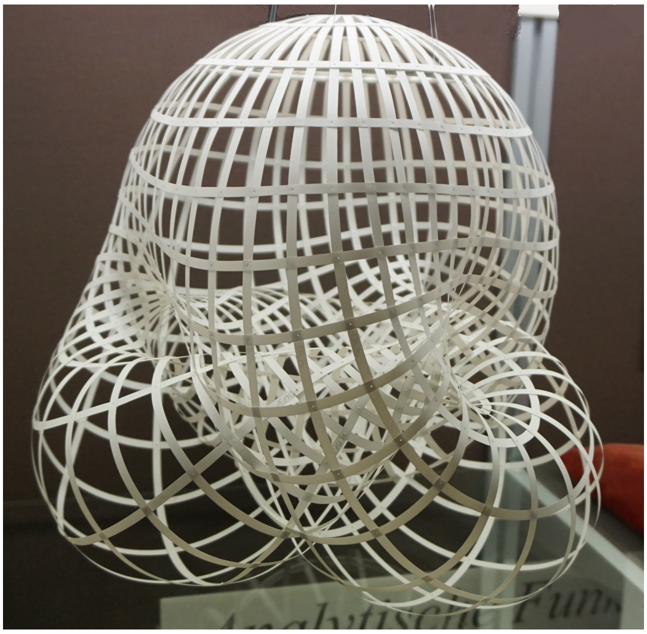
Differential Geometry II

Lecture Notes



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Preface

These are lecture notes for the lecture "Differential Geometry II" held in Regensburg in the summer term 2024. We assume that the readers of these notes and the audience of the lecture are already familiar with basic notions and results in differential and (semi-)Riemannian gemetry, as taught typically in a one-semester lecture, this includes e. g., the theorems by Hopf–Rinow, Bonnet–Myers and Cartan–Hadamard.

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https://ammann.app.uni-regensburg.de/lehre/2024s_diffgeo2/Differential_Geometry_II.pdf

I Lie groups and quotients

Tue. 16.4.

The goal of this section is to treat Lie groups, which are defined as manifolds with a compatible group structure. Important examples are $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{GL}(n,\mathbb{R})$, ...

Lie groups provide many more examples of Riemannian (and more generally semi-Riemannian) manifolds.

1 Lie groups and Lie algebras

Literature for this section: [5], [7], [1], [3], [2]

1.1 Lie groups and their homomorphisms

Definition 1.1. *A* **Lie group** *consists of a $\mathcal{C}^\infty$-manifold G together with a smooth map $\mu: G \times G \to G$, $(\sigma,\tau) \mapsto \mu(\sigma,\tau) = \sigma\tau = \sigma\cdot\tau$, called* **multiplication***, such that*

(i) (G,μ) *is a group*

(ii) $G \times G \xrightarrow{\tilde{\mu}} G$, $(\sigma,\tau) \mapsto \sigma^{-1}\tau =: \tilde{\mu}(\sigma,\tau)$ *is smooth.*

As a consequence of (ii) we see that the following maps are smooth

$$\begin{aligned}
&\ell_\sigma: G \to G, \quad \tau \mapsto \sigma\tau \quad (\textbf{left multiplication} \text{ or } \textbf{left translation} \\
&r_\sigma: G \to G, \quad \tau \mapsto \tau\sigma \quad (\textbf{right multiplication} \text{ or } \textbf{right translation}) \\
&\mathrm{inv}: G \to G, \quad \tau \mapsto \tau^{-1} \quad (\text{inversion}) \\
&\mu: G \times G \xrightarrow{\mu} G, \quad (\sigma,\tau) \mapsto \sigma\tau \quad (\text{multiplication})
\end{aligned}$$

Note also that Diff. geom. I, Exercise Sheet 3, Exercise 4 tells us that one can replace (ii) by (ii') $\mu: G \times G \xrightarrow{\mu} G, \quad (\sigma,\tau) \mapsto \sigma\tau$ is smooth

We write $\mathbb{1}$ for the neutral element of G. Then $T_{\mathbb{1}}G$ is called the **Lie algebra** of G. It is a vector space that comes with some additional structure discussed below, a "Lie bracket".

Examples 1.2.

1.) A finite-dimensional real vector space is a Lie group, if μ is the addition.

2.) $\mathbb{C}^*$, $S^1 \subset \mathbb{C}^*$, $\mathbb{R}^*$ are Lie groups, if μ is the multiplication.

3.) $\mathrm{GL}(n,\mathbb{R})$ is a Lie group, where μ is matrix multiplication. We view $\mathrm{GL}(n,\mathbb{R})$ as an open subset and thus as an n^2-dimensional submanifold of $\mathbb{R}^{n\times n}$.

4.) $\mathrm{SL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$.
In order to show that $\mathrm{SL}(n,\mathbb{R})$ is a submanifold of $\mathrm{GL}(n,\mathbb{R})$ we show that the determinant $\det: \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ is a submersion, i.e. $\mathrm{d}_A \det: T_A \mathrm{GL}(n,\mathbb{R}) \to T_{\det A}\mathbb{R}^* \cong \mathbb{R}$ is surjective for all $A \in \mathrm{GL}(n,\mathbb{R})$. It follows from this, that $\det^{-1}(t)$ is a submanifold for any $t \in \mathbb{R}^*$. For $t = 1$, this shows that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ is a submanifold.

(a) Let $B = \left(b_{ij}\right)_{ij} \in \mathrm{GL}(n,\mathbb{R})$, $C(t) := \mathbb{1} + tB = \left(c_{ij}(t)\right)_{ij} = \left(\delta_{ij} + tb_{ij}\right)_{ij}$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \det\big(\mathbb{1} + tB\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \det C(t) \\
&= \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(c_{1\sigma(1)}(t) \cdots c_{n\sigma(n)}(t)\right)}_{=0 \text{ for } \sigma \neq \mathrm{id}} \\
&\overset{(*)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\left(1 + tb_{1\sigma(1)}\right) \cdots \left(1 + tb_{n\sigma(n)}\right)\right) \\
&\overset{(+)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(1 + t(b_{1\sigma(1)} + \cdots + tb_{n\sigma(n)}) + P_{\geq 2}(t)\right) \\
&= b_{1\sigma(1)} + \cdots + tb_{n\sigma(n)} \\
&= \mathrm{tr}\, B
\end{aligned}$$

Here we used at $(*)$ and above that for $\sigma \neq \mathrm{id}$ there are $i \neq j$ with $c_{i\sigma(i)}(0) = c_{j\sigma(j)}(0) = 0$, and after $(+)$ we write $P_{\geq 2}(t)$ for a polyomial in t without constant and without a linear term, i. e., one only with monomials of degree ≥ 2.

(b) For $A \in \mathrm{GL}(n,\mathbb{R})$ we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det\big(A+tB\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det\big(A\cdot(\mathbb{1}+tA^{-1}B)\big)\\ &= \big(\det A\big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det(\mathbb{1}+tA^{-1}B)\\ &= \big(\det A\big)\cdot\mathrm{tr}(A^{-1}B)\end{aligned}$$

We conclude

$$\begin{aligned}\mathrm{d}_A\det(B) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det\big(A+tB\big)\\ &= \big(\det A\big)\cdot\mathrm{tr}(A^{-1}B).\end{aligned}$$

The linear map $\mathrm{d}_A\colon\mathbb{R}^{n\times n}\to\mathbb{R}$ is surjective as

$$\mathrm{d}_A(A) = \big(\det A\big)\,\mathrm{tr}\,\mathbb{1} = n\cdot\det A \neq 0\,.$$

Now, we now that $\mathrm{SL}(n,\mathbb{R})$ is a submanifold. Its multiplication is the restriction of the multiplication in $\mathrm{GL}(n,\mathbb{R})$, thus mutiplication is smooth as a map $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}\colon\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{GL}(n,\mathbb{R})$. The image of $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}$ is a subset of the submanifold $\mathrm{SL}(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R})$, and this implies the smoothness of $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}\colon\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{SL}(n,\mathbb{R})$.

