SURGERY AND HARMONIC SPINORS

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Abstract. Let M be a compact spin manifold with a chosen spin structure. The Atiyah-Singer index theorem implies that for any Riemannian metric on M the dimension of the kernel of the Dirac operator is bounded from below by a topological quantity depending only on M and the spin structure. We show that for generic metrics on M this bound is attained.

1. Introduction

We suppose that M is a compact spin manifold. By a *spin manifold* we will mean a smooth manifold equipped with an orientation and a spin structure. After choosing a metric g on M, one can define the spinor bundle $\Sigma^g M$ and the Dirac operator $D^g : \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$, see [8, 16, 11].

Being a self-adjoint elliptic operator, D^g shares many properties with the Hodge-Laplacian $\Delta_p^g : \Gamma(\Lambda^p T^*M) \to \Gamma(\Lambda^p T^*M)$. In particular, if M is compact, then the spectrum is discrete and real, and the kernel is finite-dimensional. Elements of $\ker \Delta_p^g$ resp. $\ker D^g$ are called *harmonic forms* resp. *harmonic spinors*.

However, the relations of Δ_p^g and D^g to topology have a slightly different character. Hodge theory tells us that the Betti numbers $\dim\ker \Delta_p^g$ only depend on the topological type of M. The dimension of the kernel of D^g is invariant under conformal changes of the metric, however it does depend on the choice of conformal structure. The first examples of this phenomenon were constructed by Hitchin [12], and it was conjectured by several people that $\dim\ker D^g$ depends on the metric for all compact spin manifolds of dimension ≥ 3.

On the other hand, the Index Theorem of Atiyah and Singer gives a topological lower bound on the dimension of the kernel of the Dirac operator. For M a compact spin manifold of dimension n this bound is [16], [3, Section 3],

$$\dim\ker D^g \geq \begin{cases} |\widehat{A}(M)|, & \text{if } n \equiv 0 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Here the $\widehat{A}$-genus $\widehat{A}(M) \in \mathbb{Z}$ and the α-genus $\alpha(M) \in \mathbb{Z}/2\mathbb{Z}$ are invariants of (the spin bordism class of) the differential spin manifold M, and g is any Riemannian metric on M.

Date: October 23, 2008.

2000 *Mathematics Subject Classification.* 53C27 (Primary) 55N22, 57R65 (Secondary).

Key words and phrases. Dirac operator, eigenvalue, surgery.

This work was initiated when the authors visited the Albert Einstein Institute, Golm, Germany, and was continued at the IECN Nancy, France. We wish to thank these institutes for their very kind hospitality and support.

It is hence natural to ask whether metrics exist for which equality holds in (1). Such metrics will be called *D-minimal*. In [17] it is proved that a generic metric on a manifold of dimension ≤ 4 is D-minimal. In [3] the same result is proved for manifolds of dimension at least 5 which are simply connected, or which have fundamental group of certain types. The argument in [3] utilizes the surgery-bordism method which has proven itself very powerful in the study of manifolds allowing positive scalar curvature metrics. In a similar fashion we will use surgery methods to prove the following.

Theorem 1.1. *Let M be a compact connected spin manifold without boundary. Then a generic metric on M is D-minimal.*

Here "generic" means that the set of all D-minimal metrics is dense in the C^∞-topology and open in the C^1-topology on the set of all metrics. The essential step in the proof is to construct one D-minimal metric on a given spin manifold. Theorem 1.1 then follows from well-known results in perturbation theory.

Since $\dim \ker D$ behaves additively with respect to disjoint union of spin manifolds while the $\hat{A}$-genus/α-genus may cancel it is easy to find disconnected manifolds with no D-minimal metric.

Let us also mention that if M is a compact Riemann surface of genus at most 2, then *all metrics* are D-minimal. The same holds for Riemann surfaces of genus 3 or 4 whose spin structures are not spin boundaries. However, if the genus is at least 5 (or equal to 3 or 4 with a spin structure which is a spin boundary), then there are also metrics with larger kernel, see [12], [4].

In order to explain the surgery-bordism method used in the proof of Theorem 1.1 we have to fix some notation.

Let M and N be spin manifolds of the same dimension. A smooth embedding $f : N \to M$ is called *spin preserving* if the pullback by f of the orientation and spin structure of M is the orientation and spin structure of N. If M is a spin manifold we denote by M^- the same manifold with the opposite orientation.

For $l \geq 1$ we denote by $B^l(R)$ the standard l-dimensional open ball of radius R and by $S^{l-1}(R)$ its boundary. We abbreviate $B^l = B^l(1)$ and $S^{l-1} = S^{l-1}(1)$. The standard Riemannian metrics on $B^l(R)$ and $S^{l-1}(R)$ are denoted by g^{flat} and g^{round}. We equip $S^{l-1}(R)$ with the *bounding spin structure*, i.e. the spin structure obtained by restricting the unique spin structure on $B^l(R)$. (If $l > 2$ the spin structure on $S^{l-1}(R)$ is unique, if $l = 2$ it is not.)

Let $f : S^k \times \overline{B^{n-k}} \to M$ be a spin preserving embedding. We define

$$\widetilde{M} = \left(M \setminus f(S^k \times \overline{B^{n-k}})\right) \cup \left(\overline{B^{k+1}} \times S^{n-k-1}\right) /\!\sim,$$

where $\sim$ identifies the boundary of $\overline{B^{k+1}} \times S^{n-k-1}$ with $f(S^k \times S^{n-k-1})$. The topological space $\widetilde{M}$ carries a differential structure and a spin structure such that the inclusions $M \setminus f(S^k \times B^{n-k}) \hookrightarrow \widetilde{M}$ and $\overline{B^{k+1}} \times S^{n-k-1} \hookrightarrow \widetilde{M}$ are spin preserving smooth embeddings. We say that $\widetilde{M}$ is obtained from M by *surgery of dimension k* or by *surgery of codimension $n-k$*.

The proof of Theorem 1.1 relies on the following surgery theorem.

