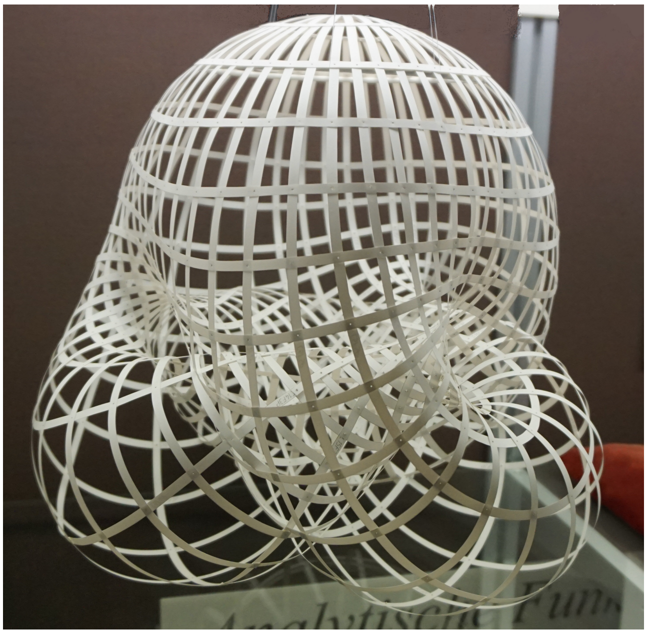
Differential Geometry II

Lecture Notes



Prof. Dr. Bernd Ammann

Summer term 2024

University of Regensburg

Public Version

Version of July 21, 2024

Contents

Preface

These are lecture notes for the lecture "Differential Geometry II" held in Regensburg in the summer term 2024. We assume that the readers of these notes and the audience of the lecture are already familiar with basic notions and results in differential and (semi-)Riemannian gemetry, as taught typically in a one-semester lecture, this includes e. g., the theorems by Hopf–Rinow, Bonnet–Myers and Cartan–Hadamard.

In some parts of the script I used previous fragments from a previous lecture in the summer term 2010 with similar content. Some of these fragments were typeset by Andreas Hermann in German language, thus I thank him for his reliable work and all the efforts he invested into this.

These lecture notes, in particular the 3rd Chapter, are also strongly influenced by previous Lecture Notes by Christian Bär, who was my PhD advisor and had an important influence on how I teach differential geometry. Mostly indirectly, the approach in Chapter 3 builds on important previous work of a strong group of German differential geometers, including Ballmann, Eschenburg, Heintze, and Karcher. Also the book of Sakai [26] was an important source of inspiration and a great reference, but also many one ideas are in there.

Bernd Ammann, Regensburg

https://ammann.app.uni-regensburg.de/lehre/2024s_diffgeo2/Differential_Geometry_II.pdf

I Lie groups and quotients

Tue. 16.4.

The goal of this section is to treat Lie groups, which are defined as manifolds with a compatible group structure. Important examples are $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{GL}(n,\mathbb{R})$, ...

Lie groups provide many more examples of Riemannian (and more generally semi-Riemannian) manifolds.

1 Lie groups and Lie algebras

Literature for this section: [20], [28], [4], [17], [15]

1.1 Lie groups and their homomorphisms

Definition 1.1. *A* **Lie group** *consists of a* $\mathcal{C}^\infty$*-manifold* G *together with a smooth map* $\mu\colon G\times G\to G$, $(\sigma,\tau)\mapsto\mu(\sigma,\tau)=\sigma\tau=\sigma\cdot\tau$, *called* **multiplication**, *such that*

(i) (G,μ) *is a group*

(ii) $G\times G\xrightarrow{\tilde{\mu}} G$, $(\sigma,\tau)\mapsto\sigma^{-1}\tau=:\tilde{\mu}(\sigma,\tau)$ *is smooth.*

As a consequence of (ii) we see that the following maps are smooth

$$\begin{aligned}
&\ell_\sigma\colon G\to G, \quad \tau\mapsto\sigma\tau \quad (\textbf{left multiplication} \text{ or } \textbf{left translation}\\
&r_\sigma\colon G\to G, \quad \tau\mapsto\tau\sigma \quad (\textbf{right multiplication} \text{ or } \textbf{right translation})\\
&\mathrm{inv}\colon G\to G, \quad \tau\mapsto\tau^{-1} \quad (\text{inversion})\\
&\mu\colon G\times G\xrightarrow{\mu} G, \quad (\sigma,\tau)\mapsto\sigma\tau \quad (\text{multiplication})
\end{aligned}$$

Note also that Diff. geom. I, Exercise Sheet 3, Exercise 4 tells us that one can replace (ii) by (ii') $\mu\colon G \times G \xrightarrow{\mu} G, \quad (\sigma,\tau) \mapsto \sigma\tau$ is smooth

We write $\mathbb{1}$ for the neutral element of G. Then $T_{\mathbb{1}}G$ is called the **Lie algebra** of G. It is a vector space that comes with some additional structure discussed below, a "Lie bracket".

Examples 1.2.

1.) A finite-dimensional real vector space is a Lie group, if μ is the addition.

2.) $\mathbb{C}^*$, $S^1 \subset \mathbb{C}^*$, $\mathbb{R}^*$ are Lie groups, if μ is the multiplication.

3.) $\mathrm{GL}(n,\mathbb{R})$ is a Lie group, where μ is matrix multiplication. We view $\mathrm{GL}(n,\mathbb{R})$ as an open subset and thus as an n^2-dimensional submanifold of $\mathbb{R}^{n\times n}$.

4.) $\mathrm{SL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A = 1\}$.
In order to show that $\mathrm{SL}(n,\mathbb{R})$ is a submanifold of $\mathrm{GL}(n,\mathbb{R})$ we show that the determinant $\det\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ is a submersion, i.e. $\mathrm{d}_A \det\colon T_A\,\mathrm{GL}(n,\mathbb{R}) \to T_{\det A}\mathbb{R}^* \cong \mathbb{R}$ is surjective for all $A \in \mathrm{GL}(n,\mathbb{R})$. It follows from this, that $\det^{-1}(t)$ is a submanifold for any $t \in \mathbb{R}^*$. For $t = 1$, this shows that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ is a submanifold.

(a) Let $B = \big(b_{ij}\big)_{ij} \in \mathrm{GL}(n,\mathbb{R})$, $C(t) := \mathbb{1} + tB = \big(c_{ij}(t)\big)_{ij} = \big(\delta_{ij} + tb_{ij}\big)_{ij}$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det\big(\mathbb{1} + tB\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det C(t) \\
&= \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\, \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(c_{1\sigma(1)}(t)\cdots c_{n\sigma(n)}(t)\big)}_{=0 \text{ for } \sigma\neq\mathrm{id}} \\
&\overset{(*)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Big(\big(1 + tb_{1\sigma(1)}\big)\cdots\big(1 + tb_{n\sigma(n)}\big)\Big) \\
&\overset{(+)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Big(1 + t(b_{1\sigma(1)} + \cdots + tb_{n\sigma(n)}) + P_{\geq 2}(t)\Big) \\
&= b_{1\sigma(1)} + \cdots + tb_{n\sigma(n)} \\
&= \mathrm{tr}\, B
\end{aligned}$$

Here we used at $(*)$ and above that for $\sigma \neq \mathrm{id}$ there are $i \neq j$ with $c_{i\sigma(i)}(0) = c_{j\sigma(j)}(0) = 0$, and after $(+)$ we write $P_{\geq 2}(t)$ for a polyomial in t without constant and without a linear term, i. e., one only with monomials of degree ≥ 2.

(b) For $A \in \mathrm{GL}(n,\mathbb{R})$ we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det\big(A+tB\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det\big(A\cdot(\mathbb{1}+tA^{-1}B)\big)\\ &= \big(\det A\big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det(\mathbb{1}+tA^{-1}B)\\ &= \big(\det A\big)\cdot \mathrm{tr}(A^{-1}B)\end{aligned}$$

We conclude

$$\begin{aligned}\mathrm{d}_A\det(B) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det\big(A+tB\big)\\ &= \big(\det A\big)\cdot\mathrm{tr}(A^{-1}B).\end{aligned}$$

The linear map $\mathrm{d}_A\colon \mathbb{R}^{n\times n}\to\mathbb{R}$ is surjective as

$$\mathrm{d}_A(A) = \big(\det A\big)\,\mathrm{tr}\,\mathbb{1} = n\cdot\det A \neq 0\,.$$

Now, we now that $\mathrm{SL}(n,\mathbb{R})$ is a submanifold. Its multiplication is the restriction of the multiplication in $\mathrm{GL}(n,\mathbb{R})$, thus mutiplication is smooth as a map $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}\colon \mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{GL}(n,\mathbb{R})$. The image of $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}$ is a subset of the submanifold $\mathrm{SL}(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R})$, and this implies the smoothness of $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}\colon\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{SL}(n,\mathbb{R})$.

Further we have

$$\mathbb{T}_{\mathbb{1}}\,\mathrm{SL}(m.\mathbb{R}) = \{A\in\mathbb{R}^{n\times n}\mid \mathrm{tr}\,A = 0\}.$$

5.) The groups $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are Lie groups, see Exercise Sheet 1, Exercise 2

6.) If G and H are Lie groups, then $G\times H$ with the product manifold structure and the product group structure

$$\begin{aligned}(G\times H)\times(G\times H)&\to G\times H\\ \big((\sigma,\tau),(\tilde\sigma,\tilde\tau)\big)&\mapsto(\sigma\tilde\sigma,\tau\tilde\tau)\end{aligned}$$

is again a Lie group.

7.) Let Γ be a discrte subgroup of $\mathbb{R}^n$, e. g., $\Gamma = \mathbb{Z}^n$ or another lattice[1] or another discrete subgroup. If we equip $\mathbb{R}^n/\Gamma$ with the usual addition of equivalence classes, called μ, then $(\mathbb{R}^n/\Gamma, \mu)$ is a Lie group.

Definition 1.3. *A* **homomorphism of Lie groups** *or a* **Lie group homomorphism** *is a smooth map $f\colon G \to H$, for G and H Lie grous, that is also a group homomorphism. The map f is a* **Lie group isomorphism** *if it is additionally a diffeomorphism, it is a* **Lie group endomorphism** *if additionally $G = H$, and it is a* **Lie group automorphism** *if $G = H$ and if f is a diffeomorphism. We write* $\mathrm{Hom}(G,H)$, $\mathrm{Iso}(G,H)$, $\mathrm{End}(G)$, $\mathrm{Aut}(G)$ *for the sets/monoid/groups of such homorphisms.*

Examples 1.4.

