Seminar on Spin Geometry

Summer term 2026

Prof. Bernd Ammann

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Goal: Introduction to spin geometry and geometric applications
Tuesday 16–18, M311

Number of sessions: **12**

Available Dates: 14.4., 21.4., 28.4., 5.5., 12.5., 19.5., 2.6., 9.6., 16.6., 23.6., 30.6., 7.7.

Special obstruction:

- May 19, Many participants absent due to schools in Stockholm and Nantes
- May 26, Pentecostal break
- July 14, Bernd Ammann away on conference

Distribution of Talks: Feb 9, 2:15 pm, M311
The seminar addresses to a rather wide range of students. Different talks will address to different groups of students. In our meeting on Feb 9 we will decide which of the following topics will be covered in the seminar, depending on the interests and on the previous knowledge of the audience.
Expected audience

- Students who have followed a lecture on differential geometry and who wish to learn about spinorial methods for applications in Riemannian and Lorentzian geometry. Talks mainly intended for this group: Talk no. 1 (Maybe even 2 talks?),
- Students with interests in the geometry of submanifolds of Euclidean space, both for surfaces in $\mathbb{R}^3$ and generalizations. Talks mainly intended for this group: Talk no. 3, Supplementary Talk no. 1, and Supplementary Talk no. 2
- Students with specific interests, connected to their master thesis project.
- PhD students within my group

The program is only preliminary and will be more refined within the next weeks.

Talks

Introduction to Spin Geometry

Talk no. 1: Spinors and the Dirac operator. *14.4.*
This talk shall serve as an introduction to spin geometry. It will be important, if there are participants, that are unfamiliar with spin geometry. If not needed, this talk may be skipped. It also moght be that we have to split it in two parts. The topics that we will need in the following are: Spinors, Dirac operators, discreteness of the spectrum, Schrödinger-Lichnerowocz formula.
Literature: Hijazi's lecture course [Hij01], Books by Lawson and Michelsohn [LM89], by Roe [Roe88], Friedrich [Fri00] or its German version [Fri97], scripts by Bär [Bär11], Ammann [Amm17], and Hanke [Han16].

Talk no. 2: Summary of spin geometry. *21.4. +28.4.*

1 Spinor representation

Talk no. 3: Spinor representation of surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$. *Currently not in the program.*
Weierstraß found an interesting method to describe minimal surfaces in Euclidean space. If $F : U \to \mathbb{R}^3$ is a conformal parametrisation of a surface, then one can express its differential dF in terms of 2 function $\psi_+, \psi_- : U \to \mathbb{C}$. These functions are holomorphic iff the image of F is a minimal surface. This Weierstraß representation has attracted the interest within surface since about 15 decades. For many decades, the data (ψ_+, ψ_-) was replaced by the Gauß map $G := \psi_+/\psi : U \to \mathbb{C} \cup \{\infty\} \cong S^2$ and the 1-form $\alpha := \psi_+ \cdot \psi \in \Omega^1(M)$. However in terms of the data (G, α) it is hard to describe other types of interesting surfaces, e.g. surfaces of constant mean curvature, or the prescription of mean curvature problem.
One gets to an essentially simpler picture by continuing Weierstrass' original approach to consider the pair (ψ_+, ψ_-). From a modern point of view one observes that (ψ_+, ψ_-) should be viewed as half-spinors on $M := F(U)$. More precisely, $\Psi := (\psi_+, \overline{\psi}_-)$ is a spinor on the surface satisfying $D\!\!\!\!/\,\Psi = H|\Psi|^2\Psi$, with respect to a metric g on M that is conformal to the metric g_0 induced from $\mathbb{R}^3$. Here H is the mean curvature of M in $\mathbb{R}^3$, and $g_0 = |\Psi|^4 g$. This gives nice (local) description of surfaces with constant or prescribed mean curvature in $\mathbb{R}^3$, known as the spinorial Weierstraß representation.
In spite of this beautiful and simple mathematical facts, the history of this problem is not so clear; and different points of view may be found in the literature. The historically first publication available to me, in which I could recognize the representation are two (unpublished) preprints by Kusner and Schmitt, apparently relying on Nick Schmitt's PhD thesis in Amherst, see Kusner-Schmitt [KS95, KS96]. A very readable approach is by Th. Friedrich [Fri98] (be aware

of the probably incorrect statement about the Lawson transform and some non-verfiable historical interpretations of the history), and a related article by Bär [Bär98] (historically slightly ahead, but with another focus). I – obviously – prefer my own presentation, given in my Habilitation [Amm03].

Supplementary Talk no. 1: Prescribing the mean curvature of a surface: a spinorial appraoch. *Currently not in the program.* This talk considers the following question: Assume (M, g) is a closed surface with a Riemannian metric g, and $H : M \to \mathbb{R}$ is a given smooth function. Is there a conformal immersion $M \to \mathbb{R}^3$ with mean curvature H?
Details of this talk will be worked out in case interest is visible in the organisational meeting.
The topic builds on work by Bernd Ammann, Ammann–Humbert, Ammann–Humbert–Ould Ahmedou, Michael Andersen, Tian Xu.

