WEIGHTED SOBOLEV SPACES AND REGULARITY FOR POLYHEDRAL DOMAINS

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Dedicated to Ivo Babuška on the occasion of his 80th birthday.

ABSTRACT. We prove a regularity result for the Poisson problem $-\Delta u = f$, $u|_{\partial\mathbb{P}} = g$ on a polyhedral domain $\mathbb{P} \subset \mathbb{R}^3$ using the Babuška–Kondratiev spaces $\mathcal{K}_a^m(\mathbb{P})$. These are weighted Sobolev spaces in which the weight is given by the distance to the set of edges [4, 29]. In particular, we show that there is no loss of $\mathcal{K}_a^m$–regularity for solutions of strongly elliptic systems with smooth coefficients. We also establish a "trace theorem" for the restriction to the boundary of the functions in $\mathcal{K}_a^m(\mathbb{P})$.

CONTENTS

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain. Then it is well known [6, 13, 22, 39] that the equation

$$\Delta u = f \in H^{m-1}(\Omega), \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

has a unique solution $u \in H^{m+1}(\Omega)$. In particular, u will be smooth on $\overline{\Omega}$ if f is smooth on $\overline{\Omega}$. This well-posedness result is especially useful in practice for the numerical approximation of the solution u of Equation (1), see for example [6, 11, 13] among many possible references.

In practice, however, it is rarely the case that Ω is smooth. For instance, if $\partial\Omega$ is *not* smooth, then the smoothness of f on $\overline{\Omega}$ does not imply that the solution u of Equation (1) is also smooth on $\overline{\Omega}$. Therefore there is a *loss of regularity* for elliptic problems on non-smooth domains. Wahlbin [41] (see also [5, 31, 42]) has shown that this leads to some inconvenience, namely that a quasi-uniform sequence of triangulations on Ω will *not* lead to optimal rates of convergence for the Galerkin approximations u_h of the solution of (1).

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The loss of regularity can be avoided, however, if one removes the singular points by "sending them to infinity" by suitably changing the metric by a conformal factor. It can be proved then that the resulting Sobolev spaces are the "Sobolev spaces with weights" considered for instance in [6, 10, 11, 29]. A related construction was considered in [25]. Let $f > 0$ be a smooth function on a domain Ω, we then define the *mth Sobolev space with weight f* by

$$\mathcal{K}_a^m(\Omega; f) := \{u,\ f^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \ |\alpha| \leq m\}, \quad m \in \mathbb{Z}_+,\, a \in \mathbb{R}. \tag{2}$$

We can then extend the regularity result for Equation (1) to polyhedral domains $\mathbb{P}$ in three dimensions with the usual Sobolev spaces replaced by the Babuška–Kondratiev spaces $\mathcal{K}_a^m(\mathbb{P}) := \mathcal{K}_a^m(\mathbb{P}; \vartheta)$, ϑ being the distance to the edges. The spaces $\mathcal{K}_a^s(\partial\mathbb{P}) := \mathcal{K}_a^s(\partial\mathbb{P}; \vartheta)$ on the boundary are defined similarly for $s \in \mathbb{Z}_+$; for $s \in \mathbb{R}_+$ they are defined using interpolation.

Theorem 1.1. *Let $\mathbb{P} \subset \mathbb{R}^3$ be a polyhedral domain. Let $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$. Assume that $u \in \mathcal{K}_{a+1}^1(\mathbb{P})$, $\Delta u \in \mathcal{K}_{a-1}^{m-1}(\mathbb{P})$, and $u|\partial\mathbb{P} \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial\mathbb{P})$, then $u \in \mathcal{K}_{a+1}^{m+1}(\mathbb{P})$ and there exists $C > 0$ independent of u such that*

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\mathbb{P})} \leq C\big(\|\Delta u\|_{\mathcal{K}_{a-1}^{m-1}(\mathbb{P})} + \|u\|_{\mathcal{K}_{a+1}^{0}(\mathbb{P})} + \|u|_{\partial\mathbb{P}}\|_{\mathcal{K}_{a+1/2}^{m+1/2}(\partial\mathbb{P})}\big).$$

The same result holds if we replace Δ with a strongly elliptic operator.

Theorem 1.1 was known in two dimensions, i. e., for polygonal domains, [29]. The corresponding result in two dimensions requires ϑ to be the distance to the vertices of the polygonal domain considered. In general, in d dimensions, one takes $\vartheta(x)$ to be the distance to the set of non-smooth boundary points of $\mathbb{P}$. In three dimensions, related results were obtained before by Mazya and Rossmann [32] using results on the Green function. See also [2, 3, 16, 18, 19, 20, 21, 27, 28, 33, 35, 36], to mention just a few papers. A result valid in all dimensions was obtained in [1] using "Lie manifolds."

In this paper we follow [1], but we use more elementary methods. We also introduce some ideas that are specific to polyhedral domains in three dimensions and may be useful in applications to Numerical Analysis. Moreover, our paper is self-contained and the references to [1] are only for historical reasons.

We would like to stress that Theorem 1.1 does *not* constitute a Fredholm (or "normal solvability") result, because the inclusion $\mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \to \mathcal{K}_{a+1}^{0}(\mathbb{P})$ is *not compact* for all m and a [1]. By contrast, if $\mathbb{P}$ is a polygon, then $P = -\Delta$ with Dirichlet boundary conditions is Fredholm precisely when a is different from $k\pi/\alpha$, where $k \in \mathbb{Z}$, $k \neq 0$, and α ranges through the angles of the polygon [29, 30].

To prove Theorem 1.1, we first introduce the weighted Sobolev spaces $\mathcal{K}_a^m(\partial\mathbb{P}, \vartheta)$ on the boundary of $\mathbb{P}$. For $m \notin \mathbb{Z}_+$, these spaces are defined by duality and interpolation. Then we provide an alternative definition of the spaces $\mathcal{K}_a^m(\mathbb{P}) := \mathcal{K}_a^m(\mathbb{P}, \vartheta)$ and $\mathcal{K}_a^m(\partial\mathbb{P}, \vartheta)$ using partitions of unity. This allows us to define a trace map $\mathcal{K}_a^m(\mathbb{P}) \to \mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P}, \vartheta)$, which extends the restriction map and is a continuous surjection, as in the case of a smooth domain. We also show that any differential operator P of order m with smooth coefficients induces a continuous map $P : \mathcal{K}_a^s(\mathbb{P}) \to \mathcal{K}_{a-m}^{s-m}(\mathbb{P}; \vartheta)$.

A new idea in this paper is to introduce an enhanced space of smooth, bounded functions $\mathcal{C}^\infty(\Sigma\mathbb{P})$, which contains the cylindrical and spherical coordinates functions and is minimal with this property. In particular, $\mathcal{C}^\infty(\overline{\Omega}) \subset \mathcal{C}^\infty(\Sigma\mathbb{P}) \subset \mathcal{C}^\infty(\Omega)$.

