Lower bounds for the Yamabe invariant

Bernd Ammann

(joint work with Mattias Dahl, Emmanuel Humbert)

In this talk I gave an overview over a series of articles [1], [2], [3], [4], joint work with M. Dahl and E. Humbert, about the (smooth) Yamabe invariant. We have seen for example that the Yamabe invariant of a simply connected compact manifold of dimension 5 is between 45 and 79. Similar estimates hold for simply connected compact manifolds of dimension 6, and for 2-connected compact manifolds of dimension at least 7 for which the KO_*-valued index of the Dirac operator vanishes.

Let us give some more details. The conformal Yamabe constant of a compact n-dimensional Riemannian manifold (M, g_0) is defined as

$$Y(M,[g_0]) := \inf_{g\in[g_0]} \frac{\int_M \mathrm{scal}^g \; dv^g}{\mathrm{vol}(M,g)^{\frac{n-2}{n}}} = \inf_{u\geq 0, u\not\equiv 0} \frac{\int_M 4\frac{n-1}{n-2}|du|^2_{g_0} + \mathrm{scal}^{g_0}\, u^2 \; dv^{g_0}}{\|u\|^2_{L^p(M,g_0)}}$$

with $p = 2n/(n-2)$. The second characterization also makes sense for non-compact M, which will be used below. The smooth Yamabe invariant of a compact manifold M is then defined as

$$\sigma(M) := \sup Y(M,[g_0])$$

where the supremum runs over all conformal classes $[g_0]$ on M.

The invariant $\sigma(M)$ is positive iff M carries a metric of positive scalar curvature. On the other hand one sees that $Y(M,[g_0])$ is bounded from above by $Y(\mathbb{S}^n) = n(n-1)\mathrm{vol}(\mathbb{S}^n)^{2/n}$, the value of the standard sphere, and thus the same upper bounds holds for $\sigma(M)$. One knows the value for $\sigma(M)$ for $M = \mathbb{R}P^3$, $M = \mathbb{C}P^2$, compact quotients of hyperbolic 3-space, and few other spaces. The proofs use Seiberg-Witten theory, the Penrose inequality from general relativity and the Ricci-flow. However, in general it is hard to calculate $\sigma(M)$. One even does not know any compact manifold M of dimension n at least 5 for which one can prove $0 < \sigma(M) < \sigma(S^n)$, although there are plenty of candidates for which such a value is expected.

In [1] we have proven a formula that estimates the behaviour of $\sigma(M)$ under performing surgery at M, namely if N is obtained by surgery of dimension $k \leq n-3$ from M, then

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_{n,k}\},$$

where $\Lambda_{n,k} > 0$ only depends on $n = \dim M$ and k.

If $k \leq n-4$ or if $n \leq 6$, then the constants $\Lambda_{n,k} > 0$ can be defined as

$$\Lambda_{n,k} = \inf_{c\in[0,1]} Y(\mathbb{M}_c), \qquad \mathbb{M}_c = H_c^{k+1} \times \mathbb{S}^{n-k-1}$$

where H_c^{k+1} is the simply connected complete Riemannian manifold with sectional curvature $-c^2$, thus it is a rescaled hyperbolic space or Euclidean space. The definition is a bit more involved in the remaining case $k+3 = n \geq 7$. Such surgery formulas immediately imply that $\min\{\sigma(M), \Lambda_{n,k}\}$ is a bordism invariant in a certain sense. In the case $k = 0$ the result holds for $\Lambda_{n,0} = Y(\mathbb{S}^n)$, i.e. $\sigma(N) \geq \sigma(M)$ in this case.

In order to get much information about $\sigma(M)$ there are several things to carry out: at first one has to obtain explicit lower estimates for $\Lambda_{n,k}$ for as many pairs (n, k) as possible. This is mainly a complicated analytical problem. As a second step one has to find lower bounds for the smooth Yamabe invariant or conformal Yamabe constant of certain building blocks, see below. Then as a third step one has use bordism theory to prove that a large class of manifolds is obtained from these building blocks by a finite number of surgeries of dimension k for which we know a lower bound for $\Lambda_{n,k}$.

In [2] we found an efficient method for a lower bound for $Y(M_1 \times M_2, [g_1 + g_2])$ in terms of $Y(M_1, [g_1])$ and $(M_2, [g_2])$ provided that both factors (M_i, g_i) have dimensions $n_i \geq 3$ and have positive conformal Yamabe constant. Namely

$$Y(M_1 \times M_2, [g_1 + g_2])^{n_1+n_2} \geq c_{n_1,n_2} Y(M_1, [g_1])^{n_1} Y(M_2, [g_2])^{n_2}$$

where c_{n_1,n_2} is a constant which is "close" to optimal in a sense that we will not explain here to be short. This result yields a strong explicit positive lower bound for $\Lambda_{n,k}$ if $2 \leq k \leq n - 4$.

