimension n. After fixing an orientation on V, one can define

$$\omega_\gamma := \gamma(e_1 \cdot \ldots \cdot e_n) \in \mathrm{End}(W)$$

where $e_1, \ldots, e_n$ is an oriented orthonormal basis on V, and where γ denotes Clifford multiplication. One easily calculates

$$\omega^2 := (-1)^{n(n+1)/2}.$$

As a consequence the eigenvalues of ω are contained in $\{-1, 1\}$ if $n \equiv 0, 3 \mod 4$, and they are contained in $\{-i, i\}$ if $n \equiv 1, 2 \mod 4$.

The case $n \equiv 0 \mod 2$.

If n is even, then Clifford multiplication by a vector $v \in V$ anticommutes with ω. Hence, if α is an eigenvalue ω, then so is $-\alpha$. One immediately obtains the well-known lemma.

Lemma 2.16. *If n is even, then ω is a complex-linear automorphism of W anticommuting with Clifford-multiplication.*

Indeed, it can even be shown that up to automorphisms there is only one irreducible representation in even dimensions n and that any endomorphism anticommuting with Clifford multiplication with vectors is a multiple of ω.

The case $n \equiv 1 \mod 4$.

The question arises, whether there is a similar endomorphism if n is odd. In this case, Clifford multiplication with a vector commutes with ω. Hence, by Schur's lemma ω has only one eigenvalue. For $n \equiv 1 \mod 4$ we have either $\omega = i\,\mathrm{Id}$ or $\omega = -i\,\mathrm{Id}$, and it can be shown, that there is exactly one irreducible representation of V with $\omega = i\,\mathrm{Id}$ denoted by (W^i, γ^i), and one with $\omega = -i\,\mathrm{Id}$, denoted by (W^{-i}, γ^{-i}). If we replace the complex structure on $W^{\pm i}$, by its complex conjugate one, then this is again a representation of the (real) Clifford algebra of V. Obviously, ω changes sign by conjugation. Hence, there is a conjugate linear isomorphism of Clifford representation $\alpha : W^i \to W^{-i}$. Another way to modify the structure of (W^i, γ^i) is to reverse the sign of Clifford multiplication by vectors. Namely, we define a Clifford multiplication $\tau^i : \mathbb{C}l(V) \to \mathrm{End}(W)$ as $\tau^i(X) := -\gamma^i(X)$ for all vectors X in V. Again, we calculate that the sign of ω changes if we replace τ^i by γ^i, and there is a complex linear isomorphism of vector spaces $\beta : W^{-i} \to W^i$ with $\beta \circ \gamma^{-i}(X) = \tau^i(X) \circ \beta$ for any vector $X \in V$. Hence, $\nu := \beta \circ \alpha : W^i \to W^i$ is a conjugate linear automorphism of vector spaces, and for vector X we have

$$\gamma^i(X) \circ \nu = -\tau^i(X) \circ \beta \circ \alpha = -\beta \circ \gamma^i(X) \circ \alpha = -\nu \circ \gamma^i(X).$$

A similar endomorphism of W^{-i} is given by $\alpha \circ \beta$. We have proven the following lemma.

Lemma 2.17. *If $n \equiv 1 \mod 4$, then there is a real vector space automorphism of $W^{\pm i}$ anticommuting with Clifford multiplication by vectors. The automorphism is conjugate linear.*

The case $n \equiv 3 \mod 4$.

The case $n \equiv 3 \mod 4$ is different. Again, we have $\omega = \mathrm{Id}$ or $\omega = -\mathrm{Id}$, and there is exactly one irreducible representation $(W^\pm, \gamma^\pm)$ in each case. However, conjugation does not exchange the representations, and the sign of ω is invariant under real automorphisms. Hence, an isomorphism as in the above lemma cannot exist.

For all $n \not\equiv 3 \mod 4$.

