A surgery formula for the smooth Yamabe invariant

Bernd Ammann
(joint work with M. Dahl, E. Humbert)

1. Historical overview

For a compact Riemannian manifold (M, g_0) of dimension $n \geq 3$ we define the conformal Yamabe invariant to be

$$Y(M, [g_0]) := \inf_{g \in [g_0]} \int_M \mathrm{Scal}_g \, dv^g$$

where $[g_0]$ denotes the set of all metrics conformal to g and of volume 1. The solution of the Yamabe problem (see e.g. [10] for a good overview) states that this infimum is always attained. The scalar curvature of the mimimzing metric satisfies $\mathrm{Scal}_g \equiv Y(M, [g_0])$. It is not difficult to show that

$$Y(M, [g_0]) \leq Y(\mathbb{S}^n) = n(n-1)\omega_n$$

where $\mathbb{S}^n$ denotes the sphere with the standard metric, and where $\omega_n = \mathrm{Vol}(\mathbb{S}^n)$.

The smooth Yamabe invariant, also called Yamabe number or Schoen's σ-invariant is then defined as

$$\sigma(M) := \sup_{[g_0]} Y(M, [g_0]) \in (-\infty, Y(\mathbb{S}^n)]$$

where the supremum runs over all conformal classes $[g_0]$ on M.

The σ-invariant is positive if and only if M carries a metric of positive scalar curvature. Calculating σ-invariants in the positive case is also difficult. The first examples have been calculated by Schoen [12, Section 2] where one sees that $\sigma(S^{n-1} \times S^1) = Y(\mathbb{S}^n)$, and the same method yields for $S^{n-1} \tilde{\times} S^1$, the non-orientable S^{n-1} bundle over S^1, that $\sigma(S^{n-1} \tilde{\times} S^1) = Y(\mathbb{S}^n)$. By Kobayashi's surgery formula [8] it is also clear that the σ-invariant of connected sums of above manifolds have the value $Y(\mathbb{S}^n)$. It is conjectured [12, Page 10, lines 6–11] that all quotient of round spheres satisfy $\sigma(S^n/\Gamma) = (\#\Gamma)^{-2/n} Y(\mathbb{S}^n)$, but this conjecture is only verified for $\mathbb{R}P^3$ [5]. An extended Aubin's Lemma enabled the complete characterization of all 3-dimensional manifolds with $\sigma(M) > \sigma(\mathbb{R}P^3)$, see [1]. The smooth Yamabe invariant of $\sigma(\mathbb{C}P^2)$ was calculated in [9] and [7]. With similar methods the σ-invariants of some several other Kähler manifolds could be determined.

As smooth Yamabe invariants are difficult to calculate explicitly, estimates are very helpful. An important class of estimates can be proven with the help of surgery formulas. Let $\Phi : S^k \times \overline{B^{n-k}} \to M$ be an embedding. We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup_\sim \overline{B^{k+1}} \times S^{n-k-1}$$

and we say that M_k^Φ *is obtained from* M *by* k*-dimensional surgery along* Φ. Connected sums are examples for 0-dimensional surgery along non-connected manifolds.

Theorem 1 ([8]). *If M_0^Φ is obtained from M by 0-dimensional surgery, then*

$$\sigma(M_0^\Phi) \geq \sigma(M).$$

The next theorem states that monotonicity still holds in the case $\sigma(M) \leq 0$ for $k \in \{1, \dots, n-3\}$.

Theorem 2 ([11]). *If M_k^Φ is obtained from M by k-dimensional surgery, $1 \leq k \leq n-3$, then*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), 0\}.$$

Using results from bordism theory and from [14] this implies that all simply connected manifolds of dimension ≥ 5 have $\sigma(M) \geq 0$.

On the other hand positivity of $\sigma(M)$ is preserved under surgeries of dimension $k \in \{1, \dots, n-3\}$.

Theorem 3 ([6],[13]). *If M_k^Φ is obtained from M by k-dimensional surgery, $1 \leq k \leq n-3$, and $\sigma(M) > 0$, then we also have $\sigma(M_k^\Phi) > 0$.*

2. New surgery formula for σ

In [4] we prove a surgery formula for the σ-invariant that is stronger than Theorems 1, 2 and 3 together.

Theorem 4. *Let $0 \leq k \leq n-3$. There is a positive constant $\Lambda_{n,k}$ depending only on n and k such that*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), \Lambda_{n,k}\}$$

if M_k^Φ is obtained from M by k-dimensional surgery. Furthermore $\Lambda_{n,0} = Y(\mathbb{S}^n)$.

We will not give a detailed proof in this short announcement, but restrict to a short sketch of the method used herein. Details will be published in [4].

The theorem directly follows from the following:

Theorem 5. *Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a family of metrics g_θ, $\theta \in [0,1]$, on M_k^Φ such that*

$$\min\{Y(M,[g]), \Lambda_{n,k}\} \leq \liminf_{\theta \to 0} Y(M_k^\Phi, [g_\theta]) \leq \limsup_{\theta \to 0} Y(M_k^\Phi, [g_\theta]) \leq Y(M,[g]).$$

These metrics are very similar to the metrics g_ρ described in [2] and the metrics g_τ described in [3].

