Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024
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Please hand in the exercises until **Tuesday, April 23, 12:00 in the letterbox no. 16.**

Exercise Sheet no. 1

1. Exercise (4 points).
Assume that $(M, g), (N, h)$ are surfaces with Riemannian metrics with negative Gauß curvature. Is it true that the product manifold $(M \times N, g + h)$ has everywhere negative sectional curvature?

2. Exercise (4 points).
Consider the following subsets of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$:

$$\begin{aligned}
\mathrm{SO}(n) &= \{A \in \mathbb{R}^{n\times n} \mid A^T A = \mathbb{1}, \det_{\mathbb{R}}(A) = 1\}\\
\mathrm{GL}(m,\mathbb{C}) &= \{A \in \mathbb{R}^{2m\times 2m} \mid AJ = JA, \det_{\mathbb{R}}(A) \neq 0\}, \qquad \text{if } m = n/2 \in \mathbb{N}\\
\mathrm{U}(m) &= \{A \in \mathrm{GL}(2m,\mathbb{C}) \mid A^* A = \mathbb{1}\}\\
\mathrm{SU}(m) &= \{A \in \mathrm{GL}(2m,\mathbb{C}) \mid A^* A = \mathbb{1}, \det_{\mathbb{C}}(A) = 1\}\\
\mathrm{Aff}(n) &= \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(k+1,\mathbb{R}) \mid A \in \mathrm{GL}(k,\mathbb{R}), b \in \mathbb{R}^k \right\}, \qquad k = n-1
\end{aligned}$$

where we used in for $m = n/2 \in \mathbb{N}$ the definition $J := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \in \mathbb{R}^{2m\times 2m}$.

a) For a matrix $A \in \mathbb{R}^{2m\times 2m}$ with $AJ = JA$ give a reasonable definition for the "complex determinant" $\det_{\mathbb{C}} A$, and show that $\det_{\mathbb{C}} A \neq 0$ if and only if $\det_{\mathbb{R}} A \neq 0$. (Bonus exercise: derive a formula for $\det_{\mathbb{R}} A$ in terms of $\det_{\mathbb{C}} A$.)

b) Show that these are Lie groups with the manifold structure induced from $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$.

c) Determine their Lie algebras (i.e. as linear subspaces of $\mathbb{R}^{n\times n}$). What is the Lie bracket?

d) Construct a Lie algebra isomorphism (i.e. an isomorphism of vector spaces preserving the Lie bracket) between the Lie algebra $\mathfrak{so}(3)$ and the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ denotes the cross product.

In the following we define the adjoint map of a Lie group G with Lie algebra $\mathfrak{g}$ as

$$\mathrm{Ad_g} := \mathrm{d}_{\mathbb{1}} C_g : \mathfrak{g} \to \mathfrak{g}$$

3. Exercise (4 points).
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and let $g \in G$. We denote with ℓ_g (respectively r_g) the left multiplication by g (respectively right multiplication by g) of the Lie group. A Riemannian metric $\langle\cdot,\cdot\rangle$ on G is called *left-invariant* (resp. *right-invariant*) if ℓ_g (resp. r_g) is an isometry of $\langle\cdot,\cdot\rangle$ for all $g \in G$. A metric is called *bi-invariant metric* if it is right- and left-invariant. Show:

a) A scalar product on the Lie algebra $\mathfrak{g}$ can be extended uniquely to a left-invariant metric on G. The same holds if we replace "left-invariant" by "right-invariant".

b) A left-invariant metric $\langle\cdot,\cdot\rangle$ is bi-invariant iff its restriction $\langle\cdot,\cdot\rangle_{\mathbb{1}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is Ad-invariant metric, i.e. if it is invariant under pullback by Ad_g for all $g \in G$.

c) The bilinear map $(A, B) \mapsto -\operatorname{tr}_{\mathbb{R}} AB$ defines an Ad-invariant metric on $\mathfrak{so}(n)$, $\mathfrak{u}(m)$, and $\mathfrak{su}(m)$, which are the Lie algebras of the corresponding groups in Exercise 2.

d) If the Lie group is compact, then there exists a bi-invariant metric on this Lie group. *Hint: Take a left-invariant metric* $\langle\cdot,\cdot\rangle$ *and show that*

$$\langle X,Y\rangle'_h := \int_G \langle d\ell_g X, d\ell_g Y\rangle_{gh} \, \mathsf{dvol}(g)$$

for $X, Y \in T_{\mathbb{1}}G$, $h \in G$ *and* dvol *a right-invariant volume form. This gives a bi-invariant metric on* G.

4. Exercise (4 points).
Let $(V, [\cdot,\cdot]_V)$ be a Lie algebra over a field $\mathbb{K}$. A linear subspace $W \subset V$ is called an *ideal* if for all $X \in W, Y \in V$ we have $[X, Y] \in W$. Show:

a) The quotient vector space V/W carries a unique Lie bracket $[\cdot,\cdot]_{V/W}$ such that the quotient map $\pi: V \to V/W$ is a Lie algebra homomorphism.

b) The kernel of a Lie algebra homomorphism is an ideal. Moreover every ideal of a Lie algebra is the kernel of a Lie algebra homomorphism.

c) Let $\mathbb{K} = \mathbb{R}$ and G, H Lie groups with $H \triangleleft G$ be a normal subgroup and submanifold. The Lie algebra of H is an ideal of the Lie algebra of G.