Any automorphism ν anticommuting with Clifford multiplication, commutes with bivectors $X \cdot Y$ where $X, Y \in V$. As the Lie algebra of $\mathrm{Spin}(n)$ is generated by elements of that form, such an isomorphisms ν is $\mathrm{Spin}(n)$-equivariant. We obtain the well-known

Proposition 2.18. *If $n \equiv 0, 1, 2 \mod 4$, then there is a real vector bundle isomorphism $\nu : \Sigma M \to \Sigma M$ anticommuting with Clifford multiplication by vectors, complex linear if n is even, and conjugate linear if $n \equiv 1 \mod 4$. Furthermore, ν is parallel.*

It follows that it anti-commutes with the Dirac operator, the Green function and the mass endomorphism.

Corollary 2.19 (Well-known, e.g. [APS76])**.** *The spectrum of the Dirac operator is symmetric in dimension $n \not\equiv 3 \mod 4$.*

Corollary 2.20. *The pointwise spectrum of the mass endomorphism is symmetric in dimension $n \not\equiv 3 \mod 4$, i.e. if λ is an eigenvalue of the mass endomorphism α_x for an $x \in M$, then $-\lambda$ is also an eigenvalue of α_x with the same multiplicity.*

Corollary 2.21. *If $\dim M = 2$, then the mass endomorphism α vanishes.*

Proof. The spectrum of α_y is symmetric and real. As α commutes with the quaternionic multiplication Q, the eigenspaces of α are quaternionic vector spaces. $\dim \Sigma_y M = 2$ implies that $\alpha_y = r_y \mathrm{Id}$ for $r_y \in \mathbb{R}$. As the spectrum of α_y is symmetric, we obtain $r_y = 0$. □

3. The estimates

Let $\psi \in \Gamma(\Sigma M)$ and define

$$J(\psi) = \frac{\left(\int_M |D\psi|^{\frac{2n}{n+1}} v_g\right)^{\frac{n+1}{n}}}{\int_M \Re e\langle D\psi, \psi\rangle v_g}$$

The first named author proved in [Amm03] that

$$\lambda^+_{\min}(M, [g], \sigma) = \inf_\psi J(\psi) \tag{4}$$

where the infimum is taken over the set of smooth spinor fields for which

$$\left(\int_M \Re e\langle D\psi, \psi\rangle v_g\right) > 0\ .$$

By adjusting some signs appropriately, one obtains by the same reasoning that

$$\lambda^-_{\min}(M, [g], \sigma) = \inf_\psi -J(\psi) \tag{5}$$

where the infimum is taken over the set of smooth spinor fields for which

$$\left(\int_M \Re e\langle D\psi, \psi\rangle v_g\right) < 0\ .$$

These two facts will be helpful to prove the following theorem

Theorem 3.1. *Assume that there exists on (M, g) a conformal metric $g_p \in [g]$ which is flat in a neighborhood of a point p. If there exists on $M \setminus \{p\}$ a spinor field ψ satisfying*

- $D\psi = 0$
- *ψ admits the following development near the point p:*

$$\psi = \frac{x}{r^n} \cdot \psi_0 + \psi_1 + \theta$$

where ψ_0 and ψ_1 are two spinors of $\Sigma_p M$ such that

$$\Re e(\langle \psi_0, \psi_1\rangle) < 0 \qquad (\textit{resp. } \Re e(\langle \psi_0, \psi_1\rangle) > 0)\ ,$$

and where $\theta = O(r)$ is an harmonic spinor field smoothly defined in a neighborhood of p.

Then we have

$$\lambda^+_{\min}(M, [g], \sigma) < \frac{n}{2}\,\omega_n^{\frac{1}{n}} \qquad \left(\textit{resp. } \lambda^-_{\min}(M, [g], \sigma) < \frac{n}{2}\,\omega_n^{\frac{1}{n}}\right)\ .$$

Proof. The proof is based on a suitable choice of a test spinor field ψ_ε to estimate $J(\psi_\varepsilon)$. We let

$$f(r) := \frac{1}{1 + r^2}\ .$$

As a first step, we consider for a given Φ the spinor defined by

$$\varphi^\pm(x) := f(|x|)^{\frac{n}{2}}(1 \mp x) \cdot \Phi \qquad \forall x \in \mathbb{R}^n.$$