1.) The inclusions $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$, $\mathrm{U}(n) \hookrightarrow \mathrm{O}(2n)$, etc. are Lie group homomorphisms

2.) $\det_{\mathbb{K}} \mathrm{GL}(n,\mathbb{K}) \to \mathbb{K}_{\neq 0}$ is a Lie group homomorphism for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

3.) For any $\sigma \in G$, **conjugation** by σ

$$\begin{aligned} C_\sigma\colon G &\longrightarrow G \\ \tau &\longmapsto \sigma\tau\sigma^{-1} \end{aligned}$$

is a Lie group automorphism, and $C_\bullet\colon G \to \mathrm{Aut}(G)$, $g \mapsto C_g$ is a group homomorphism. We obviously have

$$C_\sigma = \ell_\sigma \circ r_{\sigma^{-1}} = r_{\sigma^{-1}} \circ \ell_\sigma\,. \tag{1.1}$$

Remarks 1.5.

1.) If G is a Lie group, one might be tempted to define a Lie subgroup as a subgroup H of G such that H is a submanifold as well. However, this is not what one usually does. One says that $H \subset G$ is a **Lie subgroup**, if there is a Lie group homomorphism $f : H' \to G$, that is injective and an immersion, such that $H = \mathrm{image}(f)$. For example consider $G = \mathbb{R}^2/\mathbb{Z}^2$ and $f(t) = [t, \alpha t]$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $f : \mathbb{R} \to G$ is an injective immersion and a Lie group

[1] A lattice in $\mathbb{R}^n$ is by definition a discrete subgroup Γ of $\mathbb{R}^n$, isomorphic to $\mathbb{Z}^n$. It follows that $\mathbb{R}^n/\Gamma$ is a compact manifold (without boundary), and that there is an $A \in \mathrm{GL}(n,\mathbb{R})$ with $\Gamma = A \cdot \mathbb{Z}^n$.

homomorphism, but $H := \mathrm{image}(f)$ is not a submanifold in the usual sense: a submanifold is always a locally closed subset, but H is not a locally closed subset of G. This leads in books on Lie group, as e. g., in [28, Definition 1.27 (b)] to a slightly generalized definition of a submanifold, however we do not want to elaborate too much on this.

2.) The closed subgroup theorem, see [28, Theorem 3.42], states: Let G be a Lie group, and let H be a subgroup of G (in the sense of group theory) that is closed as a subset, then H is a submanifold of G. It follow any closed subgroup H of G is a Lie group (with induced differentiable structure and induced group structure). Although this result is rather simple to state, the proof is a bit involved. Thus we will not prove it here.

1.2 Lie algebras and their homomorphisms

Let us recall the following exercise from last semester:

Exercise 1.6 (Diff. geom. I, Exercise Sheet 7, Exercise 2)**.** *Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. $\tilde{X}, \tilde{Y}$) be (smooth) vector fields on M (resp. N). We say that X is* ***F-related*** *to $\tilde{X}$ if $dF \circ X = \tilde{X} \circ F$ holds on M.*
Show that, if X is F-related to $\tilde{X}$ and Y is F-related to $\tilde{Y}$, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

Definition 1.7. *A vector field $X \in \mathfrak{X}(G)$ is called* **left-invariant** *if for all $\sigma \in G$ we have $\mathrm{d}\ell_\sigma(X) = X \circ \ell_\sigma$, i. e., if the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ \ell_\sigma\ } & G \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle X} \\ TG & \xrightarrow{\ \mathrm{d}\ell_\sigma\ } & TG \end{array}$$

commutes. Similarly X is called **right-invariant** *if for all $\sigma \in G$ we have $\mathrm{d}r_\sigma(X) = X \circ r_\sigma$. If X is left- and right-invariant, we say X is* **bi-invariant***.*

Using the language of Exercise 1.6, we see that a vector field $X \in \mathfrak{X}(G)$ is left-invariant (right-invariant, resp.), if, and only if, it is ℓ_σ-related (r_σ-related, resp.) to itself for any $\sigma \in G$.

Remarks 1.8.

1.) For any $X_0 \in T_{\mathbb{1}}G$ there is a unique left-invariant vector field $X \in \mathfrak{X}(G)$ with $X|_{\mathbb{1}} = X_0$. The uniqueness follows from the calculation

$$X|_\sigma = X \circ \ell_\sigma(\mathbb{1}) = (\mathrm{d}\ell_\sigma \circ X)(\mathbb{1}) = \mathrm{d}\ell_\sigma\big(X|_{\mathbb{1}}\big) = \mathrm{d}\ell_\sigma\big(X_0\big). \tag{1.2}$$

On the other hand if we use (1.2) to define X, i. e., if we set $X|_\sigma := \mathrm{d}\ell_\sigma\big(X_0\big)$, then this vector field is the composition

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{id},X_0)} & G \times TG & \longrightarrow & TG \\ \sigma & \longmapsto & (\sigma, X_0) & \longmapsto & \mathrm{d}\ell_\sigma(X_0) \end{array}$$

which is obviously smooth in σ. In order to show that the vector field X thus obtained is left-invariant we calculate for any fixed $\tau \in G$

$$X \circ \ell_\tau(\sigma) = X|_{\tau\sigma} \overset{(\mathrm{def})}{=} \mathrm{d}\ell_{\tau\sigma}(X_0) \overset{(*)}{=} \mathrm{d}\ell_\tau\big(d\ell_\sigma(X_0)\big) \overset{(\mathrm{def})}{=} \mathrm{d}\ell_\tau\big(X|_\sigma\big)$$

where we used the chain rule $\mathrm{d}(f \circ g) = (\mathrm{d}f) \circ (\mathrm{d}g)$ at $(*)$, and thus we have $X \circ \ell_\tau = \mathrm{d}\ell_\tau \circ X$ for all $\tau \in G$.

2.) The analogous statement holds as well if we replace left-invariance by right-invariance.

3.) With Exercise 1.6 we see: if $X, Y \in \mathfrak{X}(G)$ are left-invariant (right-invariant, resp.) vector fields, then $[X, Y]$ is also left-invariant (right-invariant, resp.)

Definition 1.9 (Lie bracket on the Lie algebra)**.** *Let G be a Lie group with Lie algebra $T_{\mathbb{1}}G$. The vectors $X_0, Y_0 \in T_{\mathbb{1}}G$ are extended to left-invariant vector fields X and Y. We define*

$$[X_0, Y_0] := [X, Y]|_{\mathbb{1}}.$$

This defines a bilinear map $[\,\bullet\,,\bullet\,]: T_{\mathbb{1}}G \times T_{\mathbb{1}}G \to T_{\mathbb{1}}G$, called the **Lie bracket** *on the Lie algebra $T_{\mathbb{1}}G$ of G.*

Fr 19.4.

The pair $(T_{\mathbb{1}}G, [\,\bullet\,,\bullet\,])$ satisfies the defining properties of a Lie algebra over $\mathbb{R}$, which are defined as follows:

Definition 1.10 (Abstract Lie algebra). *Let K be a field and $\mathfrak{g}$ a K vector space. A bilinear map $[\bullet,\bullet]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is called a* **Lie bracket** *on $\mathfrak{g}$ if it satisfied*

(i) **Alternation**: *for all $x\in\mathfrak{g}$ we have $[x,x]=0$*

(ii) **Jacobi identity**: *for all $x,y,z\in\mathfrak{g}$ we have*

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0\,.$$

The pair $(\mathfrak{g},[\bullet,\bullet])$ is then called a **Lie algebra** *(over K).*

If the characteristic of K is not 2 – and the field $K=\mathbb{R}$ we are interested in the case that K is of characteristic 0 –, then condition (i) is equivalent to

(i') **Antisymmetry**: for all $x,y\in\mathfrak{g}$ we have $[x,y]=-[y,x]$.

(In characteristic 2 (i') still implies (i), but the converse is no longer true.)

A **Lie subalgebra** of $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ that is closed under the Lie-bracket, i. e., then it is itself a Lie algebra.

It is obvious that the Lie bracket on $T_{\mathbb{1}}G$ defined in Definition 1.9 satisfies (i') (or equivalently (i)). The Jacobi identity follows immediately in this situation from Exercise 1.6.

Usually for a Lie group the associated Lie algebra, viewed as a vector space with Lie bracket, is denoted by the the associated small fraktur (= gothic) letters, e. g.,

Lie group	G	H	$\mathrm{GL}(n,\mathbb{R})$	$\mathrm{O}(n)$	$\mathrm{SO}(n)$	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{U}(n)$
Lie algebra	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{o}(n)$	$\mathfrak{so}(n)$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{u}(n)$

We also will often write $\mathrm{Lie}(G)$ for the Lie algebra of G, e. g., $\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{h}=\mathrm{Lie}(H)$, etc.

Examples 1.11.

1.) If we consider $G:=\mathbb{R}^n$ as a Lie group with $\mu(x,y)=x+y$, then the left-invariant vector fields are the constant ones. As the Lie bracket of constant vector fields vanishes, the Lie bracket on the Lie algebra is the zero map $0:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$. Thus the Lie algebra is $(\mathbb{R}^n,0)$.

2.) Let V be a finite-dimensional real vector space. We denote the vector space automorphisms of V by $\mathrm{GL}(V)$. By choosing a basis of V, and identify $V\cong\mathbb{R}^n$,

$\mathrm{GL}(V) \cong \mathrm{GL}(n,\mathbb{R})$ we get a Lie group structure on $\mathrm{GL}(V)$, independent of the choice of basis above. Let us write $\mathrm{End}_{\mathrm{lin}}(V)$ for the vector space endomorphisms of V. We have $\mathrm{GL}(V) = \det^{-1}(\mathbb{R} \setminus \{0\})$ for $\det\colon \mathrm{End}_{\mathrm{lin}}(V) \to \mathbb{R}$, thus $\mathrm{GL}(V)$ is open in $\mathrm{End}_{\mathrm{lin}}(V)$. We obtain $\mathfrak{gl}(V) := T_{\mathrm{id}}\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$.