Supplementary Talk no. 2: A spinor representation in dimension $3+1$ and higher. *Currently not in the program.* This talk considers the following question: can we generalise the spinorial Weierstraß representation to a higher dimensional version, with similar analytic applications in reach?
There are involved attempts to derive spinorial representations of submanifolds in Euclidian spaces, in particular by Lawn and collaborators. However, in contrast to the $2+1$-dimensional case, this approach seems not to have intersting relations to conformal eigenvalue problems.
We thus generalise the spinorial Weierstraß representation to $3+1$ dimensions in another way: we consider hypersurfaces with constant mean curvature or prescribed mean curature in (formal) 4-dimensional Ricci-flat Kähler surfaces.
Work by Bernd Ammann, Andrei Moroianu, Sergiu Moroianu [AMM13], also available as arXiv: 1106.2066

Short introduction to the index theorem

Supplementary Talk no. 3: The Atiyah–Singer Index theorem. *Currently not in the program.* Most of the following talks will rely or be related to the Atiyah-Singer index theorem. In case we have participants that are unfamiliar with this theorem, the statement of this theorem shall be explained in this talk without going in to the details of its proof. Again, in case of need, this talk may even be splitt in two talks.
The topic of this talk are different versions of the index theorems, a short introduction to characteristic classes and some standard applications of the index theorem (to be speicified later), e.g. to metrics of positive scalar curvature Books by Lawson and Michelsohn [LM89], by Roe [Roe88], [Bär11], [AL24] and Hanke [Han16].

2 Prescribing the Dirac spectrum

Talk no. 4: Prescribing simple eigenvalues. *5.5.*

Describe and prove the main results of the following article by Mattias Dahl [Dah05]arXiv: math/0311172. Let M be a closed spin manifold. For a finite list of real numbers $\lambda_1 < \lambda_2 \cdots < \lambda_k$ in a finite Interval $(0, \Lambda]$ there is a Riemannian metric on M for which the eigenvalues of the Dirac operator $D\!\!\!/$ in $(0, \Lambda]$ are precisely $\lambda_1 < \lambda_2 \cdots < \lambda_k$, each one of multiplicity 1.

Talk no. 5: Simple eigenvalues are generic. *Currently not in the program.*

Show that for generic Riemannian metrics on a closed spin manifold of dimension three the Dirac operator has only simple eigenvalues (in the quaternionic sense). Follow Dahl [Dah03]arXiv: math/0204232.

Talk no. 6: The kernel of the Dirac operator. *12.5.* +(informal discussion with smaller audience on *19.5.*)+*2.6.*

The Atiyah-Singer index theorem gives a lower bound on the dimension on the kernel of the Dirac operator on a given closed manifold M. Hitchin and Bär have shown that in dimensions $n \equiv 3, 7, 0, 1 \mod 8$ and if the index of M vanishes, then there are Riemannian metrics on M with $\ker D\!\!\!/ \neq \{0\}$ [Hit74] and [Bär96]. These results were put into a more conceptual context in Dahl [Dah08]arXiv: math/0603018. The speaker hall present Dahl's proof of this statement.
Then show that for generic metrics the lower bound given by the index theorem is attained, [ADH09]arXiv: math/0606224. If time admits then mention that in dimension 2 and 4 the space of such metrics is connected [AD25]arXiv: 2508.01420.

3 A glimpse towwards gauge theory

Talk no. 7: Introduction to Seiberg-Witten invariants. *9.6.*
Morgan [Mor96], Moore [Moo96], Friedrich [Fri00, Appendix A].

Talk no. 8: Surgery for Seiberg-Witten invariants. *16.6.*
and *23.6.*
State and prove the results of Haochen Qiu, presented in arXiv: 2409.02265 and arXiv: 2411.10392. These results state that under suitable conditions, Seiberg-Witten invariants are preserved under surgeries. The results by Qiu partially rely on gluing formula from Nicolaescu's book [Nic00, Chapter 4].

4 The hyperspherical radius

Talk no. 9: The hyperspherical radius and applications. *30.6.*

Explain and prove that the first eigenvalue of $D\!\!\!/^2$ is bounded from above by a quantity that depends on the hyperspherical radius. This is the main theorem of a preprint of C. Bär [Bä24]arXiv: 2407.21704.

From this inequality many spin geometric statement can be reproved in a new way, e.g. the Llarrull theorem for the sphere, Geroch's conjecture, and a theorem for fill-ins of nonnegative scalar curvature. Variants of the main theorem provide other interesting applocations, e.g. other versions of Llarrull's theorem.

5 Recent progress on the total mean curvature

Talk no. 10: A conjecture of Gromov about an upper bound on total mean curvature – a spinor approach. *7.7.*
Bär https://arxiv.org/abs/2601.06713 arXiv: 2601.06713

Talk no. 11: A conjecture of Gromov about an upper bound on total mean curvature – a surgery approach. *next term* **(or will be skipped)**
Frenck-Hanke-Hirsch arXiv: 2601.10617

Seminar-Homepage

`https://ammann.app.uni-regensburg.de/lehre/2026s_spingeo_sem/`

Intranet, accessible with NDS account

`https://ammann.app.uni-regensburg.de/lehre/2026s_spingeo_sem/intranet/program-intranet.pdf`

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