One may compute that

$$D(\varphi^\pm) = \pm n f(|x|)\varphi^\pm\ .$$

Since the Euclidean space $(\mathbb{R}^n \setminus \{0\}, 4f^2 g_{eucl})$ and the standard sphere $(\mathbb{S}^n, g_0)$ are isometric and using the conformal covariance of D, it is well known that there exists a natural map

$$m \left| \begin{array}{ccc} \Gamma(\Sigma(\mathbb{R}^n \setminus \{0\})) & \to & \Gamma(\Sigma(\mathbb{S}^n)) \\ \varphi & \mapsto & m(\varphi) \end{array} \right.$$

such that for all vector field $\varphi \in \Gamma(\Sigma(\mathbb{R}^n \setminus \{0\}))$, we have

$$m(D(\varphi)) = 2f D_{\mathbb{S}^n}(m(\varphi))$$

As one can check the spinor field $m(\varphi^+)$ is a Killing spinor on $\mathbb{S}^n$ to the Killing constant $-\frac{1}{2}$, whereas $m(\varphi^-)$ is a Killing spinor on $\mathbb{S}^n$ to the Killing constant $+\frac{1}{2}$. One gets that

$$J_{\mathbb{R}^n}(\varphi^\pm) = \pm\frac{n}{2}\omega_n^{\frac{1}{n}}$$

where $J_{\mathbb{R}^n}$ is the functional J written on $\mathbb{R}^n$. Now, fix $\varepsilon > 0$ and let $(x_1, \cdots, x_n)$ be local coordinates on a neighborhood U of p in M. On U, we trivialize the spinor bundle via parallel transport. Using this trivialization, one may define for $x \in M$,

$$\varphi_\varepsilon^\pm := \eta_0\varphi^\pm(\frac{x}{\varepsilon})$$

where η_0 is a cut-off function equal to 1 on $B(p, \delta)$ (δ is a small number).
We would like to have test spinors $\psi_\pm$ with $\pm\int\langle D\psi_\pm, \psi_\pm\rangle > 0$ for which the *strict* inequalities

$$\pm J(\psi_\pm) < \frac{n}{2}\,\omega_n^{\frac{1}{n}}$$

hold. For a given $\varepsilon > 0$ we set

$$\xi := \varepsilon^{\frac{1}{n+1}} \qquad \varepsilon_0 := \frac{\xi^n}{\varepsilon} f(\frac{\xi}{\varepsilon})^{\frac{n}{2}} \ .$$

The test spinor we use here is the following:

$$\psi_\varepsilon^\pm := \begin{cases} f(\frac{r}{\varepsilon})^{\frac{n}{2}}(1 \mp \frac{x}{\varepsilon}) \cdot \psi_0 \mp \varepsilon_0\psi_1 & \text{if } r \le \xi \ , \\ \\ \mp\varepsilon_0(\psi - \eta\theta) + \eta f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}\psi_0 & \text{if } \xi \le\ r \le 2\xi \ , \\ \\ \mp\varepsilon_0\,\psi & \text{if } r \ge 2\xi \ , \end{cases} \tag{6}$$

where $r = |x|$, where η is a cut-off function which equals to 1 on $B(p, \xi)$, which is zero on the complement of $B(p, 2\xi)$ and which statisfies

$$|\nabla\eta| \le \frac{2}{\xi}.$$

Note that $\psi_\varepsilon^\pm$ is continuous on M.

Remark 3.2. It should be pointed out that this choice of ξ is arbitrary in the following sense. The proof of Theorem 3.1 still holds for any choice of $\xi = \varepsilon^q$ for $q \in]\frac{n-1}{n(n+1)}\ ,\ \frac{1}{n}[$.