Theorem 1.2. *Let (M, g^M) be a compact Riemannian spin manifold of dimension n. Let $\widetilde{M}$ be obtained from M by surgery of dimension k, where $0 \leq k \leq n-2$.*

Then $\widetilde{M}$ carries a metric $g^{\widetilde{M}}$ such that

$$\dim\ker D^{g^{\widetilde{M}}} \leq \dim\ker D^{g^M}.$$

Theorem 1.2 fits nicely in a hierarchy of results on surgery invariance, all of which also hold in the trivial case of codim $= 0$ surgery. The α-genus (equal to the index of the $C\ell_n$-linear Dirac operator $\mathfrak{D}$, [16, Chap. II, §7]) is invariant under all surgeries, the minimal dimension of $\ker D$ is non-increasing when codim ≥ 2 surgeries are performed, positivity of scalar curvature is preserved under codim ≥ 3 surgeries [10], and for $p \geq 1$ positivity of p-curvature is preserved under codim $\geq p+3$ surgeries [15].

In the case $n-k =$ codim ≥ 3, Theorem 1.2 is a special case of [3, Theorem 1.2], where the proof relies on the positive scalar curvature of the surgery cosphere S^{n-k-1} and the surgery result of [10]. In the codim $= 2$ case, the cosphere S^1 does not allow positive scalar curvature. However, the spin structure on this cosphere is the one that bounds a disk. The fact that the Dirac operator on S^1 with the bounding spin structure is invertible is used in an essential way in the proof of Theorem 1.2.

After the publication of the preprint version of the present article, the results and the techniques have been applied to $\mathbb{Z}$-periodic manifolds, i.e. manifolds $\widetilde{N}$ with a free $\mathbb{Z}$-action such that $N = \widetilde{N}/\mathbb{Z}$ is a smooth compact spin manifold [19, 2].

2. Preliminaries

2.1. **Spinor bundles for different metrics.** Let M be a spin manifold of dimension n and let g and g' be Riemannian metrics on M. The goal of this subsection is to identify the spinor bundles of (M,g) and (M,g') using the method of Bourguignon and Gauduchon [6].

There exists a unique endomorphism $b^g_{g'}$ of TM which is positive, symmetric with respect to g, and satisfies $g(X,Y) = g'(b^g_{g'}X, b^g_{g'}Y)$ for all $X, Y \in TM$. This endomorphism maps g-orthonormal frames at a point to g'-orthonormal frames at the same point and it gives a map $b^g_{g'} : \mathrm{SO}(M,g) \to \mathrm{SO}(M,g')$ of $\mathrm{SO}(n)$-principal bundles. If we assume that $\mathrm{Spin}(M,g)$ and $\mathrm{Spin}(M,g')$ are equivalent spin structures on M then the map $b^g_{g'}$ lifts to a map $\beta^g_{g'}$ of $\mathrm{Spin}(n)$-principal bundles,

$$\begin{tikzcd} \mathrm{Spin}(M,g) \arrow[r, "\beta^g_{g'}"] \arrow[d] & \mathrm{Spin}(M,g') \arrow[d] \\ \mathrm{SO}(M,g) \arrow[r, "b^g_{g'}"] & \mathrm{SO}(M,g') \end{tikzcd}.$$

From this we get a map between the spinor bundles $\Sigma^g M$ and $\Sigma^{g'} M$ denoted by the same symbol and defined by

$$\beta^g_{g'} : \Sigma^g M = \mathrm{Spin}(M,g) \times_\sigma \Sigma_n \to \mathrm{Spin}(M,g') \times_\sigma \Sigma_n = \Sigma^{g'} M,$$
$$\psi = [s,\varphi] \mapsto [\beta^g_{g'} s, \varphi] = \beta^g_{g'}\psi,$$

where (σ, Σ_n) is the complex spinor representation, and where $[s,\varphi] \in \mathrm{Spin}(M,g) \times_\sigma \Sigma_n$ denotes the equivalence class of $(s,\varphi) \in \mathrm{Spin}(M,g) \times \Sigma_n$ for the equivalence relation given by the action of $\mathrm{Spin}(n)$. The map $\beta^g_{g'}$ preserves fiberwise length of spinors.

We define the Dirac operator ${}^gD^{g'}$ acting on sections of the spinor bundle for g by

$$ {}^gD^{g'} = (\beta^g_{g'})^{-1} \circ D^{g'} \circ \beta^g_{g'}. $$

In [6, Thm. 20] the operator ${}^gD^{g'}$ is computed in terms of D^g and some extra terms which are small if g and g' are close. Formulated in a way convenient for us the relationship is

$$ {}^gD^{g'}\psi = D^g\psi + A^g_{g'}(\nabla^g\psi) + B^g_{g'}(\psi), \tag{2} $$

where $A^g_{g'} \in \hom(T^*M \otimes \Sigma^g M, \Sigma^g M)$ satisfies

$$ |A^g_{g'}| \leq C|g - g'|_g, \tag{3} $$

and $B^g_{g'} \in \hom(\Sigma^g M, \Sigma^g M)$ satisfies

$$ |B^g_{g'}| \leq C(|g - g'|_g + |\nabla^g(g - g')|_g) \tag{4} $$

for some constant C.

In the special case that g' and g are conformal with $g' = F^2 g$ for a positive smooth function F we have

$$ {}^gD^{g'}(F^{-\frac{n-1}{2}}\psi) = F^{-\frac{n+1}{2}} D^g\psi, \tag{5} $$

according to [12, 5, 11].

2.2. **Notations for spaces of spinors.** Throughout the article φ, ψ, and variants denote spinors, i.e. sections of the spinor bundle. If S is a closed or open subset of a manifold M we write $C^k(S)$ both for the space of k times continuously differentiable functions on S and for the space of k times continuously differentiable spinors. On $C^k(S)$ we have the norm

$$ \|\varphi\|_{C^k(S)} = \sum_{l=0}^{k} \sup_{x\in S} |\nabla^l \varphi(x)|. $$

We sometimes write $\|\varphi\|_{C^k(S,g)}$ instead of $\|\varphi\|_{C^k(S)}$ to indicate that the spinor bundle and the norm depend on g. The analogous notation is used for Schauder spaces $C^{k,\alpha}$.