Let $\rho_P(p)$ be the distance from p to the vertex P of $\mathbb{P}$ and $r_e(p)$ be the distance from p to the line determined by the edge e of $\mathbb{P}$ (for $\mathbb{P}$ non-convex we need to slightly change the definition of r_e). Let $\tilde{r}_e := \rho_A^{-1}\rho_B^{-1}r_e$ if $e = [AB]$. Then we define $r_{\mathbb{P}} = \prod_e \tilde{r}_e \times \prod_P \rho_P$, which turns out to belong to $\mathcal{C}^\infty(\Sigma\mathbb{P})$. The boundedness of the functions in $\mathcal{C}^\infty(\Sigma\mathbb{P})$ refers to the fact that

$$(r_{\mathbb{P}}\partial_x)^i(r_{\mathbb{P}}\partial_y)^j(r_{\mathbb{P}}\partial_z)^k u \in \mathcal{C}^\infty(\Sigma\mathbb{P}) \tag{3}$$

for all $u \in \mathcal{C}^\infty(\Sigma\mathbb{P})$. The consideration of $\mathcal{C}^\infty(\Sigma\mathbb{P})$ and of the derivatives of the form $r_{\mathbb{P}}\partial_x$, $r_{\mathbb{P}}\partial_y$, and $r_{\mathbb{P}}\partial_z$ is a substitute for the results on Lie manifolds used in [1]. However, the results of [1] also apply to non-compact manifolds and to a larger class of singular domains.

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2. Smooth functions and differential operators on $\mathbb{P}$

In this section, we shall introduce the space $\mathcal{C}^\infty(\Sigma\mathbb{P})$ of smooth functions on $\mathbb{P}$ and relate it to the differentials $r_{\mathbb{P}}\partial_x$, $r_{\mathbb{P}}\partial_y$, and $r_{\mathbb{P}}\partial_z$ considered in the Introduction. These vector fields have appeared also in [14].

2.1. **Polygons and polyhedral domains.**

Let us fix some terminology to be used in what follows.

A *polygon* $\mathbb{P}_0$ in a two dimensional Euclidean space is an open, connected subset whose boundary consists of finitely many straight segments (possibly of infinite length) called *sides* and having at most the end points in common. We assume that $\partial\mathbb{P}_0 = \partial\overline{\mathbb{P}}_0$. The points common to more than one straight segment of the boundary are called the vertices of $\mathbb{P}_0$. We require that each vertex belongs to exactly two sides.

We do not require the boundary of $\mathbb{P}_0$ to be connected. For simplicity, in this paper we also assume that the sides are maximally extended, so that they are not contained in larger segments contained in the boundary. This assumption is however not essential.

Similarly, a *polyhedral domain* $\mathbb{P} \subset \mathbb{R}^3$ is a connected open set such that $\partial\mathbb{P} = \partial\overline{\mathbb{P}} = \bigcup_{j=1}^N \overline{D}_j$ satisfying:

(i) each D_j is a polygon contained in an affine 2-dimensional subspace of $\mathbb{R}^3$;
(ii) the sets D_j are disjoint;
(iii) a side of D_j is a side of exactly one other D_k.

The vertices of $\mathbb{P}$ are the vertices of the polygonal domains D_j. The edges of $\mathbb{P}$ are the sides of the polygonal domains D_j. Hence an edge belongs to exactly two faces of $\mathbb{P}$. For each vertex P of $\mathbb{P}$, we choose a small open ball V_P centered in P. We assume that the neighborhoods V_P are chosen to be disjoint.

2.2. **Important functions and other notation.**

Assume for the definition of r_e, θ_e, and $\phi_{P,e}$ in this subsection that $\mathbb{P}$ is *convex*. If $\mathbb{P}$ is not convex, then we slightly change the definitions of these functions such that the new functions retain their behaviour around e, but will become smooth everywhere in space except on $\overline{e}$

[1]. The modified functions $\phi_{P,e}$ and θ_e will then be defined and smooth on $\mathbb{P}$. We include this technical construction in the Appendix, in order not to interrupt the flow of the presentation.

Let us first recall from the Introduction that we have denoted by $\rho_P(p)$ the distance from p to the vertex P of $\mathbb{P}$. Also, recall that $r_e(p)$ is the distance from p to the line determined by the edge e of $\mathbb{P}$ and by

$$r_{\mathbb{P}} := \prod_e \tilde{r}_e \times \prod_P \rho_P, \qquad \text{where } \ \tilde{r}_e := \rho_A^{-1}\rho_B^{-1} r_e \ \text{ for } \ e = [AB]. \tag{4}$$

In the above formula, the products are taken over all vertices P and all edges e of $\mathbb{P}$. The notation $e = [AB]$ means that e is the edge joining the vertices A and B. If $e = [A, \infty)$, that is, if e is a half-line, then $\tilde{r}_e := \rho_A^{-1} r_e$. Finally, if e is infinite in both directions, we let $\tilde{r}_e := r_e$.

Choose for each edge e a plane $\mathcal{P}_e$ containing one of the faces D_j of $\mathbb{P}$ such that $e \subset \overline{D}_j$. If x is not on the line defined by e, we then define θ_e to be the angle in a cylindrical coordinates system (r_e, θ_e, z) determined by the edge e and the plane $\mathcal{P}_e$. More precisely, let $q \in e$ be the foot of the perpendicular from p to e. Then $\theta_e(p)$ is the angle between pq and $\mathcal{P}_e$. Similarly, for each vertex P and edge e adjacent to P, we define $\phi_{P,e}(p)$ to be the angle between the segment pP and the edge e.

If $\mathbb{P}$ is convex, then the functions θ_e and $\phi_{P,e}$ are defined and smooth on $\mathbb{P}$ (recall that $\mathbb{P}$ is an open subset). For $\mathbb{P}$ non-convex, this property will be enjoyed by the modified functions θ_e and $\phi_{P,e}$ introduced in the Appendix. All the following definitions and constructions below are the same in the case of a non-convex domain, but using the modified θ and ϕ variables.

We shall denote by $\theta = (\theta_{e_1}, \ldots, \theta_{e_r})$ the vector variable that puts together all the θ_e functions. Similarly, we shall denote by $\phi = (\phi_{P_1,e_1}, \ldots, \phi_{P_p,e_p})$ the vector variable that puts together all the $\phi_{P,e}$ functions. We then introduce the space $W^{k,\infty}(\Sigma\mathbb{P})$ as the space of functions $u : \mathbb{P} \to \mathbb{C}$ of the form

$$u(x, y, z) = f(x, y, z, \theta, \phi) = f(x, y, z, \theta_{e_1}, \ldots, \theta_{e_r}, \phi_{P_1,e_1}, \ldots, \phi_{P_p,e_p}),$$
$$f \in W^{k,\infty}(\mathbb{P} \times (0, 2\pi)^r \times (0, \pi)^p).$$

Thus f above has k bounded weak derivatives. We let $\mathcal{C}^\infty(\Sigma\mathbb{P}) := \bigcap_k W^{k,\infty}(\Sigma\mathbb{P})$. The point of this definition is that, for example, θ_e is a smooth function on $\mathbb{P}$ which is not in $W^{k,\infty}(\mathbb{P})$ for all k because its derivatives are unbounded; on the other hand $\theta_e \in \mathcal{C}^\infty(\Sigma\mathbb{P})$.

One can show as in [1, 14] that there exists a compact space $\Sigma(\mathbb{P})$, canonically associated to $\mathbb{P}$ such that $\mathcal{C}^\infty(\Sigma(\mathbb{P})) = \mathcal{C}^\infty(\Sigma\mathbb{P})$, so our notation is justified. The construction of a space with this property is not very intuitive. However, at this point, we do not assign any significance to $\Sigma\mathbb{P}$, which should be regarded in this paper just as a symbol.