We can assume without loss of generality that $|\psi_0| = 1$. Since ψ and θ are harmonic near p, we have

$$D\psi_\varepsilon^\pm = \begin{cases} \pm\frac{n}{\varepsilon} f(\frac{r}{\varepsilon})^{\frac{n}{2}+1}(1 \mp \frac{x}{\varepsilon}) \cdot \psi_0 & \text{if } r \le \xi \ , \\ \\ \pm\varepsilon_0\nabla\eta \cdot \theta + f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}\nabla\eta \cdot \psi_0 & \text{if } \xi \le\ r \le 2\xi \ , \\ \\ 0 & \text{if } r \ge 2\xi \ . \end{cases} \tag{7}$$

Therefore, since $|(1 \mp \frac{x}{\varepsilon}) \cdot \psi_0|^2 = (1 + \frac{r^2}{\varepsilon^2})|\psi_0|^2 = f(\frac{r}{\varepsilon})^{-1}$, we have

$$|D\psi_\varepsilon^\pm|^{\frac{2n}{n+1}} = \begin{cases} \left[\frac{n}{\varepsilon} f(\frac{r}{\varepsilon})^{\frac{n+1}{2}}\right]^{\frac{2n}{n+1}} = n^{\frac{2n}{n+1}}\varepsilon^{-\frac{2n}{n+1}} f(\frac{r}{\varepsilon})^n & \text{if } r \le \xi \ , \\ \\ |\pm\varepsilon_0\nabla\eta \cdot \theta + f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}\nabla\eta \cdot \psi_0|^{\frac{2n}{n+1}} & \text{if } \xi \le\ r \le 2\xi \ , \\ \\ 0 & \text{if } r \ge 2\xi \ . \end{cases} \tag{8}$$

In the following, the notation C will stand for positive constants (eventually depending on the dimension n but not on ε) which can differ from one line to another. Equation (8) yields the following estimates:

$$\int_{B(p,\xi)} |D\psi_\varepsilon^\pm|^{\frac{2n}{n+1}} = \varepsilon^{n-\frac{2n}{n+1}} n^{\frac{2n}{n+1}} \int_{B(p,\frac{\xi}{\varepsilon})} f^n \le \varepsilon^{n-\frac{2n}{n+1}} n^{\frac{2n}{n+1}} \int_{\mathbb{R}^n} f^n$$

and

$$\begin{aligned}\int_{B(p,2\xi)\setminus B(p,\xi)} |D\psi_\varepsilon^\pm|^{\frac{2n}{n+1}} &\le C \int_{B(p,2\xi)\setminus B(p,\xi)} |\,\varepsilon_0 \nabla\eta\cdot\theta|^{\frac{2n}{n+1}} + C' \int_{B(p,2\xi)\setminus B(p,\xi)} |f(\frac{\xi}{\varepsilon})^{\frac{n}{2}} \nabla\eta\cdot\psi_0|^{\frac{2n}{n+1}} \\ &\le C\,\varepsilon^{\frac{n(2n-1)}{n+1}} + C'\,\varepsilon^{\frac{n(2n-1)}{n+1}} \le C\,\varepsilon^{\frac{n(2n-1)}{n+1}} ,\end{aligned}$$

since $\varepsilon_0 \le C\,\varepsilon^{n-1}$, $|\nabla\eta| \le \frac{2}{\xi}$, $Vol(B(p,2\xi)\setminus B(p,\xi)) \le C\xi^n$ and $|\theta| \le C\,\xi$ on $B(p,2\xi)$, as well as $f(\frac{\xi}{\varepsilon})^{\frac{n}{2}} \le C\,\varepsilon^{\frac{n^2}{n+1}}$.
Therefore

$$\left(\int_M |D\psi_\varepsilon^\pm|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{n}} \le \varepsilon^{n-1} n^2 I^{1+\frac{1}{n}} \left[1 + C\,\varepsilon^{\frac{n^2}{n+1}}\right] = \varepsilon^{n-1} n^2 I^{1+\frac{1}{n}} \left[1 + o(\varepsilon^{n-1})\right] , \tag{9}$$

