Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, July 2, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 11

1. Exercise: *Cartan-Hadamard* (4 points).
Let (M, g) be a connected and complete Riemannian manifold. We assume that the sectional curvature is non-positive, i.e. $K(M, g) \leq 0$. Let $p \in M$ and denote with $\exp_p : T_pM \to M$ the Riemannian exponential map at p.

a) Use the Proposition 4.1 from the lecture to show that there are no conjugated points.

b) Show that the Riemannian manifold $(N, h) := (T_pM, \exp_p^* g)$ is complete.

c) Show that $\exp_p : (T_pM, \exp_p^* g) \to (M, g)$ is a Riemannian covering.

2. Exercise (4 points).
Let V be a real vector space. We consider $R(t), S(t), S_0 \in \mathrm{End}(V)$ which comply to the Riccati equation

$$\begin{aligned} \dot{S}(t) &= R(t) + S^2(t), \\ S(0) &= S_0 \end{aligned} \tag{1}$$

a) Assume that $R(t)$ and S_0 is symmetric. Show that the solution $S(t)$ is also symmetric.

b) Assume moreover $\frac{\nabla}{dt} R(t) = 0$. Compute a solution of the system (1).

c) Consider the case $V = \mathbb{R}^2$ and $R(t) = \begin{pmatrix} t+2 & t-2 \\ t-2 & -t+2 \end{pmatrix}$. Find the solution of the system (1) for $S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Exercise (4 points).
Let (M, g) be a connected Riemannian manifold without any pair of conjugate points (e.g. if $K \leq 0$). For $p \in M$ define

$$C_p := \{\tilde{q} \in M \mid \text{There exists more than one shortest curve between } p \text{ and } \tilde{q}\}.$$

We also assume that $q \in Q_p$ satisfies $d(p, q) = \min\{d(p, \tilde{q}) \mid \tilde{q} \in C_p\}$.

a) Let $X_1 \neq X_2 \in T_qM \setminus \{0\}$ and assume that X_1 and X_2 are linearly independent. Show that there exists a $Y \in T_pM$ with $\langle Y, X_1 \rangle < 0$ and $\langle Y, X_2 \rangle > 0$ and $\angle(Y, X_1) = \angle(X_2, Y)$.

b) Let $\ell := d(p, q)$ and let $\gamma_i : [0, \ell] \to M$, $i = 1, 2$ be two different geodesics parametrized by arclength from p to q. Let $X_i := \dot{\gamma}_i(\ell)$. Show $X_1 = -X_2$.
Hint: Assume X_1 and X_2 were linearly independent, choose a vector Y as in a). Then show that there are variations $\gamma_{i,s}$, $s \in (-\epsilon, \epsilon)$ of γ_i, with $\gamma_{1,s}(\ell) = \gamma_{2,s}(\ell)$, and $(d/ds)|_{s=0}\gamma_{1,s}(\ell) = (d/ds)|_{s=0}\gamma_{2,s}(\ell) = Y$. Show that for small $s > 0$, we have $\hat{q}_s := \gamma_{1,s}(\ell) \in C_p$ and $d(p, \hat{q}_s) < d(p, q)$.