Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, June 18, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 9

1. Exercise (4 points).
Let (M,g) be a connected, complete Riemannian manifold and $p \in M$. Recall that we defined $\mathcal{R}_p$ as the set of all $q \in M \setminus \{p\}$ with the following properties:

- there is a unique shortest curve $\gamma_q\colon [0,1] \to M$ from p to q. We parametrize γ_q proportional to arclength, thus it is a geodesic with $\|\dot\gamma_q(0)\| = \mathrm{dist}(p,q)$.
- the endoints of γ_q are not conjugate along γ_q.

(Note that these conditions imply, in view of the 4. Exercise, that for any $0 \le a < b \le 1$, $\gamma_q(a)$ is not conjugate to $\gamma_q(b)$.) Furthermore we define

$$\mathcal{R}_p^{\tan} := \{\dot\gamma_q(0) \mid q \in \mathcal{R}_p\}.$$

We have said, that $\mathcal{R}_p^{\tan}$ is open in T_pM and that $\mathcal{R}_p$ is open in M. Finally, the *cut locus* $\mathcal{C}_p$ at p is given by the complement $M \setminus \mathcal{R}_p$. Determine $\mathcal{R}_p^{\tan}, \mathcal{R}_p$ and $\mathcal{C}_p$ for the following cases:

a) $M = \mathbb{R}^2/\Gamma$ for $\Gamma = \mathrm{span}_{\mathbb{Z}}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right)$ and the point $p = \left[\begin{pmatrix}0\\0\end{pmatrix}\right]$.

b) $M = \mathbb{R}P^n$ and the point $p = \mathbb{R}e_1$.

Remark: We consider in both cases the metric on the quotient induces from the standard metrics on $\mathbb{R}^2$ and S^2.

2. Exercise (4 points).
Let (M,g) be a complete Riemanian manifold and let γ be a *ray*, i.e. $\gamma\colon [0,\infty) \to M$ is a geodesic parameterised by arc length with $d(\gamma(t),\gamma(0)) = t$ for all $t \in [0,\infty)$.

a) Show that the function $f(q) := \lim_{t\to\infty} (d(q,\gamma(t)) - t)$ is well-defined. *Hint: Show the monotony in t and that there is a lower bound.*

b) Assume that there exists an open subset $U \subset M$ such that $f_{|U}$ is C^1. Show that the restriction $f_{|U}$ is a generalized distance function. *Hint: Imitate the proof in the lecture for the case $f(q) = d(q,A)$ for a subset $A \subset M$.*

Remark: The function f in this exercise is called Busemann function.

3. Exercise (6 points).
Let (M,g) be a connected Riemannian manifold, $p \in M$. Recall that the *injectivity radius* is given by

$$\text{injrad}(p) := \sup\{r > 0 \mid B_r^{T_pM}(0) \subset \mathcal{D}_p \text{ and } \exp|_{B_r^{T_pM}(0)} \text{ is injective}\},$$

where $\mathcal{D}_p$ is the domain of the exponential function. We want to show that the restriction of the exponential map $\exp|_{B^{T_pM}_{\text{injrad}(p)}(0)}$ is an immersion (and thus a diffeomorphism onto its image).

a) Let $\gamma\colon [-a,b) \to M$, $b > 0$ be a geodesic parametrized by arclength with $\gamma(-a) = p$, such that p is conjugate to $\gamma(0)$ along γ. Moreover let $J\colon [-a,0] \to TM$ be a non-vanishing Jacobi field along $\gamma|_{[-a,0]}$ with $J(-a) = J(0) = 0$. Consider

$$V_\epsilon(t) = \begin{cases} J(t) & \text{for } t \in [-a,-\epsilon] \\ \mathcal{P}_{\frac{1}{2}(t-\epsilon),t}\Big(J\big(\tfrac{1}{2}(t-\epsilon)\big)\Big) & \text{for } t \in [-\epsilon,\epsilon] \end{cases}$$

for $\mathcal{P}_{s,t}\colon T_{\gamma(s)}M \to T_{\gamma(t)}M$ the parallel transport w.r.t the curve γ. Show that $V_\epsilon(t)$ is a vector field along $\gamma|_{[-a,\epsilon]}$ and that it is the variational vector field of a piecewise C^1-variation of $\gamma|_{[-a,\epsilon]}$ with fixed endpoints.

b) Let $\delta > 0$. Show that by renormalizing J and choosing an appropriate ϵ we may achieve that $\|J'(t)\| \geq 1$ and $\|J(t)\| \leq \delta$ holds for $t \in [-\epsilon, 0]$.

c) Show:

$$\int_{-\epsilon}^{\epsilon} \|V'(t)\|^2 dt = \frac{1}{2}\int_{-\epsilon}^{0} \|J'(t)\|^2 dt\,.$$

d) Show the estimate:

$$\Big|\int_{-\epsilon}^{\epsilon} \langle R(V,\partial_t)\partial_t, V\rangle dt - \int_{-\epsilon}^{\epsilon} \langle R(J,\partial_t)\partial_t, J\rangle dt\Big| \leq 3\cdot\epsilon\cdot\delta^2\cdot \sup_M |K(M,g)|\,.$$

e) Use the second variation formula for the energy to show that $\gamma|_{[-a,\epsilon]}$ is not a shortest curve between its endpoints.

f) Show that $a < \text{injrad}(p)$ leads to a contradiction. Conclude that $\exp|_{B^{T_pM}_{\text{injrad}(p)}(0)}$ is an immersion.