Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, April 30, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 2

1. Exercise (4 points).
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We recall:

$$\begin{aligned} c\colon G \times G &\to G, (g,h) \mapsto ghg^{-1}, && \text{(conjugation)} \\ \mathsf{Ad}\colon G &\to \mathrm{Aut}(\mathfrak{g}), g \mapsto d_{\mathbb{1}} c(g,\cdot), && \text{(adjoint representation)} \\ \mathsf{ad} = d_{\mathbb{1}}\,\mathsf{Ad}\colon \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}), && \text{(adjoint map).} \end{aligned}$$

Determine all these maps explicitly in the case of a matrix group $G \subset \mathbb{R}^{n\times n}$. In particular show $\mathsf{ad}_X(Y) = [X,Y]$.

2. Exercise (4 points).
Let G and H be two Lie groups and $\varphi\colon G \to H$ is a Lie group homomorphism. Show:

a) The differential $d_{\mathbb{1}}\varphi : \mathfrak{g} \to \mathfrak{h}$ is surjective if and only if φ is a submersion.

b) The differential $d_{\mathbb{1}}\varphi : \mathfrak{g} \to \mathfrak{h}$ is bijective if and only if φ is a local diffeomorphism.

c) Assume that H is connected and the differential $d_{\mathbb{1}}\varphi : \mathfrak{g} \to \mathfrak{h}$ is surjective. Show that φ is surjective. *Hint: Show that the image of the map φ is closed and open.*

d) If $\varphi\colon G \to H$ is a local diffeomorphism, then it is a covering map.

3. Exercise (4 points).
Show that there does not exist a matrix $A \in \mathfrak{gl}(2,\mathbb{R})$, such that $\exp(A) = \begin{pmatrix} -\pi & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$ holds. Deduce a contradiction by considering the following cases:

a) A is diagonalizable.

b) A is triagonalizable, but not diagonalizable.

c) A has no real eigenvalues.

Moreover show that the following maps are well-defined:

$$\exp : \mathfrak{u}(n) \to \mathrm{U}(n), \tag{1}$$
$$\exp : \mathfrak{so}(n) \to \mathrm{SO}(n), \tag{2}$$
$$\exp : \mathrm{Der}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}),$$

where $\mathrm{Der}(\mathfrak{g})$ denotes the derivation of a Lie algebra $\mathfrak{g}$. Show that the maps (1) and (2) are surjective.

4. Exercise (4 points).
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

a) Let $\langle\cdot,\cdot\rangle$ be a bi-invariant metric on G. Show that

$$\langle \mathsf{ad}_\xi \eta, \zeta\rangle + \langle \eta, \mathsf{ad}_\xi \zeta\rangle = 0$$

holds for all $\xi, \eta, \zeta \in \mathfrak{g}$.

b) Show that the Levi-Civita connection of a bi-invariant metric on G is given by

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for all left-invariant vector fields X, Y on G. *Hint: Use the Koszul formula. You may use that* $\mathsf{ad}_\xi \eta = [\xi, \eta]$ *for all* $\xi, \eta \in \mathfrak{g}$.

c) Show that the geodesics of a bi-invariant metric on G passing through $\mathbb{1} \in G$ are exactly the 1-parameter subgroups of G.

d) Conclude from the previous statements that the exponential map for a connected, compact Lie group is surjective.