Further we have

$$\mathbb{T}_{\mathbb{1}}\,\mathrm{SL}(m.\mathbb{R}) = \{A\in\mathbb{R}^{n\times n}\mid \mathrm{tr}\,A = 0\}.$$

5.) The groups $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are Lie groups, see Exercise Sheet 1, Exercise 2

6.) If G and H are Lie groups, then $G\times H$ with the product manifold structure and the product group structure

$$\begin{aligned}(G\times H)\times(G\times H) &\to G\times H\\ \big((\sigma,\tau),(\tilde\sigma,\tilde\tau)\big) &\mapsto (\sigma\tilde\sigma,\tau\tilde\tau)\end{aligned}$$

is again a Lie group.

7.) Let Γ be a discrte subgroup of $\mathbb{R}^n$, e.g., $\Gamma = \mathbb{Z}^n$ or another lattice[1] or another discrete subgroup. If we equip $\mathbb{R}^n/\Gamma$ with the usual addition of equivalence classes, called μ, then $(\mathbb{R}^n/\Gamma, \mu)$ is a Lie group.

Definition 1.3. *A* **homomorphism of Lie groups** *or a* **Lie group homomorphism** *is a smooth map $f: G \to H$, for G and H Lie grous, that is also a group homomorphism. The map f is a* **Lie group isomorphism** *if it is additionally a diffeomorphism, it is a* **Lie group endomorphism** *if additionally $G = H$, and it is a* **Lie group automorphism** *if $G = H$ and if f is a diffeomorphism. We write* $\mathrm{Hom}(G,H)$, $\mathrm{Iso}(G,H)$, $\mathrm{End}(G)$, $\mathrm{Aut}(G)$ *for the sets/monoid/groups of such homorphisms.*

Examples 1.4.

1.) The inclusions $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$, $\mathrm{U}(n) \hookrightarrow \mathrm{O}(2n)$, etc. are Lie group homomorphisms

2.) $\det_{\mathbb{K}} \mathrm{GL}(n,\mathbb{K}) \to \mathbb{K}_{\neq 0}$ is a Lie group homomorphism for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

3.) For any $\sigma \in G$, **conjugation** by σ

$$\begin{aligned} C_\sigma : G &\longrightarrow G \\ \tau &\longmapsto \sigma\tau\sigma^{-1} \end{aligned}$$

is a Lie group automorphism, and $C_\bullet : G \to \mathrm{Aut}(G)$, $g \mapsto C_g$ is a group homomorphism. We obviously have

$$C_\sigma = \ell_\sigma \circ r_{\sigma^{-1}} = r_{\sigma^{-1}} \circ \ell_\sigma \,. \tag{1.1}$$

Remarks 1.5.

1.) If G is a Lie group, one might be tempted to define a Lie subgroup as a subgroup H of G such that H is a submanifold as well. However, this is not what one usually does. One says that $H \subset G$ is a **Lie subgroup**, if there is a Lie group homomorphism $f : H' \to G$, that is injective and an immersion, such that $H = \mathrm{image}(f)$. For example consider $G = \mathbb{R}^2/\mathbb{Z}^2$ and $f(t) = [t, \alpha t]$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $f : \mathbb{R} \to G$ is an injective immersion and a Lie group

[1] A lattice in $\mathbb{R}^n$ is by definition a discrete subgroup Γ of $\mathbb{R}^n$, isomorphic to $\mathbb{Z}^n$. It follows that $\mathbb{R}^n/\Gamma$ is a compact manifold (without boundary), and that there is an $A \in \mathrm{GL}(n,\mathbb{R})$ with $\Gamma = A \cdot \mathbb{Z}^n$.

homomorphism, but $H := \mathrm{image}(f)$ is not a submanifold in the usual sense: a submanifold is always a locally closed subset, but H is not a locally closed subset of G. This leads in books on Lie group, as e. g., in [7, Definition 1.27 (b)] to a slightly generalized definition of a submanifold, however we do not want to elaborate too much on this.

2.) The closed subgroup theorem, see [7, Theorem 3.42], states: Let G be a Lie group, and let H be a subgroup of G (in the sense of group theory) that is closed as a subset, then H is a submanifold of G. It follow any closed subgroup H of G is a Lie group (with induced differentiable structure and induced group structure). Although this result is rather simple to state, the proof is a bit involved. Thus we will not prove it here.

1.2 Lie algebras and their homomorphisms

Let us recall the following exercise from last semester:

Exercise 1.6 (Diff. geom. I, Exercise Sheet 7, Exercise 2)**.** *Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. $\tilde{X}, \tilde{Y}$) be (smooth) vector fields on M (resp. N). We say that X is* **F-related** *to $\tilde{X}$ if $dF \circ X = \tilde{X} \circ F$ holds on M.*
Show that, if X is F-related to $\tilde{X}$ and Y is F-related to $\tilde{Y}$, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

Definition 1.7. *A vector field $X \in \mathfrak{X}(G)$ is called* **left-invariant** *if for all $\sigma \in G$ we have $\mathrm{d}\ell_\sigma(X) = X \circ \ell_\sigma$, i. e., if the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ \ell_\sigma\ } & G \\ {\scriptstyle X}\big\downarrow & & \big\downarrow{\scriptstyle X} \\ TG & \xrightarrow{\ \mathrm{d}\ell_\sigma\ } & TG \end{array}$$

commutes. Similarly X is called **right-invariant** *if for all $\sigma \in G$ we have $\mathrm{d}r_\sigma(X) = X \circ r_\sigma$. If X is left- and right-invariant, we say X is* **bi-invariant***.*

Using the language of Exercise 1.6, we see that a vector field $X \in \mathfrak{X}(G)$ is left-invariant (right-invariant, resp.), if, and only if, it is ℓ_σ-related (r_σ-related, resp.) to itself for any $\sigma \in G$.

Remarks 1.8.

1.) For any $X_0 \in T_{\mathbb{1}}G$ there is a unique left-invariant vector field $X \in \mathfrak{X}(G)$ with $X\big|_{\mathbb{1}} = X_0$. The uniqueness follows from the calculation

$$X\big|_\sigma = X \circ \ell_\sigma(\mathbb{1}) = (\mathrm{d}\ell_\sigma \circ X)(\mathbb{1}) = \mathrm{d}\ell_\sigma\big(X\big|_{\mathbb{1}}\big) = \mathrm{d}\ell_\sigma\big(X_0\big). \tag{1.2}$$

On the other hand if we use (1.2) to define X, i. e., if we set $X\big|_\sigma := \mathrm{d}\ell_\sigma\big(X_0\big)$, then this vector field is the composition

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{id}, X_0)} & G \times TG & \longrightarrow & TG \\ \sigma & \longmapsto & (\sigma, X_0) & \longmapsto & \mathrm{d}\ell_\sigma(X_0) \end{array}$$

which is obviously smooth in σ. In order to show that the vector field X thus obtained is left-invariant we calculate for any fixed $\tau \in G$

$$X \circ \ell_\tau(\sigma) = X\big|_{\tau\sigma} \overset{(\mathrm{def})}{=} \mathrm{d}\ell_{\tau\sigma}(X_0) \overset{(*)}{=} \mathrm{d}\ell_\tau\big(d\ell_\sigma(X_0)\big) \overset{(\mathrm{def})}{=} \mathrm{d}\ell_\tau\big(X\big|_\sigma\big)$$

where we used the chain rule $\mathrm{d}(f \circ g) = (\mathrm{d}f) \circ (\mathrm{d}g)$ at $(*)$, and thus we have $X \circ \ell_\tau = \mathrm{d}\ell_\tau \circ X$ for all $\tau \in G$.