Another method was developed in [4] which provides a lower bound for $Y(\mathbb{M}_c)$ in terms of $Y(\mathbb{M}_0) = Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$. It relies on the volume preserving diffeomorphism $H_c^{k+1} \to \mathbb{R}^{k+1}$ which has in polar coordinates the form $(r, \phi) \mapsto (f(r), \phi)$. This diffeomorphism allows us to use the Yamabe minimizer of $\mathbb{M}_c$ — which exists due to [6] — as a test function defined on $\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1}$. As result we obtain an explicit positive lower bound for $\Lambda_{n,k}$ as soon as there is an explicit positive lower bound for $Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$. Luckily such a lower bound was calculated in [7] for $(n, k) = (4, 1)$, and we are very grateful to Petean and Ruiz, that they adapted in [8] their method for $(n, k) \in \{(5, 1), (5, 2), (9, 1), (10, 1)\}$. Their method can probably be adapted to all dimensions, but these dimensions play an important role when we apply bordism theory to draw conclusions.

In the second step one should find lower bounds for the smooth Yamabe invariant of simple building blocks. Important buidling blocks are total spaces T of bundles whose fibers are quaternionic projective planes $\mathbb{H}P^2$. Assume that we have a sequence of metrics g_i on such a fixed T, such that all fibers are isometric for all basepoints and all i. We also assume that the bundle map $(T, g_i) \to (B, a_i h)$ is a Riemannian submersion, where h is a metric on the base B and where $a_i \in \mathbb{R}^+$ tend to ∞. This is often called an adiabatic limit. Then the conformal Yamabe constants $Y(T, [g_i])$ converge to $Y(\mathbb{H}P^2 \times \mathbb{R}^{n-8})$ and the latter quantity is positively bounded from below by Obata's theorem for $n = 8$, by further work of Petean in cases $n = 9$ and $n = 10$ and by the product formula explained above for $n \geq 11$. These fiber bundles are important as they generate the spin bordism classes of simply connected manifolds with vanishing index, see [9]. For simply connected oriented non-spin manifold, $\mathbb{C}P^2$-bundles play a similar role, see [5]. Another example of an important building block is SU(3)/SO(3) whose bi-invariant metric g_{bi} is Einstein, and thus Obata's theorem determines the value of

$Y(\mathrm{SU}(3)/\mathrm{SO}(3),[g_{bi}]) > 0$, As a consequence Obata's theorem yields an explicit positive lower bound for $\sigma(\mathrm{SU}(3)/\mathrm{SO}(3))$. This manifold is important as it represents the only non-trivial class in the oriented bordism group in dimension 5. Combinded with further arguments, in particular results about possible decompositions of bordisms into elementary bordisms which correspond to surgeries, we finally obtain for instance that $\sigma(M) > 45$ for any simply connected manifold M of dimension 5. In current work we prove an analogous bound for simply-connected manifolds of dimension 6.

A similar statement holds for 2-connected manifolds with vanishing KO-index of dimension different from 4, see [4] for details, in particular Table 2 gives explicit numbers for dimension at most 11.

There are many other conclusions, for example:

- The set $\{\sigma(\Sigma) \mid \Sigma \text{ is a homotopy-}S^7\}$ has at most 4 values.
- For many fundamental groups Γ one can show that there is no strictly decreasing sequence in

$$\{\sigma(M) \mid \pi_1(M) = \Gamma, \dim M = n\} \cap [0, \min\{\Lambda_{n,1}, \dots, \Lambda_{n,n-3}\}].$$

- Assume that M is a 5-dimensional manifold with $0 \le \sigma(M) < 45$. Then for relatively prime natural numbers r_1 and r_2 we have $\sigma(\#^{r_1} M) = \sigma(M)$ or $\sigma(\#^{r_2} M) = \sigma(M)$.

The last application raises the question whether such an M exists, obviously it would be not simply-connected. As the explicit calculation of $\sigma(M)$ is very difficult and only possible in very rare cases, no such examples are known. It is conjectured that $M = S^5/(\mathbb{Z}/_m)$, $m \ge 5$, provides such an example, namely it is conjectured that the supremum in the definition of $\sigma(M)$ is attained by the round metric which would imply $\sigma(M) = \sigma(S^5)/m^{2/5} \approx 78.996/m^{2/5}$.

References

[1] B. Ammann, M. Dahl, and E. Humbert, *Smooth Yamabe invariant and surgery*, Preprint, 2008, `http://arxiv.org/abs/0804.1418`, to appear in J. Diff. Geom.

[2] ———, *The conformal Yamabe constant of product manifolds*, Preprint, 2011, `http://arxiv.org/abs/1103.1826`, to appear in Proc. AMS.

[3] ———, *Square-integrability of solutions of the Yamabe equation*, Preprint, 2011, `http://arxiv.org/abs/1111.2780`.

[4] ———, *Low-dimensional surgery and the Yamabe invariant*, Preprint, 2012, `http://arxiv.org/abs/1204.1197`.

[5] S. Führing, *Bordismus und $\mathbb{C}P^2$-Bündel*, Diplomarbeit, TU München, 2008.

[6] N. Große, *The Yamabe equation on manifolds of bounded geometry*, (2009), `http://arxiv.org/abs/0912.4398`.

[7] J. Petean and J. M. Ruiz, *Isoperimetric profile comparisons and Yamabe constants*, Ann. Global Anal. Geom. **40** (2011), 177–189.

[8] ———, *On the Yamabe constants of $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$*, Preprint, 2012, `http://arxiv.org/abs/1202.1022`.

[9] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **136** (1992), no. 3, 511–540.