The left-invariant extension of $X_0 \in T_{\mathrm{id}}\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$ is $X|_A := A \mapsto A \circ X_0 \in T_A\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$, $X \in \mathfrak{X}\big(\mathrm{GL}(V)\big)$. We proceed similarly for $Y_0 \in T_{\mathrm{id}}\,\mathrm{GL}(V)$ and $Y \in \mathfrak{X}\big(\mathrm{GL}(V)\big)$. Then

$$\begin{aligned}
\partial_X Y|_A &= A \circ \partial_{X_0}|_A\big(B \mapsto B \circ Y_0\big) = A \circ X_0 \circ Y_0\\
\partial_Y X|_A &= A \circ \partial_{Y_0}|_A\big(B \mapsto B \circ X_0\big) = A \circ Y_0 \circ X_0\\
[X,Y]|_A &= \partial_X Y|_A - \partial_Y X|_A = A \circ (X_0 \circ Y_0 - Y_0 \circ X_0)\\
[X_0,Y_0] &= [X,Y]|_{\mathrm{id}} = X_0 \circ Y_0 - Y_0 \circ X_0\,.
\end{aligned}$$

Thus the Lie algebra structure on $T_{\mathrm{id}}\,\mathrm{GL}(V) \cong \mathrm{End}_{\mathrm{lin}}(V)$ is given by $(X_0,Y_0) \mapsto_0 \circ Y_0 - Y_0 \circ X_0$, i. e., $[\,\bullet\,,\bullet\,]$ is the usual commutator in $\mathrm{End}_{\mathrm{lin}}(V)$, usually denoted by $[\,\bullet\,,\bullet\,]$ as well.

Definition 1.12 (Lie algebra homomorphism)**.** *Let* $(\mathfrak{g},[\,\bullet\,,\bullet\,]_{\mathfrak{g}})$ *and* $(\mathfrak{h},[\,\bullet\,,\bullet\,]_{\mathfrak{h}})$ *be Lie algebras. A* **homomorphism of Lie algebras** *or a* **Lie algebra homomorphism** *is a linear map* $f\colon \mathfrak{g} \to \mathfrak{h}$ *such that for all* $x,y \in \mathfrak{g}$*:*

$$f\big([x,y]_{\mathfrak{g}}\big) = \big[f(x),f(y)\big]_{\mathfrak{h}}\,.$$

Writing $\mathfrak{g}$ *for* $(\mathfrak{g},[\,\bullet\,,\bullet\,]_{\mathfrak{g}})$ *and* $\mathfrak{h}$ *for* $(\mathfrak{h},[\,\bullet\,,\bullet\,]_{\mathfrak{h}})$*, we denote by* $\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ *the set of all Lie algebra homomorphisms. And similarly to Definition 1.3 we define isomorphisms, endomorphisms, automorphisms and* $\mathrm{Iso}(\mathfrak{g},\mathfrak{h})$*,* $\mathrm{End}(\mathfrak{g})$ *and* $\mathrm{Aut}(\mathfrak{g})$*.*

Proposition 1.13. *Let* G *and* H *be Lie groups and let* $f\colon G \to H$ *be a Lie group homomorphism. Then*

$$\mathrm{d}_{\mathbb{1}} f\colon \mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism.

Proof: Assume $X_0, Y_0 \in \mathfrak{g}$. We extend X_0 (resp. Y_0) to a left-invariant vector field $X \in \mathfrak{X}(G)$ (resp. $Y \in \mathfrak{X}(G)$), i. e., $X|_\sigma = \mathrm{d}_{\mathbb{1}}\ell_\sigma(X_0)$ for all $\sigma \in G$. Also extend $\widehat{X}_0 := \mathrm{d}_{\mathbb{1}} f(X_0) \in \mathfrak{h}$ to a left-invariant vector field $\widehat{X} \in \mathfrak{X}(H)$, and define similarly $\widehat{Y}_0$ and $\widehat{Y}$. Thus $\widehat{X}|_\sigma = \mathrm{d}_{\mathbb{1}}\ell_\sigma(\widehat{X}_0)$ for all $\sigma \in H$.

For $\sigma, \tau \in G$ we have $(f \circ \ell_\sigma)(\tau) = f(\sigma\tau) = f(\sigma)f(\tau) = \ell_{f(\sigma)}\big(f(\tau)\big)$, thus $f \circ \ell_\sigma = \ell_{f(\sigma)} \circ f$. We calculate for $\sigma \in G$.

$$\begin{aligned}(\mathrm{d}_\sigma f)\Big(X\big|_\sigma\Big) &= \big(\mathrm{d}_\sigma f \circ \mathrm{d}_{\mathbb{1}}\ell_\sigma\big)(X_0) = \mathrm{d}_{\mathbb{1}}\big(f \circ \ell_\sigma\big)(X_0) \\ &= \mathrm{d}_{\mathbb{1}}(\ell_{f(\sigma)} \circ f\big)(X_0) = \mathrm{d}_{\mathbb{1}}\ell_{f(\sigma)} \circ \mathrm{d}_{\mathbb{1}} f(X_0) \\ &= \mathrm{d}_{\mathbb{1}}\ell_{f(\sigma)}\widehat{X}_0 = \widehat{X}\big|_{f(\sigma)}.\end{aligned}$$

As a result $\mathrm{d}f \circ X = \widehat{X} \circ f$. And similarly we get $\mathrm{d}f \circ Y = \widehat{Y} \circ f$. Thus we have just shown that

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle X,Y}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{X},\widehat{Y}} \\ TG & \xrightarrow{\quad \mathrm{d}f \quad} & TH \end{array}$$

commutes. This means that X resp. Y is f-related to $\widehat{X}$ resp. $\widehat{Y}$ – in the language of Exercise 1.6. It follows from this exercise that $[X, Y]$ is also f-related to $[\widehat{X}, \widehat{Y}]$. Thus

$$\begin{aligned}\mathrm{d}_{\mathbb{1}} f([X_0, Y_0]) &= \big(\mathrm{d}f \circ [X, Y]\big)\big|_{\mathbb{1}} \\ &= \big([\widehat{X}, \widehat{Y}] \circ f\big)\big|_{\mathbb{1}} = [\widehat{X}, \widehat{Y}]\big|_{\mathbb{1}} \\ &= \big[\widehat{X}_0, \widehat{Y}_0\big] = \big[\mathrm{d}f_{\mathbb{1}}(X_0), \mathrm{d}f_{\mathbb{1}}(Y_0)\big],\end{aligned}$$

which is the statement of the proposition. ∎

Corollary of Proposition 1.13. *Assume that V is a finite-dimensional real vector space. Let G be a subgroup and submanifold of $\mathrm{GL}(V)$. Let $\mathfrak{g}$ be the Lie algebra of G. Then the Lie-bracket on $\mathfrak{g}$ is the commutator bracket on $\mathrm{End}(V)$.*

Proof: We have seen in Example 1.11 2.) that the Lie bracket on $\mathfrak{gl}(V)$ is the commutator bracket of $\mathrm{End}_{\mathrm{lin}}(V)$. The Lie group homomorphism $i: G \to \mathrm{GL}(V)$ induces an injective Lie algebra homomorphism $\mathrm{d}i: \mathfrak{g} \to \mathfrak{gl}(V)$, thus $\mathfrak{g}$ the Lie brackt of $\mathfrak{gl}(V)$ restricts to the one on $\mathfrak{g}$. ∎

1.3 Adjoint representations

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_{\mathbb{1}}G$. For a given $\sigma \in G$ we differentiate $C_\sigma: G \to G$ at $\mathbb{1}$ and we obtain $\mathrm{Ad}_\sigma := \mathrm{d}_{\mathbb{1}} C_\sigma: \mathfrak{g} \to \mathfrak{g}$, which is obviously a linear map. For $\sigma, \tau \in G$ differentiating $C_{\sigma\tau} = C_\sigma \circ C_\tau$ implies $\mathrm{Ad}_{\sigma\tau} = \mathrm{Ad}_\sigma \circ \mathrm{Ad}_\tau$.

Lemma 1.14. *For $\sigma \in G$ the map $\mathrm{Ad}_\sigma: \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism.*

Proof: Apply Proposition 1.13 to the Lie group homomorphism $C_\sigma: G \to G$. ∎

Definition 1.15 (The adjoint representation of a Lie group)**.** *The group homomorphism obtained this way*

$$\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$$

is called the **adjoint representation of the Lie group** *G.*

Remarks 1.16.

1.) One can show that $\mathrm{Aut}(\mathfrak{g})$ is itself a Lie-group, in fact a Lie subgoup of the group $\mathrm{GL}(\mathfrak{g})$ of vector space automorphisms.

2.) The Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$. A linear map $D: \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, is called a **derivation** of $\mathfrak{g}$, if for all $x, y \in \mathfrak{g}$ we have

$$D([x,y]) = [D(x), y] + [x, D(y)].$$

Thus we have $\mathfrak{aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

We will not prove these statements here, as they will not be used in what follows and they are easier to prove later.

Definition 1.17 (The adjoint representation of a Lie algebra)**.** *The differential at $\mathbb{1}$ of $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$, namely*

$$\mathrm{ad} := \mathrm{d}_{\mathbb{1}}\,\mathrm{Ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}_X = \mathrm{d}_{\mathbb{1}}(\sigma \mapsto \mathrm{Ad}_\sigma)(X)$$

is called the **adjoint representation of the Lie algebra** *$\mathfrak{g}$.*

Accoding to Remarks 1.16 the adjoint representation of a Lie algebra is in fact a

Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})\,.$$

Lemma 1.18. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group. Then the adjoint map* ad *satisfies.* $\mathrm{ad}_X(Y) = [X,Y]$

The proof will be given later. Tue. 23.4.

1.4 The exponential map

In the following $t \in \mathbb{R}$, so $\partial_t := \frac{\mathrm{d}}{\mathrm{d}t}$ is the positively oriented vector field on $\mathbb{R}$ of constant length 1. For a smooth map $f\colon \mathbb{R} \to M$ we also write $\dot f(t) = \mathrm{d}f(\partial_r|_t)$. We write $\mathrm{Diff}(M)$ for the group of diffeomorphisms of M.

Definition 1.19. *Let M be a manifold and $X \in \mathfrak{X}(M)$. A curve $\gamma\colon I \to M$ is called* **integral curve** *of X or* **flow line** *of X, if for all $t \in I$ we have*

$$\dot\gamma(t) = X|_{\gamma(t)}\,.$$

The theorem of Picard-Lindelöf implies: For any $p \in M$ there is an integral curve γ_p of X with $\gamma_p(0) = p$ and we assume that γ_p is defined on its maximal domain I_p, and this maximal solution is unique. We say that X is **complete** *if $I_p = \mathbb{R}$ for all $p \in M$. We also define $\Phi^X_t(p) := \gamma_p(t)$. Thus if X is complete, then we have a group homomorphism $\Phi^X_\bullet\colon \mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \Phi^X_t$, called the* **flow** *of X.*

We encourage the reader to check that $t \mapsto \Phi^X_t$ is indeed a group homomorphism.

