Seminar on Pseudo-holomorphic curves and Floer homology

Summer term 2025

Prof. Bernd Ammann

Tuesday, 16-18, M101

Number of sessions: **12**

Available Dates: 29.4., 6.5., 13.5., 20.5., 27.5., 3.6., 17.6., 24.6., 1.7., 8.7., 15.7., 22.7.

Special obstruction:

- June 10, Pentecoastal Tuesday

Talk no. 1: Summary about symplectic geometry. *29.4.* N.N..
Give an introduction to symplectic geometry, depending on the audience. Content will be discussed after distrbution of talks. In particular, for a closed Riemannian manifold (M, ω) discuss the group of its symplectomorphisms and its subgroup of Hamiltonian symplectomorphisms. Also explain the Arnold conjecture, e.g. [Sal99, Section 1.1]. Further references [Cie10, Sections 6 and 8 (up to page 77)], [MS98, Chapters 1-3] and [CdS01].

Supplementary Talk no. 1: More on symplectic geometry N.N..
Whether this talk will be included depends on the audience.

Talk no. 2: Compatible almost complex structures and pseudo-holomorphic curves. *6.5.* N.N..
Explain how to associate almost complex structures to symplectic manifolds. Discuss and prove Gromov's Schwarz- und monotonicty lemmata. Main reference [Hum97, Chapter I and II]. Also explain how to the the first Chern class of a symplectic manifold and the monotonicity condtion [Sal99, Section 1.2]. Additional references [LA94, Pages 41-48] and [MS98, Chapter 4]

Talk no. 3: Gromov's compactness theorem for pseudo-holomorphic curves. *13.5.* N.N..
The main reference of this talk are Chapters III–V in Hummel's book [Hum97]. Explain Gromov's compactness theorem for pseudo-holomorphic curves [Hum97, V. Theorem 1.2] and sketch the main ideas in the proof.

Talk no. 4: Morse-Smale-Witten complex and Morse homology. *20.5.* N.N..
Introduce the Morse-Smale-Witten complex and discuss the isomorphism between Morse-Smale-Witten homology and singular homology. Deduce the Morse inequalities [Sal99, Section 1.3]. As far as time allows, discuss further aspects of the Morse complex and Morse homology, following Part I in [AD14].

Talk no. 5: Closed geodesics and its connecting orbits. *27.5.* N.N..
Discuss the symplectic action functional. Show that any 1-periodic Hamiltonian defines a (closed) 1-form on the loop space of the symplectic manifold M. Explain how compatible almost complex structures may be used to describe the connecting orbits. Study the moduli space of connecting orbits. Main literature: [Sal99, Sections 1.4 and 1.6 (before Remark 1.25)]. Further literature: [AD14, Chapter 9].

Supplementary Talk no. 2: More on J-holomorphic curves N.N..
Energy identity, unique continuation, critical points, somewhere injective curves, adjunction inequality [MS04, Sections 2.2 to 2.6]

Talk no. 6: Regularity theory for connecting orbits. *3.6.* N.N..
We linearize the equations for pseudo-holomrphic curves and study its Fredholm properties. As a result the moduli space of connecting orbits is generically a smooth manifold [Sal99, Sections 2.1 to 2.3]. The index of the associated Fredholm operator is related to the Conley-Zehder index, see [Sal99, Theorem 2.2 in Section 2.2]. Additional literature: [LA94, Pages 165–174] and [MS04, Parts of Chapter 3].

Talk no. 7: Conley-Zehnder index and spectral flow. *17.6.* N.N..
Introduce Maslov cycles, Maslov and Conley-Zehnder index. Discuss how the spectral flow of the linearized J-holomorphicity operator is related to the Conley-Zehnder index [Sal99, Sections 2.4–2.5].

Talk no. 8: Transversality and exponential convergence. *24.6.* N.N..
Discuss [Sal99, Sections 2.6 and 2.7]. Additional literature: [MS04, Chapter 3, in particular Section 3.2].

Talk no. 9: Definition of the Floer complex. *1.7.* N.N..
Discuss compactness of the moduli space modulo bubbling. Then we have the necessary ingredients to define Floer homology. [Sal99, Sections 3.1 and 3.2]. Explain Floer's Theorems [Sal99, Theorems 3.5–3.7] without proofs.

Talk no. 10: Floer's gluing theorem. *8.7.* N.N..
Prove Floer's gluing theorem and use it to prove Theorem 3.5 [Sal99, Section 3.3].

Talk no. 11: Invariance of Floer homology. *15.7.* N.N..
Present [Sal99, Sections 3.4 and 3.5]. These results provide proofs of Theorems 3.6 and 3.7 and finally the Arnold conjecture.

Talk no. 12: Outlook. *22.7.* N.N..
Depending on how fast we have advanced so far, we will have some outlook now. Natural contents of this talk could be the remaining parts of [Sal99, chapter 3], connections to Gromov-Witten invariants and quantum cohomology [MS94] or holomorphic curves in symplectic manifolds of low dimensions [Wen18]. This also might be the subject of a follow-up seminar.

Seminar-Homepage:

https://ammann.app.uni-regensburg.de/floer/

Literatur

[AD14] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.

[CdS01] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[Cie10] Kai Cieliebak. Symplectic geometry – lecture notes, Part A. 2010.

[Hum97] Christoph Hummel. *Gromov's compactness theorem for pseudo-holomorphic curves*, volume 151 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[LA94] Jacques Lafontaine and Michèle Audin. Introduction: applications of pseudo-holomorphic curves to symplectic topology. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 1–14. Birkhäuser, Basel, 1994.

[MS94] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and quantum cohomology*, volume 6 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1994.

[MS98] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1998. Third edition is probably also fine or even better.

[MS04] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Sal99] Dietmar Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999. available here.

[Wen18] Chris Wendl. *Holomorphic curves in low dimensions*, volume 2216 of *Lecture Notes in Mathematics*. Springer, Cham, 2018. From symplectic ruled surfaces to planar contact manifolds.