Block Seminar Fall 2026
Recent developments in the positive mass theorem

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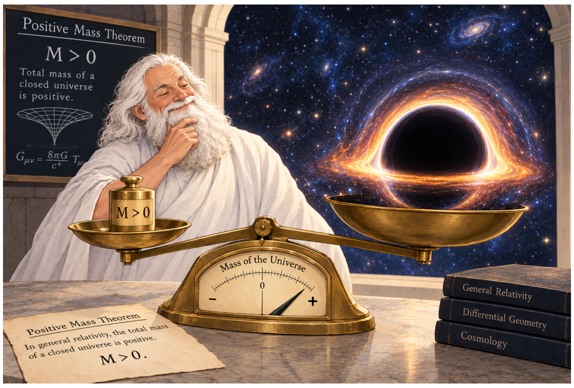

Overview:

The positive mass theorem (PMT) states that every asymptotically flat Riemannian manifold with nonnegative scalar curvature has nonnegative total mass and that the total mass vanishes only in Euclidean space. From a physical standpoint, this confirms the fundamental principle that the total gravitational energy of an isolated system cannot be negative. Mathematically, the theorem has had far-reaching consequences. It played a decisive role in solving the Yamabe problem and provides rigidity results for scalar curvature. The theorem has also inspired many developments in geometric analysis, including the study of minimal hypersurfaces, spin geometry, and geometric flows.

Witten has verified the positive mass theorem for spin manifolds based on the spectral theory of Dirac operators. For non-spin manifolds, it has been verified in low dimensions based on the Schoen-Yau descent method. However, in high dimensions, this method produces singularities that are difficult to handle.

Brendle and Wang recently announced a proof of the positive mass theorem for non-spin manifolds in all dimensions, see https://arxiv.org/abs/2604.08473. This preprint uses a conformal blow-up argument to push the singular points to infinity, an idea that goes back to Bi, Hao, He, Shi, and Zhu, see https://arxiv.org/abs/2603.02769.

When:

- Arrival: Sunday Nov 8, 2026, before dinner.
- Departure: Friday Nov 13, 2026, after lunch.

Participation is only possible for the entire seminar.

Where:

The block seminar takes place in Kloster Frauenwörth, on the island "Fraueninsel" in the lake Chiemsee, Germany

How:

Regular talks should not exceed 60 minutes (plus time for discussion). Please adhere to this time limit and take it into account when planning your presentations. Just in case, you should identify sections in advance that can be omitted without significantly affecting the rest of the presentation. For the short talks, we plan a duration of 40 minutes plus time for discussion.

Since there is no blackboard, the presentations will be given using two document cameras connected to projectors. The presentation should be in the style of a blackboard talk and developed live during the talk.

Who:

To participate effectively, one must have a solid understanding of Riemannian geometry.

The seminar is run by the CRC 1785 *Generalized motivic methods in geometry*. It is also supported by the Universities of Regensburg, Augsburg and Potsdam, as well as by the European Union (ERC Starting Grant 101116001 – COMSCAL) and by the Swedish Research Council (Grant No. 2025-04149).

What:

0. Introduction *(Bernd Ammann)*:

   Overview of the topic and goals of the seminar

1. The mass according to Arnowitt-Deser-Misner I (short talk):

   This and the next talk should be given by speakers who are familiar with the subject and the relevant literature. Please divide the two talks between the two speakers, as you see fit: Definition of asymptotically Euclidean metrics (in the usual sense) and asymptotically Schwarzschild metrics (equal to *asymptotically flat end* in the sense of Brendle-Wang). Definition of the AMD mass. Statement of the positive mass conjecture. Show that a proof of the positive mass conjecture for asymptotically Schwarzschild metrics implies the proof for asymptotically Euclidean metrics.

2. The mass according to Arnowitt-Deser-Misner II:

   Second part of previous talk.

3. First and second variation of the volume functional (short talk):

   Prove the first and second variation formulas for the volume functional of hypersurfaces with respect to compactly supported perturbations. Prove the Schoen-Yau identitiy for minimal hypersurfaces in arbitrary dimensions.

