Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, May 28, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 6

1. Exercise (4 points).
Let $0 < m < n$ and $\mathrm{Gr}(m,n) := \mathrm{Gr}_m(\mathbb{R}^n) = \{V \subset \mathbb{R}^n \mid V \text{ is } m\text{−dimenionsal subvector space}\}$ be the m-Grassmannian of $\mathbb{R}^n$, i.e. the collection of all m-dimensional subvector spaces of $\mathbb{R}^n$.

a) Show that $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ act transitively on $\mathrm{Gr}(m,n)$.

b) Determine the stabilizers of these actions at the point $\mathbb{R}^m \times \{0\} \in \mathrm{Gr}(m,n)$ and write $\mathrm{Gr}(m,n)$ as a homogeneous space G/H with $G \in \{\mathrm{GL}(n,\mathbb{R}), \mathrm{O}(n)\}$.

c) Can you give an interpretation of the following homogeneous spaces

i) $\mathrm{O}(n)/\left(\mathrm{O}(m) \times \mathrm{O}(n-m)\right)$
ii) $\mathrm{SO}(n)/\left(\mathrm{SO}(m) \times \mathrm{SO}(n-m)\right)$
iii) $\mathrm{GL}(n,\mathbb{R})/\left(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n-m,\mathbb{R})\right)$
iv) $\mathrm{GL}_+(n,\mathbb{R})/\left(\mathrm{GL}_+(m,\mathbb{R}) \times \mathrm{GL}_+(n-m,\mathbb{R})\right)$
v) $\mathrm{O}(n)/\left(\mathrm{O}(n_1) \times \ldots \times \mathrm{O}(n_k)\right)$ with $n = n_1 + \ldots + n_k$.
vi) $\mathrm{U}(n)/\left(\mathrm{U}(n-m) \times \mathrm{U}(m)\right)$

2. Exercise: *Symmetric spaces* (8 points).
Let (M,g) be a complete, simply-connected Riemannian manifold. In the following you may use – without proof – a theorem by Myers and Steenrod, that says that the isometry group $\mathrm{Isom}(M,g)$ of (M,g) is a Lie group that acts smoothly on M. You also may use that any closed subgroup of a Lie group is a submanifold. Show that the following statements are equivalent:

i) For any $p \in M$, there exists an isometry $\sigma_p\colon M \to M$ with $\sigma_p(p) = p$ and $d_p\sigma_p = -\mathrm{id}_{T_pM}$

ii) There is a simply-connected Lie group G that acts transitively and isometrically on M and a closed connected subgroup $H \subset G$ such that for some $p \in M$, the map $G/H \to M$, $gH \mapsto g \cdot p$ is a well-defined G-equivariant diffeomorphism. Moreover we have Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ such that

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h},$$
$$[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p},$$
$$[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$$

holds. Furthermore the canonical isomorphism $T_pM \to \mathfrak{p}$ induces a scalar product on $\mathfrak{p}$ for which the isotropy representation $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{p})$ is orthogonal (i.e. $\mathrm{Ad}(H) \subset \mathrm{O}(\mathfrak{p})$).

Hints for proving i)⇒ii): You may take as G the universal covering of $\mathrm{Isom}(M,g)$*, and for some fixed* $p \in M$*, take the isotropy group* $H := G_p$*. A possible way to prove transitivity is to use the Hopf–Rinow theorem. For topological statements the long exact sequence* $\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G)$ *may be helpful. Show that conjugation* $C_{\sigma_p} : G \to G$ *is the identity on* $H = G_p$*. Establish a relation betwween the map* $G/H \to G/H$ *induced from* C_{σ_p} *and* σ_p *itself. Consider* $I := \mathrm{Ad}_{\sigma_p}$ *which is a Lie algebra automorphism of* $\mathfrak{g}$*. Show that* $\mathfrak{h} = \ker(I - \mathrm{id})$*. Use* $I^2 = \mathrm{id}$ *in order to define the complement* $\mathfrak{p}$*.*

Hints for proving ii)⇒i): You may construct a Lie group automorphism $A : G \to G$ *with* $A^2 = \mathrm{id}$ *and* $\ker\big(d_{\mathbb{1}} A - \mathrm{id}_{\mathfrak{g}}\big) = \mathfrak{h}$*. Show that the map* $G/H \to G/H$*,* $gH \mapsto A(g)H$ *(why well-defined?) defines an isometry* $\sigma_H : G/H \to G/H$ *for a suitable* G*-invariant Riemannian metric on* G/H*. This isometry fixes* $\mathbb{1}\,H \cong p$*, and defines* σ_p*. Then define* σ_q *for* $q \in M$*,* $q \neq p$*.*

3. Exercise (2 bonus points).
Let (M,g) as in the previous exercise, satisfying i) or equivalently ii). Show that $\nabla R = 0$.

4. Exercise (2 bonus points).
Which of the examples in Exercise 1 have a Riemannian metric such that the universal covering statisfies all the assumptions in Exercise 2.