where $I := \int_{\mathbb{R}^n} f^n$.
If we set

$$\nu := \langle \psi_0, \psi_1 \rangle ,$$

we have

$$\begin{aligned}\Re e\langle D\psi_\varepsilon^\pm, \psi_\varepsilon^\pm\rangle_{|B(p,\xi)} &= \Re e\langle \pm\frac{n}{\varepsilon} f(\frac{r}{\varepsilon})^{\frac{n}{2}+1}(1 \mp \frac{x}{\varepsilon})\cdot\psi_0\,, f(\frac{r}{\varepsilon})^{\frac{n}{2}}(1 \mp \frac{x}{\varepsilon})\cdot\psi_0 \mp \varepsilon_0\,\psi_1\rangle \\ &= \pm\frac{n}{\varepsilon} f(\frac{r}{\varepsilon})^n - \frac{n}{\varepsilon}\varepsilon_0\, f(\frac{r}{\varepsilon})^{\frac{n}{2}+1}\,\Re e(\nu) \pm \frac{n}{\varepsilon}\varepsilon_0\, f(\frac{r}{\varepsilon})^{\frac{n}{2}+1}\,\Re e\langle \frac{x}{\varepsilon}\cdot\psi_0, \psi_1\rangle\end{aligned}$$

and hence, since by symmetry the last term vanishes when integrating over $B(p,\xi)$, we have

$$\int_{B(p,\xi)} \Re e\langle D\psi_\varepsilon^\pm, \psi_\varepsilon^\pm\rangle = n\,\varepsilon^{n-1}\Big[\pm \int_{B(p,\frac{\xi}{\varepsilon})} f(r)^n - \Re e(\nu)\varepsilon_0 \int_{B(p,\frac{\xi}{\varepsilon})} f(r)^{\frac{n}{2}+1}\Big] .$$

Moreover,

$$\int_{B(p,\frac{\xi}{\varepsilon})} f(r)^n = I - \omega_{n-1}\int_{\frac{\xi}{\varepsilon}}^{+\infty} r^{n-1} f(r)^n \mathrm{d}r ,$$

Since

$$\int_{\frac{\xi}{\varepsilon}}^{+\infty} r^{n-1} f(r)^n \mathrm{d}r \le \int_{\frac{\xi}{\varepsilon}}^{+\infty} r^{-(n+1)} \mathrm{d}r \le C\,\varepsilon^{\frac{n^2}{n+1}}$$

and since $\varepsilon_0 \sim \varepsilon^{n-1}$ when $\varepsilon \to 0$, we have, for $\Re e(\nu) < 0$,

$$\int_{B(p,\xi)} \Re e\langle D\psi_\varepsilon^+, \psi_\varepsilon^+\rangle \ge n\,\varepsilon^{n-1}\Big[I - C_0\,\Re e(\nu)\varepsilon^{n-1} + o(\varepsilon^{n-1})\Big] , \tag{10}$$

and for $\Re e(\nu) > 0$,

$$-\int_{B(p,\xi)} \Re e\langle D\psi_\varepsilon^-, \psi_\varepsilon^-\rangle \ge n\,\varepsilon^{n-1}\Big[I + C_0\,\Re e(\nu)\varepsilon^{n-1} + o(\varepsilon^{n-1})\Big] , \tag{11}$$

where

$$C_0 = \int_{\mathbb{R}^n} f(r)^{\frac{n}{2}+1}$$

We also have

$$\begin{aligned}\Re e\langle D\psi_\varepsilon^\pm, \psi_\varepsilon^\pm\rangle_{|B(p,2\xi)\setminus B(p,\xi)} &= \Re e\langle \pm\varepsilon_0\nabla\eta\cdot\theta + f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}\nabla\eta\cdot\psi_0, \mp\varepsilon_0(\psi - \eta\theta) + \eta f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}\psi_0\rangle \\ &= \Re e\langle \pm\varepsilon_0\nabla\eta\cdot\theta + f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}\nabla\eta\cdot\psi_0, \mp\varepsilon_0\,\psi\rangle\end{aligned}$$

since $\Re e\langle\nabla\eta\cdot\theta,\theta\rangle = 0$, $\Re e\langle\nabla\eta\cdot\psi_0,\psi_0\rangle = 0$ and

$$\Re e\langle\nabla\eta\cdot\psi_0,\theta\rangle + \Re e\langle\nabla\eta\cdot\theta,\psi_0\rangle = \Re e\langle\nabla\eta\cdot\psi_0,\theta\rangle - \Re e\overline{\langle\nabla\eta\cdot\psi_0,\theta\rangle} = 0\ .$$