   Literature: [7, Prop. 2.12] and [10, Equations (1.1) – (1.8)]

4. The positive mass theorem in dimension 3:

   Present the classical proof of the positive mass theorem in dimension 3 due to Schoen-Yau.

   Literature: [6, Theorem 2.5 in the special case $n = 3$]

   Note that the classical proof of this statement was given in [11].

5. Positive mass theorem for BW 3-manifolds:

   Introduce Brendle-Wang (BW) n-manifold for $n \geq 3$ (these are called *n-data sets* in [3]). Call the function ρ the *weight function* and Q the *generalized scalar curvature function*. Show the PMT for BW 3-manifolds by reduction to the case of Riemannian 3-manifolds.

   Literature: [3, Section 2]

6. Introduction to μ-bubbles:

   The next step of the proof of [3] uses the μ-bubble method. In this talk, introduce this method in the classical context without weight function. This will be generalized in later talks.

   Literature: [4, Section 6.2 and Proposition 6.12 for $n \leq 7$]

7. Excursion: Width estimates as an application of μ-bubbles:

   Apply μ-bubbles to prove band width estimates in positive scalar curvature geometry. Introduce bands and formulate and prove Main Theorem (A) in Section 2 of [9], see in particular Theorem 5.1.

   Literature: [9]

8. The Lesourd-Unger-Yau construction (short talk):

   This is the beginning of the inductive step of the proof of the main result in [3]. As a prepration we need a result of Lesourd-Unger-Yau that will be important to apply their shielding principle for the construction of generalized μ-bubbles. The goal of the talk is to construct a appropriate penalty function Φ.

   Literature: [3, Lemma 3.1]

9. Construction of the modified weight function $\hat{\rho}$ (short talk):

   Later we will need a modified weight function for the generalized μ-bubble method. The goal of this talk is to explain the first step of this modification by resolving a coercivity estimate and then solving a boundary value problem for a function $\hat{v} = v_0 + v$.

   Literature: [3, Section 3.2]

10. Discussion of barrier hypersurfaces (short talk):

    The construction of generalized μ-bubbles requires appropriate barriers. Construct and discuss these barriers at infinity, called $N_\lambda^\pm$. Show that the cylinders W_s intersect transversally for large s.

    Literature: [3, Section 3.3]

11. The μ-bubble construction of Brendle-Wang I:

    Please divide this and the next talk between two speakers, as you see fit: The goal of these talks is to find a weighted μ-bubble within the barriers in [3, Section 3.3]. The penalty function Φ constructed in [3, Lemma 3.1] avoids that the μ-bubbles hits the singularities of the ambient space. But it can introduce new singularities, which will be treated in later talks. Necessary results from geometric measure theory can be used without proof.

    Literature: [3, Section 3.4]

12. The μ-bubble construction of Brendle-Wang II:

    Second part of previous talk.

13. Stability inequalities for weighted μ-bubbles:

    The classical stability inequalities for minimal hypersurfaces and μ-bubbles are extended to the weighted and noncompact setting. For proving the stability inequality, one decomposes the region E_{slab} which is controlled by [3, Prop. 3.26] into a bounded part controlled by [3, Lemma 3.18] and an unbounded part which is controlled by combining the two controls before. In a limit this provides the inequality in [3, Prop. 3.27]. The final stability inequality is obtained in [3, Cor. 3.28].

    Literature: [3, Section 3.5]

14. A generalized Schoen-Yau identity (short talk):

    In this talk we generalize the classical Schoen-Yau identity to the weighted setup. This identity will lead to an upper bound for the generalized scalar curvature of the μ-bubble [3, Prop. 3.30]. Combining this with the stability inequality for weighted μ-bubbles [3, Cor. 3.28], this proves the coercivitiy estimate [3, Cor. 3.31]. If the μ-bubble has no singularities, in particular for $n \leq 7$, this completes the induction step.

