Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, June 11, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 8

1. Exercise (4 points).
Let V be a $n+1$-dimensional real vector space.

a) Let g and g' be non-degenerated symmetric bilinearforms of index 1 of V. Assume that for all $v \in V$ we have $g(v,v) = 0$ if and only if $g'(v,v) = 0$. Show that there exists a constant $\lambda \in \mathbb{R}$ with $g' = \lambda g$.

b) Show that the constant from a) is positive if $n+1 \geq 3$.

c) Let $A: \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ be an isomorphism of vector spaces for $n+1 \geq 3$, which maps lightlike vectors to lightlike vectors. Show that there exists a constant $\lambda > 0$ such that $\lambda \cdot A \in \mathrm{O}(n,1)$ holds.

d) Show that in the case $n+1 = 2$ the conclusion of c) does not hold.

2. Exercise (4 points).
Let $\mathbb{R}^{n,1}$ be the Minkowski space. We call a linear map $A: \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ *self-adjoint* if

$$\langle Av, w \rangle_{n,1} = \langle v, Aw \rangle_{n,1}$$

holds for all $v, w \in \mathbb{R}^{n,1}$. We call two linear maps $A, B: \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ *similar* if there exists a $U \in \mathrm{O}(n,1)$ such that $A = UBU^{-1}$ holds. Show that every self-adjoint map $A: \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ is similar to one of the following types: Either it is 1) a diagonal matrix D_{n+1} or

$$2) \begin{pmatrix} D_{n-1} & 0 \\ 0 & \begin{matrix} a & b \\ -b & a \end{matrix} \end{pmatrix} \qquad 3) \begin{pmatrix} D_{n-1} & 0 \\ 0 & \begin{matrix} \lambda & 0 \\ \epsilon & \lambda \end{matrix} \end{pmatrix} \qquad 4) \begin{pmatrix} D_{n-1} & 0 \\ 0 & \begin{matrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{matrix} \end{pmatrix},$$

where $a, b, \lambda \in \mathbb{R}$ with $b \neq 0$ and $\epsilon = \pm 1$.

3. Exercise (4 points).
Let $\mathcal{L} := \{x \in \mathbb{R}^{n+1,1} \mid \langle x, x \rangle_{n+1,1} = 0\}$ be the set of all lightlike vectors in Minkowski space. Show:

a) The map $\iota: S^n \to \mathcal{L}, x \mapsto (x, 1)$ is a smooth embedding of smooth manifolds. The scalar multiplication of the ambient Minkowski space gives rise to a bijection $S^n \to \mathcal{L} \to \mathcal{L}/\mathbb{R}_{\neq 0}$. We call the inverse of this map $\pi: \mathcal{L} \to S^n$. This map is smooth.

b) Let $x \in \mathcal{L}$ and $v \in T_x\mathcal{L}$ be spacelike. We consider the positively oriented 2-plane $E_v := \mathrm{span}\{x, v\}$. We take two oriented planes $E = E_v$ and $F = E_w$ for some spacelike vectors $v, w \in T_x\mathcal{L}$. The enclosed angle

$$\angle(E, F) := \arccos \frac{\sqrt{\langle v, w \rangle}}{\sqrt{\langle v, v \rangle}\sqrt{\langle w, w \rangle}}$$

is independent on the choice of v, w and takes values in the interval $[0, \pi]$. This angle can also be computed by $\angle(E, F) = \angle(d_x\pi(v), d_x\pi(w))$, where the latter is angled taken in the standard sphere (S^n, g_{std}).

c) Let $A \in \mathrm{O}(n+1, 1)$. The restriction of the map A to $\mathcal{L}$ gives rise to a commutative square

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{A} & \mathcal{L} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ S^n & \xrightarrow{\bar{A}} & S^n \end{array}$$

of diffeomorphisms, where $\bar{A}$ is defined $\pi \circ A \circ \pi^{-1}$. The map $\bar{A}$ is a *conformal diffeomorphism*, i.e. there exists a function $f: S^n \to \mathbb{R}_{>0}$ such that $\bar{A}^* g_{\text{std}} = f^2 g_{\text{std}}$ holds.

4. Exercise: *(Conformal compactification of Minkowski space)* (4 points).
Let $(M, g) = (\mathbb{R}^{n,1}, \langle\cdot,\cdot\rangle_{n,1})$ be the Minkowski space. We choose spherical coordinates $(t, r, x) \in \mathbb{R} \times (0, \infty) \times S^{n-1} \cong M \smallsetminus (\mathbb{R} \times \{0\})$ on the complement of the wordline $\mathbb{R} \times \{0\}$ in Minkowski space. We also consider null coordinates $u = t - r, v = t + r$ and $U = \arctan(u), V = \arctan(v)$ as well as $T = U + V, R = V - U$. Show:

a) The map $M \smallsetminus (\mathbb{R} \times \{0\}) \to \mathbb{R}^2 \times S^{n-2}, (t, rx) \mapsto (T, R, x)$ defines a diffeomorphism onto its image. Give an explicit form of the image $U \subset \mathbb{R}^2 \times S^{n-2}$.

b) Determine the pulled-back metric $\tilde{g}$ on U of the conformally transformed metric $\Omega^2 g := \frac{4}{(1+u^2)(1+v^2)} g$ on M.

c) The space $(M \smallsetminus (\mathbb{R} \times \{0\}), \Omega^2 g) \cong (U, \tilde{g})$ isometrically embeds into $(\mathbb{R} \times S^n, -dt^2 + g_{S^n})$ via the map $\mathbb{R}^2 \times S^{n-1} \to \mathbb{R} \times S^n, (T, R, x) \mapsto (T, \sin(R)x, \cos(R))$ and that this extends to an isometric embedding of $(M, \Omega^2 g)$.

d) The closure of the image of M in $\mathbb{R} \times S^n$ is compact.

e) Determine the following subsets of the closure of the images:

$$\begin{aligned} \mathcal{I}^0 &= \left\{ \lim_{t\to\infty} \gamma(t) \mid \gamma: \mathbb{R} \to M \text{ is a spacelike geodesic} \right\} \\ \mathcal{I}^\pm &= \left\{ \lim_{t\to\infty} \gamma(t) \mid \gamma: \mathbb{R} \to M \text{ is a future timelike geodesic} \right\} \\ \mathcal{J}^\pm &= \left\{ \lim_{t\to\infty} \gamma(t) \mid \gamma: \mathbb{R} \to M \text{ is a future lightlike geodesic} \right\}. \end{aligned}$$