2.) The analogous statement holds as well if we replace left-invariance by right-invariance.

3.) With Exercise 1.6 we see: if $X, Y \in \mathfrak{X}(G)$ are left-invariant (right-invariant, resp.) vector fields, then $[X, Y]$ is also left-invariant (right-invariant, resp.)

Definition 1.9 (Lie bracket on the Lie algebra)**.** *Let G be a Lie group with Lie algebra $T_{\mathbb{1}}G$. The vectors $X_0, Y_0 \in T_{\mathbb{1}}G$ are extended to left-invariant vector fields X and Y. We define*

$$[X_0, Y_0] := [X, Y]\big|_{\mathbb{1}}.$$

This defines a bilinear map $[\,\cdot\,,\,\cdot\,]: T_{\mathbb{1}}G \times T_{\mathbb{1}}G \to T_{\mathbb{1}}G$, called the **Lie bracket** *on the Lie algebra $T_{\mathbb{1}}G$ of G.*

Fr. 19.4.

The pair $(T_{\mathbb{1}}G, [\,\cdot\,,\,\cdot\,])$ satisfies the defining properties of a Lie algebra over $\mathbb{R}$, which are defined as follows:

Definition 1.10 (Abstract Lie algebra). *Let K be a field and $\mathfrak{g}$ a K vector space. A bilinear map $[\bullet,\bullet]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is called a* **Lie bracket** *on $\mathfrak{g}$ if it satisfied*

(i) **Alternation***: for all $x\in\mathfrak{g}$ we have $[x,x]=0$*

(ii) **Jacobi identity***: for all $x,y,z\in\mathfrak{g}$ we have*

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0\,.$$

The pair $(\mathfrak{g},[\bullet,\bullet])$ is then called a **Lie algebra** *(over K).*

If the characteristic of K is not 2 – and the field $K=\mathbb{R}$ we are interested in the case that K is of characteristic 0 –, then condition (i) is equivalent to

(i') **Antisymmetry**: for all $x,y\in\mathfrak{g}$ we have $[x,y]=-[y,x]$.

(In characteristic 2 (i') still implies (i), but the converse is no longer true.)

A **Lie subalgebra** of $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ that is closed under the Lie-bracket, i. e., then it is itself a Lie algebra.

It is obvious that the Lie bracket on $T_{\mathbb{1}}G$ defined in Definition 1.9 satisfies (i') (or equivalently (i)). The Jacobi identity follows immediately in this situation from Exercise 1.6.

Usually for a Lie group the associated Lie algebra, viewed as a vector space with Lie bracket, is denoted by the the associated small fraktur (= gothic) letters, e. g.,

Lie group	G	H	$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{O}(n)$	$\mathrm{SO}(n)$	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{U}(n)$
Lie algebra	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{o}(n)$	$\mathfrak{so}(n)$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{u}(n)$

We also will often write $\mathrm{Lie}(G)$ for the Lie algebra of G, e. g., $\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{h}=\mathrm{Lie}(H)$, etc.

Examples 1.11.

1.) If we consider $G:=\mathbb{R}^n$ as a Lie group with $\mu(x,y)=x+y$, then the left-invariant vector fields are the constant ones. As the Lie bracket of constant vector fields vanishes, the Lie bracket on the Lie algebra is the zero map $0:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$. Thus the Lie algebra is $(\mathbb{R}^n,0)$.

2.) Let V be a finite-dimensional real vector space. We denote the vector space automorphisms of V by $\mathrm{GL}(V)$. By choosing a basis of V, and identify $V\cong\mathbb{R}^n$,

$\mathrm{GL}(V) \cong \mathrm{GL}(n,\mathbb{R})$ we get a Lie group structure on $\mathrm{GL}(V)$, independent of the choice of basis above. Let us write $\mathrm{End}_{\mathrm{lin}}(V)$ for the vector space endomorphisms of V. We have $\mathrm{GL}(V) = \det^{-1}(\mathbb{R} \smallsetminus \{0\})$ for $\det\colon \mathrm{End}_{\mathrm{lin}}(V) \to \mathbb{R}$, thus $\mathrm{GL}(V)$ is open in $\mathrm{End}_{\mathrm{lin}}(V)$. We obtain $\mathfrak{gl}(V) := T_{\mathrm{id}}\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$.

The left-invariant extension of $X_0 \in T_{\mathrm{id}}\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$ is $X\big|_A := A \mapsto A \circ X_0 \in T_A\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$, $X \in \mathfrak{X}\big(\mathrm{GL}(V)\big)$. We proceed similarly for $Y_0 \in T_{\mathrm{id}}\,\mathrm{GL}(V)$ and $Y \in \mathfrak{X}\big(\mathrm{GL}(V)\big)$. Then

$$\begin{aligned}
\partial_X Y\big|_A &= A \circ \partial_{X_0}\big|_A\big(B \mapsto B \circ Y_0\big) = A \circ X_0 \circ Y_0\\
\partial_Y X\big|_A &= A \circ \partial_{Y_0}\big|_A\big(B \mapsto B \circ X_0\big) = A \circ Y_0 \circ X_0\\
[X,Y]\big|_A &= \partial_X Y\big|_A - \partial_Y X\big|_A = A \circ (X_0 \circ Y_0 - Y_0 \circ X_0)\\
[X_0,Y_0] &= [X,Y]\big|_{\mathrm{id}} = X_0 \circ Y_0 - Y_0 \circ X_0\,.
\end{aligned}$$

Thus the Lie algebra structure on $T_{\mathrm{id}}\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$ is given by $(X_0,Y_0) \mapsto_0 \circ Y_0 - Y_0 \circ X_0$, i.e., $[\,\bullet\,,\bullet\,]$ is the usual commutator in $\mathrm{End}_{\mathrm{lin}}(V)$, usually denoted by $[\,\bullet\,,\bullet\,]$ as well.

Definition 1.12 (Lie algebra homomorphism)**.** *Let* $(\mathfrak{g},[\,\bullet\,,\bullet\,]_{\mathfrak{g}})$ *and* $(\mathfrak{h},[\,\bullet\,,\bullet\,]_{\mathfrak{h}})$ *be Lie algebras. A* **homomorphism of Lie algebras** *or a* **Lie algebra homomorphism** *is a linear map* $f\colon \mathfrak{g} \to \mathfrak{h}$ *such that for all* $x,y \in \mathfrak{g}$*:*

$$f\big([x,y]_{\mathfrak{g}}\big) = \big[f(x),f(y)\big]_{\mathfrak{h}}\,.$$

Writing $\mathfrak{g}$ *for* $(\mathfrak{g},[\,\bullet\,,\bullet\,]_{\mathfrak{g}})$ *and* $\mathfrak{h}$ *for* $(\mathfrak{h},[\,\bullet\,,\bullet\,]_{\mathfrak{h}})$*, we denote by* $\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ *the set of all Lie algebra homomorphisms. And similarly to Definition 1.3 we define isomorphisms, endomorphisms, automorphisms and* $\mathrm{Iso}(\mathfrak{g},\mathfrak{h})$, $\mathrm{End}(\mathfrak{g})$ *and* $\mathrm{Aut}(\mathfrak{g})$.

