Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, June 25, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 10

1. Exercise (4 points).

a) Let $F_\leq, F_\geq : I \to \mathbb{R}$ be two C^1 functions defined on a common interval I and $\kappa \in \mathbb{R}$. Assume that we have

$$F_\geq'(t) \geq \kappa + F_\geq(t)^2,$$
$$F_\leq'(t) \leq \kappa + F_\leq(t)^2,$$

for all times $t \in I$. Let $t_0 \in \mathbb{R}$. Show:

$$\frac{d}{dt}\left[(F_\geq(t) - F_\leq(t)) \cdot \exp\left(-\int_{t_0}^{t}(F_\geq(s) + F_\leq(s))ds\right)\right] \geq 0.$$

Assume moreover that if $F_\geq(t_0) \geq F_\leq(t_0)$ holds, then we have $F_\geq(t) \geq F_\leq(t)$ for all $t \geq t_0$.

b) Let (M^m, g) be a Riemannian manifold with $\mathrm{Ric}^g(X,X) \geq (m-1)g(X,X)$ for all $X \in TM$. Let $f : M \to \mathbb{R}$ be a generalized distance function and H the mean curvature of a hypersurface $N_p = f^{-1}(f(p))$ for a point $p \in M$. Show:

$$\partial_{\mathrm{grad}\, f} H \geq 1 + H^2.$$

c) Let $\gamma : (a,b) \to M$ be an integral curve of $\mathrm{grad}\, f$ with $0 \in (a,b)$ and $H(\gamma(0)) = 0$. Show that $a \geq -\frac{\pi}{2}$ and $b \leq \frac{\pi}{2}$. *Hint: Use a) with $\kappa = 1$ and $F_\leq = \tan$.*

d) Assume $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. What can we say about $\lim\limits_{t \to \pm\frac{\pi}{2}} H(t)$?

2. Exercise (4 points).
Let $\gamma : [0,b) \to M$ be a geodesic in a Riemannian manifold. Show that

$$\{t \in [0,b) \,|\, t \text{ is not conjugate to } 0\}$$

is open and dense in $[0,b)$.

3. Exercise (4 points).

a) Let (M, g) be a Riemannian manifold and $K_0 > 0$. Assume that we have a bound on the sectional curvature $|K(M, g)| \leq K_0$. Let $J\colon [0, b] \to TM$ be a non-trivial Jacobi field along a geodesic $\gamma\colon [0, b] \to M$ with $J(0) = J(b) = 0$, i.e. $\gamma(0)$ and $\gamma(b)$ are conjugate points. Show that $b \geq \frac{\pi}{\sqrt{K_0}}$ holds. *Hint: Derive the function $h(t) = \|J(t)\|$ twice and write down a differential inequality of second order. Consider the substitution $F(t) = \frac{h'(t)}{h(t)}$ and find a differential inequality for F like in Exercise 1a).*

b) We equip $\mathbb{C}P^n$ with the Fubini-Study metric g_{FS}, i.e. the metric such that the Hopf fibration $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion. Describe the geodesics in $(\mathbb{C}P^n, g_{FS})$. *Hint: You may use the fact that horizontal lifts $\tilde{\gamma}$ of $\gamma : I \to \mathbb{C}P^n$ are geodesics, iff γ is a geodesic.*

c) Compute the cut locus of $\mathcal{C}_p$ for a point $p \in \mathbb{C}P^n$ in $(\mathbb{C}P^n, g_{FS})$.

4. Exercise (4 points).
Consider a Riemannian manifold $(I \times N, ds^2 + g_s)$ for an interval $I \subset \mathbb{R}$ and a family of Riemannian metrics $(g_s)_{s \in I}$, which is smooth in s. Show:

a) The second fundamental form is given by $\mathrm{II}_{N_s} = -\frac{1}{2}\dot{g}_s$, where $N_s := \{s\} \times N$ and the dot-Notation denotes the derivation w.r.t. the parameter s.

b) Consider the hyperbolic space

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid -x_{n+1}^2 + \sum_{k=1}^{n} x_k^2 = -1, x_{n+1} > 0\}$$

and show that the distance function $f(x) = \mathrm{dist}(x, e_{n+1})$ is given by $f(x_1, \ldots, x_{n+1}) = \operatorname{arcosh}(x_{n+1})$.

c) A hypersurface $N_s = f^{-1}(s)$ for $s \in (0, \infty)$ is smoothly diffeomorphic to a sphere.

d) The second fundamental form of $N_s \subset \mathbb{H}^n$ is given by $\mathrm{II}_{N_s} = -\coth(s) \cdot g_{\text{sphere}}$. In particular, the Weingarten map is given by $S = -\coth(s)\, \mathbb{1}$.

e) Conclude that the Riccati inequality for mean curvature (Theorem 2.14 from the lecture) is an equality.