2.3. **Vector fields and $\mathcal{C}^\infty(\Sigma\mathbb{P})$.**

We now establish several technical properties of the functions in $\mathcal{C}^\infty(\Sigma\mathbb{P})$, especially in relation to the vector fields (differentials) $r_{\mathbb{P}}\partial_x$, $r_{\mathbb{P}}\partial_y$, and $r_{\mathbb{P}}\partial_z$.

Lemma 2.1. *Let P be a vertex of $\mathbb{P}$, then $\rho_P \in \mathcal{C}^\infty(\Sigma\mathbb{P})$. Similarly, let $e = [AB]$ be the edge of $\mathbb{P}$ joining the vertices A and B, then $\tilde{r}_e := \rho_A^{-1}\rho_B^{-1} r_e \in \mathcal{C}^\infty(\Sigma\mathbb{P})$. In particular, $r_{\mathbb{P}} := \prod_e \tilde{r}_e \times \prod_P \rho_P \in \mathcal{C}^\infty(\Sigma\mathbb{P})$.*

This is proved using polar coordinates. Assume P belongs to the edge e, then $\rho_P = (\sin\phi_{P,e}\cos\theta_e)^{-1}x$, where this is defined (x stands for the first component variable). Similar formulas for ρ_P in terms of y and z then combine, using a partition of unity on $\mathbb{R}^3 \smallsetminus \{P\}$ with functions in $\mathcal{C}^\infty(\Sigma\mathbb{P})$, to define ρ_P globally as an element in $\mathcal{C}^\infty(\Sigma\mathbb{P})$.

Lemma 2.2. *Let $\vartheta(p)$ be the distance from p to the union of the edges of $\mathbb{P}$. Then there exists $C > 0$ such that $C^{-1}\vartheta(p) \le r_{\mathbb{P}} \le C\vartheta(p)$ for all $p \in \mathbb{P}$.*

This lemma is proved using the homogeneity properties of the functions ϑ and $r_{\mathbb{P}}$ close to the vertices and edges of $\mathbb{P}$. Using a compactness argument, it is enough to prove that the ratio $r_{\mathbb{P}}/\vartheta$ is bounded and bounded away from zero in the neighborhood of each point. This allows us to assume that $\mathbb{P}$ is either a dihedral angle or an infinite cone. If $\mathbb{P}$ is the dihedral angle $0 < \theta < \alpha$, with α fixed, then $r_{\mathbb{P}}/\vartheta = 1$. If $\mathbb{P}$ is a cone with center the origin, let α_t be the dilation with center the origin and ratio t. Then $r_{\mathbb{P}}(\alpha_t(p)) = tr_{\mathbb{P}}(p)$ and $\vartheta(\alpha_t(p)) = t\vartheta(p)$. This shows that the ratio $r_{\mathbb{P}}(p)/\vartheta(p)$ depends only on $p/|p|$. This allows us to reduce the problem to the case of a spherical polygon, which is treated like the case of a dihedral angle.

Lemma 2.3. *The functions $r_e\partial_x\theta_e$, $r_e\partial_y\theta_e$, $r_e\partial_z\theta_e$, $\rho_P\partial_x\phi_{P,e}$, $\rho_P\partial_y\phi_{P,e}$, $\rho_P\partial_z\phi_{P,e}$, $\partial_x r_e$, $\partial_y r_e$, $\partial_z r_e$, $\partial_x\rho_P$, $\partial_y\rho_P$, and $\partial_z\rho_P$ are in $\mathcal{C}^\infty(\Sigma\mathbb{P})$.*

To prove this, let us notice first that we can use any linear system of coordinates (x, y, z). In particular, for each of the above calculations, we can assume that our cylindrical or spherical coordinate system is aligned to the coordinate system (x, y, z). Then the result is simply an exercise in the calculation of the partial derivatives of the cylindrical coordinates θ and r and of the spherical coordinates ϕ and ρ.

Corollary 2.4. *We have $\partial_x r_{\mathbb{P}}, \partial_y r_{\mathbb{P}}, \partial_z r_{\mathbb{P}} \in \mathcal{C}^\infty(\Sigma\mathbb{P})$.*

Proof. Let us concentrate on ∂_x. We use the product rule to compute the derivative of $r_{\mathbb{P}}$. A summand containing $\partial_x\rho_P$ is in $\mathcal{C}^\infty(\Sigma\mathbb{P})$ by Lemma 2.3. Let $e = [AB]$. The other products are obtained by replacing $\tilde{r}_e := \rho_A^{-1}\rho_B^{-1}r_e$ with

$$\partial_x(\tilde{r}_e) = \rho_A^{-1}\rho_B^{-1}\partial_x(r_e) - \rho_A^{-1}\partial_x(\rho_A)\tilde{r}_e - \rho_B^{-1}\partial_x(\rho_B)\tilde{r}_e.$$

The factors of ρ_A^{-1} and ρ_B^{-1} then cancel out and all the remaining factors are in $\mathcal{C}^\infty(\Sigma\mathbb{P})$ by Lemma 2.3. □

Proposition 2.5. *If $u \in \mathcal{C}^\infty(\Sigma\mathbb{P})$, then the functions $r_{\mathbb{P}}\partial_x u$, $r_{\mathbb{P}}\partial_y u$, and $r_{\mathbb{P}}\partial_z u$ are in $\mathcal{C}^\infty(\Sigma\mathbb{P})$.*

Proof. This follows from Lemma 2.3 and $r_{\mathbb{P}} \in r_e\mathcal{C}^\infty(\Sigma\mathbb{P}) \cap \rho_P\mathcal{C}^\infty(\Sigma\mathbb{P})$. □

Let us denote by $\mathrm{Diff}_0^m(\mathbb{P})$ the differential operators of order m on $\mathbb{P}$ linearly generated by differential operators of the form

$$u(r_{\mathbb{P}}\partial)^\alpha := u(r_{\mathbb{P}}\partial_x)^{\alpha_1}(r_{\mathbb{P}}\partial_y)^{\alpha_2}(r_{\mathbb{P}}\partial_z)^{\alpha_3}, \quad |\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \le m,\ u \in \mathcal{C}^\infty(\Sigma\mathbb{P}).$$

For $m = 0$ we agree that $\mathrm{Diff}_0^m(\mathbb{P}) := \mathcal{C}^\infty(\Sigma\mathbb{P})$. We shall denote by $\mathrm{Diff}_0^\infty(\mathbb{P}) := \bigcap_m \mathrm{Diff}_0^m(\mathbb{P})$. To get more insight into the structure of $\mathrm{Diff}_0^\infty(\mathbb{P})$, we shall need two simple calculations that we formalize in the following lemma whose proof is based on the fact that $\partial_j r_{\mathbb{P}} \in \mathcal{C}^\infty(\Sigma\mathbb{P})$.