Lemma 1.20. *For a left-invariant vector field X on a Lie group we have:*

(1) *X is complete,*

(2) *If γ is an integral curve of X, and $\sigma \in G$, then $\ell_\sigma \circ \gamma$ is an integral curve of X as well,*

(3) *$\Phi^X_t(\sigma\tau) = \sigma\Phi^X_t(\tau)$ for $t \in \mathbb{R}$, $\sigma, \tau \in G$.*

(4) *$\Phi^{\lambda X}_t = \Phi^X_{\lambda t}$ for all $\lambda, t \in \mathbb{R}$.*

In the proof we use the conventions $\infty + t := \infty$ and $-\infty + t = -\infty$ for all $t \in \mathbb{R}$.

Proof: Let G be a Lie group and let $X \in \mathfrak{X}(G)$ be a left-invariant vector field. Consider the integral curve $\gamma_{\mathbb{1}}: I_{\mathbb{1}} \to G$, with $\gamma_{\mathbb{1}}(0) = \mathbb{1}$, $I_{\mathbb{1}} = (\alpha, \omega)$. For any $\sigma \in G$ we calculate that the curve $\ell_\sigma \circ \gamma_{\mathbb{1}}$ is also an integral curve of X:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ell_\sigma \circ \gamma_{\mathbb{1}}(t)) = \mathrm{d}\ell_\sigma(\dot{\gamma}_{\mathbb{1}}(t)) = \mathrm{d}\ell_\sigma\Big(X\big|_{\gamma_{\mathbb{1}}(t)}\Big) = X\big|_{\ell_\sigma \circ \gamma_{\mathbb{1}}(t)}.$$

Thus $\gamma_\sigma := \ell_\sigma \circ \gamma_{\mathbb{1}}: (\alpha, \omega) \to G$ is the intergral curve with $\gamma_\sigma(0) = \sigma$. This already shows (2).

Now for $t_0 \in (\alpha, \omega)$ we have

$$\gamma_{\mathbb{1}}(t_0) = \gamma_{\gamma(t_0)}(t_0 - t_0),$$

thus $\gamma_{\mathbb{1}}$ and $\gamma_{\gamma_{\mathbb{1}}(t_0)}(\bullet - t_0)$ coincide, including their maximal domains. Hence $(\alpha, \omega) = (\alpha + t_0, \omega + t_0)$, hence $\alpha = -\infty$ and $\omega = \infty$. This proves the completeness, i. e., (1).

The statement (3) follows from the facts that both $t \mapsto \Phi_t^X(\sigma\tau)$ and $t \mapsto \sigma\Phi_t^X(\tau)$ are integral lines for X and that they coincide for $t = 0$.

In the notation above, and for any $\sigma \in G$ we have $\Phi_{\lambda t}^X(\sigma) = \gamma_\sigma(\lambda t)$. We calculate with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_\sigma(\lambda t) = \lambda\dot{\gamma}_\sigma(\lambda t) = \lambda\Big(X\big|_{\gamma_\sigma(\lambda t)}\Big) = (\lambda X)\big|_{\gamma_\sigma(\lambda t)}$$

Thus $\mapsto \Phi_{\lambda t}^X(\sigma)$ is the integral curve of λX that attains σ for $t = 0$. Thus, by definition of $\Phi_t^{\lambda X}$, we have (4). ∎

Definition 1.21. *A homomorphism $f: \mathbb{R} \to G$ is called a* **1-parameter subgroup** *of G.*

Remark. Note that in general $f(\mathbb{R})$ is in general not a submanifold (in the usual sense[2]) of G, but it is a submanifold in the generalized sense of [28], see Remark 1.5 1.). This explains the usage of the word "subgroup".

Proposition 1.22. *Let G be a Lie group. Then the 1-parameter subgroups are the integral curves of some left-invariant vector field through $\mathbb{1}$. More precisely:*

(1) *Let $f: \mathbb{R} \to G$ be a 1-parameter subgroup, and take the left-invariant vector field $X \in \mathfrak{X}(G)$ such that $\dot{f}(0) = X|_{\mathbb{1}}$. Then f is the integral curve of X with $f(0) = \mathbb{1}$.*

[2] i. e., in the sense of Analysis IV, Differential Geometry I, etc

(2) *Let X be a left-invariant vector field and $f:\mathbb{R}\to G$ an integral curve of X with $f(0)=\mathbb{1}$. Then f is a 1-parameter subgroup.*

It follows, that two 1-parameter subgroups $f_1, f_2:\mathbb{R}\to G$ coincide if and only if $\dot f_1(0)=\dot f_2(0)$.

Proof:

"(1)": Obviously $f(0)=\mathbb{1}$. As f is a homomorphism $f(t+\bullet)=\ell_{f(t)}\circ f$. Thus

$$\dot f(t)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(t+\bullet)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \ell_{f(t)}\circ f=\mathrm{d}\ell_{f(t)}(\dot f(0))=\mathrm{d}\ell_{f(t)}(X|_{\mathbb{1}})=X|_{f(t)}.$$

Thus f is an integral curve of X.

"(2)": Obviously f is smooth. It is defined on $\mathbb{R}$ due to Lemma 1.20 (1). By definition of the flow we have $f(t)=\Phi_t^X(\mathbb{1})$, and thus we calculate, using Lemma 1.20 (1) at $(*)$

$$f(t+s)=f(s+t)=\Phi_{s+t}^X(\mathbb{1})=\Phi_s^X(\Phi_t^X(\mathbb{1}))=\Phi_s^X(f(t))\overset{(*)}{=}f(t)\Phi_s^X(\mathbb{1})=f(t)f(s).$$

Thus f is a Lie group homomorphism. ■

Definition 1.23. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. We write X for the left-invariant vector field extending $X_0\in\mathfrak{g}$. The* **exponential map** exp *is defined as the map*

$$\exp:\mathfrak{g}\to G,\quad X_0\mapsto\Phi_1^X(\mathbb{1}).$$

WARNING 1.24. *This exponential map is in general not the same as the Riemannian exponential map, even if we know that the metric is left- or right-invariant.[3]. As a consequence this map is also called the* **Lie group exponential map** *in order to distinguish it from the* **(semi-)Riemannian exponential map**. *It does however – as will be shown in the exercises – coincide with the Riemannian one for bi-invariant metrics on Lie groups.*

Theorem 1.25 (Properties of the exponential map)**.** *Let G be a Lie group with Lie algebra fg, and $X\in\mathfrak{g}$, $t,s\in\mathbb{R}$. Then we have*

[3] The notions of left-, right-, and bi-invariant Riemannian metrics are defined in the exercises.

(1) $\exp$ *is smooth and with our usual identification* $T_0\mathfrak{g} \cong \mathfrak{g}$*, its differential* $\mathrm{d}_0 \exp$ *is the identity of* $\mathfrak{g}$*. As a consequence there is an open neighborhood* U *of* 0*, such that* $\exp\big|_U : U \to \exp(U)$ *is a parametrization.*

(2) $\exp(tX) = \Phi_t^X(\mathbb{1})$

(3) $t \mapsto \exp(tX) =: f_X(t)$*,* $\mathbb{R} \to G$ *is a* 1*-parameter subgroup of* G *and any* 1*-parameter subgroup is of that form for some* $X \in \mathfrak{g}$*. Furthermore* $\dot{f}_X(0) = X$*.*

(4) *The integral curves of the left-invariant vector field associated to* X *are, the curves* $t \mapsto \sigma \exp(tX)$ *for* $\sigma \in G$

(5) *If* $\overline{X}$ *is the left-invariant vector that extends* $X \in \mathfrak{g}$*, then for all* $t \in \mathbb{R}$ *we have*

$$\Phi_t^{\overline{X}} = r_{\exp(tX)} .$$

Proof:

"(1)": The smoothness of exp follows from the smooth dependence on the initial conditions in the theorem of Picard–Lindelöf. We calculate for the left-invariant vector fields $\overline{X} \in \mathfrak{X}(G)$ extending $X \in \mathfrak{g}$.

$$\big(\mathrm{d}_{\mathbb{1}} \exp\big)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\exp(tX)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi_t^{\overline{X}}(\mathbb{1}) = \overline{X}\big|_{\mathbb{1}} = X .$$

Thus $\mathrm{d}_{\mathbb{1}} \exp = \mathrm{id}_{\mathfrak{g}}$.

"(2)": It follows from Lemma 1.20 (4) that $\exp(tX) = \Phi_1^{tX}(\mathbb{1}) = \Phi_t^X(\mathbb{1})$.

"(3)": This immediately follows from Proposition 1.22.

"(4)": The integral curves in this item are $t \mapsto \Phi_t^X(\sigma)$ and we have seen in Lemma 1.20 (4) that $\Phi_t^X(\sigma) = \ell_\sigma\big(\Phi_t^X(\mathbb{1})\big) = \ell_\sigma \circ \exp(tX)$.

(5) immediately follows from (4) and the definition of $\Phi_t^{\overline{X}}$. ∎

Example 1.26 (Exponential map of matrix groups)**.** We consider again the Lie group $\mathrm{GL}(V)$ for a finite-dimensional real vector space V. We have already seen in Example 1.11 2.) that the left-invariant extension of $X_0 \in \mathfrak{g}$ is $X \in \mathfrak{X}(\mathrm{GL}(V))$ with $X\big|_A = A \cdot X_0$, $A \in \mathrm{GL}(V)$.

For $A \in \mathfrak{gl}(V)$ we know from the theory of ordinary differential equations, that the series

$$\mathrm{EXP}(A) := \sum_{i=0}^{\infty} \frac{1}{i!} A^i \tag{1.3}$$

converges (uniformly on compact sets and also all derivatives converge uniformly on compact sets). We obtain a map $\mathrm{EXP}\colon \mathfrak{gl}(V) \to \mathrm{GL}(V)$ such that for $t, s \in \mathbb{R}$, $A \in \mathfrak{gl}(V)$

$$\mathrm{EXP}\big((t+s)A\big) = \mathrm{EXP}(tA)\,\mathrm{EXP}(sA), \quad \mathrm{EXP}(0) = \mathbb{1}, \quad \mathrm{EXP}(-A) = \mathrm{EXP}(A)^{-1}\,.$$

Thus $t \mapsto \mathrm{EXP}(tA)$ a 1-parameter subgroup, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{EXP}(tA) = A$$

It follows from Proposition 1.25 (3) that $\mathrm{EXP}(A) = \exp(A)$. So we will write exp instead of EXP from now on. The same holds if G is a submanifold and subgroup of $\mathrm{GL}(V)$.