The spaces of square-integrable functions and spinors are denoted by $L^2(S) = L^2(S, g)$ and equipped with the norm

$$ \|\varphi\|^2_{L^2(S,g)} = \int_S |\varphi|^2 \, dv^g. $$

Further, $H^2_k(S) = H^2_k(S, g)$ denotes the Sobolev spaces of functions and spinors equipped with the norm

$$ \|\varphi\|^2_{H^2_k(S,g)} = \sum_{l=0}^{k} \int_S |\nabla^l \varphi|^2 \, dv^g. $$

Let U be an open subset of M. The set of locally C^1-spinors $C^1_{\mathrm{loc}}(U)$ carries a topology such that $\varphi_i \to \varphi$ in $C^1_{\mathrm{loc}}(U)$ if and only if $\varphi_i \to \varphi$ in $C^1(K)$ for any compact subset $K \subset U$.

2.3. **Regularity and elliptic estimates.** In this section M is not assumed to be compact.

Lemma 2.1. *Let (M,g) be a Riemannian manifold and let ψ be a spinor in $L^2(M)$. If ψ is* weakly harmonic, *that is*

$$\int_M \langle \psi, D\varphi \rangle \, dv^g = 0$$

for all compactly supported smooth spinors φ, then ψ is smooth.

Lemma 2.2. *Let (M,g) be a Riemannian manifold and let $K \subset M$ be a compact subset. Then there is a constant $C = C(K,M,g)$ such that*

$$\|\psi\|_{C^2(K,g)} \le C\|\psi\|_{L^2(M,g)}$$

for all harmonic spinors ψ on (M,g).

Proof of the lemmata. The assumptions of Lemma 2.1 imply that $\int_M \langle \psi, D^2\varphi \rangle \, dv^g = 0$ for any compactly supported smooth spinor φ. Writing down the equation in local coordinates, one can use standard results on elliptic partial differential equations (see for example [9, Theorem 8.13]) to derive via recursion that ψ is contained in $H_k^2(K,g)$ for any $k \in \mathbb{N}$ and any compact $K \subset M$ with smooth boundary. Further,

$$\|\psi\|_{H_k^2(K,g)} \le C\|\psi\|_{L^2(M,g)}. \tag{6}$$

The Sobolev embedding $H_k^2(K,g) \to C^1(K,g)$ for $k > n/2+1$ (see [1, Theorem 6.2]) tells us that $\psi \in C^1(K,g)$ and gives us an estimate for $\|\psi\|_{C^1(K,g)}$ analogous to (6). Now one can use Schauder estimates (as in [9, Theorem 6.6]) to conclude that ψ is smooth on any compact set K' contained in the interior of K, and similarly we derive the C^2 estimate of Lemma 2.2. □

In the next lemma $K \subset M$ again denotes a compact subset.

Lemma 2.3 (Ascoli's theorem, [1, Theorem 1.30 and 1.31])**.** *Let φ_i be a bounded sequence in $C^{1,\alpha}(K)$. Then a subsequence of φ_i converges in $C^1(K)$.*

2.4. **Removal of singularities lemma.** In the proof of Theorem 1.2 we will need the following lemma.

Lemma 2.4. *Let (M,g) be an n-dimensional Riemannian spin manifold and let $S \subset M$ be a compact submanifold of dimension $k \le n-2$. Assume that φ is a spinor field such that $\|\varphi\|_{L^2(M)} < \infty$ and $D^g\varphi = 0$ weakly on $M \setminus S$. Then $D^g\varphi = 0$ holds weakly also on M.*

Proof. Let ψ be a smooth spinor compactly supported in M. We have to show that

$$\int_M \langle \varphi, D^g\psi \rangle \, dv^g = 0. \tag{7}$$

Let $U_S(\delta)$ be the set of points of distance at most δ to S. For a small $\delta > 0$ we choose a smooth function $\eta : M \to [0,1]$ such that $\eta = 1$ on $U_S(\delta)$, $|\mathrm{grad}\eta| \le 2/\delta$, and $\eta = 0$ outside $U_S(2\delta)$. We rewrite the left hand side of (7) as

$$\begin{aligned}\int_M \langle \varphi, D^g\psi\rangle \, dv^g &= \int_M \langle \varphi, D^g((1-\eta)\psi + \eta\psi)\rangle \, dv^g \\ &= \int_M \langle \varphi, D^g((1-\eta)\psi)\rangle \, dv^g \\ &\quad + \int_M \langle \varphi, \eta D^g\psi\rangle \, dv^g + \int_M \langle \varphi, \mathrm{grad}\eta \cdot \psi\rangle \, dv^g.\end{aligned}$$

As $D^g\varphi = 0$ weakly on $M \setminus S$ the first term vanishes. The absolute value of the second term is bounded by

$$\|\varphi\|_{L^2(U_S(2\delta))}\|D^g\psi\|_{L^2(U_S(2\delta))}$$

which tends to 0 as $\delta \to 0$. Finally, the absolute value of the third term is bounded by

$$\begin{aligned}\frac{2}{\delta}\|\varphi\|_{L^2(U_S(2\delta))}\|\psi\|_{L^2(U_S(2\delta))} &\leq \frac{C}{\delta}\|\varphi\|_{L^2(U_S(2\delta))}\mathrm{Vol}(U_S(2\delta)\cap \mathrm{supp}(\psi))^{\frac{1}{2}} \\ &\leq C\|\varphi\|_{L^2(U_S(2\delta))}\delta^{\frac{n-k}{2}-1}.\end{aligned}$$

Since $n - k \geq 2$, the third term also tends to 0 as $\delta \to 0$. □

2.5. **Products with spheres.**

Lemma 2.5. *Let λ be an eigenvalue of $(D^{g^{\mathrm{round}}})^2$ on S^l for some $l \geq 1$. Then $\lambda \geq l^2/4$.*

When $l = 1$ it is important that S^1 is equipped with the spin structure which bounds the disk, for the non-bounding spin structure there are harmonic spinors and the estimate is not true.

Proof. First we look at the case $l = 1$. Since S^1 carries the bounding spin structure it follows that the action of S^1 on itself by multiplication does not lift to ΣS^1. However the action of the non-trivial double covering of S^1—which is again S^1—lifts to ΣM, turning $\Gamma(\Sigma M)$ into an S^1-representation. The irreducible components have odd weight. From this one concludes that $\lambda = (k + \frac{1}{2})^2$ for some $k \in \mathbb{Z}$, and the lemma follows.