Lemma 2.6. *Let $\lambda \in \mathbb{R}$ and let ∂_j and ∂_k stand for either of ∂_x, ∂_y, or ∂_z. Then $r_{\mathbb{P}}^{-\lambda}(r_{\mathbb{P}}\partial_j)r_{\mathbb{P}}^{\lambda} - r_{\mathbb{P}}\partial_j = \lambda\partial_j(r_{\mathbb{P}}) \in \mathcal{C}^{\infty}(\Sigma\mathbb{P})$, and*

$$[r_{\mathbb{P}}\partial_j, r_{\mathbb{P}}\partial_k] := (r_{\mathbb{P}}\partial_j)(r_{\mathbb{P}}\partial_k) - (r_{\mathbb{P}}\partial_k)(r_{\mathbb{P}}\partial_j) = \partial_j(r_{\mathbb{P}})r_{\mathbb{P}}\partial_k - \partial_k(r_{\mathbb{P}})r_{\mathbb{P}}\partial_j \in \mathrm{Diff}_0^1(\mathbb{P}).$$

Then we have the following simple but basic result.

Proposition 2.7. *We have $\mathrm{Diff}_0^k(\mathbb{P})\mathrm{Diff}_0^m(\mathbb{P}) \subset \mathrm{Diff}_0^{k+m}(\mathbb{P})$ and hence $\mathrm{Diff}_0^{\infty}(\mathbb{P})$ is an algebra.*

Proof. We shall prove by induction on $k+m$ that $\mathrm{Diff}_0^k(\mathbb{P})\mathrm{Diff}_0^m(\mathbb{P}) \subset \mathrm{Diff}_0^{k+m}(\mathbb{P})$. Indeed, if $k+m=0$, then $k=m=0$ and the statement is clearly true because $\mathcal{C}^{\infty}(\Sigma\mathbb{P})$ is closed under products. Let us assume then that $k+m>0$. We need to show that $u(r_{\mathbb{P}}\partial)^{\alpha}v(r_{\mathbb{P}}\partial)^{\beta} \in \mathrm{Diff}_0^{k+m}(\mathbb{P})$ if $u, v \in \mathcal{C}^{\infty}(\Sigma\mathbb{P})$ and $|\alpha| := \alpha_1+\alpha_2+\alpha_3 = k$, $|\beta| := \beta_1+\beta_2+\beta_3 = m$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$.

If $m = 0$, then the relation

$$u(r_{\mathbb{P}}\partial)^{\alpha}v = \sum u(r_{\mathbb{P}}\partial)^{\alpha'}\left[r_{\mathbb{P}}\partial_j(v)\right](r_{\mathbb{P}}\partial)^{\alpha''}$$

for suitable α', α'' with $|\alpha'|+|\alpha''| = k-1$, together with the induction hypothesis and with Proposition 2.5, shows that $u(r_{\mathbb{P}}\partial)^{\alpha}v \in \mathrm{Diff}_0^k(\mathbb{P})$.

Let now m be arbitrary. We shall proceed by a second induction on m. The same argument as in the paragraph above allows us to assume that $v = 1$. We can also assume that the monomial $(r_{\mathbb{P}}\partial)^{\alpha}(r_{\mathbb{P}}\partial)^{\beta}$ is already ordered in the standard way. Then, using Lemma 2.6, we commute $r_{\mathbb{P}}\partial_j$, the last derivative in $(r_{\mathbb{P}}\partial)^{\alpha}$, with $r_{\mathbb{P}}\partial_k$, the first derivative in $(r_{\mathbb{P}}\partial)^{\beta}$. Induction on $k+m$ for the terms containing $\partial_j(r_{\mathbb{P}})r_{\mathbb{P}}\partial_k$ and $\partial_k(r_{\mathbb{P}})r_{\mathbb{P}}\partial_j$ and induction on m or $k+m$ for the term containing $(r_{\mathbb{P}}\partial_k)(r_{\mathbb{P}}\partial_j)$ then complete the proof of the fact that $\mathrm{Diff}_0^k(\mathbb{P})\mathrm{Diff}_0^m(\mathbb{P}) \subset \mathrm{Diff}_0^{k+m}(\mathbb{P})$. ☐

The above proposition gives the following useful corollary.

Corollary 2.8. *If P is a differential operator of order m with smooth coefficients, then $r_{\mathbb{P}}^m P \in \mathrm{Diff}_0^m(\mathbb{P})$.*

Proof. It is enough to show that $r_{\mathbb{P}}^m\partial^{\alpha} \in \mathrm{Diff}_0^m(\mathbb{P})$ if $\partial^{\alpha} = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}$ with $|\alpha| = m$. We shall again proceed by induction on m. The case $m = 1$ is obvious. Let ∂_j be the first derivative in ∂^{α}, so that $\partial^{\alpha} = \partial_j\partial^{\alpha'}$. Then Lemma 2.6 and Corollary 2.4 give

$$r_{\mathbb{P}}^m\partial^{\alpha} - (r_{\mathbb{P}}\partial_j)(r_{\mathbb{P}}^{m-1}\partial^{\alpha'}) = -(m-1)\partial_j(r_{\mathbb{P}})(r_{\mathbb{P}}^{m-1}\partial^{\alpha'}) \in \mathrm{Diff}_0^{m-1}(\mathbb{P}).$$

Then Proposition 2.7 shows that $\mathrm{Diff}_0^1(\mathbb{P})\mathrm{Diff}_0^{m-1}(\mathbb{P}) \subset \mathrm{Diff}_0^m(\mathbb{P})$. This and the induction hypothesis allows us to complete the proof. ☐

The proof of the above corollary also shows that

$$r_{\mathbb{P}}^m\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3} - (r_{\mathbb{P}}\partial_x)^{\alpha_1}(r_{\mathbb{P}}\partial_y)^{\alpha_2}(r_{\mathbb{P}}\partial_z)^{\alpha_3} \in \mathrm{Diff}_0^{m-1}(\mathbb{P}), \quad |\alpha| = m. \tag{5}$$

3. Function spaces on $\mathbb{P}$

In this section, we shall introduce and study the Babuška–Kondratiev spaces $\mathcal{K}_a^m(\mathbb{P})$ and $\mathcal{K}_a^m(\partial\mathbb{P}) := \mathcal{K}_a^m(\partial\mathbb{P}; \vartheta)$. These spaces are weighted Sobolev spaces with weight given by ϑ, the distance to the set of edges of $\mathbb{P}$, as in Equation (2).

3.1. **The Babuška–Kondratiev spaces.** We define

$$W_{BK}^{k,p,a}(\mathbb{P}) = \{u : \mathbb{P} \to \mathbb{C},\ r_{\mathbb{P}}^{|\alpha|-a}\partial^\alpha u \in L^p(\mathbb{P}),\ \text{for all } |\alpha| \le k\}, \quad k \in \mathbb{Z}_+. \tag{6}$$

If $p = 2$, we denote $\mathcal{K}_a^k(\mathbb{P}) := W_{BK}^{k,2,a}(\mathbb{P})$, which coincides with the definition in the Introduction (Equation 2).

We similarly define

$$W_{BK}^{m,p,a}(\partial\mathbb{P}) = \{u : \partial\mathbb{P} \to \mathbb{C},\ r_{\mathbb{P}}^{k-a}P(u|_{D_j}) \in L^p(D_j), \quad \text{for all } k \le m$$
$$\text{and all differential operators } P \text{ of order } k \text{ on } D_j,\ k \le m\}, \quad m \in \mathbb{Z}_+.$$

We let $\mathcal{K}_a^k(\partial\mathbb{P}) := W_{BK}^{m,2,a}(\partial\mathbb{P})$. Thus $\mathcal{K}_a^k(\partial\mathbb{P}) \simeq \bigoplus \mathcal{K}_a^k(D_j, \vartheta)$ is thus a direct sum of weighted Sobolev spaces. Note that we require *no compatibility conditions* for the resulting functions on the faces D_j.