Now, we write that on $B(p,2\xi)\setminus B(p,\xi)$, $\varepsilon_0|\psi| \le C\varepsilon^{n-1}\xi^{-(n-1)}$, $\varepsilon_0|\nabla\eta||\theta| \le C\varepsilon^{n-1}$ and $f(\frac{\xi}{\varepsilon})^{\frac{n}{2}}|\nabla\eta| \le C\varepsilon^{n-1}$. This leads to

$$\Re e\langle D\psi_\varepsilon^\pm,\psi_\varepsilon^\pm\rangle_{|B(p,2\xi)\setminus B(p,\xi)} \quad\le\quad C\varepsilon^{2n-2}\xi^{1-n}\ ,$$

which yields

$$\int_{B(p,2\xi)\setminus B(p,\xi)} \Re e\langle D\psi_\varepsilon^\pm,\psi_\varepsilon^\pm\rangle = O(\varepsilon^{2n-2+\frac{1}{n+1}}) = o(\varepsilon^{2n-2})\ . \tag{12}$$

Therefore, since

$$\Re e\langle D\psi_\varepsilon^\pm,\psi_\varepsilon^\pm\rangle_{|M\setminus B(p,2\xi)} \quad=\quad 0\ ,$$

from Equations (10), (11), (12), we have for $\Re e(\nu) < 0$,

$$\int_M \Re e\langle D\psi_\varepsilon^+,\psi_\varepsilon^+\rangle \ge n\,\varepsilon^{n-1} I\Big[1 - C_0\,\Re e(\nu)\varepsilon^{n-1} + o(\varepsilon^{n-1})\Big]\ ,$$

and for $\Re e(\nu) > 0$,

$$-\int_M \Re e\langle D\psi_\varepsilon^-,\psi_\varepsilon^-\rangle \ge n\,\varepsilon^{n-1} I\Big[1 + C_0\,\Re e(\nu)\varepsilon^{n-1} + o(\varepsilon^{n-1})\Big]\ ,$$

Together with (9), we then have, for $\Re e(\nu) < 0$,

$$J(\psi_\varepsilon^+) \le n\,I^{\frac{1}{n}}\,\frac{1 + o(\varepsilon^{n-1})}{1 - C_0\,\Re e(\nu)\varepsilon^{n-1} + o(\varepsilon^{n-1})}\ ,$$

and for $\Re e(\nu) > 0$,

$$-J(\psi_\varepsilon^-) \le n\,I^{\frac{1}{n}}\,\frac{1 + o(\varepsilon^{n-1})}{1 + C_0\,\Re e(\nu)\varepsilon^{n-1} + o(\varepsilon^{n-1})}\ .$$

For ε small enough, we obtain

$$J(\psi_\varepsilon^+) < nI^{\frac{1}{n}} \text{ and } -J(\psi_\varepsilon^-) < nI^{\frac{1}{n}}$$

Recall now the following fact: let p be any point of the round sphere $\mathbb{S}^n$. Then $\mathbb{S}^n\setminus\{p\}$ is isometric to $\mathbb{R}^n$ with the metric

$$g_{\mathbb{S}} = 4f^2 g_{\mathrm{eucl}}\ .$$

Therefore

$$2^n I = \int_{\mathbb{R}^n} 2^n f^n = \omega_n\ ,$$

which yields, for $\Re e(\nu) < 0$ (resp. for $\Re e(\nu) > 0$)

$$J(\psi_\varepsilon^+) < \frac{n}{2}\,\omega_n^{\frac{1}{n}} \qquad (\text{resp.} \quad -J(\psi_\varepsilon^-) < \frac{n}{2}\,\omega_n^{\frac{1}{n}})\ .$$