In the case $l \geq 2$ the scalar curvature of $(S^l, g^{\mathrm{round}})$ is $\mathrm{scal}^{g^{\mathrm{round}}} = l(l-1)$ and thus Friedrich's bound [7] implies

$$\lambda \geq \frac{l}{4(l-1)} \min_{x\in S^l} \mathrm{scal}^{g^{\mathrm{round}}} = l^2/4.$$

□

If (M, g) and (N, h) are compact Riemannian spin manifolds then the Dirac operator D^{g+h} on $(M \times N, g + h)$ can be written as $D^{\mathrm{vert}} + D^{\mathrm{hor}}$ where D^{vert} is the part of D which only contains derivatives along M, and D^{hor} is the part which only contains derivatives along N. These two operators anticommute, and thus

$$(D^{g+h})^2 = (D^{\mathrm{vert}})^2 + (D^{\mathrm{hor}})^2.$$

It is easy to see that D^{vert} has the same spectrum as D^g, but with infinite multiplicities. The same holds for D^{hor} and D^h. We conclude the following.

Proposition 2.6. *Let (M, g) be a compact spin manifold and $l \geq 1$. Then the spectrum of $(D^{g+g^{\mathrm{round}}})^2$ on $M \times S^l$ is bounded from below by $l^2/4$.*

3. Proof of Theorem 1.2

For Theorem 1.2 we have given a compact Riemannian spin manifold (M,g) of dimension n. The manifold $\widetilde{M}$ is the result of surgery on M using an embedding $f : S^k \times \overline{B^{n-k}} \to M$. We assume $n-k \geq 2$.

We will here use a slightly more detailed description of the surgery data which better suits our geometric constructions. Let $i : S^k \to M$ be an embedding and denote by S the image of i. Let $\pi^\nu : \nu \to S$ be the normal bundle of S in (M,g). We assume that a trivialization of ν is given in the form of a vector bundle map $\iota : S^k \times \mathbb{R}^{n-k} \to \nu$ with $(\pi^\nu \circ \iota)(p,0) = i(p)$ for $p \in S^k$. We also assume that ι is fiberwise an isometry when the fibers $\mathbb{R}^{n-k}$ of $S^k \times \mathbb{R}^{n-k}$ are given the standard metric and the fibers of ν have the metric induced by g. We then get an embedding as above by setting $f = \exp^\nu \circ \iota : S^k \times \overline{B^{n-k}(R)} \to M$ for sufficiently small R. Here $\exp^\nu$ denotes the normal exponential map of S. For small R we define open neighbourhoods $U_S(R)$ of S by

$$U_S(R) = (\exp^\nu \circ \iota)(S^k \times B^{n-k}(R)).$$

3.1. **Approximation by a metric of product form near S.** In the following $r(x)$ denotes the distance from the point x to S with respect to the metric g. We denote by h the restriction of g to the tangent bundle of S.

Lemma 3.1. *For sufficiently small $R > 0$ there is a constant $C > 0$ so that*

$$G = g - ((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)$$

satisfies

$$|G(x)| \leq Cr(x), \quad |\nabla G(x)| \leq C$$

on $U_S(R)$.

Note that in this lemma the function $r(x)$ is by definition the distance of x to S with respect to g but it coincides with the distance of x to S with respect to the metric $((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)$.

Proof. Since $x \mapsto \nabla G(x)$ is continuous on a neighborhood of S we can find a constant C such that $|\nabla G(x)| \leq C$ for sufficiently small $R > 0$. Now, let $x \in S$. First we notice that the spaces $T_x S$ and ν_x are orthogonal with respect to both the scalar products $g(x)$ and $((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)(x)$. It is also clear that these scalar products coincide on $T_x S$. Since the differential $d(\exp^\nu \circ \iota)$ is an isometry, they coincide also on ν_x. This implies that $g(x) = ((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h)(x)$ and hence that $G(x) = 0$. We obtain that G vanishes on S. Since G is C^1, $|G|$ is 1-lipschitzian and thus there exists $C > 0$ such that $|G(x)| \leq Cr(x)$. □

The following proposition allows us to assume that the metric g has product form close to the surgery sphere S.

Proposition 3.2. *Let (M,g) and S be as above. Then there is a metric $\tilde{g}$ on M and $\delta_0 > 0$ such that $d^g(x,S) = d^{\tilde{g}}(x,S)$, $\tilde{g}$ has product form on $U_S(\delta_0)$, and*

$$\dim \ker D^{\tilde{g}} \leq \dim \ker D^g.$$

For $\delta > 0$ let η be a smooth cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $U_S(\delta)$, $\eta = 0$ on $M \setminus U_S(2\delta)$, and $|d\eta|_g \leq 2/\delta$. We set

$$g_\delta = \eta((\exp^\nu \circ \iota)^{-1})^*(g^{\text{flat}} + h) + (1-\eta)g.$$

Then g_δ has product form on $U_S(\delta)$ and $d^g(x,S) = d^{g_\delta}(x,S) = r(x)$. Through a series of lemmas we will prove Proposition 3.2 for $\tilde{g} = g_\delta$ with sufficiently small δ.

In the following estimates C denotes a constant whose value might vary from one line to another, which is independent of δ and η but might depend on M, g, S. Terms denoted by $o_i(1)$ tend to zero when $i \to \infty$.

