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Mini-Workshop: Dirac Operators in Differential and Noncommutative Geometry

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Abstract.

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Introduction by the Organisers

Workshop: Mini-Workshop: Dirac Operators in Differential and Noncommutative Geometry

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Abstracts

The enlargeability obstruction to positive scalar curvature

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In this talk we explain the enlargeability obstruction to positive scalar curvature due to Gromov and Lawson [2], [3], [6, IV, 5+6]. One of the main goals of this talk is to show that tori (of any dimension) belong to a large class of manifolds that do not admit metrics of positive scalar curvature.

A compact riemannian manifold M of dimension n is said to be enlargeable if there is for any $\epsilon > 0$ an orientable riemannian covering $\widehat{M} \to M$ together with an ϵ-contracting map $f : \widehat{M} \to S^n$ which is constant at infinity and of non-zero degree. One easily sees that this definition is an invariant of the homotopy type of M, in particular, it is independent of the metric on M. The simplest examples of enlargeable manifolds are tori. It is not very hard to see that compact manifolds with non-positive sectional curvature are enlargeable. With some more effort, it can be shown that compact quotients of a solvable Lie group by a discrete subgroup are enlargeable.

Theorem (Gromov, Lawson [2, 3])
An enlargeable spin manifold cannot carry a metric of positive scalar curvature.

The proof of this remarkable theorem is based on the index theorem. We argue by contradiction and assume that M is an enlargeable manifold equipped with a metric of positive scalar curvature. We can assume (by possibly taking the product with S^1) that M has even dimension n. The enlargeability condition provides the existence of a riemannian covering $\widehat{M} \to M$ together with an ϵ-contracting map $f : \widehat{M} \to S^n$. For any complex vector bundle E over S^n, let $D_E : \Gamma(\Sigma\widehat{M} \otimes f^*E) \to \Gamma(\Sigma\widehat{M} \otimes f^*E)$ be the Dirac operator on $\widehat{M}$ acting on spinors twisted by f^*E. As explained in the talk of C. Bär, the completeness of $\widehat{M}$ implies that D_E has a self-adjoint extension, and we have the Lichnerowicz formula

$$D_E^2 = \nabla_E^*\nabla_E + \frac{1}{4}\mathrm{scal} + \frac{1}{2}F^{f^*E},$$

where F^{f^*E} is the curvature of f^*E acting on twisted spinors by Clifford action.

In particular, if E_1 is a trivial bundle (with a trivial connection), then the Dirac operator D_{E_1} has no spectrum in the interval $(-\sqrt{s}/2, \sqrt{s}/2)$ where $s := \min \mathrm{scal} > 0$. If E_2 is a non-trivial bundle over S^n then the curvature term $F^{f^*E_2}$ is bounded by a constant times ϵ^2, and hence for small $\epsilon > 0$ it is dominated by the scalar curvature term. Hence, in this case we obtain for small $\epsilon > 0$ that D_{E_2} has no spectrum in $(-\sqrt{s}/4, \sqrt{s}/4)$. In particular, the indices of (the positive parts of) D_{E_1} and D_{E_2} vanish.

As mentioned in the talk of A. Strohmaier, usual index theory extends to the possibly non-compact manifold $\widehat{M}$. In particular, the Dirac operators are Fredholm and a relative version of the index theorem holds:

$$\text{ind}(D_{E_1}) - \text{ind}(D_{E_2}) = \{(\text{ch}\, f^*(E_1) - \text{ch}\, f^*(E_2)) \cup \hat{A}(TM)\}[M]. \tag{1}$$

We have already seen that the left hand side is zero. We can choose E_2 such that $\alpha := c_{n/2}(E_2) \neq 0$ and such that the fibers of E_1 and E_2 have the same dimension. This implies that $\text{ch}(E_1) - \text{ch}(E_2) \in H^*(S^n)$ is trivial in all degrees except in degree n where it is $-\frac{1}{((n/2)-1)!}\alpha \neq 0$. Hence, on the right hand side of equation (1), $\hat{A}(TM)$ only contributes in order zero, and we obtain

$$\begin{aligned} \{(\text{ch}\, f^*(E_1) - \text{ch}\, f^*(E_2)) \cup \hat{A}(TM)\}[M] &= -\frac{1}{((n/2)-1)!}(f^*\alpha)[M] \\ &= -\frac{1}{((n/2)-1)!}(\deg f)\alpha[S^n] \neq 0. \end{aligned}$$

The theorem follows from this contradiction.

