Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
Prof. Dr. Bernd Ammann, Julian Seipel
This sheet will have to be solved **independently** in the 20th calendar week.

Exercise Sheet no. 4

1. Exercise (0 points).
Let G be a connected topological group.

a) Let $\gamma_1, \gamma_2 \colon [0,1] \to G$ be two paths in G with $\gamma_1(0) = \gamma_2(0) = \gamma_1(1) = \gamma_2(1) = \mathbb{1}$. Show the two concatenations

$$\gamma_1 * \gamma_2 := \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

and $\gamma_2 * \gamma_1$ defined analogously are homotopic with fixed endpoints. *Hint: To construct a homotopy, try modifications of the map* $(t,s) \mapsto \gamma_1(t) \cdot \gamma_2(s+t)$ *where we define* $\gamma_1(t) = \gamma_2(t) = \mathbb{1}$ *for* $t \notin [0,1]$.

b) Deduce that the fundamental group $\pi_1(G)$ is abelian.

c) Let Γ be a discrete normal subgroup of the Lie group H. Then Γ is abelian.

Hint: you may prove c) directly, and you may use c) to get b). Or you by apply b) to $G := \Gamma\backslash H$ *and argue that covering space theory then provides a surjective homomorphism* $\pi_1(G) \to \Gamma$ *and then get c).*

2. Exercise (0 points).
We consider the two Lie groups

$$G = \left\{ g(t,a,b) := \begin{pmatrix} \cos(t) & -\sin(t) & a \\ \sin(t) & \cos(t) & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; t,a,b \in \mathbb{R} \right\}.$$

$$\tilde{G} = \left\{ \tilde{g}(t,a,b) := \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) & a \\ 0 & 0 & \sin(t) & \cos(t) & b \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; t,a,b \in \mathbb{R} \right\}$$

Show:

a) The projection $\pi \colon \tilde{G} \to G$ $\tilde{g}(t,a,b) \to g(t,a,b)$ is the universal covering of G.

b) The exponential map for G is surjective.

c) The exponential map for $\tilde{G}$ is not surjective.

3. Exercise (0 points).
Let G be topological group and denote by l_g (respectively r_g) the left (right) multiplication by a group element g on the group.

a) Show that an automorphism $\varphi: G \to G$ of the topological group, which commutes with all right multiplication, is given by a left multiplication.

b) Assume that G is a compact Lie group and $H \subset G$ is a subgroup and a submanifold. If the subgroup H is isomorphic to G as an abstract group, then $H = G$.

4. Exercise (0 points).
Let A be a connected abelian Lie group. Show that:

a) The exponential map $\exp_A$ is surjective.

b) The kernel $\Gamma_A := \ker(\exp_A)$ is a discrete subgroup of the vector space $\mathfrak{a} = \mathrm{Lie}(A)$.

c) There exists $k, l \in \mathbb{N}$ such that $\mathfrak{a}/\Gamma_A \cong \mathbb{R}^k \times T^l$ and in particular the quotient map $\pi_A: \mathfrak{a} \to \mathfrak{a}/\Gamma_A$ is a smooth map.

d) The exponential map $\exp_A$ factors through a diffeomorphism $\varphi: \mathfrak{a}/\Gamma_A \to A$.

e) There exists a Lie group isomorphism $A \cong \mathbb{R}^k \times T^l$.