Proposition 1.13. *Let* G *and* H *be Lie groups and let* $f\colon G \to H$ *be a Lie group homomorphism. Then*

$$\mathrm{d}_{\mathbb{1}} f\colon \mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism.

Proof: Assume $X_0, Y_0 \in \mathfrak{g}$. We extend X_0 (resp. Y_0) to a left-invariant vector field $X \in \mathfrak{X}(G)$ (resp. $Y \in \mathfrak{X}(G)$), i.e., $X\big|_\sigma = \mathrm{d}_{\mathbb{1}}\ell_\sigma(X_0)$ for all $\sigma \in G$. Also extend $\widehat{X}_0 := \mathrm{d}_{\mathbb{1}} f(X_0) \in \mathfrak{h}$ to a left-invariant vector field $\widehat{X} \in \mathfrak{X}(H)$, and define similarly $\widehat{Y}_0$ and $\widehat{Y}$. Thus $\widehat{X}\big|_\sigma = \mathrm{d}_{\mathbb{1}}\ell_\sigma(\widehat{X}_0)$ for all $\sigma \in H$.

For $\sigma, \tau \in G$ we have $(f \circ \ell_\sigma)(\tau) = f(\sigma\tau) = f(\sigma)f(\tau) = \ell_{f(\sigma)}\big(f(\tau)\big)$, thus $f \circ \ell_\sigma = \ell_{f(\sigma)} \circ f$. We calculate for $\sigma \in G$.

$$\begin{aligned}\big(\mathrm{d}_\sigma f\big)\Big(X\big|_\sigma\Big) &= \big(\mathrm{d}_\sigma f \circ \mathrm{d}_{\mathbb{1}}\ell_\sigma\big)(X_0) = \mathrm{d}_{\mathbb{1}}\big(f \circ \ell_\sigma\big)(X_0) \\ &= \mathrm{d}_{\mathbb{1}}\big(\ell_{f(\sigma)} \circ f\big)(X_0) = \mathrm{d}_{\mathbb{1}}\ell_{f(\sigma)} \circ \mathrm{d}_{\mathbb{1}} f(X_0) \\ &= \mathrm{d}_{\mathbb{1}}\ell_{f(\sigma)}\widehat{X}_0 = \widehat{X}\big|_{f(\sigma)}.\end{aligned}$$

As a result $\mathrm{d}f \circ X = \widehat{X} \circ f$. And similarly we get $\mathrm{d}f \circ Y = \widehat{Y} \circ f$. Thus we have just shown that

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle X,Y}\Big\downarrow & & \Big\downarrow{\scriptstyle \widehat{X},\widehat{Y}} \\ TG & \xrightarrow{\quad \mathrm{d}f \quad} & TH \end{array}$$

commutes. This means that X resp. Y is f-related to $\widehat{X}$ resp. $\widehat{Y}$ – in the language of Exercise 1.6. It follows from this exercise that $[X, Y]$ is also f-related to $[\widehat{X}, \widehat{Y}]$. Thus

$$\begin{aligned}\mathrm{d}_{\mathbb{1}} f([X_0, Y_0]) &= \big(\mathrm{d}f \circ [X, Y]\big)\big|_{\mathbb{1}} \\ &= \big([\widehat{X}, \widehat{Y}] \circ f\big)\big|_{\mathbb{1}} = [\widehat{X}, \widehat{Y}]\big|_{\mathbb{1}} \\ &= \big[\widehat{X}_0, \widehat{Y}_0\big] = \big[\mathrm{d}f_{\mathbb{1}}(X_0), \mathrm{d}f_{\mathbb{1}}(Y_0)\big],\end{aligned}$$

which is the statement of the proposition. ∎

Corollary of Proposition 1.13. *Assume that V is a finite-dimensional real vector space. Let G be a subgroup and submanifold of* $\mathrm{GL}(V)$. *Let $\mathfrak{g}$ be the Lie algebra of G. Then the Lie-bracket on $\mathfrak{g}$ is the commutator bracket on* $\mathrm{End}(V)$.

Proof: We have seen in Example 1.11 2.) that the Lie bracket on $\mathfrak{gl}(V)$ is the commutator bracket of $\mathrm{End}_{\mathrm{lin}}(V)$. The Lie group homomorphism $i: G \to \mathrm{GL}(V)$ induces an injective Lie algebra homomorphism $\mathrm{d}i: \mathfrak{g} \to \mathfrak{gl}(V)$, thus $\mathfrak{g}$ the Lie brackt of $\mathfrak{gl}(V)$ restricts to the one on $\mathfrak{g}$. ∎

1.3 Adjoint representations

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_{\mathbb{1}}G$. For a given $\sigma \in G$ we differentiate $C_\sigma\colon G \to G$ at $\mathbb{1}$ and we obtain $\mathrm{Ad}_\sigma := \mathrm{d}_{\mathbb{1}} C_\sigma\colon \mathfrak{g} \to \mathfrak{g}$, which is obviously a linear map. For $\sigma, \tau \in G$ differentiating $C_{\sigma\tau} = C_\sigma \circ C_\tau$ implies $\mathrm{Ad}_{\sigma\tau} = \mathrm{Ad}_\sigma \circ \mathrm{Ad}_\tau$.

Lemma 1.14. *For $\sigma \in G$ the map $\mathrm{Ad}_\sigma\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism.*

Proof: Apply Proposition 1.13 to the Lie group homomorphism $C_\sigma\colon G \to G$. ∎

Definition 1.15 (The adjoint representation of a Lie group)**.** *The group homomorphism obtained this way*

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$$

is called the **adjoint representation of the Lie group** G*.*

Remarks 1.16.

1.) One can show that $\mathrm{Aut}(\mathfrak{g})$ is itself a Lie-group, in fact a Lie subgoup of the group $\mathrm{GL}(\mathfrak{g})$ of vector space automorphisms.

2.) The Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$. A linear map $D\colon \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, is called a **derivation** of $\mathfrak{g}$, if for all $x, y \in \mathfrak{g}$ we have

$$D([x,y]) = [D(x), y] + [x, D(y)]\,.$$

Thus we have $\mathfrak{aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

We will not prove these statements here, as they will not be used in what follows and they are easier to prove later.

Definition 1.17 (The adjoint representation of a Lie algebra)**.** *The differential at* $\mathbb{1}$ *of* $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$*, namely*

$$\mathrm{ad} := \mathrm{d}_{\mathbb{1}}\,\mathrm{Ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}_X = \mathrm{d}_{\mathbb{1}}(\sigma \mapsto \mathrm{Ad}_\sigma)(X)$$

is called the **adjoint representation of the Lie algebra** $\mathfrak{g}$*.*

Accoding to Remarks 1.16 the adjoint representation of a Lie algebra is in fact a

Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})\,.$$

Lemma 1.18. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group. Then the adjoint map* ad *satisfies.* $\mathrm{ad}_X(Y) = [X, Y]$

The proof will be given later. Tue. 23.4.

1.4 The exponential map

In the following $t \in \mathbb{R}$, so $\partial_t := \frac{\mathrm{d}}{\mathrm{d}t}$ is the positively oriented vector field on $\mathbb{R}$ of constant length 1. For a smooth map $f\colon \mathbb{R} \to M$ we also write $\dot{f}(t) = \mathrm{d}f(\partial_r|_t)$. We write $\mathrm{Diff}(M)$ for the group of diffeomorphisms of M.