Hence, by (4) and (5), the proof of Theorem 3.1 is now complete. □

4. Proofs of Theorems 1.2 and 1.3

Let $p\in M$. Up to a conformal change of metric, we may assume that g is flat near p. Assume that the mass endomorphism α_p possesses a non-zero eigenvalue ν. Let $\psi_0\in\Sigma_p(M)$ be an eigenvector associated to ν. Then, we set

$$\psi = -\omega_{n-1} G_D(x,p)\psi_0$$

The spinor field ψ then satisfies the assumptions of Theorem 3.1 with $\psi_1 = \nu\psi_0$. Theorem 3.1 implies that if $\nu > 0$ then $\lambda^+_{\min}(M,g,\theta) < \frac{n}{2}\,\omega_n^{\frac{1}{n}}$ and if $\nu < 0$, $\lambda^-_{\min}(M,g,\theta) < \frac{n}{2}\,\omega_n^{\frac{1}{n}}$. This proves Theorem 1.2. Now, if $n \not\equiv 3 \mod 4$, the spectrum of the mass endomorphism is symmetric and hence if $\alpha_p \neq 0$, ν can be chosen positive or negative. This proves Theorem 1.3.

References

[ADM62] R. L. Arnowitt, S. Deser, and C. W. Misner, *Canonical analysis of general relativity*, Recent developments in general relativity, Pergamon, Oxford, 1962, pp. 127–136.

[Amm03] B. Ammann, *A spin-conformal lower bound of the first positive Dirac eigenvalue*, Diff. Geom. Appl. **18** (2003), 21–32.

[Amm03a] B. Ammann, *The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions*, Preprint, ArXiv math.DG/0309061

[Amm03b] B. Ammann, *A variational problem in conformal spin Geometry*, Habilitationsschrift, Universität Hamburg, May 2003.

[AH03] B. Ammann, E. Humbert, *Positive mass theorem for the Yamabe problem on spin manifolds.* Preprint

[APS76] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99.

[Aub76] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.* J. Math. Pur. Appl., IX. Ser., 55:269–296, 1976.

[Bär92] C. Bär, Lower eigenvalue estimates for Dirac operators. *Math. Ann.*, 293, 1992.

[Dah03] M. Dahl, *Prescribing eigenvalues of the Dirac operator*, Preprint, ArXiv math.DG/0311172

[Fri00] T. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Providence, RI, 2000.

[Heb97] E. Hebey, *Introduction à l'analyse non-linéaire sur les variétés,* Diderot Éditeur, Arts et sciences, 1997.

[Hij86] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing Spinors*, Comm. Math. Phys., **1**04 151–162, 1986

[Hij91] O. Hijazi, *Première valeur propre de l'opérateur de Dirac et nombre de Yamabe*, C. R. Acad. Sci. Paris, Série I **313**, (1991), 865–868.

[Hit74] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1–55.

[LM89] H.-B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton 1989.

[LP87] J. M. Lee and T. H. Parker. *The Yamabe problem.* Bull. Am. Math. Soc., New Ser., **17** 37–91, 1987.

[Lot86] J. Lott, *Eigenvalue bounds for the Dirac operator.* Pacific J. of Math., **125**, 117–126, 1986.

[Sch84] R. Schoen. *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Diff. Geom., **20**, 479–495, 1984.

[Tru68] N.S. Trudinger. *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds.* Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser., **22**, 265–274, 1968.

[Yam60] H. Yamabe. *On a deformation of Riemannian structures on compact manifolds.* Osaka Math. J., **12**, 21–37, 1960.

Author's addresses:

Bernd Ammann, Emmanuel Humbert, and Bertrand Morel,
Institut Élie Cartan BP 239
Université de Nancy 1
54506 Vandoeuvre-lès -Nancy Cedex
France

E-Mail: `ammann@iecn.u-nancy.fr, humbert@iecn.u-nancy.fr` and `morel@iecn.u-nancy.fr`