Equation (5) and Lemma 2.2 then give immediately the following lemma.

Lemma 3.1. *We have* $\mathcal{K}_a^m(\mathbb{P}) = \{u,\ \vartheta^{-a}Pu \in L^2(\mathbb{P}),$ *for all* $P \in \mathrm{Diff}_0^k(\mathbb{P})\}$. *A similar result holds for* $\mathcal{K}_a^m(\partial\mathbb{P})$ *and for* $W_{BK}^{k,p,a}(\mathbb{P})$.

Next, Proposition 2.5 and Corollary 2.4, together with a straightforward calculation, show the following.

Lemma 3.2. *The multiplication map* $W_{BK}^{m,\infty,b} \times \mathcal{K}_a^m(\mathbb{P}) \ni (u, f) \to uf \in \mathcal{K}_{a+b}^m(\mathbb{P})$ *is continuous. We also have* $\mathcal{C}^\infty(\Sigma\mathbb{P}) \subset W_{BK}^{m,\infty,0}(\mathbb{P})$ *and* $r_{\mathbb{P}}^b \in W_{BK}^{m,\infty,b}(\mathbb{P})$, *and hence the map* $\mathcal{K}_a^m(\mathbb{P}) \ni u \to r_{\mathbb{P}}^b u \in \mathcal{K}_{a+b}^m(\mathbb{P})$ *is a continuous isomorphism of Banach spaces.*

From this lemma we obtain right away the following result.

Proposition 3.3. *Let* $k \ge m$. *Each* $P_0 \in \mathrm{Diff}_0^m(\mathbb{P})$ *defines a continuous map* $P_0 : \mathcal{K}_a^k(\mathbb{P}) \to \mathcal{K}_a^{k-m}(\mathbb{P})$. *The family* $r_{\mathbb{P}}^{-\lambda}P_0 r_{\mathbb{P}}^{\lambda}$ *is a family of bounded operators* $\mathcal{K}_a^k(\mathbb{P}) \to \mathcal{K}_a^{k-m}(\mathbb{P})$ *depending continuously on* λ.

Similarly, if P *is a differential operator with smooth coefficients on* $\mathbb{P}$, *then* $r_{\mathbb{P}}^{-\lambda}Pr_{\mathbb{P}}^{\lambda}$ *defines a continuous family of bounded operators* $\mathcal{K}_a^k(\mathbb{P}) \to \mathcal{K}_{a-m}^{k-m}(\mathbb{P})$.

Proof. The first part follows from Lemma 3.1. The second part follows from the last part of Proposition 2.7. □

We define the spaces $\mathcal{K}_a^{-k}(\mathbb{P})$, $k \in \mathbb{Z}_+$, by duality. More precisely, let $\overset{\circ}{\mathcal{K}}{}_a^k(\mathbb{P})$ be the closure of $\mathcal{C}_c^\infty(\mathbb{P})$ in $\mathcal{K}_a^k(\mathbb{P})$. Then we define $\mathcal{K}_{-a}^{-k}(\mathbb{P})$ to be the dual of $\overset{\circ}{\mathcal{K}}{}_a^k(\mathbb{P})$, the duality pairing being an extension of the bilinear form $(u, v) \mapsto \int_{\mathbb{P}} uv\, \mathrm{dvol}$. With this definition, we can drop the requirement that $k \ge m$ in Proposition 3.3.

Let us also note that the resulting weighted Sobolev spaces on the polygons D_j are *different* from the weighted Sobolev spaces obtained by using the distance to the vertices of these polygons. A regularity theorem on D_j would involve the latter weight (as in Kondratiev's theorem [29] mentioned in the Introduction). A consequence of this is that the spaces $\mathcal{K}_a^k(\partial\mathbb{P})$ behave more like the Sobolev spaces defined on a smooth manifold without boundary than like the Sobolev spaces defined on a bounded domain with (smooth) boundary. In particular, we define $\mathcal{K}_{-a}^{-k}(\partial\mathbb{P})$ as the dual of $\mathcal{K}_a^k(\partial\mathbb{P})$. The spaces $\mathcal{K}_a^s(\partial\mathbb{P})$, $s \notin \mathbb{Z}$, can be defined by interpolation, although in this paper we shall use a different definition using partitions of unity (see the following subsection; the two definitions are equivalent, although we shall not need a proof of this fact in this article).

3.2. **Definition of Sobolev spaces using partitions of unity.** As in [1], it is important to define the spaces $\mathcal{K}^a_m(\mathbb{P})$ using partitions of unity. Similar constructions were used in [17, 37, 38, 40]. This construction is possible because the spaces $\mathcal{K}^m_{3/2}(\mathbb{P})$ are the Sobolev spaces associated to the metric $r_{\mathbb{P}}^{-2} g_E$, where g_E is the usual Euclidean metric.

We shall need the following lemma. Recall that $\vartheta(p)$ denotes the distance from p to the edges of $\mathbb{P}$. In view of Lemma 2.2, in all estimates involving ϑ, we can replace ϑ with $r_{\mathbb{P}}$, although not the other way around, because ϑ is not smooth.

Let $\partial_{\mathrm{sing}}\mathbb{P}$ be the union of the edges of $\mathbb{P}$ and $\mathbb{P}' := \overline{\mathbb{P}} \smallsetminus \partial_{\mathrm{sing}}\mathbb{P}$.

Lemma 3.4. *There is $\epsilon_0 \in (0,1)$, an integer κ, and a sequence $C_k > 0$ of constants such that, for any $\epsilon \in (0, \epsilon_0]$, there is a sequence of points $\{x_j\} \subset \mathbb{P}' := \overline{\mathbb{P}} \smallsetminus \partial_{\mathrm{sing}}\mathbb{P}$ and a partition of unity $\phi_j \in \mathcal{C}^\infty_c(\mathbb{P}')$ with the following properties:*

(i) *either $B(x_j, \epsilon\vartheta(x_j)/4)$ is contained in $\mathbb{P}$* or *$x_j \in \partial\mathbb{P}$, $\vartheta(x_j) > 0$, and the ball $B(x_j, \epsilon\vartheta(x_j))$ intersects only the face D_k to which x_j belongs;*

(ii) $\mathrm{supp}(\phi_j) \subset B(x_j, \epsilon\vartheta(x_j)/2)$ *if $x_j \in \partial\mathbb{P}$ and* $\mathrm{supp}(\phi_j) \subset B(x_j, \epsilon\vartheta(x_j)/8)$ *otherwise;*

(iii) $\phi_j(x_j) = 1$ *and $\|(r_{\mathbb{P}}\partial)^\alpha \phi_j\|_{L^\infty(\mathbb{P})} \leq C_{|\alpha|}\epsilon^{-|\alpha|}$; and*

(iv) *a point $x \in \mathbb{P}$ can belong to at most κ of the sets $B(x_j, \epsilon\vartheta(x_j))$.*

Let us notice that $\overline{B(x_j, \epsilon\vartheta(x_j))}$ does not intersect any edge of $\mathbb{P}$ because $\epsilon < 1$. Moreover, the conditions that $\|r_{\mathbb{P}}\nabla\phi_j\|_{L^\infty} \leq C_1\epsilon^{-1}$ and $\phi_j(x_j) = 1$ guarantee that the support of ϕ_j is comparable in size with $\epsilon\vartheta(x_j)$. This is reminiscent of the conditions appearing in the definition of the Generalized Finite Element spaces [7, 8, 9].