The theorem as presented here was generalized by Gromov and Lawson in many directions. At first the condition that f has non-trivial degree can be weakened to the condition that f has non-trivial $\hat{A}$-degree. Here we define the $\hat{A}$-degree of $f : M^n \to S^{n-4k}$ as $\hat{A} - \deg(f) := \hat{A}(f^{-1}(p))$ where p is a generic point of S^{n-4k}, and as the $\hat{A}$-numbers are bordism invariants, the $\hat{A}$-degree does not depend on the choice of p. As before, Gromov and Lawson prove that $\hat{A}$-enlargeable manifolds do not carry metrics of positive scalar curvature.

The argument still works if we replace the fact that f is contractible by the weaker condition

$$|f^*(\omega)| \leq \epsilon^2|\omega| \qquad \forall\omega \in \Lambda^2T^*S^{n-4k},$$

and we obtain the notion of $\hat{A}$-area-enlargeability. In a similar way as before, $\hat{A}$-area-enlargeable manifolds do not carry metrics of positive scalar curvature.

For a further generalization, so-called "weak enlargeability which applies to non-compact manifolds, we refer directly to the literature [3], [6, Chap. IV §6].

Another spinorial obstruction to positive scalar curvature is the Mishchenko-Fomenko index theorem. In this index theorem one twist spinors by an appropriate infinite dimensional bundle, and the index of the resulting Dirac operator is an element of $KO_n(C^*_{\max}\pi_1(M))$, see for example [9] for an introduction. If the compact manifold M admits a positive scalar curvature metric, then this index has to vanish [7]. Thus, this index is an obstruction to positive scalar curvature as well. In two recent articles [4, 5] B. Hanke and T. Schick prove that area-enlargeable spin manifolds have non-trivial Mishchenko-Fomenko index. It is believed that the same construction also works for $\hat{A}$-area-enlargeability, but the construction has not yet been carried out. Hence, the enlargeability obstruction to positive scalar curvature can be reduced in some cases to the Mishchenko-Fomenko obstruction.

However, if one wants to prove that a given manifold does not admit a metric of positive scalar curvature, then in some cases the enlargeability condition is simpler to verify than the calculation of the Mishchenko-Fomenko index. Hence, enlargeability provides an efficient mean for finding examples with non-vanishing Mishchenko-Fomenko index. Using fiber bundle techniques, one can then construct new classes of manifolds that do not admit positive scalar curvature metrics [4, Section 6].

Finally, we want to mention that Schoen and Yau [8] have developed an alternative method to prove the non-existence of positive scalar curvature metrics on another class of manifolds containing tori. Their method is based on the construction of a minimal hypersurface in a riemannian manifold of positive scalar curvature and is strongly linked to their proof of the positive mass conjecture. As such minimal hypersurfaces may have singularities in codimension ≥ 7, the Schoen-Yau method does not extend directly to manifolds of dimension ≥ 7. However, in a recent preprint Christ and Lohkamp [1] explain, how to overcome this difficulty.

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Metrics with few harmonic spinors

Bernd Ammann

(joint work with M. Dahl and E. Humbert)

Let M be a fixed compact manifold with spin structure, $n = \dim M$. For any Riemannian metric g on M we denote by D^g the Dirac operator and by k_g the (complex) dimension of the kernel of D^g acting on complex spinors. The number k_g is finite because of the ellipticity of D^g.
The Atiyah-Singer index theorem provides a lower bound on k_g, namely

