Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, July 16, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 13

1. Exercise (4 points).
Let $n \in \mathbb{N}_{\geq 2}$. Show that there exists a complete Riemannian metric g on $\mathbb{R}^n$ with sectional curvature $K(M,g) > 0$.

2. Exercise (4 points).
Let G be a finite generated group and $\Gamma \subset G$ be a finite generating set. Let dist' be a left-invariant metric on G. We denote with $\mathrm{dist}^{G,\Gamma}$ the word metric with respect to Γ. Show:

a) There exists a constant $C > 0$ such that

$$\mathrm{dist}'(g,h) \leq C \cdot \mathrm{dist}^{G,\Gamma}(g,h)$$

holds for all $g,h \in G$.

b) There exists a constant $C > 0$ such that

$$N_{\mathrm{dist}'}(C \cdot R) := \#\{h \in G \mid \mathrm{dist}'(h, \mathbb{1}_G) \leq C \cdot R\} \geq N_\Gamma(R)$$

for all $R \in \mathbb{N}$ holds. The function $N_\Gamma(R)$ is given by $\#\{g \in G \mid \mathrm{dist}^{G,\Gamma}(g, \mathbb{1}_G) \leq R\}$.

3. Exercise (4 points).
Consider the upper half-plane model of the hyperbolic space, i.e.

$$\mathbb{H}^n := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with metric $g_x^{\mathbb{H}^n} = \frac{1}{x_n^2} \cdot g_{\mathrm{eucl}}$ for $x \in \mathbb{H}^n$. Show:

a) The map $f\colon \mathbb{H}^n \to \mathbb{R}, x = (x_1,\dots,x_n) \mapsto \log(x_n)$ is a generalized distance function.

b) The map f from $b)$ is the Busemann function for all geodesics of the form $\gamma(t) = (x_1, x_2, \dots, x_{n-1}, x_n \cdot e^t)$ for $x = (x_1,\dots,x_n) \in \mathbb{H}^n$.

4. Exercise (4 points).
We consider the 3-dimensional discrete Heisenberg group, i.e.

$$\mathcal{H}_3 := \mathcal{H}_3(\mathbb{Z}) := \left\{ M_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{Z} \right\}$$

Denote with $X = M_{1,0,0}, Y = M_{0,1,0}$ and $Z = M_{0,0,1}$ and consider the subset $\Gamma = \{X, Y\}$ of $\mathcal{H}_3$. Show:

a) We have $X^a = M_{a,0,0}, Y^b = M_{0,b,0}, Z = M_{0,0,c}$ and $X^a Y^b Z^{c-ab} = M_{a,b,c}$. Conclude that Γ is a generating set of $\mathcal{H}_3$. *Bonus: Write down the Cayley graph* $Cay^{\mathbb{H}_3,\Gamma}$.

b) Let $r, s, t \in \mathbb{Z}$. We have $[X^r, Y^s] = Z^{rs}$. Conclude from that $d^{G,\Gamma}(Z^{rs+t}, \mathbb{1}_{\mathcal{H}_3}) \leq 2|r| + 2|s| + 4|t|$.

c) Let $k, a, b, c \in \mathbb{Z}$ and choose an $r \in \mathbb{N}$ with $r^2 \leq |k| \leq (r+1)^2$. We write $k = \pm r^2 + t$ with $|t| \leq \frac{|2r+1|}{2}$. Conclude that there exists constants $C, D \in \mathbb{R}$ such that

$$d^{G,\Gamma}(X^a Y^b Z^k, \mathbb{1}_G) \leq C \cdot (|a| + |b| + \sqrt{|k|}) + D.$$

d) Conclude from the statements above that, for $R > 0$ large enough, we have

$$\left\{ X^a Y^b Z^k \mid |a|, |b|, \sqrt{|k|} \leq \frac{R}{2C} \right\} \subset \bar{B}_\Gamma(R)$$

Show that there is a constant $c > 0$ such that $N_\Gamma(R) \geq cR^4$ for sufficiently large R.

e) (B2 bonus points) Show that there is a constant $C' > 0$ with the property: if $M_{a,b,k} \in \bar{B}_\Gamma(R)$, then $|a| + |b| \leq R$ and $|k| \leq C' \cdot R^2$. Then show that N_Γ grows polynomially of degree 4.