A proof of this lemma will be given in the Appendix. It is essentially a result that, in the case of non-compact manifolds, goes back to Aubin. It was subsequently used by Gromov and in [1, 37, 38, 40]. We shall fix $\epsilon = \epsilon_0$ in what follows and a sequence x_j and a partition of unity ϕ_j as in the lemma.

Lemma 3.5. *Let $u_k = \sum_{j=1}^k \phi_j u$, for $u \in \mathcal{K}^m_a(\mathbb{P})$. Then $u_k \to u$ in $\mathcal{K}^m_a(\mathbb{P})$.*

Proof. Let $\Phi_k := \sum_{j=1}^k \phi_j$. We have that the sequence $(r_{\mathbb{P}}\partial)^\alpha\Phi_k$ is bounded in the 'sup'-norm and converges to 0 pointwise everywhere if $\alpha \neq 0$. Similarly, Φ_k is bounded and converges to 1 pointwise everywhere. The result then follows from this using also the Lebesgue dominated convergence theorem. □

Denote by $\alpha_j(x) = x_j + \vartheta(x_j)(x - x_j)$ be the dilation of center x_j and ratio $\vartheta(x_j)$. Let J be the set of indices j such that $x_j \in \partial\mathbb{P}$. Also, denote by

$$\begin{aligned} (7) \quad \nu_{m,a}(u)^2 &:= \sum_j \vartheta(x_j)^{3-2a}\|(\phi_j u)\circ\alpha_j\|^2_{H^m} \\ &:= \sum_{j\notin J} \vartheta(x_j)^{3-2a}\|(\phi_j u)\circ\alpha_j\|^2_{H^m(\mathbb{R}^3)} + \sum_{j\in J} \vartheta(x_j)^{3-2a}\|(\phi_j u)\circ\alpha_j\|^2_{H^m(\mathbb{R}^3_+)}. \end{aligned}$$

We agree that $\|(\phi_j u)\circ\alpha_j\|_{H^m} = \infty$ if $(\phi_j u)\circ\alpha_j \notin H^m(\mathbb{R}^3)$ (or if $(\phi_j u)\circ\alpha_j \notin H^m(\mathbb{R}^3_+)$, respectively). Note that the functions $(\phi_j u)\circ\alpha_j$ will all have support contained in a fixed ball, namely, the ball $B(0, \epsilon_0/2)$ of radius $\epsilon_0/2$ and center the origin. Moreover, all derivatives $\partial^\alpha(\phi_j \circ \alpha_j)$ are bounded for each fixed α and arbitrary j.

Proposition 3.6. *We have $u \in \mathcal{K}_a^m(\mathbb{P})$, $m \in \mathbb{Z}$, if, and only if, $\nu_{m,a}(u) < \infty$. Moreover, $\nu_{m,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^m(\mathbb{P})$.*

The proof of this Proposition is standard (see [12, Lemma 2.4], [1], or [40]); for $m < 0$ one also has to check that both definitions are compatible with duality. We include a brief sketch below.

Proof. Let us also introduce

$$\tilde{\nu}_{m,a}(u)^2 := \sum_j \|(\phi_j u)\|^2_{\mathcal{K}_a^m(\mathbb{P})}.$$

Then the fact that $\vartheta(x)/\vartheta(x_j)$ and $\vartheta(x_j)/\vartheta(x)$ are bounded by $(1-\epsilon)^{-1}$ on the ball $B(x_j, \epsilon\vartheta(x_j))$, for $\epsilon \in (0,1)$ and a change of variables shows that $\tilde{\nu}_{m,a}$ and $\nu_{m,a}$ define equivalent norms. It is then enough to prove that $\tilde{\nu}_{m,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^m(\mathbb{P})$. For $m = 0$, this follows from the inequalities

$$\|r_{\mathbb{P}}^{-a} u\|^2_{L^2(\mathbb{P})} \leq \kappa \tilde{\nu}_{0,a}^2 \leq \kappa \|r_{\mathbb{P}}^{-a} u\|^2_{L^2(\mathbb{P})}.$$

For arbitrary m, we use induction on m and the fact that $\sum_j |(r_{\mathbb{P}}\partial)^\alpha \phi_j(p)|$ is bounded uniformly in $p \in \mathbb{P}$ for all α. □

We proceed in the same way to study the spaces $\mathcal{K}_a^s(\partial\mathbb{P})$, $s \in \mathbb{R}$. Let us identify the plane containing each face D_k of $\mathbb{P}$ with a copy of $\mathbb{R}^2$. Then let

$$\mu_{s,a}(u)^2 := \sum_{j \in J} \vartheta(x_j)^{2-2a} \|(\phi_j u) \circ \alpha_j\|^2_{H^s(\mathbb{R}^2)}, \quad s \in \mathbb{R}_+. \tag{8}$$

Only the indices j for which $x_j \in \partial\mathbb{P}$ are needed above. Then we have an analogous description of the spaces $\mathcal{K}_a^s(\partial\mathbb{P})$, $s \in \mathbb{Z}$.

Proposition 3.7. *We have $u \in \mathcal{K}_a^s(\partial\mathbb{P})$ if, and only if, $\nu_{s,a}(u) < \infty$. Moreover, $\nu_{s,a}(u)$ defines an equivalent norm on $\mathcal{K}_a^s(\partial\mathbb{P})$, $s \in \mathbb{Z}$.*

We can then define $\mathcal{K}_a^s(\mathbb{P})$, $s \in \mathbb{R}$, as the space of functions u for which $\mu_{s,a}(u) < \infty$ with the induced norm. From this we obtain, by reducing to the Euclidean case, the following Trace Theorem. Let $\partial_{\text{sing}}\mathbb{P}$ be the union of the edges of $\mathbb{P}$ and $\mathbb{P}' := \overline{\mathbb{P}} \smallsetminus \partial_{\text{sing}}\mathbb{P}$, as above.

Theorem 3.8. *The space $\mathcal{C}_c^\infty(\mathbb{P}')$ is dense in $\mathcal{K}_a^m(\mathbb{P})$, $m \in \mathbb{Z}_+$. The restriction to the boundary extends to a continuous, surjective map $\mathcal{K}_a^m(\mathbb{P}) \to \mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P})$ for $m \geq 1$. For $m = 1$, the kernel of this map is the closure of $\mathcal{C}_c^\infty(\mathbb{P})$ in $\mathcal{K}_a^1(\mathbb{P})$.*

Proof. Clearly $\mathcal{C}_c^\infty(\mathbb{P}') \subset \mathcal{K}_a^m(\mathbb{P})$, for any $m \in \mathbb{Z}_+$ and any $a \in \mathbb{R}$. To prove that it is a dense subspace, let $u \in \mathcal{K}_a^m(\mathbb{P})$. By Lemma 3.5, we may assume that the support of u does not intersect $\partial_{\text{sing}}\mathbb{P}$ (replace u with u_k for some k large). Then we use the fact that $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $H^m(\Omega)$ for Ω a smooth, bounded domain and the fact that the H^m-norm is equivalent to the norm on $\mathcal{K}_a^m(\mathbb{P})$ when restricted to functions with support in a fixed compact K such that K does not intersect any edge of $\mathbb{P}$ (i.e., $K \cap \partial_{\text{sing}}\mathbb{P} = \emptyset$).