Definition 1.19. *Let M be a manifold and $X \in \mathfrak{X}(M)$. A curve $\gamma\colon I \to M$ is called* **integral curve** *of X or* **flow line** *of X, if for all $t \in I$ we have*

$$\dot{\gamma}(t) = X|_{\gamma(t)}\,.$$

The theorem of Picard-Lindelöf implies: For any $p \in M$ there is an integral curve γ_p of X with $\gamma_p(0) = p$ and we assume that γ_p is defined on its maximal domain I_p, and this maximal solution is unique. We say that X is **complete** *if $I_p = \mathbb{R}$ for all $p \in M$. We also define $\Phi_t^X(p) := \gamma_p(t)$. Thus if X is complete, then we have a group homomorphism $\Phi_\bullet^X\colon \mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \Phi_t^X$, called the* **flow** *of X.*

We encourage the reader to check that $t \mapsto \Phi_t^X$ is indeed a group homomorphism.

Lemma 1.20. *For a left-invariant vector field X on a Lie group we have:*

(1) *X is complete,*

(2) *If γ is an integral curve of X, and $\sigma \in G$, then $\ell_\sigma \circ \gamma$ is an integral curve of X as well,*

(3) $\Phi_t^X(\sigma\tau) = \sigma\Phi_t^X(\tau)$ *for $t \in \mathbb{R}$, $\sigma, \tau \in G$.*

(4) $\Phi_t^{\lambda X} = \Phi_{\lambda t}^X$ *for all $\lambda, t \in \mathbb{R}$.*

In the proof we use the conventions $\infty + t := \infty$ and $-\infty + t = -\infty$ for all $t \in \mathbb{R}$.

Proof: Let G be a Lie group and let $X \in \mathfrak{X}(G)$ be a left-invariant vector field. Consider the integral curve $\gamma_{\mathbb{1}}: I_{\mathbb{1}} \to G$, with $\gamma_{\mathbb{1}}(0) = \mathbb{1}$, $I_{\mathbb{1}} = (\alpha, \omega)$. For any $\sigma \in G$ we calculate that the curve $\ell_\sigma \circ \gamma_{\mathbb{1}}$ is also an integral curve of X:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ell_\sigma \circ \gamma_{\mathbb{1}}(t)\right) = \mathrm{d}\ell_\sigma\left(\dot{\gamma}_{\mathbb{1}}(t)\right) = \mathrm{d}\ell_\sigma\Big(X\big|_{\gamma_{\mathbb{1}}(t)}\Big) = X\big|_{\ell_\sigma \circ \gamma_{\mathbb{1}}(t)}.$$

Thus $\gamma_\sigma := \ell_\sigma \circ \gamma_{\mathbb{1}}: (\alpha, \omega) \to G$ is the intergral curve with $\gamma_\sigma(0) = \sigma$. This already shows (2).

Now for $t_0 \in (\alpha, \omega)$ we have

$$\gamma_{\mathbb{1}}(t_0) = \gamma_{\gamma(t_0)}(t_0 - t_0),$$

thus $\gamma_{\mathbb{1}}$ and $\gamma_{\gamma_{\mathbb{1}}(t_0)}(\bullet - t_0)$ coincide, including their maximal domains. Hence $(\alpha, \omega) = (\alpha + t_0, \omega + t_0)$, hence $\alpha = -\infty$ and $\omega = \infty$. This proves the completeness, i. e., (1).

The statement (3) follows from the facts that both $t \mapsto \Phi_t^X(\sigma\tau)$ and $t \mapsto \sigma\Phi_t^X(\tau)$ are integral lines for X and that they coincide for $t = 0$.

In the notation above, and for any $\sigma \in G$ we have $\Phi_{\lambda t}^X(\sigma) = \gamma_\sigma(\lambda t)$. We calculate with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_\sigma(\lambda t) = \lambda\dot{\gamma}_\sigma(\lambda t) = \lambda\Big(X\big|_{\gamma_\sigma(\lambda t)}\Big) = (\lambda X)\big|_{\gamma_\sigma(\lambda t)}$$

Thus $\mapsto \Phi_{\lambda t}^X(\sigma)$ is the integral curve of λX that attains σ for $t = 0$. Thus, by definition of $\Phi_t^{\lambda X}$, we have (4). ■

Definition 1.21. *A homomorphism $f: \mathbb{R} \to G$ is called a* **1-parameter subgroup** *of G.*

Remark. Note that in general $f(\mathbb{R})$ is in general not a submanifold (in the usual sense[2]) of G, but it is a submanifold in the generalized sense of [7], see Remark 1.5 1.). This explains the usage of the word “subgroup”.

Proposition 1.22. *Let G be a Lie group. Then the 1-parameter subgroups are the integral curves of some left-invariant vector field through $\mathbb{1}$. More precisely:*

(1) *Let $f: \mathbb{R} \to G$ be a 1-parameter subgroup, and take the left-invariant vector field $X \in \mathfrak{X}(G)$ such that $\dot{f}(0) = X\big|_{\mathbb{1}}$. Then f is the integral curve of X with $f(0) = \mathbb{1}$.*

[2] i. e., in the sense of Analysis IV, Differential Geometry I, etc

(2) *Let X be a left-invariant vector field and $f\colon \mathbb{R} \to G$ an integral curve of X with $f(0) = \mathbb{1}$. Then f is a 1-parameter subgroup.*

It follows, that two 1-parameter subgroups $f_1, f_2\colon \mathbb{R} \to G$ coincide if and only if $\dot{f}_1(0) = \dot{f}_2(0)$.

Proof:

"(1)": Obviously $f(0) = \mathbb{1}$. As f is a homomorphism $f(t + \bullet) = \ell_{f(t)} \circ f$. Thus

$$\dot{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(t + \bullet) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \ell_{f(t)} \circ f = \mathrm{d}\ell_{f(t)}(\dot{f}(0)) = \mathrm{d}\ell_{f(t)}(X\big|_{\mathbb{1}}) = X\big|_{f(t)}.$$

Thus f is an integral curve of X.

"(2)": Obviously f is smooth. It is defined on $\mathbb{R}$ due to Lemma 1.20 (1). By definition of the flow we have $f(t) = \Phi_t^X(\mathbb{1})$, and thus we calculate, using Lemma 1.20 (1) at $(*)$

$$f(t+s) = f(s+t) = \Phi_{s+t}^X(\mathbb{1}) = \Phi_s^X(\Phi_t^X(\mathbb{1})) = \Phi_s^X(f(t)) \overset{(*)}{=} f(t)\Phi_s^X(\mathbb{1}) = f(t)f(s).$$

Thus f is a Lie group homomorphism. ∎

Definition 1.23. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. We write X for the left-invariant vector field extending $X_0 \in \mathfrak{g}$. The* **exponential map** $\exp$ *is defined as the map*

$$\exp\colon \mathfrak{g} \to G, \quad X_0 \mapsto \Phi_1^X(\mathbb{1}).$$

WARNING. *This exponential map is in general not the same as the Riemannian exponential map, even if we know that the metric is left- or right-invariant.*[3]*. As a consequence this map is also called the* **Lie group exponential map** *in order to distinguish it from the* **(semi-)Riemannian exponential map***. It does however – as will be shown in the exercises – coincide with the Riemannian one for bi-invariant metrics on Lie groups.*

Theorem 1.24 (Properties of the exponential map)**.** *Let G be a Lie group with Lie algebra fg, and $X \in \mathfrak{g}$, $t, s \in \mathbb{R}$. Then we have*

[3] The notions of left-, right-, and bi-invariant Riemannian metrics are defined in the exercises.

