Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, May 21, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 5

1. Exercise (4 points).
Let G be a compact Lie group and $H \subset G$ be a closed subgroup. Endow the Lie group with a bi-invariant metric. Show that the closed subgroup H is a totally geodesic submanifold of G, i.e. every geodesic of H is also a geodesic of G. *Bonus: Construct a compact Lie group with a left-invariant metric and a subgroup which is not totally geodesic.*

2. Exercise (4 points).
Let G be a Lie group which acts smoothly on a connected manifold M. Assume that the action is transitive, i.e. for all points $x, y \in M$ there exists a group element $g \in G$ with $x = g \cdot y$. Let $x \in M$ and denote with $G_x := \{g \in G \mid g \cdot x = x\}$ the stabilizer at the point x. Show that:

a) The action restricted to the connected component of the identity G_0 is still transitive.

b) Let $x \in M$. The connected component G_0 is a normal subgroup in G and we have isomorphisms $\pi_0(G) \cong G/G_0 \cong G_x/(G_x \cap G_0)$.

c) There exists an example of a non-trivial group action of a connected Lie group G on a connected manifold such that G_x is not connected. If G_x is connected for a point $x \in M$,ne then G is already connected.

3. Exercise (4 points).
Let G be a connected Lie group which acts smoothly on a connected manifold M. We denote the group action by $\alpha\colon G \times M \to M$ and $\alpha_x := \alpha(\cdot, x)\colon G \to M$ for a point $x \in M$. Show that:

a) If the differential $d_{\mathbb{1}}\alpha_x\colon \mathfrak{g} \to T_xM$ is surjective, then every orbit of the action is an open subset.

b) The group action is transitive if and only if the differential $d_{\mathbb{1}}\alpha_x\colon \mathfrak{g} \to T_xM$ is surjective.

4. Exercise (4 points).
We endow $\mathbb{CP}^n$ we the Fubini-Study metric g_{FS}. In particular the quotient map $\pi\colon S^{2n+1} \to \mathbb{CP}^n, x \mapsto [x]$ is a Riemannian submersion.

a) Show that the canonical $\mathrm{U}(n+1)$ action on $\mathbb{CP}^n$ is an isometric action.

b) Compute the A-tensor of this Riemannian Submersion.

c) Use the O'Neill formula to compute the Riemannian curvature tensor and show

$$K^{(\mathbb{CP}^n, g_{\mathrm{FS}})}(\mathrm{span}\{d\pi(X), d\pi(Y)\}) = K^{S^{2n+1}}(\mathrm{span}\{X, Y\}) + \frac{3}{4}\|A(X,Y)\|^2$$

for all horizontal vectors X, Y of the Riemannian Submersion. What is the numerical range of the sectional curvature of the Fubini-Study metric?