We have

$$\begin{aligned}\mu_{m-1/2,a-1/2}(u|_{\partial\mathbb{P}})^2 &:= \sum_{j \in J} \vartheta(x_j)^{3-2a} \|(\phi_j u) \circ \alpha_j|_{\partial\mathbb{P}}\|^2_{H^{m-1/2}(\mathbb{R}^2)} \\ &\leq C \sum_{j \in J} \vartheta(x_j)^{3-2a} \|(\phi_j u) \circ \alpha_j\|^2_{H^m(\mathbb{R}^3)} \leq C\nu_{m,a}(u),\end{aligned}$$

and hence the restriction map $\mathcal{K}_a^m(\mathbb{P}) \to \mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P})$ is defined and continuous for $m \geq 1$, by Propositions 3.6 and 3.7. To prove that this map is continuous, let us fix a continuous extension operator $E : H^{m-1/2}(\mathbb{R}^2) \to H^m(\mathbb{R}^3)$. By rotation and translation, we extend this definition to an extension operator $E : H^{m-1/2}(V) \to H^m(\mathbb{R}^3)$, for any two dimensional subspace $V \subset \mathbb{R}^3$.

Let then $v : \partial\mathbb{P} \to \mathbb{C}$ be a function in $\mathcal{K}_{a-1/2}^{m-1/2}(\partial\mathbb{P})$. Let us fix a function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with support in the ball $B(0,\epsilon_0)$ of radius ϵ_0 and center at the origin such that $\psi = 1$ on $B(0,\epsilon_0/2)$. Let $v_j(p) = \phi_j(\alpha_j(p))u(\alpha_j(p))$, which is defined on a subspace of $\mathbb{R}^3$ of dimension 2. We then define

$$u = \sum_j \big(\psi E(v_j)\big) \circ \alpha_j^{-1}.$$

Then $u \in \mathcal{K}_a^m(\mathbb{P})$ and $u|_{\partial\mathbb{P}} = v$.

Finally, let $u \in \mathcal{K}_a^1(\mathbb{P})$ such that $u|_{\partial\mathbb{P}} = 0$. Let u_k be as in Lemma 3.5. Then $u_k|_{\partial\mathbb{P}} = 0$, and hence, to complete the proof, we can again use the equivalence of the H^1 and $\mathcal{K}_a^1(\mathbb{P})$-norms on functions with support in a fixed compact set K such that $K \cap \partial_{\text{sing}}\mathbb{P} = \emptyset$. □

4. Proof of the regularity theorem

We include in this section the proof of Theorem 1.1. Its proof is reduced to the Euclidean case using a partition of unity argument ϕ_j satisfying the conditions of Lemma 3.4 for $\epsilon = \epsilon_0$, as in the previous section.

Proof. (of Theorem 1.1.) The trace theorem, Theorem 3.8 allows us to assume that $u|_{\partial\mathbb{P}} = 0$. We then notice that, locally, Theorem 1.1 is a well known statement. Namely, let us consider a function v with support in the ball of radius ϵ_0. We assume that either $v \in H^1(\mathbb{R}^3)$ or $v \in H_0^1(\mathbb{R}^3_+)$ (that is, $v = 0$ on $\mathbb{R}^2$, the boundary of $\mathbb{R}^3_+ = \{z \geq 0\}$). Then there exists a constant $C > 0$ such that

$$\|v\|^2_{H^{m+1}(\mathbb{R}^3)} \leq C_r\big(\|\Delta v\|^2_{H^{m-1}(\mathbb{R}^3)} + \|v\|^2_{L^2(\mathbb{R}^3)}\big). \tag{9}$$

or, respectively,

$$\|v\|^2_{H^{m+1}(\mathbb{R}^3_+)} \leq C_r\big(\|\Delta v\|^2_{H^{m-1}(\mathbb{R}^3_+)} + \|v\|^2_{L^2(\mathbb{R}^3_+)}\big). \tag{10}$$

The constant C_r in the two equations above depends only on ϵ_0.

We shall proceed by induction on $m \geq 0$. For $m = 0$, the result is tautologically true. Let now $\{\phi_j\}$ be the partition of unity and α_j be dilations appearing in Equation (7). In particular, the partition of unity ϕ_j satisfies the conditions of Lemma 3.4, which implies that $\text{supp}(\phi_j) \subset B(x_j, \epsilon_0\vartheta(x_j)/2)$ if $x_j \in \partial\mathbb{P}$ and $\text{supp}(\phi_j) \subset B(x_j, \epsilon_0\vartheta(x_j)/8)$ otherwise. Let $\eta_j = \psi \circ \alpha_j^{-1}$, where $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ has support in $B(0,\epsilon_0)$ and is equal to 1 on $B(0,\epsilon_0/2)$, as before. We also have that all derivatives of order $\leq k$ of the functions $\phi_j \circ \alpha_j$ are bounded. This implies in turn that the commutator $P_j := [\Delta, \phi_j \circ \alpha_j] := \Delta(\phi_j \circ \alpha_j) - (\phi_j \circ \alpha_j)\Delta$ is a differential operator all of whose coefficients have bounded derivatives.

Let $\|v\|_{H^m}$ denote either $\|v\|_{H^m(\mathbb{R}^3)}$ or $\|v\|_{H^m(\mathbb{R}^3_+)}$, depending on where the function v is defined. Then Equations (9) and (10) and the above remarks give

$$\nu_{m+2,a}(u)^2 := \sum_j \vartheta(x_j)^{3-2a} \|(\phi_j u) \circ \alpha_j\|^2_{H^{m+2}}$$
$$\leq C_r \sum_j \vartheta(x_j)^{3-2a} \Big(\|\Delta[(\phi_j u) \circ \alpha_j]\|^2_{H^m} + \|(\phi_j u) \circ \alpha_j\|^2_{L^2} \Big)$$
$$\leq C \sum_j \vartheta(x_j)^{3-2a} \Big(\|(\phi_j \circ \alpha_j)\Delta(u \circ \alpha_j)\|^2_{H^m} + \|P_j(u \circ \alpha_j)\|^2_{H^m} + \|(\phi_j u) \circ \alpha_j\|^2_{L^2} \Big)$$
$$\leq C \sum_j \vartheta(x_j)^{3-2a} \Big(\vartheta(x_j)^4 \|(\phi_j \Delta u) \circ \alpha_j)\|^2_{H^m} + \|(\eta_j u) \circ \alpha_j\|^2_{H^{m+1}} + \|(\phi_j u) \circ \alpha_j\|^2_{L^2} \Big)$$
$$\leq C\big(\nu_{m,a-2}(\Delta u)^2 + \sum_j \nu_{m+1,a}(\eta_j u)^2 + \nu_{0,a}(u)^2\big).$$

Since no more than κ of the functions $\eta_j u$ are non-zero at any given point of $\mathbb{P}$ and all the derivatives $(r_{\mathbb{P}}\partial)^\alpha \eta_j$ are bounded for all fixed $|\alpha|$, we obtain that $\sum_j \nu_{m+1,a}(\eta_j u)^2 \leq C\nu_{m+1,a}(u)^2$. This then gives

$$\nu_{m+2,a}(u)^2 \leq C\big(\nu_{m,a-2}(\Delta u)^2 + \nu_{m+1,a}(u)^2\big).$$

By induction on m we then obtain

$$\nu_{m+2,a}(u)^2 \leq C\big(\nu_{m,a-2}(\Delta u)^2 + \nu_{0,a}(u)^2\big).$$

The result then follows from Proposition 3.6, which states that the norms $\|\cdot\|_{\mathcal{K}^t_a(\mathbb{P})}$ and $\nu_{t,a}$ are equivalent. $\square$

See [15] for applications of these results, especially of the above theorem.

