Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, July 9, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 12

1. Exercise (4 points).
Let (M,g) be a complete Riemannian manifold and $(W,h)\subset(M,g)$ a compact Riemannian submanifold. We denote with $\mathcal{N}_M W = (TW)^\perp \subset TM$ the normal bundle of W in M. Moreover we denote with $\exp^{\mathcal{N}_M W} := \exp^M_{|\mathcal{N}_M W}\colon \mathcal{N}_M W \to M$ the exponential map associated to the normal bundle.

a) Show that there exists an $\epsilon > 0$, such that $d_X \exp^{\mathcal{N}_M W}$ is invertible for all $\|X\| < \epsilon$.

b) Assume moreover that $W^{n-1} \subset M^n$ is a hypersurface of M. Let ν be a unit normal field and S the shape operator of the embedding. Let $c\colon(-\epsilon,\epsilon)\to W$ be a smooth curve in W with $c(0) = p \in W$ and $\dot c(0) = X \in T_pW$. Show that the Jacobi field $J = \frac{d}{ds}\gamma_s$ associated to the geodesic variation

$$\gamma_s(t) := \exp^{\mathcal{N}_M W}_{c(s)}(t\cdot \nu_{c(s)})$$

satisfies $J(0) = X$ and $\frac{\nabla}{dt}J(0) = -S(X)$.

2. Exercise (4 points).
Let (M,g) be a complete Riemannian manifold and $(W,h)\subset(M,g)$ a compact submanifold. Denote $\mathcal{N}_M W$ and $\exp^{\mathcal{N}_M W}$ as in Exercise 1.

a) Show that there exists an $\epsilon > 0$ and an open neighbourhood $W \subset U$, such that $\exp^{\mathcal{N}_M W}$ maps the set $\{X \in \mathcal{N}_M W \mid \|X\| < \epsilon\}$ to U.

b) Assume that $W^{n-1}\subset M^n$ is a hypersurface with unit normal field ν and (M,g) is flat. Show that $d_{\lambda^{-1}\nu_p}\exp^{\mathcal{N}_M W}$ is not invertible, where $p \in W$ is a point and λ is an eigenvalue of the shape operator S_p at p.

3. Exercise (4 points).
Let (M,g) be a Riemannian manifold, whose sectional curvature $K = K(M,g)$ satisfies the bounds:

$$0 < \Lambda_1 \le K \le \Lambda_2$$

Let $\gamma\colon[0,l]\to M$ be a geodesic parametrized by arclength. We define

$$d := \min\{t > 0 \mid \gamma(t) \text{ is conjugated to } \gamma(0) \text{ along } \gamma_{|[0,t]}\}.$$

Use the Rauch Comparison Theorem to show:

$$\frac{\pi}{\sqrt{\Lambda_2}} \le d \le \frac{\pi}{\sqrt{\Lambda_1}}$$

4. Exercise (4 points).
Let (M,g) be a compact Riemannian manifold, which has no conjugated points. Recall that we have defined the function

$$c\colon SM \to (0,\infty],$$
$$X \in S_pM \mapsto \sup\{t>0 \mid d(p,\exp^M_p(tX)) = t\}.$$

Show that there exits a closed geodesic of length $2\cdot\min_{X\in SM} c(X)$. *Hint: Use a similar argument as in Exercise 3 of sheet 11.*