Furthermore from the theory of ordinary differential equations we know that for $t \in \mathbb{R}$, $A, B \in \mathfrak{gl}(V)$, $M \in \mathrm{GL}(V)$ we have

$$\exp(MAM^{-1}) = M \exp(A) M^{-1} \tag{1.4}$$

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A), \text{ if } [A,B] = 0 \tag{1.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(tA)) = \exp(tA)A = A\exp(tA) \tag{1.6}$$

We would like to have similar properties in adapted form for arbitrary Lie groups. We already have an adapted form of the first equality of (1.6) which is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)) = \big(\mathrm{d}\ell_{\exp(tX)}\big)(X)$$

i. e., $t \mapsto \exp(tX)$ is an integral curve of the left-invariant extension of X.

Lemma 1.27. *Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$. Then*

$$\mathrm{Ad}_{\exp(tX)}(X) = X\,.$$

Proof: One easily checks $C_{\exp(tX)}(\exp(tX)) = \exp(tX)$. If we derive this at $t = 0$ one gets the equation stated in the lemma. ∎

It immediately follows that $\big(\mathrm{d}\ell_{\exp(tX)}\big)(X) = \big(\mathrm{d}r_{\exp(tX)}\big)(X)$, and we get the following corollary that generalizes (1.6). **Fr 26.4.**

Corollary 1.28.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)) = \big(\mathrm{d}\ell_{\exp(tX)}\big)(X)) = \big(\mathrm{d}r_{\exp(tX)}\big)(X)$$

Lemma 1.29. *If $f: G \to H$ is a homomorphism of Lie groups. Let $\exp^G: \mathfrak{g} \to G$ and $\exp^H: \mathfrak{h} \to H$ be the exponential maps of G and H. Then the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ \uparrow {\scriptstyle \exp^G} & & \uparrow {\scriptstyle \exp^H} \\ \mathfrak{g} & \xrightarrow{\quad \mathrm{d}_{\mathbb{1}} f \quad} & \mathfrak{h} \end{array} \tag{1.7}$$

commutes, i. e., $f \circ \exp^G = \exp^H \circ \mathrm{d}_{\mathbb{1}} f$.

Proof: Let $X \in \mathfrak{g}$. Then $t \mapsto \exp^G(tX)$ is a 1-parameter subgroup of G. Thus $t \mapsto f \circ \exp^G(tX)$ is a 1-parameter subgroup of G. We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \exp^G(tX) = \mathrm{d}_{\mathbb{1}} f\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp^G(tX)\Big) = \mathrm{d}_{\mathbb{1}} f(X)\,.$$

Thus this is the 1-parameter subgroup $t \mapsto \exp^H\big(t\,\mathrm{d}_{\mathbb{1}} f(X)\big)$, i. e., $f \circ \exp^G(tX) = \exp^H\big(t\,\mathrm{d}_{\mathbb{1}} f(X)\big)$ for all $t \in \mathbb{R}$ which implies the statement. ∎

As a corollary we get the Lie group analogon of equation (1.4):

Corollary 1.30. *For a Lie G and $\sigma \in G$ we get $C_\sigma \circ \exp = \exp \circ \mathrm{Ad}_\sigma$, i. e., the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\quad C_\sigma \quad} & G \\ \uparrow {\scriptstyle \exp} & & \uparrow {\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\quad \mathrm{Ad}_\sigma \quad} & \mathfrak{g} \end{array}$$

commutes.

Proof: Apply Lemma 1.29 to $H = G$, $f = C_\sigma$ and thus $\mathrm{d}_{\mathbb{1}} f = \mathrm{Ad}_\sigma$. ∎

1.5 Proof of Lemma 1.18

We now provide the proof of Lemma 1.18 which is still missing.

Let us recall the following exercise from last semester:

Exercise 1.31 (Diff. geom. I, Exercise Sheet 6, Exercise 4 with changed notation)**.** *Let M be a smooth, not necessarily compact, manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on M, i. e., φ is smooth with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s,t \in \mathbb{R}$. Let ξ be the associated tangent vector field on M, defined as*

$$\xi|_x := \frac{d}{dt}\Big|_{t=0}(\varphi_t(x)),$$

see also Diff. geom. I, Exercise Sheet 5, Exercise 3. Show that, for any smooth tangent vector field Y on M and point $p \in M$ it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left((\varphi_t)_*\eta\right)\big|_p = -[\xi,\eta]\big|_p,$$

where, for any diffeomorphism $\psi : M \to M$, the term $\psi_*\eta$ denotes the pushforward tangent vector field of η defined by $\psi_*\eta := \mathrm{d}\psi \circ \eta \circ \psi^{-1}$.

Proof of Lemma 1.18: [4] Let $X, Y \in \mathfrak{g}$ with left-invariant extensions $\overline{X}$ and $\overline{Y}$. At first, we calculate for $t \in \mathbb{R}$:

$$\begin{aligned}
\mathrm{Ad}_{\exp(tX)}(Y) &= \mathrm{d}r_{\exp(-tX)} \circ \mathrm{d}\ell_{\exp(tX)}(\overline{Y})\big|_{\mathbb{1}} \\
&= \mathrm{d}r_{\exp(-tX)}(\overline{Y})\big|_{\exp(tX)} \\
&\overset{(*)}{=} \mathrm{d}\Phi^{\overline{X}}_{-t}(\overline{Y})\big|_{\Phi^{\overline{X}}_{-t}(\mathbb{1})} \\
&\overset{(+)}{=} \left(\Phi^{\overline{X}}_{-t}\right)_*(\overline{Y})\big|_{\mathbb{1}}
\end{aligned}$$

where we used at $(*)$ Proposition (1) (5), and where we used at $(+)$ the pushforward of vector fields from the preceding exercise. We derive this with respect to t at $t = 0$, and use the results of the exercise above at $(\dagger)$ for $\xi = -\overline{X}$, $\eta = \overline{Y}$ and $\varphi_t = \Phi^{\overline{X}}_{-t}$. This gives

$$\begin{aligned}
\mathrm{ad}_X Y &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Ad}_{\exp(tX)}(Y) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\Phi^{\overline{X}}_{-t}\right)_*(\overline{Y})\big|_{\mathbb{1}}
\end{aligned}$$

[4] We roughly follow [28, 3.46].

$$\stackrel{(\dagger)}{=} [\overline{X}, \overline{Y}]\big|_{\mathbb{1}} = [X, Y]$$

■

1.6 Commuting elements in Lie groups and Lie algebras

Definition 1.32. *Two elements $\sigma, \tau \in G$ in a Lie group* **commute**, *if $\sigma\tau = \tau\sigma$. Two elements $X, Y \in \mathfrak{g}$ in a Lie algebra* **commute**, *if $[x, y]$.*

We want to relate commutativity in a Lie group to commutativity in its Lie algebra.

We start by some considerations on arbitrary manifolds M and N.

Lemma 1.33. *Let $f: M \to N$ be a smooth map, and let $X \in \mathfrak{X}(M)$ be f-related to $Y \in \mathfrak{X}(N)$, i. e., $\mathrm{d}f \circ X = Y \circ f$. Then the flows Φ_t^X and Φ_t^Y of X and Y satisfy*

$$\Phi_t^Y \circ f = f \circ \Phi_t^X .$$

Proof: For $p \in M$ we will show that $t \mapsto \gamma(t) := f \circ \Phi_t^X(p) \in N$ is an integral curve of Y. As one easily checks $\gamma(0) = f(p)$, this proves the statement.

$$\begin{aligned}
\dot{\gamma}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(f \circ \Phi_t^X(p)\Big) \\
&= \mathrm{d}f \circ \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^X(p)\Big) \\
&= \mathrm{d}f \circ \Big(X\big|_{\Phi_t^X(p)}\Big) \\
&= (\mathrm{d}f \circ X)\big|_{\Phi_t^X(p)}\Big) \\
&= (Y \circ f)\big|_{\Phi_t^X(p)}\Big) \\
&= Y\big|_{\gamma(t)} .
\end{aligned}$$

■

Proposition 1.34. *Let X and Y be vector fields on M with flows $\Phi_\bullet^X$ and $\Phi_\bullet^Y$. Then*

$$[X, Y] = 0 \Longleftrightarrow \forall s, t \in \mathbb{R}: \ \Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X .$$

Proof:

"⇐": We apply $\frac{d}{ds}\big|_{s=0}$ to

$$\Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t$$

and using $\frac{d}{ds}\big|_{s=0}\Phi^Y_s = Y$ we obtain

$$\mathrm{d}\big(\Phi^X_t\big) \circ Y = Y \circ \Phi^X_t$$

which means $\big(\Phi^X_t\big)_* Y = Y$. We apply Exercise 1.31 for $\xi = X$ and thus $\varphi_t = \Phi^X_t$, so we obtain by deriving with respect to t at $t = 0$:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} Y \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\Phi^X_t\big)_* Y \\ &= -[X, Y] \end{aligned}$$

"⇒": For $p \in M$ and for $s \in \mathbb{R}$ we define

$$v_p(t) := \Big(\big(\Phi^X_t\big)_* Y\Big)\Big|_p = \mathrm{d}\big(\Phi^X_t\big) \circ Y \circ \Phi^X_{-t}(p) \in T_pM\,.$$

We may differentiate this in the sense of Analysis II, and we write this differential as $v'_p(t)$. Exercise 1.31 tells us that $v'_p(0) = -[X, Y] = 0$ for all $p \in M$. We set $p = \Phi^X_{-s}(q)$ and we get

$$\begin{aligned} v_{\Phi^X_{-s}(q)}(t) &= \Big(\mathrm{d}\big(\Phi^X_t\big) \circ Y \circ \Phi^X_{-t-s}\Big)\Big|_q \\ &= \Big(\mathrm{d}\big(\Phi^X_{-s}\big) \circ \Big(\mathrm{d}\big(\Phi^X_{t+s}\big) \circ Y \circ \Phi^X_{-t-s}\Big)\Big)\Big|_q \\ &= \Big(\mathrm{d}\big(\Phi^X_{-s}\big) \circ \big(\Phi^X_{t+s}\big)_* Y\Big)\Big|_q \\ &= \mathrm{d}\big(\Phi^X_{-s}\big)\,\big(v_q(t+s)\big) \end{aligned}$$

Deriving this with respect to t at $t = 0$ yields

$$0 = v'_{\Phi^X_{-s}(q)}(0) = \mathrm{d}\big(\Phi^X_{-s}\big)\,\big(v'_q(s)\big)$$

and as $\mathrm{d}\big(\Phi^X_{-s}\big)$ is an isomorphism, this gives $v'_q(s) = 0$ for all $s \in \mathbb{R}$ and all $q \in M$. Thus we have $v_q(t) = v_q(0) = Y$ for all $q \in M$ and $t \in \mathbb{R}$. We have thus proven

$$\mathrm{d}\big(\Phi^X_t\big) \circ Y = Y \circ \Phi^X_t$$

which means that Y is Φ_t^X-related to itself. Using Lemma 1.33 for $M = N$, X replaced by Y, $f = \Phi_t^X$ and t replaced by s we get

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X .$$

■

Corollary 1.35. *Let G be a Lie group, $X, Y \in \mathfrak{g} = \mathrm{Lie}(G)$.*

(1) *If* $[X,Y] = 0$ *then* $\exp(X)\exp(Y) = \exp(Y)\exp(X) = \exp(X+Y)$.