By contrast, it is known that in the framework of the usual Sobolev spaces $H^m(\mathbb{P})$, the smoothness of the solution of (1) is limited [20, 23, 24, 26, 34].

Appendix A. Additional constructions

In this appendix we explain how to modify the constructions of the functions θ_e and $\phi_{P,e}$ introduced in Section 2 and how to construct a partition of unity satisfying the conditions of Lemma 3.4.

A.1. **The modified functions θ, ϕ, and r_e.** We continue to denote by $\rho_P(p)$ the distance from p to the vertex P. By a dilation, we can assume that each edge of $\mathbb{P}$ has length at least 4. Let $e = [AB]$ and $\psi_1 : [0,\pi] \to [0, 7\pi/8]$ be a smooth, non-decreasing function such that $\psi_1(x) = x$ for $0 \leq x \leq 3\pi/4$ and $\psi_1(x) = 7\pi/8$ for $x \geq 7\pi/8$. Also, let $\psi_2 : [0,\infty) \to [0,1]$ be a smooth, non-increasing function such that $\psi_2(x) = 1$ for $0 \leq x \leq 1$ and $\psi_2(x) = 0$ for $2 \leq x$. Then we replace $\phi_{A,e}$ with $\psi_1(\phi_{A,e})\psi_2(\rho_A)$. This modifies the function $\phi_{A,e}$ to make it smooth everywhere except on $\overline{e}$.

We similarly modify θ_e by replacing it with $\psi_3(\phi_{A,e})\psi_3(\phi_{B,e})\theta_e$, where $\psi_3 : [0,\pi] \to [0,1]$ is a smooth, non-increasing function such that $\psi_3(x) = 1$ for $x \in [0, 3\pi/4]$ and $\psi_3(x) = 0$ for $x \geq 7\pi/8$. This again will make θ_e defined and smooth everywhere in space except on $\overline{e}$.

Finally, let ψ_3 and $e = [AB]$ are as in the above paragraph. We then replace r_e with $\psi_3(\phi_{A,e})\psi_3(\phi_{B,e})r_e + (1 - \psi_3(\phi_{A,e}))\rho_A + (1 - \psi_3(\phi_{B,e}))\rho_B$.

A.2. **The partition of unity.** Our partition of unity will depend on parameters (a, b, c) that will be specified below.

First of all, let us denote by $B(P, n)$ the open ball of center P and radius $2^{-n}a$. By choosing a small enough, we can assume that the balls $B(P, 1)$ do not intersect. Then let $E_{e,n}$ be the set of points $p \in \mathbb{P}$ that do not belong to any $B(P, 2)$ and are at distance $\leq 2^{-n}ab$ to the edge e. By choosing b small enough, we can assume that the sets $E_{e,1}$ do not intersect. Let Ω_1 be obtained from $\mathbb{P}$ by removing the sets $B(P, 2)$ and $E_{e,2}$.

For each edge e, let N_e be the plane normal to e. Project $E_{e,1} \smallsetminus E_{e,2}$ onto N_e. The projection will be the intersection of an annulus with an angle. Denote this projection by C_e. We shall cover C_e with disks of radius $c/2$ and radius $c/8$ such that the disks of radius $c/2$ have the center on the straight sides of C_e (the ones obtained from the angle) and the disks of radius $c/8$ that have centers in the interior of C_e at distance at least $c/4$ to the angle defining C_e. This yields the disks $D_1, \ldots, D_N$ with centers $q_1, \ldots, q_N$.

Let z be the variable along the line containing e. Then we cover $E_{e,k+1} \smallsetminus E_{e,k+2}$, $k \in \mathbb{Z}_+$, with balls of radius $2^{-k}c$ and centers of the form $(2^{-k}q_j, 2^{-k-3}c)$, if $2^{-k}q_j$ is on one of the faces of $\mathbb{P}$ and is inside $E_{e,1}$. Otherwise, we consider the ball of radius $2^{-k-2}c$ with centers of the form $(2^{-k}q_j, 2^{-k-3}c)$ as long as the center is still inside $E_{e,1}$.

Let us cover

$$\Omega_1 := \mathbb{P} \smallsetminus \Big(\bigcup_P B(P, 2) \cup \bigcup_e E_{e,2} \Big)$$

with finitely many balls of radius $c/2$ or radius $c/8$ with centers in Ω_1 such that the balls of radius $c/2$ have the centers on the faces of $\mathbb{P}$ and the balls of radius $c/8$ are at distance at least $c/4$ to the faces of $\mathbb{P}$.

Let $D_{P,1}, \ldots, D_{P,N}, \ldots$ be the balls already constructed with centers in $B(P, 1) \smallsetminus B(P, 2)$. Then consider also the balls $2^{-k}D_{P,1}, \ldots, 2^{-k}D_{P,N}, \ldots$ obtained by dilations of ratio 2^{-k} and center P. We repeat this construction for all vertices P and all $k \in \mathbb{Z}_+$. We consider all the balls $D_1, D_2, \ldots,$ constructed so far (relabeled into a sequence) from the coverings of $E_{e,1}$, Ω_1, and from the dilations of ratio 2^{-k} for all the vertices P, as already explained. If we choose c small enough (after the above choices of a and b as explained), then the sequence of these balls is locally finite, the center of each ball is either on the faces of $\mathbb{P}$ or the closure of the ball is inside $\mathbb{P}$. Moreover, for any such ball D with center p and radius r, we have that $r/\vartheta(p)$ is bounded from above and bounded from below from zero, say $r/\vartheta(p) \in [\epsilon_0, \epsilon_0^{-1}]$, for some $\epsilon_0 \in (0, 1)$. There is an integer κ such that no $\kappa + 1$ balls constructed have a common point.

To any ball D of center q and radius r we associate the *bump function* $\psi_D(p) := \psi(|p - q|/r)$, where $\psi : [0, \infty) \to [0, 1]$ is equal to 1 in a neighborhood of 0, is equal to 0 in a neighborhood of $[1, \infty)$, and is > 0 on $[0, 1)$. Then we let $\eta = \sum \psi_{D_j}$ and $\phi_j = \psi_{D_j}/\eta$. By further decreasing c, if necessary, we see that our partition of unity (together with the points x_j obtained as the centers of our balls) satisfies the conditions of Lemma 3.4 for the ϵ_0 chosen above.

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