Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, June 4, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 7

1. Exercise: *Inverse Cauchy-Schwarz inequality* (4 points).
Let $n \geq 1$ and $\langle \cdot, \cdot \rangle$ be a non-degenerated symmetric bilinearform on a real $n+1$-dimensional vector space of signature $(n, 1)$. We call a vector $v \in V \setminus \{0\}$ *causal* if $\langle v, v \rangle \leq 0$ holds. Show:

a) We have

$$\langle v, w \rangle^2 \geq \langle v, v \rangle \langle w, w \rangle \tag{1}$$

for all causal $v, w \in V$.

b) The subset of all causal vectors in $V \setminus \{0\}$ has exactly two connected components. If two causal vectors $v, w \in V \setminus \{0\}$ belong to the same component, then $\langle v, w \rangle \leq 0$.

c) If $v, w \in V$ belong to the same component of causal vectors, then the *inverse triangle equality* holds:

$$\sqrt{-\langle v + w, v + w \rangle} \geq \sqrt{-\langle v, v \rangle} + \sqrt{-\langle w, w \rangle}. \tag{2}$$

When does equality hold in (1) or (2)?

2. Exercise: *No Hopf-Rinow in Lorentzian geometry* (2 points).
We consider the coordinates (x, y) on $\mathbb{R}^2 \setminus \{0\}$ and the Lorentzian metric

$$g(x, y) = \frac{1}{x^2 + y^2} (dx \otimes dy + dy \otimes dx).$$

a) Show that g is a Lorentzian metric.

b) Show that the Lorentzian metric is invariant under the group action

$$\mathbb{R} \times \left(\mathbb{R}^2 \setminus \{0\}\right) \to \mathbb{R}^2 \setminus \{0\}$$
$$(t, (x, y)) \mapsto (e^t x, e^t y)$$

and there is an induced Lorentzian metric $\bar{g}$ on the quotient manifold $M := \mathbb{R}^2 \setminus \{0\} / \mathbb{R}$.

c) Show that the quotient is diffeomorphic to a 2-torus.

d) Determine the geodesic $\gamma: I_{\max} \to M$ with initial conditions

$$\gamma(0) = [(1, 0)],$$
$$\dot{\gamma}(0) = \frac{\partial}{\partial x}_{|\gamma(0)},$$

where $I_{\max}$ is the maximal domain of definition.
Hint: prove and use the fact that $\dot{\gamma}(t)$ is lightlike for all t, this reduces the geodesic equation to a simple ODE.

e) Conclude that $(M, \bar{g})$ is not geodesically complete, i.e. not every geodesic may be extended so that is defined on all of $\mathbb{R}$. Conclude that there is no analog of the Hopf–Rinow theorem in Lorentzian signature.

3. Exercise (4 points).
Let $m > 0$ and $n \geq 3$. Consider the *Schwarzschild metric*

$$g(t,r,x) = -\left(1 - \frac{2m}{r^{n-2}}\right) dt^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} dr^2 + r^2 g_{S^{n-1}}(x)$$

on $M = \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1}$. Show that (M, g) solves the *vacuum Einstein equation*, i.e. $G_g + \Lambda g = 0$, where

$$G_g = \mathrm{Ric}^g - \frac{1}{2} \mathrm{scal}^g \, g$$

is the *Einstein tensor* and $\Lambda \in \mathbb{R}$ is called the *cosmological constant*.

4. Exercise: *de Sitter and anti-de-Sitter space* (4 points).
Let $\langle \cdot, \cdot \rangle_{n,k}$ be the standard scalar product on $\mathbb{R}^{n,k}$. Consider the *pseudosphere*

$$\mathbb{S}^{n-1,k} := \{x \in \mathbb{R}^{n,k} \mid \langle x, x \rangle_{n,k} = 1\}$$

and the *pseudohyperbolic space*

$$\mathbb{H}^{n-1,k} := \{x \in \mathbb{R}^{n,k} \mid \langle x, x \rangle_{n,k} = -1\}.$$

Show:

a) $\mathbb{S}^{n-1,k}$ is diffeomorphic to $S^{n-1} \times \mathbb{R}^k$.

b) $\mathbb{H}^{n-1,k}$ is diffeomorphic to $\mathbb{R}^n \times S^{k-1}$.

c) Show that $\mathbb{S}^{n-1,1}$ and $\mathbb{H}^{n-1,1}$ with the induced metric from $\mathbb{R}^{n,1}$ are solutions of the vacuum Einstein equation. Determine the constant Λ in both situations.

d) Show that there exists a closed timelike geodesic in $\mathbb{H}^{n-1,1}$.

e) (Bonus question) Are there closed timelike geodesics on the universal covering of $\mathbb{H}^{n-1,1}$?

Note: $\mathbb{S}^{n-1,1}$ is called de-Sitter space, and the universal covering of $\mathbb{H}^{n-1,1}$ is called anti-de-Sitter space.