$$k_g \geq \begin{cases} |\widehat{A}(M)|, & \text{if } n \equiv 0 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Hence we have a lower bound on k_g in terms of a topological quantity, that only depends on the differential structure of M in the case $n \equiv 0 \mod 4$ and only depends on the differential structure and the spin structure of M in the cases $n \equiv 1$ and $n \equiv 2 \mod 8$.
Motivated by comparing D^g to other generalized Dirac operators as the Gauss-Bonnet-Chern operator, the signature operator or the Dolbeault operator, it is natural to ask whether the dimension k_g has topological significance on its own. Hitchin [8, Prop. 1.3] proved that k_g is a conformal invariant, i.e. $k_g = k_{\tilde{g}}$ for conformal metrics g and $\tilde{g}$. However, k_g depends on the conformal class on M.
To start with an example, let M be a Riemann surface with a spin structure. Because of the quaternionic structure of the spinor bundle which commutes with the Dirac operator, we know that k_g is even, and one can show $k_g/2 \mod 2 = \alpha(M)$. Furthermore, one knows that on a Riemann surface of genus γ there is the bound $k_g \leq \gamma + 1$ [8, 2.1, Remark (4)]. If there is a spin 3-manifold with boundary W such that $\partial W = M$ in the sense of spin manifolds, then $\alpha(M) = 0$; we say M is a spin-boundary, or the spin structure is spin-bounding. If W is not a spin-boundary, one can show that $\alpha(W) = 1 \in \mathbb{Z}/2\mathbb{Z}$. The unique spin structure on S^2 is the boundary of the 3-disk, all other compact Riemann surfaces carry spin-bounding and non-spin-bounding spin structures. Now it is clear that in the spin-bounding case, $k_g = 0$ for $\gamma \leq 2$ and all possible metrics g. Similarly, in the non-spin-bounding case we deduce $k_g = 2$ for $\gamma \in \{1, 2, 3, 4\}$. In all other cases of Riemann spin surfaces there are metrics with large k_g, see [5], but because of the main result of this talk, there are also Riemann surface with $k_g \in \{0, 2\}$, see [9].
We visualize this phenomenon for a Riemann surface M of genus 4. Karsten Große-Brauckmann has constructed a smooth family $F_t : M \to \mathbb{R}^3/\mathbb{Z}^3$, $t \in (-\epsilon, \epsilon)$ of embeddings such that $F_t(M)$ has constant mean curvature h_t, $h_0 = 0$, and $dh_t/dt > 0$, see Figure 1. Furthermore $F_t(M)$ separates the torus $\mathbb{R}^3/\mathbb{Z}^3$ into two connected components and is the boundary of each of them. The spin structure on M induced from the embedding F_t is hence spin-bounding. Let g_t be the metric induced from F_t, i.e. the maps F_t are isometric. The spinorial Weierstrass correspondence of Kusner-Schmitt in the form of [3] tells us that we can restrict a parallel spinor on $\mathbb{R}^3/\mathbb{Z}^3$ of constant length 1 to $F_t(M) \cong M$, and this restricted spinor $\phi_t \in \Gamma(\Sigma^{g_t} M)$ satisfies $D^{g_t}\phi_t = h_t\phi_t$, $|\phi_t| \equiv 1$. We see $k_{g_0} \neq 0$, and for $t \to 0$ at least one eigenvalue of D^{g_t} converges to 0 which forces k_{g_t} to jump at $t = 0$.
On the other hand we know from the above considerations that k_{g_t} is either 4 or 0. Hence, it is clear that $k_{g_0} = 4$ and $k_{g_t} = 0$ for $t \neq 0$ close to 0. And we have now seen that k_g depends on the conformal structure in this example.
The same arguments would also apply for genus 3 by considering similar deformations of the Schwarz minimal surface.
Other examples for the dependence of k_g on the conformal class are manifolds of dimension $1, 3, 7, 0 \mod 8$ that admit positive scalar curvature metrics. On these manifolds we have $k_{g_1} = 0$ for the positive scalar curvature metric, but on

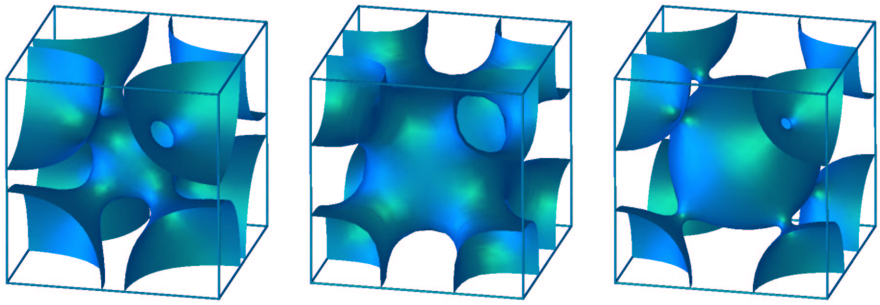

FIGURE 1. The surface $F_t(M)$ for $t < 0$, $t = 0$ and $t > 0$. We thank Karsten Große-Brauckmann for producing these images.

the other hand Hitchin [8] and Bär [2] proved the existence of a metric g_2 with $k_{g_2} > 0$.
These examples suggest to conjecture two statements (for $n \geq 3$).

- For any compact spin manifold M and any $N \in \mathbb{N}$ there is a metric g with $k_g \geq N$.
- For generic metrics g on M one has equality in inequality (1).

Important progress was done during the recent years to confirm cases of the first item [8],[2],[10],[6].
Our main result here is that the second item holds true. In the following we call a metric *D-minimal* if we have equality in (1).
Theorem A (D-minimality theorem, [1]).
Generic metrics on connected compact spin manifolds are D-minimal.
In this theorem, "generic" means that the set of all D-minimal metrics is dense in the C^∞-topology and open in the C^1-topology.
The proof of the theorem has a quite rich and interesting history. As remarked by S. Maier in [9], it suffices to construct one D-minimal metric on M in order to prove the theorem. In the same article Maier used perturbation methods to construct such a D-minimal metric on manifolds on manifolds of dimension at most 4.
The construction of a D-minimal metric is strongly linked to the question of constructing metrics of positive scalar curvature. Important progress in the construction of metrics with positive scalar curvature was achieved by Gromov-Lawson [7] and Stolz [11]. Using techniques from surgery theory, Gromov and Lawson showed that any connected, simply connected non-spin manifold of dimension at least 5 carries a metric of positive scalar curvature. Later Stolz combined these surgery techniques with some very delicate spectral sequence arguments in order to conclude that a connected, simply connected spin manifold of dimension at least 5 carries a metric of positive scalar curvature if and only if the right hand side of inequality (1) vanishes. The Lichnerowicz formula implies that any metric g with positive scalar curvature has $k_g = 0$, and hence it is D-minimal.

