Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, May 7, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 3

1. Exercise (4 points).
Let $\mathrm{SL}(2,\mathbb{R})$ be the Lie group of matrices of determinant 1 and its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Moreover let $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ be the exponential map and let $k(x) = \frac{1}{2}\operatorname{tr}(x^2)$ for an element $x \in \mathfrak{sl}(2,\mathbb{R})$. As proven in the lecture, the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is given by matrices of vanishing trace. Show:

a) The exponential is given by

$$\exp(x) = C(k(x))\,\mathbb{1}_2 + S(k(x))x$$

for all $x \in \mathfrak{sl}(2,\mathbb{R})$. The functions $C, S\colon \mathbb{R} \to \mathbb{R}$ are given by

$$C(t) = \begin{cases} \cosh(\sqrt{t}) & \text{for } t \geq 0 \\ \cos(\sqrt{-t}) & \text{for } t < 0 \end{cases} \qquad S(t) = \begin{cases} \frac{\sinh(\sqrt{t})}{\sqrt{t}} & \text{for } t > 0 \\ 1 & \text{for } t = 0 \\ \frac{\sin(\sqrt{-t})}{\sqrt{-t}} & \text{for } t < 0 \end{cases}$$

b) A 1-parameter subgroup $t \mapsto \exp(tx)$ of $\mathrm{SL}(2,\mathbb{R})$ has a compact image if and only if $k(x) < 0$ holds.

2. Exercise (4 points).
Let G be a Lie group which acts on a manifold M. Show that the following statements are equivalent:

i) G acts *proper* on M, i.e. the shear map $\mathsf{shear}\colon G \times M \to M \times M, (g,x) \mapsto (gx, x)$ satisfies that preimages of compact subsets are compact again.

ii) For all compact subsets $K \subset M$ the set $G_K := \{g \in G \mid g \cdot K \cap K \neq \varnothing\}$ is compact.

iii) Let $(p_i)_{i\in\mathbb{N}}$ be a sequence in M and $(g_i)_{i\in\mathbb{N}}$ a sequence in G such that the sequences $(p_i)_{i\in\mathbb{N}}$ and $(g_i \cdot p_i)_{i\in\mathbb{N}}$ converge. Then we find a convergent subsequence of $(g_i)_{i\in\mathbb{N}}$.

3. Exercise (4 points).
Let G be a discrete group which acts on a manifold M. We call the group action *properly discontinuous* if for all points p and q in M there exist neighbourhoods U_p and U_q such that

$$\forall g \in G\colon \big(p \neq g \cdot q \Rightarrow U_p \cap (g \cdot U_q) = \varnothing\big)$$

holds.

a) Assume moreover that the group action is free. Show that the group action is proper if and only if the action is properly discontinuous.

b) Consider the group action

$$\begin{aligned} \mathbb{Z} \times \mathbb{R}^n &\to \mathbb{R}^n \\ (k, x) &\mapsto 2^k x. \end{aligned}$$

Is this action properly discontinuous on $M_1 = \mathbb{R}^n$ or $M_2 = \mathbb{R}^n \setminus \{0\}$ or $M_3 = (0, \infty) \times (0, \infty) \times \mathbb{R}^{n-2}$?

c) We write $p_i \colon M_i \to M_i/\mathbb{Z}$ for the canonical maps. We equip each $M_i/\mathbb{Z}$ with the quotient topology (i.e. $U \subset M_i/\mathbb{Z}$ is open in $M_i/\mathbb{Z}$ iff $p_i^{-1}(U)$ is open in M_i). Are the quotients $M_i/\mathbb{Z}$ Hausdorff or compact for $i = 1, 2, 3$?

4. Exercise (4 points).
We define the 3-*dimensional Heisenberg group* H_3 as

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

This is a submanifold and subgroup of $\mathrm{GL}(3, \mathbb{R})$, thus a Lie group.

a) Show that the Lie algebra $\mathfrak{h}_3$, the 3-*dimensional Heisenberg Lie algebra*, is given by matrices as follows:

$$\mathfrak{h}_3 := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

b) Calculate $\exp \colon \mathfrak{h}_3 \to H_3$, and show that it is a diffeomorphism.

c) Show that $\log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A, B]$ for all $A, B \in \mathfrak{h}_3$, where log is the inverse function of exp.

d) Show that $\mathfrak{h}_3$ is 2-step nilpotent, i.e. $[A, [B, C]] = 0$ for $A, B, C \in \mathfrak{h}_3$.