Lemma 3.3. *Let δ_i be a sequence with $\delta_i \to 0$ as $i \to \infty$. Let φ_i be a sequence of spinors on (M, g_{δ_i}) such that $D^{g_{\delta_i}}\varphi_i = 0$ and $\int_M |\varphi_i|^2\, dv^{g_{\delta_i}} = 1$. Then the sequence $\beta_g^{g_{\delta_i}}\varphi_i$ is bounded in $H_1^2(M,g)$.*

Proof. As $\int |\beta_g^{g_{\delta_i}}\varphi_i|^2\, dv^g = 1 + o_i(1)$ we have to show that

$$\alpha_i = \sqrt{\int_M |\nabla^g(\beta_g^{g_{\delta_i}}\varphi_i)|_g^2\, dv^g}$$

is bounded. Suppose that the opposite is true, that is $\alpha_i \to \infty$, and set $\psi_i = \alpha_i^{-1}\beta_g^{g_{\delta_i}}\varphi_i$. Then we have ${}^gD^{g_{\delta_i}}\psi_i = 0$ since $\beta^g_{g_{\delta_i}} \circ \beta_g^{g_{\delta_i}} = \mathrm{Id}$, so the Schrödinger-Lichnerowich formula [16, Theorem 8.8, p. 160] together with (2) tells us that

$$\begin{aligned}
1 &= \int_M |\nabla^g\psi_i|_g^2\, dv^g \\
&= \int_M (|D^g\psi_i|^2 - \frac{1}{4}\mathrm{scal}^g|\psi_i|^2)\, dv^g \\
&= \int_M (|A^g_{g_{\delta_i}}(\nabla^g\psi_i) + B^g_{g_{\delta_i}}(\psi_i)|^2 - \frac{1}{4}\mathrm{scal}^g|\psi_i|^2)\, dv^g \\
&\le \int_M (2|A^g_{g_{\delta_i}}(\nabla^g\psi_i)|^2 + 2|B^g_{g_{\delta_i}}(\psi_i)|^2 - \frac{1}{4}\mathrm{scal}^g|\psi_i|^2)\, dv^g.
\end{aligned}$$

Using (3), (4), Lemma 3.1, and the fact that g and g_{δ_i} coincide outside $U_S(2\delta_i)$ we get

$$\begin{aligned}
1 &\le C\delta_i^2 \int_{U_S(2\delta_i)} |\nabla^g\psi_i|_g^2\, dv^g + C\int_{U_S(2\delta_i)} |\psi_i|^2\, dv^g + C\int_M |\psi_i|^2\, dv^g \\
&\le C\delta_i^2 + C\int_{U_S(2\delta_i)} |\psi_i|^2\, dv^g + \alpha_i^{-2}(1 + o_i(1)) \\
&\le C\int_{U_S(2\delta_i)} |\psi_i|^2\, dv^g + o_i(1). \\
&\le C\alpha_i^{-2} \underbrace{\int_M |\varphi_i|^2\, dv^g}_{=1+o_i(1)} + o_i(1) \to 0
\end{aligned}$$

which is obviously a contradiction. $\square$

Lemma 3.4. *Again let δ_i be a sequence with $\delta_i \to 0$ as $i \to \infty$ and let φ_i be a sequence of spinors on (M, g_{δ_i}) such that $D^{g_{\delta_i}}\varphi_i = 0$ and $\int_M |\varphi_i|^2\, dv^{g_{\delta_i}} = 1$. Then, after passing to a subsequence, $\beta_g^{g_{\delta_i}}\varphi_i$ converges weakly in $H_1^2(M,g)$ and strongly in $L^2(M,g)$ to a harmonic spinor on (M,g).*

Proof. According to Lemma 3.3 the sequence $\beta_g^{g_{\delta_i}}\varphi_i$ is bounded in $H_1^2(M,g)$ and hence a subsequence converges weakly in $H_1^2(M,g)$. After passing to a subsequence once again we obtain strong convergence in $L^2(M,g)$. We denote the limit spinor by φ.

$$0 < \rho << r_0 < r_1/2 << R_{\max}$$

FIGURE 1. Hierarchy of variables

Lemma 2.2 implies that $\beta_g^{g_{\delta_i}} \varphi_i$ is bounded in $C^2(M \setminus U_S(\varepsilon))$ for any $\varepsilon > 0$, and Lemma 2.3 then implies that a subsequence converges in $C^1(M \setminus U_S(\varepsilon))$. Hence the limit φ is in $C^1_{\mathrm{loc}}(M \setminus S)$ and satisfies $D^g \varphi = 0$ on $M \setminus S$. Since φ is in $L^2(M, g)$ it follows from Lemma 2.4 that φ is a weak solution of $D\varphi = 0$ on (M, g). By Lemma 2.1 it then follows that φ is a strong solution and a harmonic spinor on (M, g). □

Proof of Proposition 3.2. Let $m = \liminf_{\delta \to 0} \dim \ker D^{g_\delta}$. For sufficiently small δ let $\varphi_\delta^1, \ldots, \varphi_\delta^m \in \ker D^{g_\delta}$ be spinors such that

$$\int_M \langle \varphi_\delta^j, \varphi_\delta^k \rangle \, dv^{g_\delta} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases} \tag{8}$$

According to Lemma 3.4 there are spinors $\varphi^1, \ldots, \varphi^m \in \ker D^g$ and a sequence $\delta_i \to 0$ such that $\beta_g^{g_{\delta_i}} \varphi_{\delta_i}^j$ converges to φ^j weakly in $H_1^2(M, g)$ and strongly in $L^2(M, g)$ for $j = 1, \ldots, m$. Because of strong L^2-convergence the orthogonality relation (8) is preserved in the limit so $\dim \ker D^g \geq m$. Hence there is a $\delta_0 > 0$ so that $\dim \ker D^{g_{\delta_0}} = m \leq \dim \ker D^g$ and the Proposition is proved with $\tilde{g} = g_{\delta_0}$. □

3.2. **Proof for metrics of product form near S.** We now assume that g is a product metric on $U_S(R_{\max})$ for some $R_{\max} > 0$. This we may do by Proposition 3.2. In polar coordinates $(r, \Theta) \in (0, R_{\max}) \times S^{n-k-1}$ on $B^{n-k}(R_{\max})$ we have

$$g = g^{\mathrm{flat}} + h = dr^2 + r^2 g^{\mathrm{round}} + h.$$

Let $\rho > 0$ be a small number which we will finally let tend to 0 (see also Figure 1). We decompose M into three parts

(1) $M \setminus U_S(R_{\max})$,
(2) $U_S(R_{\max}) \setminus U_S(\rho/2) = [\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$,
(3) $U_S(\rho/2) = B^{n-k}(\rho/2) \times S^k$.