(1) $\exp$ *is smooth and with our usual identification* $T_0\mathfrak{g} \cong \mathfrak{g}$*, its differential* $\mathrm{d}_0 \exp$ *is the identity of* $\mathfrak{g}$*. As a consequence there is an open neighborhood* U *of* 0*, such that* $\exp\big|_U : U \to \exp(U)$ *is a parametrization.*

(2) $\exp(tX) = \Phi_t^X(\mathbb{1})$

(3) $t \mapsto \exp(tX) =: f_X(t)$*,* $\mathbb{R} \to G$ *is a* 1*-parameter subgroup of* G *and any* 1*-parameter subgroup is of that form for some* $X \in \mathfrak{g}$*. Furthermore* $\dot{f}_X(0) = X$*.*

(4) *The integral curves of the left-invariant vector field associated to* X *are, the curves* $t \mapsto \sigma \exp(tX)$ *for* $\sigma \in G$

(5) *If* $\overline{X}$ *is the left-invariant vector that extends* $X \in \mathfrak{g}$*, then for all* $t \in \mathbb{R}$ *we have*

$$\Phi_t^{\overline{X}} = r_{\exp(tX)} \,.$$

Proof:

"(1)": The smoothness of exp follows from the smooth dependence on the initial conditions in the theorem of Picard–Lindelöf. We calculate for the left-invariant vector fields $\overline{X} \in \mathfrak{X}(G)$ extending $X \in \mathfrak{g}$.

$$\big(\mathrm{d}_{\mathbb{1}} \exp\big)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\exp(tX)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t^{\overline{X}}(\mathbb{1}) = \overline{X}\big|_{\mathbb{1}} = X \,.$$

Thus $\mathrm{d}_{\mathbb{1}} \exp = \mathrm{id}_{\mathfrak{g}}$.

"(2)": It follows from Lemma 1.20 (4) that $\exp(tX) = \Phi_1^{tX}(\mathbb{1}) = \Phi_t^X(\mathbb{1})$.

"(3)": This immediately follows from Proposition 1.22.

"(4)": The integral curves in this item are $t \mapsto \Phi_t^X(\sigma)$ and we have seen in Lemma 1.20 (4) that $\Phi_t^X(\sigma) = \ell_\sigma\big(\Phi_t^X(\mathbb{1})\big) = \ell_\sigma \circ \exp(tX)$.

(5) immediately follows from (4) and the definition of $\Phi_t^{\overline{X}}$. ■

Example 1.25 (Exponential map of matrix groups)**.** We consider again the Lie group $\mathrm{GL}(V)$ for a finite-dimensional real vector space V. We have already seen in Example 1.11 2.) that the left-invariant extension of $X_0 \in \mathfrak{g}$ is $X \in \mathfrak{X}(\mathrm{GL}(V))$ with $X\big|_A = A \cdot X_0$, $A \in \mathrm{GL}(V)$.

For $A \in \mathfrak{gl}(V)$ we know from the theory of ordinary differential equations, that the series

$$\mathrm{EXP}(A) := \sum_{i=0}^{\infty} \frac{1}{i!} A^i \tag{1.3}$$

converges (uniformly on compact sets and also all derivatives converge uniformly on compact sets). We obtain a map $\mathrm{EXP}\colon \mathfrak{gl}(V) \to \mathrm{GL}(V)$ such that for $t, s \in \mathbb{R}$, $A \in \mathfrak{gl}(V)$

$$\mathrm{EXP}\big((t+s)A\big) = \mathrm{EXP}(tA)\,\mathrm{EXP}(sA), \quad \mathrm{EXP}(0) = \mathbb{1}, \quad \mathrm{EXP}(-A) = \mathrm{EXP}(A)^{-1}.$$

Thus $t \mapsto \mathrm{EXP}(tA)$ a 1-parameter subgroup, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{EXP}(tA) = A$$

It follows from Proposition 1.24 (3) that $\mathrm{EXP}(A) = \exp(A)$. So we will write exp instead of EXP from now on. The same holds if G is a submanifold and subgroup of $\mathrm{GL}(V)$.

Furthermore from the theory of ordinary differential equations we know that for $t \in \mathbb{R}$, $A, B \in \mathfrak{gl}(V)$, $M \in \mathrm{GL}(V)$ we have

$$\exp(MAM^{-1}) = M \exp(A) M^{-1} \tag{1.4}$$

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A), \text{ if } [A,B] = 0 \tag{1.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(tA)) = \exp(tA)A = A\exp(tA) \tag{1.6}$$

We would like to have similar properties in adapted form for arbitrary Lie groups. We already have an adapted form of the first equality of (1.6) which is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)) = \big(\mathrm{d}\ell_{\exp(tX)}\big)(X)$$

i. e., $t \mapsto \exp(tX)$ is an integral curve of the left-invariant extension of X.

Lemma 1.26. *Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$. Then*

$$\mathrm{Ad}_{\exp(tX)}(X) = X.$$

Proof: One easily checks $C_{\exp(tX)}(\exp(tX)) = \exp(tX)$. If we derive this at $t = 0$ one gets the equation stated in the lemma. ■

It immediately follows that $\big(\mathrm{d}\ell_{\exp(tX)}\big)(X) = \big(\mathrm{d}r_{\exp(tX)}\big)(X)$, and we get the following corollary that generalizes (1.6). Fr 26.4.

Corollary 1.27.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)) = \big(\mathrm{d}\ell_{\exp(tX)}\big)(X)) = \big(\mathrm{d}r_{\exp(tX)}\big)(X)$$

Lemma 1.28. *If $f\colon G \to H$ is a homomorphism of Lie groups. Let $\exp^G\colon \mathfrak{g} \to G$ and $\exp^H\colon \mathfrak{h} \to H$ be the exponential maps of G and H. Then the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle \exp^G}\uparrow & & \uparrow{\scriptstyle \exp^H} \\ \mathfrak{g} & \xrightarrow{\quad \mathrm{d}_{\mathbb{1}} f \quad} & \mathfrak{h} \end{array} \tag{1.7}$$

commutes, i. e., $f \circ \exp^G = \exp^H \circ \mathrm{d}_{\mathbb{1}} f$.

Proof: Let $X \in \mathfrak{g}$. Then $t \mapsto \exp^G(tX)$ is a 1-parameter subgroup of G. Thus $t \mapsto f \circ \exp^G(tX)$ is a 1-parameter subgroup of G. We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \exp^G(tX) = \mathrm{d}_{\mathbb{1}} f\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp^G(tX)\Big) = \mathrm{d}_{\mathbb{1}} f(X)\,.$$

Thus this is the 1-parameter subgroup $t \mapsto \exp^H\big(t\, \mathrm{d}_{\mathbb{1}} f(X)\big)$, i. e., $f \circ \exp^G(tX) = \exp^H\big(t\, \mathrm{d}_{\mathbb{1}} f(X)\big)$ for all $t \in \mathbb{R}$ which implies the statement. ∎

As a corollary we get the Lie group analogon of equation (1.4):

Corollary 1.29. *For a Lie G and $\sigma \in G$ we get $C_\sigma \circ \exp = \exp \circ \mathrm{Ad}_\sigma$, i. e., the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\quad C_\sigma \quad} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\quad \mathrm{Ad}_\sigma \quad} & \mathfrak{g} \end{array}$$

commutes.