(2) *Conversely, if*

$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX) \text{ for all } t, s \in \mathbb{R},$$

then $[X,Y] = 0$.

The first part of the Corollary provides a Lie group analogon of equation (1.5).

Proof: We extend $X, Y \in \mathfrak{g}$ to left-invariant vector fields, also denoted by X and Y.[5]

At first, let us assume $[X,Y] = 0$. As we have $\exp(X) = \Phi_1^X(\mathbb{1})$, the statement $\exp(X)\exp(Y) = \exp(Y)\exp(X)$ follows from "⇒" in Proposition 1.34. We also have $[sX, tY] = st[X,Y] = 0$. Thus, we already know $\exp(sX)\exp(tY) = \exp(tY)\exp(sX)$ and this yields

$$\begin{aligned}\exp\big((t+s)X\big)\exp\big((t+s)Y\big) &= \exp(tX)\exp(sX)\exp(tY)\exp(sY)\\ &= \Big(\exp(tX)\exp(tY)\Big)\Big(\exp(sX)\exp(sY)\Big),\end{aligned}$$

thus $t \mapsto \gamma(t) := \exp(tX)\exp(tY)$ is 1-parameter subgroup of G, and $\gamma'(0) = \mathrm{d}r_{\exp(0)}X + \mathrm{d}\ell_{\exp(0)}Y = X + Y$. Thus implies $\exp\big(t(X+Y)\big) = \exp(tX)\exp(tY)$ which gives the remaining statement for $t = 1$.

The converse statement immediately follows from "⇐" in Proposition 1.34. ■

We have seen that $\exp\colon \mathfrak{g} \to G$ satisfies $\mathrm{d}_0 \exp = \mathrm{id}$. Thus, the local reversal theorem tells us that there is an open neighborhood U_0 of 0 and and open neighborhood V_0

[5] We assume it is clear from the context, when a vector is meant, and when we denote a vector field.

of $\mathbb{1}$ such that $\exp|_{U_0}: U_0 \to V_0$ is a diffeomorphism. Using continuity of multiplication and inversion, we see that there is an open neighborhood U_1 of 0 such that $U_1 \subset U_0$, such that U_1 is starshaped with respect to 0, satisfying $X \in U_1 \Longleftrightarrow -X \in U_1$ and $X, Y \in U_1 \Rightarrow X + Y \in U_0$. We put $V_1 := \exp(U_1)$ and by shrinking U_1 and V_1 further we can achieve additionally $\mu(V_1 \times V_1) \subset V_0$ and we already have that inversion maps V_1 to itself.

Let $\gamma: [0, b] \to G$ be a continous path. For any $t \in [0, b]$ we define W_t as the connected component of $\{s \in [0, b] \mid \gamma(t)^{-1}\gamma(s) \in V_1\}$ that contains t. Then $(W_t)_{t \in [0,b]}$ is an open cover[6] of $[0, b]$. An elemantary compactness argument for $[0, b]$, treated under the name **Lebesgue number** ε, says: there is an $\varepsilon > 0$ if we have a partition

$$0 = t_0 \le t_1 \le \cdots \le t_k = b, \quad \forall i \in \{1, 2, \ldots, k\} : \; t_i - t_{i-1} < \varepsilon \tag{1.8}$$

then

$$\forall i \in \{1, 2, \ldots, k\} : \; t_i \in W_{t_{i-1}}$$

and thus

$$\forall i \in \{1, 2, \ldots, k\} : \forall s \in [t_{i-1}, t_i] : \; \gamma(s)^{-1}\gamma(t_{i-1}) \in V_1 \text{ and } \gamma(t_{i-1})^{-1}\gamma(s) \in V_1 \,. \tag{1.9}$$

Corollary 1.36. *Let G be a connected Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. Then the following are quivalent:*

(i) *G is abelian, i. e., $\sigma\tau = \tau\sigma$ for all $\sigma, \tau \in G$,*

(ii) *$\mathfrak{g}$ is abelian, i. e., $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$,*

(iii) *$\exp: (\mathfrak{g}, +) \to (G, \mu)$ is a group homomorphism.*

Proof:

"(i)⇒(ii)": This follows immediately from the second part of Corollary 1.35.

"(ii)⇒(iii)": This follows immediately from the first part of Corollary 1.35.

"(iii)⇒(i)": For $\sigma \in G$ we choose a continuous path $\gamma: [0, b] \to G$ from $\mathbb{1}$ to σ. We choose a subdivision as (1.8)/(1.9). Then $\sigma_i := \gamma(t_i)^{-1}\gamma(t_{i-1}) \in V_1$ satisfies $\sigma = \sigma_1\sigma_2\cdots\sigma_k$. We write $\sigma_i = \exp(X_i)$, $X_i \in U_1$. Similarly we decompose $\tau = \tau_1\tau_2\cdots\tau_\ell$, $\tau_j = \exp(Y_j)$, $Y_j \in U_1$. Condition (iii) implies that $\sigma_i\tau_j = \tau_j\sigma_i$ for all i, j and thus

[6] in German: "Überdeckung", **nicht** "Überlagerung", the two terms have different meanings, but are denoted with the same words "cover" and "covering" in English, but they are properly distinguished in German

$\sigma\tau = \tau\sigma$.

■

1.7 The Baker–Campbell–Hausdorff Formula

We have seen that $\exp\colon \mathfrak{g} \to G$ satisfies $d_0 \exp = \mathrm{id}$, and in the discussion following Corollary 1.35 we have discussed the diffeomorphism $\exp|_{U_0}\colon U_0 \to V_0$, and also had the smaller open neighborhoods $U_1 \subset U_0$ of 0 and $V_1 \subset V_0$ of $\mathbb{1}$. In particular multiplication restricts to a map $V_1 \times V_1 \to V_0$ and inversion maps V_1 to itself. We write $\log\colon U_0 \to V_0$. In this language, it follows from Corollary 1.35 (1) for all $X, Y \in U_1$:

$$\text{if } [X,Y] = 0, \text{ then } \log\big(\exp(X)\exp(Y)\big) = X + Y\,.$$

On the other hand it is clear from (the proof of) Corollary 1.35 (2) that this formula no longer holds, if $\mathfrak{g}$ is not abelian. The Baker–Campbell–Hausdorff formula, says that this can be repaired by adding commutator terms.

Exercise 1.37. *We define the* **3-dimensional Heisenberg group** $\mathcal{H}_3(R)$ *with coefficients in the ring* R *as*

$$\mathcal{H}_3(R) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in R \right\}.$$

For $R = \mathbb{R}$ *this is a submanifold and a subgroup of* $\mathrm{GL}(3,\mathbb{R})$, *thus a Lie group.*

(a) *Show that its Lie algebra* $\mathfrak{h}_3$, *the* **3-dimensional Heisenberg Lie algebra** *is given by matrices as follows:*

$$\mathfrak{h}_3 := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

(b) *Calculate* $\exp\colon \mathfrak{h}_3 \to \mathcal{H}_3(\mathbb{R})$, *and show that it is a diffeomorphism.*

(c) *Show that* $\log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A,B]$.

(d) *Show that* $[X,[Y,Z]] = 0$ *for all* $X, Y, Z \in \mathfrak{h}_3$, *i.e.* $\mathfrak{h}_3$ *is 2-step nilpotent.*

Tu 30.4.

The formula in (c) of this exercise is simple as higher order commutators vanish in the sense of (d) . In general one has to work with a power series.

Theorem 1.38 (Baker–Campbell–Hausdorff formula)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a power series* BCH *whose term of degree $k \in \mathbb{N}_0$ is a homogeneous polynomial of degree k*

$$\mathrm{BCH}_k\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that

(1) $\mathrm{BCH} = \sum_{k=0}^{\infty} \mathrm{BCH}_k$ *converges*[7] *on a neighborhood U_2 of* 0*. (We assume $U_2 \subset U_1$ for the U_1 defined above.)*

(2) $\log\big(\exp(X)\exp(Y)\big) = \mathrm{BCH}(X,Y)$ *for all $X, Y \in U_2$.*

(3) *The first terms are* $\mathrm{BCH}_0(X,Y) = 0$, $\mathrm{BCH}_1(X,Y) = X + Y$, $\mathrm{BCH}_2(X,Y) = \frac{1}{2}[X,Y]$, $\mathrm{BCH}_3(X,Y) = \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]]$.

(4) BCH_k *can be expressed by a formula, which only uses the vector space operations of $\mathfrak{g}$ and* $[\,\bullet\,,\,\bullet\,]$.

(5) *The formula for BCH_k is the same formula for any Lie group/algebra: obviously the bracket $[\,\bullet\,,\,\bullet\,]$ is given by $\mathfrak{g}$, but using this bracket, the formula no longer depends on $\mathfrak{g}$ (or G). This property can also be expressed as follows: if $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then*

$$\mathrm{BCH}_k\big(\varphi(\,\bullet\,), \varphi(\,\bullet\,)\big) = \varphi\Big(\mathrm{BCH}_k(\,\bullet\,,\,\bullet\,)\Big).$$

In other words, we have

$$\log\big(\exp(X)\exp(Y)\big) = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \textit{higher order terms (with at least 3 commutator terms in each summand)}$$

for X, Y sufficiently close to 0*.*

We do not prove this theorem here, see [15, Sections 3.1–3.5] for a proof.

[7] uniformly on any compactum in U_2, and also all derivaties converge uniformly on such a compactum)

1.8 From Lie algebra homomorphisms to Lie group homomorphisms

Theorem 1.39 (Lifting Lie algebra homomorphism to Lie group homomorphisms)**.** *Let G and H be Lie groups, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$. Assume that G is simply-connected[8]. Then for any Lie algebra homomorphism $f{:}\,\mathfrak{g} \to \mathfrak{h}$ there is a unique Lie group homomorphism $F{:}\,G \to H$, such that $\mathrm{d}_{\mathbb{1}}F = f$.*

A full proof of this theorem is carried out in [28, Theorem 3.27], building on the Frobenius theorem. Another approach, using the Baker–Campbell–Hausdorff formula, is worked out in [15] where the above theorem is Theorem 3.7. We sketch the latter approach.