The manifold $\widetilde{M}$ is obtained by removing part (3) and by gluing in $S^{n-k-1} \times B^{k+1}$, that is $\widetilde{M}$ is the union of

(1) $M \setminus U_S(R_{\max})$,
(2) $U_S(R_{\max}) \setminus U_S(\rho/2) = [\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$,
(3') $S^{n-k-1} \times B^{k+1}$.

We now define a sequence of metrics g_ρ on $\widetilde{M}$. The metrics g_ρ will coincide with g on part (1), but will be modified on part (2) in order to close up nicely on part (3').

Let r_0, r_1 be fixed such that $2\rho < r_0 < r_1/2 < R_{\max}/2$. Define g_ρ on $\widetilde{M}$ by

(1) $g_\rho = g$ on $M \setminus U_S(R_{\max})$,
(2) $g_\rho = F^2(dr^2 + r^2 g^{\mathrm{round}} + f_\rho^2 h)$ on $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$, where F and f_ρ satisfy

$$F(r) = \begin{cases} 1, & \text{if } r_1 < r < R_{\max}; \\ 1/r, & \text{if } r < r_0, \end{cases} \quad \text{and} \quad f_\rho(r) = \begin{cases} 1, & \text{if } r > 2\rho; \\ r, & \text{if } r < \rho\,. \end{cases}$$

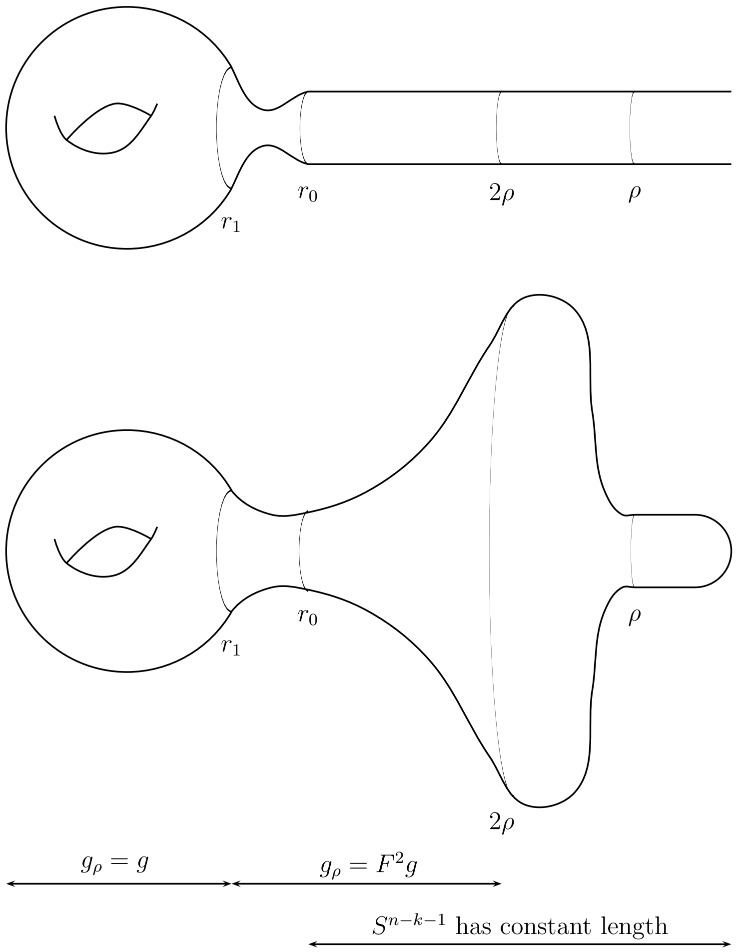

FIGURE 2. The metric g_ρ.

(3') $g_\rho = g^{\mathrm{round}} + \gamma_\rho$ on $S^{n-k-1} \times B^{k+1}$ where γ_ρ is some metric so that g_ρ is smooth.

In order to visualize the metric g_ρ two projections are drawn in Figure 2. In both projections the horizontal direction represents $-\log r$. In the first projection the vertical direction indicates the size of the cosphere S^{n-k-1}. In the second projection the vertical direction indicates the size of S which is fiberwise homothetic to $(S \cong S^k, h)$.

Our goal is to prove that

$$\dim \ker D^{g_\rho} \leq \dim \ker D^g \tag{9}$$

for small $\rho > 0$. Before we are able to do that we need some estimates.

For $\alpha \in (0, R_{\max})$ let $\widetilde{U}(\alpha) = \widetilde{M} \setminus (M \setminus U_S(\alpha))$ so that $M \setminus U_S(\alpha) = \widetilde{M} \setminus \widetilde{U}(\alpha)$.

Proposition 3.5. *Let $s \in (0, r_0/2)$. Let ψ_ρ be a harmonic spinor on $(\widetilde{M}, g_\rho)$. Then for $2\rho \in (0, s)$ it holds that*

$$\int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |F^{\frac{n-1}{2}}\psi_\rho|^2 \, dv^g \leq 32 \int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} |F^{\frac{n-1}{2}}\psi_\rho|^2 \, dv^g.$$

Proof. Let $\eta \in C^\infty(\widetilde{M})$ be a cut-off function with $0 \leq \eta \leq 1$, $\eta = 1$ on $\widetilde{U}(s)$, $\eta = 0$ on $\widetilde{M} \setminus \widetilde{U}(2s)$, and

$$|d\eta|_g \leq \frac{2}{s}. \tag{10}$$

The spinor $\eta\psi_\rho$ is compactly supported in $\widetilde{U}(2s)$. Moreover, the metric g_ρ can be written as $g_\rho = g^{\mathrm{round}} + h_\rho$ on $\widetilde{U}(2s)$ where the metric h_ρ is equal to $r^{-2}dr^2 + r^{-2}f_\rho^2 h$ on $\widetilde{U}(2s)\setminus\widetilde{U}(\rho/2)$ and is equal to γ_ρ on $S^{n-k-1} \times B^{k+1} = \widetilde{U}(\rho/2)$. Hence $(\widetilde{U}(2s), g_\rho)$ is isometric to an open subset of a manifold of the form $S^{n-k-1} \times N$ equipped with a product metric $g^{\mathrm{round}} + g_N$, where N is compact. By Proposition 2.6 the squared eigenvalues of the Dirac operator on this product manifold are greater than or equal to $(n-k-1)^2/4 \geq 1/4$. Writing the Rayleigh quotient of $\eta\psi_\rho$ we obtain

$$\frac{1}{4} \leq \frac{\int_{\widetilde{U}(2s)} |D^{g_\rho}(\eta\psi_\rho)|^2 \, dv^{g_\rho}}{\int_{\widetilde{U}(2s)} |\eta\psi_\rho|^2 \, dv^{g_\rho}}. \tag{11}$$