The inequality

$$\limsup_{\theta \to 0} Y(M_k^\Phi, [g_\theta]) \leq Y(M,[g])$$

in Theorem 5 can be shown by standard cut-off arguments. The other inequality, namely

$$\min\{Y(M,[g]), \Lambda_{n,k}\} \leq \liminf_{\theta \to 0} Y(M_k^\Phi, [g_\theta])$$

is much harder to show, but it is the important one for proving Theorem 4.

For showing this inequality we study minimzers of the Yamabe functional of $(M_k^\Phi, [g_\theta])$ for a sequence of $\theta \to 0$. A sophisticated blow-up analysis for such minimizers is worked out. The arguments of [3] can only be applied for $2k < n-2$, hence completely new tools have to be developped to treat the general case. We see that the sequence either concentrates in the part $(M_k^\Phi, [g_\theta])$ that is locally conformal to $(M \setminus \Phi(S^k \times \{0\}), g)$. In this case we obtain convergence to a solution of the Yamabe equation on $(M \setminus \Phi(S^k \times \{0\}), g)$ that can be extended to a solution on (M, g) and we derive

$$Y(M, [g]) \leq \liminf_{\theta \to 0} Y(M_k^\Phi, [g_\theta]).$$

Or the minimizers concentrate on the new "handle" and then blowup analysis provides a solution of

$$Lu = \lambda |u|^{\frac{2}{n-2}} u \tag{1}$$

on $(P(n,k), H_K) := \mathbb{S}^{n-k-1} \times (B^{k+1}, h_K)$ for a suitable $K \in [-1, 0]$. Here h_K is "the" complete metric of sectional curvature $K \in [-1, 0]$ on the ball B^{k+1}. For example (B^{k+1}, h_0) is isometric to euclidean space and (B^{k+1}, h_{-1}) is isometric to the hyperbolic space.

We define

$$Y(P(n,k), H_K) := \inf \lambda \in [0, \infty]$$

where λ runs over all values $\lambda \in (0, \infty)$ for which there is a positive function $u \in L^\infty(P(n,k)) \cap C^2_{\mathrm{loc}}(P(n,k))$, $\|u\|_{L^{\frac{2n}{n-2}}} = 1$ solving (1) on $(P(n,k), H_K)$. and we obtain

$$Y(P(n,k), H_K) \leq \liminf_{\theta \to 0} Y(M_k^\Phi, [g_\theta]).$$

Other blowup arguments show

$$\Lambda_{n,k} := \inf_{K \in [-1,0]} Y(P(n,k), H_K) > 0$$

and Theorem 5 follows.

As $(P(n,k), H_K)$ is conformal to $\mathbb{S}^n \setminus \{p_1, p_2\}$, one can see that $\Lambda_{n,0} = Y(\mathbb{S}^n)$.

3. Topological applications

We consider pairs (M, f) where M is a compact spin manifold and where $f : M \to B\Gamma$ is continuous. Two such pairs (M_1, f_1) and (M_2, f_2) are spin bordant if there exists an $n+1$-dimensional manifold W with boundary $-M_1 \cup M_2$ with a map $F : W \to B\Gamma$ such that the restriction of F to the boundary yields f_1 and f_2. It is implicitly required that the boundary carries the induced orientation and spin structure and $-M_1$ denotes M_1 with reversed orientation. Bordance is an equivalence relation, the corresponding equivalence classes are denoted by $[(M, f)]$ and the set of equivalence classes is $\Omega_n(B\Gamma)$. Disjoint union defines a sum on $\Omega_n(B\Gamma)$ which turns it into a group.

We say that a tuple (M, f) with $f : M \to B\Gamma$ is a *π_1-bijective representative of* $[(M, f)]$ if M is connected and if the induced map $f_* : \pi_1(M) \to \Gamma$ is a bijection. Any equivalence class in $\Omega_n(B\Gamma)$ has a π_1-bijective representative.

Now we define

$$\Lambda_n := \min\{\Lambda_{n,1}, \ldots \Lambda_{n,n-3}\} > 0,$$

$$\bar{\sigma}(M) := \min\{\sigma(M), \Lambda_n\}.$$

Proposition. *Let $n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be compact spin manifolds with maps $f_i : M_i \to B\Gamma$. If (M_1, f_1) and (M_2, f_2) are spin bordant and if f_2 is π_1-bijective, then*

$$\bar{\sigma}(M_1) \leq \bar{\sigma}(M_2).$$

Hence $\bar{\sigma}$ defines a well-defined map $\Omega_n(B\Gamma) \to \mathbb{R}$. We conclude from the surgery formula.

Theorem 6. *Let $t \geq 0$. $n \geq 5$. Then the sets*

$$G(t) := \{[x] \in \Omega_n(B\Gamma) \,|\, \bar{\sigma}(x) > t\}$$

and

$$\overline{G}(t) := \{[x] \in \Omega_n(B\Gamma) \,|\, \bar{\sigma}(x) \geq t\}$$

are subgroups of $\Omega_n(B\Gamma)$.

The theorem admits — among other interesting conclusions — the following application. For $p \in \mathbb{N}^*$ we write $p\#M$ for $M\# \cdots \#M$ where M appears p times. We already know $\bar{\sigma}(p\#M) \geq \bar{\sigma}(M)$ if $\bar{\sigma}(M) \geq 0$.

Corollary. *Suppose M is a compact spin manifold of dimension at least 5 with $\sigma(M) \in (0, \Lambda_n)$. Let p and q be two relatively prime positive integers. If $\sigma(p\#M) > \sigma(M)$, then $\sigma(q\#M) = \sigma(M)$.*

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