Proof: Apply Lemma 1.28 to $H = G$, $f = C_\sigma$ and thus $\mathrm{d}_{\mathbb{1}} f = \mathrm{Ad}_\sigma$. ∎

1.5 Proof of Lemma 1.18

We now provide the proof of Lemma 1.18 which is still missing.

Let us recall the following exercise from last semester:

Exercise 1.30 (Diff. geom. I, Exercise Sheet 6, Exercise 4 with changed notation)**.** *Let M be a smooth, not necessarily compact, manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on M, i. e., φ is smooth with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s,t \in \mathbb{R}$. Let ξ be the associated tangent vector field on M, defined as*

$$\xi|_x := \frac{d}{dt}\Big|_{t=0}(\varphi_t(x)),$$

see also Diff. geom. I, Exercise Sheet 5, Exercise 3. Show that, for any smooth tangent vector field Y on M and point $p \in M$ it is

$$\frac{d}{dt}\Big|_{t=0}((\varphi_t)_*\eta)\big|_p = -[\xi,\eta]\big|_p,$$

where, for any diffeomorphism $\psi : M \to M$, the term $\psi_*\eta$ denotes the pushforward tangent vector field of η defined by $\psi_*\eta := \mathrm{d}\psi \circ \eta \circ \psi^{-1}$.

Proof of Lemma 1.18: [4] Let $X, Y \in \mathfrak{g}$ with left-invariant extensions $\overline{X}$ and $\overline{Y}$. At first, we calculate for $t \in \mathbb{R}$:

$$\begin{aligned}
\mathrm{Ad}_{\exp(tX)}(Y) &= \mathrm{d}r_{\exp(-tX)} \circ \mathrm{d}\ell_{\exp(tX)}(\overline{Y})\big|_{\mathbb{1}} \\
&= \mathrm{d}r_{\exp(-tX)}(\overline{Y})\big|_{\exp(tX)} \\
&\overset{(*)}{=} \mathrm{d}\Phi^{\overline{X}}_{-t}(\overline{Y})\big|_{\Phi^{\overline{X}}_{-t}(\mathbb{1})} \\
&\overset{(+)}{=} \left(\Phi^{\overline{X}}_{-t}\right)_*(\overline{Y})\big|_{\mathbb{1}}
\end{aligned}$$

where we used at $(*)$ Proposition (1) (5), and where we used at $(+)$ the pushforward of vector fields from the preceding exercise. We derive this with respect to t at $t = 0$, and use the results of the exercise above at $(\dagger)$ for $\xi = -\overline{X}$, $\eta = \overline{Y}$ and $\varphi_t = \Phi^{\overline{X}}_{-t}$. This gives

$$\begin{aligned}
\mathrm{ad}_X Y &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\Phi^{\overline{X}}_{-t}\right)_*(\overline{Y})\big|_{\mathbb{1}}
\end{aligned}$$

[4] We roughly follow [7, 3.46].

$$\overset{(\dagger)}{=} [\overline{X}, \overline{Y}]\big|_{\mathbb{1}} = [X, Y]$$ ■

1.6 Commuting elements in Lie groups and Lie algebras

Definition 1.31. *Two elements $\sigma, \tau \in G$ in a Lie group* **commute**, *if $\sigma\tau = \tau\sigma$. Two elements $X, Y \in \mathfrak{g}$ in a Lie algebra* **commute**, *if $[x, y]$.*

We want to relate commutativity in a Lie group to commutativity in its Lie algebra.

We start by some considerations on arbitrary manifolds M and N.

Lemma 1.32. *Let $f\colon M \to N$ be a smooth map, and let $X \in \mathfrak{X}(M)$ be f-related to $Y \in \mathfrak{X}(N)$, i. e., $\mathrm{d}f \circ X = Y \circ f$. Then the flows Φ_t^X and Φ_t^Y of X and Y satisfy*

$$\Phi_t^Y \circ f = f \circ \Phi_t^X .$$

Proof: For $p \in M$ we will show that $t \mapsto \gamma(t) := f \circ \Phi_t^X(p) \in N$ is an integral curve of Y. As one easily checks $\gamma(0) = f(p)$, this proves the statement.

$$\begin{aligned}
\dot\gamma(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(f \circ \Phi_t^X(p)\Big) \\
&= \mathrm{d}f \circ \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^X(p)\Big) \\
&= \mathrm{d}f \circ \Big(X\big|_{\Phi_t^X(p)}\Big) \\
&= (\mathrm{d}f \circ X)\big|_{\Phi_t^X(p)}\Big) \\
&= (Y \circ f)\big|_{\Phi_t^X(p)}\Big) \\
&= Y\big|_{\gamma(t)} .
\end{aligned}$$ ■

Proposition 1.33. *Let X and Y be vector fields on M with flows $\Phi_\bullet^X$ and $\Phi_\bullet^Y$. Then*

$$[X, Y] = 0 \iff \forall s, t \in \mathbb{R}:\ \Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X .$$

Proof:

"⇐": We apply $\frac{d}{ds}\big|_{s=0}$ to

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$$

and using $\frac{d}{ds}\big|_{s=0}\Phi_s^Y = Y$ we obtain

$$\mathrm{d}\left(\Phi_t^X\right) \circ Y = Y \circ \Phi_t^X$$

which means $\left(\Phi_t^X\right)_* Y = Y$. We apply Exercise 1.30 for $\xi = X$ and thus $\varphi_t = \Phi_t^X$, so we obtain by deriving with respect to t at $t = 0$:

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} Y \\ &= \frac{d}{dt}\Big|_{t=0} \left(\Phi_t^X\right)_* Y \\ &= -[X, Y] \end{aligned}$$

"⇒": For $p \in M$ and for $s \in \mathbb{R}$ we define

$$v_p(t) := \left(\left(\Phi_t^X\right)_* Y\right)\Big|_p = \mathrm{d}\left(\Phi_t^X\right) \circ Y \circ \Phi_{-t}^X(p) \in T_pM\,.$$

We may differentiate this in the sense of Analysis II, and we write this differential as $v_p'(t)$. Exercise 1.30 tells us that $v_p'(0) = -[X, Y] = 0$ for all $p \in M$. We set $p = \Phi_{-s}^X(q)$ and we get

$$\begin{aligned} v_{\Phi_{-s}^X(q)}(t) &= \left(\mathrm{d}\left(\Phi_t^X\right) \circ Y \circ \Phi_{-t-s}^X\right)\Big|_q \\ &= \left(\mathrm{d}\left(\Phi_{-s}^X\right) \circ \left(\mathrm{d}\left(\Phi_{t+s}^X\right) \circ Y \circ \Phi_{-t-s}^X\right)\right)\Big|_q \\ &= \left(\mathrm{d}\left(\Phi_{-s}^X\right) \circ \left(\Phi_{t+s}^X\right)_* Y\right)\Big|_q \\ &= \mathrm{d}\left(\Phi_{-s}^X\right)\left(v_q(t+s)\right) \end{aligned}$$

Deriving this with respect to t at $t = 0$ yields

$$0 = v'_{\Phi_{-s}^X(q)}(0) = \mathrm{d}\left(\Phi_{-s}^X\right)\left(v_q'(s)\right)$$

and as $\mathrm{d}\left(\Phi_{-s}^X\right)$ is an isomorphism, this gives $v_q'(s) = 0$ for all $s \in \mathbb{R}$ and all $q \in M$. Thus we have $v_q(t) = v_q(0) = Y$ for all $q \in M$ and $t \in \mathbb{R}$. We have thus proven

$$\mathrm{d}\left(\Phi_t^X\right) \circ Y = Y \circ \Phi_t^X$$

which means that Y is Φ_t^X-related to itself. Using Lemma 1.32 for $M = N$, X replaced by Y, $f = \Phi_t^X$ and t replaced by s we get

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X .$$

■

Corollary 1.34. *Let G be a Lie group, $X, Y \in \mathfrak{g} = \mathrm{Lie}(G)$.*

(1) *If $[X,Y] = 0$ then $\exp(X)\exp(Y) = \exp(Y)\exp(X) = \exp(X+Y)$.*

(2) *Conversely, if*

$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX) \text{ for all } t, s \in \mathbb{R},$$

then $[X,Y] = 0$.