    Literature: [3, Section 3.6]

15. The singular sets of μ-bubbles (short talk):

    Introduce the Minkowski dimension and state without proof the Cheeger-Naber estimate of singular sets in μ-bubbles, see [3, Theorem 3.34] and the references mentioned in this context. Present the Simons cone in $\mathbb{R}^8$ as an example of a minimizing hypersurface with a point singularity.

    Literature: [3, Theorem 3.34] and [8]

16. Conformal blow-up near the singular set I:

    The strategy to remove the singularities in the μ-bubble is to send the singularities to infinity by a conformal change of the metric near the singular set. Construct the corresponding scaling function $w = 1 + \varepsilon_0 \cdot \Psi$. Such blow ups were first used by [1] in order to prove the PMT for manifolds of dimension ≤ 19.

    Literature: [3, Section 3.7]

17. Conformal blow-up near the singular set II:

    We now have a singular μ-bubble with a singular set $\mathcal{S}$ and a regular part Σ. The metric on the ambient space induces a metric Σ and by multiplying by $w^{\frac{n+1}{n-3}}$ we obtain a complete metric $\tilde{g}$ on Σ with an asymptotically flat end and other ends corresponding to the singular set. One also defines an associated weight function $\tilde{\rho}$ and generalized scalar curvature $\tilde{Q}$ on Σ. The final step of the induction is to show that $(\Sigma, \tilde{g}, \tilde{\rho}, \tilde{Q})$ is a BW $(n-1)$-manifold with mass equal to 0.

    Literature: [3, Section 3.8]

18. The Geroch conjecture in arbitrary dimensions *(Yuchen Bi)*

    Literature: [2]

19. Hyperboloidal and spacetime positive mass theorem *(Sven Hirsch)*

    Literature: [5]

Literature:

[1] Yuchen Bi; Tianze Hao; Shihang He; Yuguang Shi; Jintian Zhu: A proof for the Riemannian positive mass theorem up to dimension 19. arXiv:2603.02769 (2026).

[2] Bi, Y.; Zhu, J.: Positive Scalar Curvature Obstructions via Singular Dimension Descent. arXiv:2606.20528 (2026).

[3] Brendle, S.; Wang, Y.: A dimension descent scheme for the positive mass theorem in arbitrary dimension. arXiv:2604.08473 (2026).

[4] Chodosh, O.: *Stable minimal surfaces and positive scalar curvature*. Lecture Notes. Stanford University, 2021. URL: `https://web.stanford.edu/~ochodosh/Math258-min-surf.pdf`.

[5] Sven Hirsch; Marcus Khuri; Martin Lesourd; Yiyue Zhang: The Hyperboloidal and Spacetime Positive Mass Theorem in All Dimensions. arXiv:2604.24746 (2026).

[6] Huang, L.-H.: *A tour of the positive mass theorem*. Lecture Notes, 2025. URL: `https://lhhuang-math.media.uconn.edu/wp-content/uploads/sites/1744/2025/06/2025-Copenhagen.pdf?utm_source=chatgpt.com`.

[7] Lee, D. A.: *Geometric relativity*. Graduate Studies in Mathematics, vol. 201. American Mathematical Society (AMS), Providence, RI, 2019. DOI: 10.1090/gsm/201.

[8] De Philippis, G.; Paolini, E.: A short proof of the minimality of Simons cone. *Rend. Semin. Mat. Univ. Padova*. **121** (2009), 233–241. DOI: 10.4171/RSMUP/121-14. URL: `https://eudml.org/doc/108759`.

[9] Räde, D.: Scalar and mean curvature comparison via μ-bubbles. *Calculus of Variations and Partial Differential Equations*. **62** (2023), 39. DOI: 10.1007/s00526-023-02520-8.

[10] Schoen, R.; Yau, S.-T.: On the structure of manifolds with positive scalar curvature. *Manuscripta Mathematica*. **28** (1979), 159–183. DOI: 10.1007/BF01647970.

[11] ———: On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*. **65** (1979), 45–76. DOI: 10.1007/BF01940959.