Sketch of Proof:

(a) On $\mathfrak{g}$ we choose an open neighborhood U_2 of 0 as in Corollary 1.36. On $\mathfrak{h}$ we choose $U_2^{\mathfrak{h}} \ni \mathbb{1}$, $U_2^{\mathfrak{h}} \Subset \mathfrak{h}$ analogously. We set $U_3 := U_2 \cap f^{-1}(U_2^{\mathfrak{h}})$. We define $V_3 := \exp(U_3)$. We set $F_3{:}\,V_3 \to H$ as $F_3 := \exp^H \circ f \circ \log^G$ where exp^H is the exponential map of the Lie group H, and $\log^G$ the local inverse of the exponential map of G. For $\sigma, \tau \in U_3$ we calculate using the Baker–Campbell–Hausdorff formula, more precisely Theorem 1.38 (2) for $X = \log(\sigma)$ and $Y = \log(\tau)$ at $(*)$, Theorem 1.38 (5) at $(+)$ and Theorem 1.38 (2) for $X = f \circ \log(\sigma)$ and $Y = f \circ \log(\tau)$ at $(\dagger)$:

$$\begin{aligned}
F_3(\mathbb{1}) &= \exp^H \circ f \circ \underbrace{\log^G(\mathbb{1})}_{=0} = \exp^H(0) = \mathbb{1} \\
F_3(\sigma\tau) &= \exp^H \circ f \circ \log^G(\sigma\tau) \\
&\overset{(*)}{=} \exp^H \circ f \left(\mathrm{BCH}\big(\log^G(\sigma), \log^G(\tau)\big)\right) \\
&\overset{(+)}{=} \exp^H\left(\mathrm{BCH}\big(f \circ \log^G(\sigma), f \circ \log^G(\tau)\big)\right) \\
&\overset{(\dagger)}{=} \left(\exp^H \circ f \circ \log^G(\sigma)\right) \cdot \left(\exp^H \circ f \circ \log^G(\tau)\right) \\
&= F_3(\sigma)F_3(\tau)\,.
\end{aligned} \tag{1.10}$$

(b) For a given $\sigma \in G$ we choose a path $\gamma{:}\,[0,1] \to G$ with $\gamma(0) = \mathbb{1}$ and $\gamma(1) = \sigma$. This is possible, as G is connected and thus path-connected. We restrict the open neighborhood U_3 further to some star-shaped open neighborhood U_4 of 0 that is symmetric with respect to 0, i. e., $X \in U_4 \Longleftrightarrow -X \in U_4$, and such that $\mu(U_4 \times U_4) \subset U_3$. We define $V_4 := \exp(U_4)$. We choose a subdvision SUB: $0 = t_0 \le t_1 \le \cdots \le t_k = 1$ as

[8] We use the convention that the definition of simply-connectedness includes connectedness

in (1.8) and (1.9) with V_4 instead of V_1. We define

$$F_{\mathrm{path}}(\gamma,\mathrm{SUB}) := \underbrace{F_3\big(\gamma(t_0)^{-1}\gamma(t_1)\big)}_{F_3(\gamma(t_1))} F_3\big(\gamma(t_1)^{-1}\gamma(t_2)\big)\cdots \underbrace{F_3\big(\gamma(t_{k-1})^{-1}\gamma(t_k)\big)}_{F_3(\gamma(t_{k-1})^{-1}\sigma)} \tag{1.11}$$

If SUB' is a subdivision of SUB, then follows from (1.10) that

$$F_{\mathrm{path}}(\gamma,\mathrm{SUB'}) = F_{\mathrm{path}}(\gamma,\mathrm{SUB})\,.$$

If SUB_1 and SUB_2 are two subdivisions, then we choose SUB' to be a refinement of both of them, and we argue

$$F_{\mathrm{path}}(\gamma,\mathrm{SUB}_1) = F_{\mathrm{path}}(\gamma,\mathrm{SUB'}) = F_{\mathrm{path}}(\gamma,\mathrm{SUB}_2)\,.$$

Thus we now write $F_{\mathrm{path}}(\gamma)$, as this does not depend on the subdivision SUB.

(c) Now, one shows: if γ' is another path as above, and if $\mathcal{H}\colon[0,1]\times[0,1]$ is a homotopy from γ to γ' with fixed endpoints, then $F_{\mathrm{path}}(\gamma) = F_{\mathrm{path}}(\gamma')$.

For this purpose one chooses a $k\in\mathbb{N}$ (such a number is given again by a “Lebesgue number”, whose existence again relies on a compactness argument) , such

$$\mathcal{H}\left(\left[i-\tfrac{1}{k}\right]\times\left[j-\tfrac{1}{k}\right]\right)\subset V_4$$

for all $i,j\in\{1,\dots,k\}$. Now one passes from γ to γ' in k^2 steps by replacing in each step a piece of the curve described by the square $\left[i-\frac{1}{k}\right]\times\left[j-\frac{1}{k}\right]$, see the drawing in the lecture. This proves the claim in this item.

(d) As g is simply-connected, we see that there is a map $F\colon G\to H$, such that $F(\sigma) = F_{\mathrm{path}}(\gamma)$ if $\gamma(0)=\mathbb{1}$ and $\gamma(1)=\sigma$. The smoothness of F follows from the smoothness of F_3.

(e) Now let $\sigma,\tau\in G$, we choose paths $\gamma,\rho\colon[0,1]\to G$ with $\gamma(0)=\rho(0)=\mathbb{1}$, $\gamma(1)=\sigma$, $\rho(1)=\tau$. Then $\ell_\sigma\circ\rho$ is a path from σ to $\sigma\tau$. Thus the concatenation $\gamma*(\ell_\sigma\circ\rho)$, defined as

$$\gamma*(\ell_\sigma\circ\rho)(t) = \begin{cases}\gamma(2t) & \text{for } 0\le t\le\frac{1}{2}\\ \sigma\cdot\rho(2t-1) & \text{for } \frac{1}{2}\le t\le 1\end{cases}$$

is a path from $\mathbb{1}$ to $\sigma\tau$, and one easily checks[9] that (1.11) yields

$$F_{\text{path}}(\gamma * (\ell_\sigma \circ \rho)) = F_{\text{path}}(\sigma) \cdot F_{\text{path}}(\tau)\,.$$

This gives the homomorphism property $F(\sigma\tau) = F(\sigma) \cdot F(\tau)$. ∎

2 Actions of groups on spaces

Fr 3.5.

2.1 Definitions for groups actions and examples

In this section all topological spaces are assumed to be Hausdorff spaces.

Definitions 2.1.

1.) *A* **topological group** *is a topological space with a map* $\mu\colon G \times G \to G$ *such that* (G, μ) *is a group and such that* $(\sigma, \tau) \mapsto \mu(\sigma, \tau^{-1})$ *is continuous*[10]*. We write* $\sigma\tau := \mu(\sigma, \tau)$*. We denote the unit element by* $\mathbb{1}$*.*

2.) *A (continuous)* **left action** *of a topological group* G *on a topological space* X *is a continuous map* $a\colon G \times X \to X$ *such that*

(i) $a(\mathbb{1}, x) = x$ *for all* $x \in X$*,*

(ii) $a(\sigma\tau, x) = a(\sigma, a(\tau, x))$ *for all* $\sigma, \tau \in G$*,* $\forall x \in X$*.*

One also says that X *is a* **G-space***. We often write* σx *for* $a(\sigma, x)$*, condition (ii) then reads as* $(\sigma\tau)x = \sigma(\tau x)$*, so we can omit the parentheses. As a symbol we write* $G \curvearrowright X$*. Note that a group action induces a group homorphism* $G \to \mathrm{Homeo}(X)$*, where* $\mathrm{Homeo}(X)$ *denotes the group of homeomorphisms from* X *to* X*. However, not every group hommorphism* $G \to \mathrm{Homeo}(X)$ *defines an action, in general.*

3.) *In order to get the definition of a* **right action** *we replace condition (ii) by*

(ii') $a(\sigma\tau, x) = a(\tau, a(\sigma, x))$ *for all* $\sigma, \tau \in G$*,* $\forall x \in X$*.*

One then writes $x\sigma$ *and we have* $x(\sigma\tau) = (x\sigma)\tau$*.*

[9] in fact one has to be careful with the order!

[10] This is equivalent to claiming that μ and $\sigma \mapsto \sigma^{-1}$ are continuous.

4.) *The action of G on X is called* **discrete** *if G carries the discrete topology. Then continuity in 1.) is trivially satisfied, and the only conditions on the continuity in 2.) is that for all $\sigma \in G$, $\ell_\sigma : X \to X$ is continuous. So a discrete group action of G on X is the same as a group homomorphism $G \to \mathrm{Homeo}(X)$.*

5.) *An action is* **smooth**, *if G is a Lie group, if X is a smooth manifold, and if a is a smooth map. We then say that X is a* **smooth G-space**.

6.) *An action is* **free**, *if*

$$\forall \sigma \in G \setminus \{\mathbb{1}\} : \forall x \in X : \sigma x \neq x \,.$$

An action is **effective** *or* **faithful** *if*

$$\forall \sigma \in G \setminus \{\mathbb{1}\} : \exists x \in X : \sigma x \neq x \,.$$

An action is **transitive** *if*

$$\forall x, y \in X : \exists \sigma \in G : \sigma x = y \,.$$

(For right actions the obvious modification should be done in each definition.)

7.) *The orbit of x is $Gx := \{\sigma x \mid \sigma \in G\}$. (We then have: G acts transitively $\iff$ $Gx = X$ for all $x \in X \overset{X \neq \varnothing}{\iff} Gx = X$ for some $x \in X$.)*

8.) *The* **stabilizer** *or* **isotropy group** *at $x \in X$ is*

$$G_x := \{\sigma \in G \mid \sigma x = x\} \,.$$

This is a closed subgroup of G (obvious). For smooth actions it is a submanifold (more involved, no proof here).

9.) *The* **quotient space** *is*

$$G \backslash X := \{Gx \mid x \in X\} \,.$$

We will clarify its topology and its smooth structure, if it exists.

Examples 2.2.

1.) Let $X = G$ be a topological group (or even a Lie group)

(i) $a(\sigma,\tau) = \mu(\sigma,\tau) = \sigma\tau$

(ii) $a(\sigma,\tau) = \mu(\tau,\sigma^{-1})$

(iii) $a(\sigma,\tau) = \mu(\tau,\sigma)$

(iv) $a(\sigma,\tau) = \mu(\sigma^{-1},\tau)$

(i) and (ii) are left actions, while (iii) and (iv) are right actions.