Since $D^{g_\rho}\psi_\rho = 0$ we have $D^{g_\rho}(\eta\psi_\rho) = \mathrm{grad}^{g_\rho}\eta \cdot \psi_\rho$ so

$$|D^{g_\rho}(\eta\psi_\rho)|^2 = |\mathrm{grad}^{g_\rho}\eta \cdot \psi_\rho|^2 = |d\eta|^2_{g_\rho}|\psi_\rho|^2_{g_\rho}. \tag{12}$$

By definition $d\eta$ is supported in $\widetilde{U}(2s) \setminus \widetilde{U}(s)$. On $\widetilde{M} \setminus \widetilde{U}(2\rho)$ we have $g_\rho = F^2 g$. Moreover, by Relation (10) and since $F = 1/r$ on the support of $d\eta$, we have

$$|d\eta|^2_{g_\rho} = r^2 |d\eta|^2_g \leq \frac{4r^2}{s^2}$$

and hence

$$|D^{g_\rho}(\eta\psi_\rho)|^2 \leq \frac{4r^2}{s^2}|\psi_\rho|^2.$$

Since $g_\rho = r^{-2}g$ on $\widetilde{U}(2s) \setminus \widetilde{U}(s)$ we have $dv^{g_\rho} = r^{-n}\, dv^g$. Using equation (12) it follows that

$$\begin{aligned}\int_{\widetilde{U}(2s)} |D^{g_\rho}(\eta\psi_\rho)|^2 \, dv^{g_\rho} &\leq \frac{4}{s^2}\int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} r^{2+(n-1)-n}|r^{-\frac{n-1}{2}}\psi_\rho|^2 \, dv^g \\ &\leq \frac{8}{s}\int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} |F^{\frac{n-1}{2}}\psi_\rho|^2 \, dv^g,\end{aligned} \tag{13}$$

where we also use that $r \leq 2s$ on the domain of integration. Since $\eta \in [0,1]$ on $\widetilde{U}(2s) \setminus \widetilde{U}(s)$, since $\eta = 1$ on $\widetilde{U}(s)$, and since $g_\rho = r^{-2}g$ on $\widetilde{U}(s) \setminus \widetilde{U}(2\rho)$, we have

$$\begin{aligned}\int_{\widetilde{U}(2s)} |\eta\psi_\rho|^2 \, dv^{g_\rho} &\geq \int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |\psi_\rho|^2 \, dv^{g_\rho} \\ &= \int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} r^{(n-1)-n}|r^{-\frac{n-1}{2}}\psi_\rho|^2 \, dv^g \\ &\geq \frac{1}{s}\int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |F^{\frac{n-1}{2}}\psi_\rho|^2 \, dv^g,\end{aligned} \tag{14}$$

where we use that $r \leq s$ in the last inequality. Plugging (13) and (14) into (11) we get

$$\frac{1}{4} \leq \frac{\frac{8}{s}\int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} |F^{\frac{n-1}{2}}\psi_\rho|^2\, dv^g}{\frac{1}{s}\int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |F^{\frac{n-1}{2}}\psi_\rho|^2\, dv^g}$$

and hence Proposition 3.5 follows. □

Proof of Theorem 1.2. We are now ready to prove (9). For a contradiction suppose that this relation does not hold. Then there is a strictly decreasing sequence $\rho_i \to 0$ such that $\dim\ker D^g < \dim\ker D^{g_{\rho_i}}$ for all i. To simplify the notation for subsequences we define $E = \{\rho_i : i \in \mathbb{N}\}$. We have $0 \in \overline{E}$ and passing to a subsequence of ρ_i means passing to a subset $E' \subset E$ with $0 \in \overline{E'}$.

Let $m = \dim\ker D^g + 1$. For all $\rho \in E$ we can find D^{g_ρ}-harmonic spinors $\psi_\rho^1, \dots, \psi_\rho^m$ on $(\widetilde{M}, g_\rho)$ such that

$$\int_{M\setminus U_S(s)} \langle \varphi_\rho^j, \varphi_\rho^k\rangle\, dv^g = \int_{\widetilde{M}\setminus\widetilde{U}(s)} \langle \varphi_\rho^j, \varphi_\rho^k\rangle\, dv^g = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k, \end{cases} \tag{15}$$

where $\varphi_\rho^j = F^{\frac{n-1}{2}}\psi_\rho^j$ and $s \leq r_0 < r_1/2$ is fixed as above. The spinor fields φ_ρ^j are defined on $M \setminus U_S(2\rho)$ and by (5) they are D^g-harmonic there.

Step 1. *Let $\delta \in (0, R_{\max})$. For $j = 1, \dots, m$ and $\rho \in E$ small enough we have*

$$\int_{M\setminus U_S(\delta)} |\varphi_\rho^j|^2\, dv^g \leq 33. \tag{16}$$

By Proposition 3.5 we have

$$\int_{U_S(s)\setminus U_S(2\rho)} |\varphi_\rho^j|^2\, dv^g \leq 32 \int_{U_S(2s)\setminus U_S(s)} |\varphi_\rho^j|^2\, dv^g,$$

and hence if $2\rho \leq \delta$ it follows that

$$\int_{U_S(s)\setminus U_S(\delta)} |\varphi_\rho^j|^2\, dv^g \leq 32 \int_{M\setminus U_S(s)} |\varphi_\rho^j|^2\, dv^g.$$

We conclude that

$$\begin{aligned}\int_{M\setminus U_S(\delta)} |\varphi_\rho^j|^2\, dv^g &= \int_{M\setminus U_S(s)} |\varphi_\rho^j|^2\, dv^g + \int_{U_S(s)\setminus U_S(\delta)} |\varphi_\rho^j|^2\, dv^g \\ &\leq (1 + 32)\int_{M\setminus U_S(s)} |\varphi_\rho^j|^2\, dv^g,\end{aligned}$$

and (16) follows from (15).