The first part of the Corollary provides a Lie group analogon of equation (1.5).

Proof: We extend $X, Y \in \mathfrak{g}$ to left-invariant vector fields, also denoted by X and Y.[5]

At first, let us assume $[X,Y] = 0$. As we have $\exp(X) = \Phi_1^X(\mathbb{1})$, the statement $\exp(X)\exp(Y) = \exp(Y)\exp(X)$ follows from "⇒" in Proposition 1.33. We also have $[sX, tY] = st[X,Y] = 0$. Thus, we already know $\exp(sX)\exp(tY) = \exp(tY)\exp(sX)$ and this yields

$$\begin{aligned}\exp\big((t+s)X\big)\exp\big((t+s)Y\big) &= \exp(tX)\exp(sX)\exp(tY)\exp(sY)\\ &= \Big(\exp(tX)\exp(tY)\Big)\Big(\exp(sX)\exp(sY)\Big),\end{aligned}$$

thus $t \mapsto \gamma(t) := \exp(tX)\exp(tY)$ is 1-parameter subgroup of G, and $\gamma'(0) = \mathrm{d}r_{\exp(0)}X + \mathrm{d}\ell_{\exp(0)}Y = X + Y$. Thus implies $\exp\big(t(X+Y)\big) = \exp(tX)\exp(tY)$ which gives the remaining statement for $t = 1$.

The converse statement immediately follows from "⇐" in Proposition 1.33. ■

We have seen that $\exp\colon \mathfrak{g} \to G$ satisfies $\mathrm{d}_0 \exp = \mathrm{id}$. Thus, the local reversal theorem tells us that there is an open neighborhood U_0 of 0 and and open neighborhood V_0

[5] We assume it is clear from the context, when a vector is meant, and when we denote a vector field.

of $\mathbb{1}$ such that $\exp|_{U_0}: U_0 \to V_0$ is a diffeomorphism. Using continuity of multiplication and inversion, we see that there is an open neighborhood U_1 of 0 such that $U_1 \subset U_0$, such that U_1 is starshaped with respect to 0, satisfying $X \in U_1 \iff -X \in U_1$ and $X, Y \in U_1 \Rightarrow X + Y \in U_0$. We put $V_1 := \exp(U_1)$ and by shrinking U_1 and V_1 further we can achieve additionally $\mu(V_1 \times V_1) \subset V_0$ and we already have that inversion maps V_1 to itself.

Let $\gamma: [0, b] \to G$ be a continous path. For any $t \in [0, b]$ we define W_t as the connected component of $\{s \in [0, b] \mid \gamma(t)^{-1}\gamma(s) \in V_1\}$ that contains t. Then $(W_t)_{t\in[0,b]}$ is an open cover[6] of $[0, b]$. An elemantary compactness argument for $[0, b]$, treated under the name **Lebesgue number** ε, says: there is an $\varepsilon > 0$ if we have a partition

$$0 = t_0 \le t_1 \le \cdots \le t_k = b, \quad \forall i \in \{1, 2, \ldots, k\}: \ t_i - t_{i-1} < \varepsilon \tag{1.8}$$

then

$$\forall i \in \{1, 2, \ldots, k\}: \ t_i \in W_{t_{i-1}}$$

and thus

$$\forall i \in \{1, 2, \ldots, k\} : \forall s \in [t_{i-1}, t_i]: \ \gamma(s)^{-1}\gamma(t_{i-1}) \in V_1 \text{ and } \gamma(t_{i-1})^{-1}\gamma(s) \in V_1 \,. \tag{1.9}$$

Corollary 1.35. *Let G be a connected Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. Then the following are quivalent:*

(i) *G is abelian, i. e., $\sigma\tau = \tau\sigma$ for all $\sigma, \tau \in G$,*

(ii) *$\mathfrak{g}$ is abelian, i. e., $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$,*

(iii) *$\exp: (\mathfrak{g}, +) \to (G, \mu)$ is a group homomorphism.*

Proof:

"(i)⇒(ii)": This follows immediately from the second part of Corollary 1.34.

"(ii)⇒(iii)": This follows immediately from the first part of Corollary 1.34.

"(iii)⇒(i)": For $\sigma \in G$ we choose a continuous path $\gamma: [0, b] \to G$ from $\mathbb{1}$ to σ. We choose a subdivision as (1.8)/(1.9). Then $\sigma_i := \gamma(t_i)^{-1}\gamma(t_{i-1}) \in V_1$ satisfies $\sigma = \sigma_1\sigma_2\cdots\sigma_k$. We write $\sigma_i = \exp(X_i)$, $X_i \in U_1$. Similarly we decompose $\tau = \tau_1\tau_2\cdots\tau_\ell$, $\tau_j = \exp(Y_j)$, $Y_j \in U_1$. Condition (iii) implies that $\sigma_i\tau_j = \tau_j\sigma_i$ for all i, j and thus

[6] in German: "Überdeckung", **nicht** "Überlagerung", the two terms have different meanings, but are denoted with the same words "cover" and "covering" in English, but they are properly distinguished in German

$\sigma\tau = \tau\sigma$.

■

1.7 The Baker–Campbell–Hausdorff Formula

We have seen that $\exp\colon \mathfrak{g} \to G$ satisfies $\mathrm{d}_0 \exp = \mathrm{id}$, and in the discussion following Corollary 1.34 we have discussed the diffeomorphism $\exp|_{U_0}\colon U_0 \to V_0$, and also had the smaller open neighborhoods $U_1 \subset U_0$ of 0 and $V_1 \subset V_0$ of $\mathbb{1}$. In particular multiplication restricts to a map $V_1 \times V_1 \to V_0$ and inversion maps V_1 to itself. We write $\log\colon U_0 \to V_0$. In this language, it follows from Corollary 1.34 (1) for all $X, Y \in U_1$:

$$\text{if } [X,Y] = 0, \text{ then } \log\big(\exp(X)\exp(Y)\big) = X + Y\,.$$

On the other hand it is clear from (the proof of) Corollary 1.34 (2) that this formula no longer holds, if $\mathfrak{g}$ is not abelian. The Baker–Campbell–Hausdorff formula, says that this can be repaired by adding commutator terms.

Exercise 1.36. *We define the* **3-dimensional Heisenberg group** H_3 *as*

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

This is a submanifold and subgroup of $\mathrm{GL}(3,\mathbb{R})$*, thus a Lie group.*

(a) *Show that its Lie algebra* $\mathfrak{h}_3$*, the* **3-dimensional Heisenberg Lie algebra** *is given by matrices as follows:*

$$\mathfrak{h}_3 := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

(b) *Calculate* $\exp\colon \mathfrak{h}_3 \to H_3$*, and show that it is a diffeomorphism.*

(c) *Show that* $\log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A,B]$.

(d) *Show that* $[X,[Y,Z]] = 0$ *for alle* $X, Y, Z \in \mathfrak{h}_3$.

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