A key step in Gromov-Lawson's approach is to prove that if M carries a metric with positive scalar curvature and if $M^{\#}$ arises from M by surgery of codimension at least 3, then $M^{\#}$ carries a metric of positive scalar curvature as well. This surgery statement is no longer true for surgery of codimension 2.
A similar surgery theory was then built by C. Bär and M. Dahl [4] for D-minimal metrics, in order to prove the D-minimality theorem for all connected, simply connected manifolds of dimension at least 5. A key step in their article is to prove the following: If M carries a D-minimal metric and if $M^{\#}$ arises from M by surgery of codimension at least 3, then $M^{\#}$ carries a D-minimal metric as well. As scalar curvature is local invariant whereas D-minimality is a global invariant, essential steps in the analysis had to be redevelopped. Some other fundamental groups can be treated with the same method.
The breakthrough of [1] is based on a construction, that shows the following theorem:
Theorem B *If M carries a D-minimal metric and if $M^{\#}$ arises from M by surgery of codimension 2, then $M^{\#}$ carries a D-minimal metric as well.*
The fact that we can control surgery in codimension 2 admits much stronger conclusions. In particular, if M and N are compact spin-manifold that are spin-bordant, and if N is connected, then the existence of a D-minimal metric on M implies the existence of such a metric on N. In fact, using standard methods in surgery theory, one can simplify a given bordism between M and N into elementary bordisms, that correspond to surgeries in dimensions $0, 1, 2, \dots, n-2$. Hence starting with a D-minimal metric on M one can successively construct a D-minimal metric on the manifolds after the corresponding surgeries, and finally obtain a D-minimal metric on N.
What remains to show is that any spin bordism class contains a representative with a D-minimal metric. This will be explained here by following arguments of Bär and Dahl in [4].
The first step amounts to finding D-minimal metrics in the following cases

- $M = S^1$ with the non-spin-bounding spin structure $\alpha(M) \neq 0$: the standard metric,
- $M = S^1 \times S^1$ with the non-spin-bounding spin structure $\alpha(M) \neq 0$: the standard metric,
- M is a $K3$ surface ($n = 4$, $\widehat{A}(M) = 2$): a Calabi-Yau metric,
- M is a Bott manifold B ($n = 8$, $\widehat{A}(M) = 1$): a metric with holonomy Spin(7).

Let g_1 (resp. g_2) be D-minimal metrics on M_1 (resp. M_2). The product metric on $M_1 \times M_2$ and the disjoint union metric on $M_1 \dot{\cup} M_2$ is not always D-minimal, but one easily sees the following:

- If M_1 and M_2 have teh same dimension, if this dimension is divisible by 4, and if $\widehat{A}(M_1) > 0$ and $\widehat{A}(M_2) > 0$, then the disjoint sum $(M_1 \dot{\cup} M_2, g_1 \dot{\cup} g_2)$ is D-minimal.
- If M_2 is the Bott manifold, then $(M_1 \times M_2, g_1 \times g_2)$ is D-minimal as well.

- $(-M_1, g_1)$ is D-minimal, where $-M_1$ denotes M_1 with the opposite orientation and the same spin structure.

By induction one obtains a list $\mathcal{L}$ of manifolds with D-minimal metrics.
Now, given an arbitrary connected compact spin manifold M, it is easy to find a manifold $P \in \mathcal{L}$ such that

- $\widehat{A}(M \dot{\cup} -P) = 0$ if $n \equiv 0 \mod 4$,
- $\alpha(M \dot{\cup} -P) = 0$ if $n \equiv 1, 2 \mod 8$,
- $P = \emptyset$ otherwise

It follows from a result of Stolz [11] that the manifold $M \dot{\cup} -P$ is spin-bordant to a manifold T which is the total space of a bundle whose fiber is the 2-dimensional quaternionic projective space. In particular, T carries a metric with positive scalar curvature. This defines a D-minimal metric on $T \dot{\cup} P$. The bordism from T to $M \dot{\cup} -P$ can also be seen as a bordism from $T \dot{\cup} P$ to M, and hence our considerations above imply the existence of a D-minimal metric on M.
More details can be found in [1] and on the website
http://www.berndammann.de/publications.

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