Step 2. *There exists $E' \subset E$ with $0 \in \overline{E'}$ and spinors $\Phi^1, \dots, \Phi^m \in C^1(M \setminus S)$, D^g-harmonic on $M \setminus S$ such that φ_ρ^j tend to Φ^j in $C^1_{\mathrm{loc}}(M \setminus S)$ as $\rho \to 0$, $\rho \in E'$.*

Let $Z \in \mathbb{N}$ be an integer, $Z > 1/s$. By (16) the sequence $\{\varphi_\rho^j\}_{\rho\in E}$ is bounded in $L^2(M \setminus U_S(1/Z))$. By Lemma 2.2 it follows that $\{\varphi_\rho^j\}_{\rho\in E}$ is bounded in $C^2(M \setminus U_S(2/Z))$ for all sufficiently large Z. For a fixed $Z_0 > 1/s$ we apply Lemma 2.3 and conclude that for any j there is a subsequence $\{\varphi_\rho^j\}_{\rho\in E_0}$ of $\{\varphi_\rho^j\}_{\rho\in E}$ that converges in $C^1(M \setminus U_S(2/Z_0))$ to a spinor Φ_0^j. Similarly we construct further and further subsequences $\{\varphi_\rho^j\}_{\rho\in E_i}$ converging to Φ_i^j in $C^1(M \setminus U_S(2/(Z_0 + i)))$ with $E_i \subset E_{i-1} \subset \cdots \subset E_0 \subset E$, $0 \in \overline{E_i}$. Obviously Φ_i^j extends Φ_{i-1}^j. Define $E' \subset E$

as consisting of one ρ_i from each E_i chosen so that $\rho_i \to 0$ as $i \to \infty$. Then the sequence $\{\varphi^j_\rho\}_{\rho \in E'}$ converges in $C^1_{\mathrm{loc}}(M \setminus S)$ to a spinor Φ^j. As φ^j_ρ is D^g-harmonic on $(M \setminus U_S(2\rho))$ the $C^1_{\mathrm{loc}}(M \setminus S)$-convergence implies that $D^g\Phi^j = 0$ on $M \setminus S$. We have proved Step 2.

Step 3. *Conclusion.*

By (16) we conclude that

$$\int_{M \setminus S} |\Phi^j|^2 \, dv^g \leq 33,$$

and hence $\Phi^j \in L^2(M)$ for $j = 1, \dots, m$. By Lemmas 2.4 and 2.1 we then conclude that Φ^j is harmonic and smooth on all of (M, g). Since $M \setminus U_S(s)$ is a relatively compact subset of $M \setminus S$ the normalization (15) is preserved in the limit $\rho \to 0$ and hence

$$\int_{M \setminus U_S(s)} \langle \Phi^j, \Phi^k \rangle \, dv^g = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

This proves that $\Phi^1, \dots, \Phi^m$ are linearly independent harmonic spinors on (M, g) and hence $\dim \ker D^g \geq m$ which contradicts the definition of m. This proves Relation (9) and Theorem 1.2. □

4. Proof of Theorem 1.1

The proof will follow the argument of [3] and we introduce notation in accordance with that paper. For a compact spin manifold M the space of smooth Riemannian metrics on M is denoted by $\mathcal{R}(M)$ and the subset of D-minimal metrics is denoted by $\mathcal{R}_{\min}(M)$.

From standard results in perturbation theory it follows that $\mathcal{R}_{\min}(M)$ is open in the C^1-topology on $\mathcal{R}(M)$ and if $\mathcal{R}_{\min}(M)$ is not empty then it is dense in $\mathcal{R}(M)$ in all C^k-topologies, $k \geq 1$, see for example [17, Prop. 3.1]. We define the word generic to mean these open and dense properties satisfied by $\mathcal{R}_{\min}(M)$ if non-empty. Theorem 1.1 is then equivalent to the following.

Theorem 4.1. *Let M be a compact connected spin manifold. Then there is a D-minimal metric on M.*

Before we start the proof we note the following consequence of Theorem 1.2.

Proposition 4.2. *Let N be a compact spin manifold which has a D-minimal metric and suppose that M is obtained from N by surgery of codimension ≥ 2. Then M has a D-minimal metric.*

Proof. This follows from Theorem 1.2 since the left hand side of (1) is the same for M and N while the right hand side may only decrease. □

From the theory of handle decompositions of bordisms we have the following.

Proposition 4.3. *Suppose that M is connected,* $\dim M \geq 3$*, and that M is spin bordant to a manifold N. Then M can be obtained from N by a sequence of spin-compatible surgeries of codimension ≥ 2.*

Proof. If $\dim M = 3$ the statement follows from [13, VII Theorem 3]. If $n = \dim M \geq 4$, then we can do surgery in dimensions 0 and 1 on a given spin bordism between M and N and obtain a connected, simply connected spin bordism between M and N. It then follows from [14, VIII Proposition 3.1] that one can obtain M from N by surgeries of dimensions $0, \ldots, n-2$. $\square$

Proof of Theorem 4.1. From the solution of the Gromov-Lawson conjecture by Stolz [20] together with knowledge of some explicit manifolds with D-minimal metrics one can show that any compact spin manifold is spin bordant to a manifold with a D-minimal metric. This is worked out in detail in [3, Prop. 3.9]. We may thus assume that the given manifold M is spin bordant to a manifold N equipped with a D-minimal metric. If $\dim M \geq 3$ the Theorem then follows from Propositions 4.2 and 4.3.

Next we assume $\dim M = 2$. If $\alpha(M) = 0$ then M is obtained by surgeries of dimension 0 on S^2. We let the 2-torus T^2 be equipped with the spin structure for which $\alpha(T^2) \neq 0$, then if $\alpha(M) \neq 0$ we know that M is obtained by surgeries of dimension 0 on T^2. The canonical metrics on S^2 and T^2 are D-minimal, so Theorem 4.1 follows from Proposition 4.2. $\square$

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