2.) Let $X = G$. **Conjugation**: $a(\sigma,\tau) = \sigma\tau\sigma^{-1} =: C_\sigma(\tau)$ is a left action.

3.) $\mathrm{Ad}_\sigma\colon \mathfrak{g} \to \mathfrak{g}$ defines a left action $a(\sigma, X) := \mathrm{Ad}_\sigma(X)$ on $\mathfrak{g}$.

4.) $\mathrm{O}(n+1)$ acts on $S^n \subset \mathbb{R}^{n+1}$ transitively and smoothly. Let $e_{n+1} := (0,0,\cdots,0,1)^\mathsf{T}$ Then the stabilizer of $\mathrm{O}(n+1)$ at e_{n+1} is

$$\mathrm{O}(n+1)\big|_{e_{n+1}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| A \in \mathrm{O}(n) \right\} \cong \mathrm{O}(n)$$

5.) $\{\pm 1\}$ acts on $S^n \subset \mathbb{R}^{n+1}$ by multiplication. This is a free, discrete smooth action. The quotient $\mathbb{R}\mathrm{P}^n = {}_{\{\pm 1\}}\backslash^{S^n}$ is the **real projective space**.

6.) $\mathrm{U}(1) = S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by multiplication. This is a free, smooth action, called the **Hopf action**. The quotient $\mathbb{C}\mathrm{P}^n = {}_{S^1}\backslash^{S^{2n+1}}$ is the **complex projective space**.

2.2 Proper maps and proper actions

Definitions 2.3.

1.) *A continuous map $f\colon X_1 \to X_2$ is* **proper** *if $F^{-1}(K)$ is compact for all compacta $K \subset X_2$.*

2.) *An action of G on X is defined to be* **proper** *if*

$$\begin{aligned} G \times X &\xrightarrow{\Theta} X \times X \\ (\sigma, x) &\mapsto (\sigma x, x) \end{aligned}$$

is a proper map. The map Θ is called the associated **shear map**.

Example 2.4. Assume that G acts on X continuously. If G is compact, then the action is proper. In order to prove this, let $K \subset X \times X$ be compact. We write

$\mathrm{pr}_i\colon X \times X \to X$ for the projection to the i-th factor. Then $\widehat{K} := \mathrm{pr}_2(K) \subset X$ is also compact. As a consequence $\Theta^{-1}(K)$ is a closed subset of the compact set $G \times \widehat{K}$, and thus compact as well.

Note: if the action map $a\colon G \times X \to X$ is a proper map, then the action is proper (i. e., Θ is a proper map). The converse is *not true*. In fact let $a\colon G \times X \to X$ be a compact map. For some $x_0 \in X$ consider the compact set $A := a^{-1}(\{x_0\}) = \{(g, g^{-1}x_0) \mid g \in G\}$. Then pr_1 defines a continuous surjective map $A \to G$, thus G is compact. Thus it is too restrictive to claim that $a\colon G \times X \to X$ is a proper map.

As an example, consider the action of $G = (\mathbb{R}, +)$ on $X = \mathbb{R}$ given by $a(\sigma, x)0\sigma + x$. One easily checks, that this action is proper (as an action), but as $\mathbb{R}$ is not compact, the map a is not proper.

For $K \subset X$ we define $\sigma K := \{\sigma k \mid k \in K\}$ and

$$G_K := \{\sigma \in G \mid \sigma K \cap K \neq \varnothing\}.$$

In particular, for $x \in X$, $G_{[x]} = G_x$ is the isotropy group of x.

Proposition 2.5. *The action of G on X is proper if, and only if, G_K is compact for all compact sets $K \subset X$.*

In the special case that G acts smoothly on the manifold X, this is the equivalence of i) and ii) of Exercise Sheet 3, Exercise 2.

Proof:
"$\Rightarrow$": Let $\Theta\colon G \times X \to X \times X$ be proper and K compact. Then

$$\begin{aligned} G_K &= \{\sigma \in G \mid \exists x \in K \text{ with } \sigma x \in K\} \\ &= \{\sigma \in G \mid \exists x \in K \text{ with } \Theta(\sigma, x) \in K \times K\} \\ &= \underbrace{\mathrm{pr}_G\Big(\underbrace{\Theta^{-1}(\underbrace{K \times K}_{\text{compact}})}_{\text{compact}}\Big)}_{\text{compact}} \end{aligned}$$

"$\Leftarrow$": Assume $L \subset X \times X$ is compact. We set $K := \mathrm{pr}_1(L) \cup \mathrm{pr}_2(L)$ which is compact. Thus $L \subset K \times K$. Then

$$\Theta^{-1}(L) \subset \Theta^{-1}(K \times K) = \{(\sigma, x) \mid \sigma x \in K \text{ and } x \in K\} \subset G_K \in K$$

and both subsets are closed. Thus as by assumption G_K is compact, we get the compactness of $\Theta^{-1}(L)$. ∎

Recall the following from the beginners' lectures:

Theorem 2.6 (Bolzano–Weierstrass)**.** *Let X be a metrizable topological space (i. e., a space whose topology is induced from a metric). Then*

$$X \text{ is compact} \iff X \text{ is sequentially compact}$$

where a space X is called **sequentially compact** *if X is a Hausdorff space in which any sequence has a convergent subsequence.*

All smooth manifolds are metrizable.

Proposition 2.7. *Let G and X be metrizable, let X be locally compact, and let G act continuously on X. Then the following are equivalent*

(i) *the action of G on X is proper.*

(ii) *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in M and $(\sigma_i)_{i\in\mathbb{N}}$ a sequence in G such that the sequences $(x_i)_{i\in\mathbb{N}}$ and $(\sigma_i \cdot x_i)_{i\in\mathbb{N}}$ converge. Then we find a convergent subsequence of $(\sigma_i)_{i\in\mathbb{N}}$.*

In the special case that G acts smoothly on the manifold X, this is the equivalence of ii) and iii) of Exercise Sheet 3, Exercise 2. We will thus skip the proof.

Lemma 2.8. *Let $F\colon X \to Y$ be a continuous, proper map between (topological) manifolds. Then F is closed.*

Note: The Lemma is still correct if one does not require X and Y to be topological manifolds, but to require instead that X and Y are locally compact and metrizable (Hausdorff) spaces. However, an adapted proof is required in this generality.

Proof: Let $A \subset X$ be closed, and take $p \in \overline{F(A)}$. We have to show that $p \in F(A)$.

We choose a chart $U \xrightarrow{y} V$ of Y, containing p, $y(p) = 0$. We choose an $\varepsilon > 0$ with $\overline{B_\varepsilon(0)} \subset V$.

There is a sequence $(q_i)_{i\in\mathbb{N}}$ in A such that $\lim_{i\to\infty} F(q_i) = p$. After removing finitely many exceptions from the sequence, we get $F(q_i) \in y^{-1}\big(\overline{B_\varepsilon(0)}\big) =: K$ for all i. Obviously, K is compact, and as F is proper $F^{-1}(K)$ is a compact subset of X, hence $K' := F^{-1}(K) \cap A$ is also compact, and we have $q_i \in K'$.

Thus after passing to a subsequence, $q_\infty := \lim_{i\to\infty} q_i$ exists in $K' \subset A$. Then

$$F(q_\infty) = F\Big(\lim_{i\to\infty} q_i\Big) = \lim_{i\to\infty} F(q_i) = p\,.$$

Hence $p \in \operatorname{image} F$. ∎

3 Topological quotients

Motivation: Let G be a Lie group acting on (smooth) manifold M. We try to find good conditions, such that $G\backslash M$ is a (smooth) manifold.

We now consider the topology on such quotients, admitting a setting that is a bit more general.

Definition and Lemma 3.1. *Let X be a topological space. Let $f\colon X \to Y$ be surjective. Then Y has exactly one topology such that*

(i) *f is continuous*

(ii) *for any topological space Z and any continuous map $g\colon X \to Z$ we have: if there is a map $\overline{g}\colon Y \to Z$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \overline{g}} & \\ & Z & \end{array}$$

commutes, then $\overline{g}$ is already continuous.

This topology is called the **quotient topology** *on Y. A surjective map such that Y carries the quotient topology is called a* **topological quotient***.*
Moreover the quotient topology is characterized by the following:

$$U \subset Y \text{ is open} \iff f^{-1}(U) \subset X \text{ is open.}$$

Proof:

(a) **Uniqueness of the topology:**

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on Y with properties (i) and (ii). Then we get the following commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f} \\ (Y,\mathcal{O}_1) & \xleftrightarrow[\text{id}]{} & (Y,\mathcal{O}_2) \end{array} \quad .$$

As the maps $\mathrm{id}\colon (Y,\mathcal{O}_1) \to (Y,\mathcal{O}_2)$ and $\mathrm{id}\colon (Y,\mathcal{O}_2) \to (Y,\mathcal{O}_1)$ are both continuous, we have $\mathcal{O}_1 = \mathcal{O}_2$. The topology is thus unique (if it exists).

(b) **Existence of such a topology:**

We define: a subset $U \subset Y$ is *open* iff $f^{-1}(U)$ is open. This is a topology:

(i) $\varnothing$ and Y are open in Y, as $\varnothing = f^{-1}(\varnothing)$ and $X = f^{-1}(Y)$ are open in X.

(ii) Assume that $U_1, \ldots, U_k$ are open in Y. Then

$$f^{-1}\Big(\bigcap_{i=1}^{k} U_i\Big) = \bigcap_{i=1}^{k}\Big(f^{-1}\left(U_i\right)\Big)$$

is also open.

(iii) Assume that U_i $i \in I$ are open in Y. Then

$$f^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I}\Big(f^{-1}\left(U_i\right)\Big)$$

is also open.

f is continuous: obvious!

Continuity of maps $\overline{g}$:

Assume Z and $\overline{g}$ as in (ii). Let W be open in Z. Then $g^{-1}(W)$ is open in X. Note that $g^{-1}(W) = f^{-1}\big(\overline{g}^{-1}(W)\big)$. By the definition of the topology on Y at the beginning of this step, this holds only if $\overline{g}^{-1}(W)$ is open in Y. ∎

Examples 3.2.

1.) $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x$, is a topological quotient. The map f_1 is an open map, i. e., it maps open subsets to open subsets.

2.) $f_2 : \mathbb{R} \to \mathbb{R}$,

$$t \mapsto \begin{cases} t & \text{if } t < 0 \\ 0 & \text{if } 0 \le t \le 1 \\ t-1 & \text{if } 1 < t \end{cases}$$

is a topological quotient. The map f_2 is not an open map, as it maps to open set $(0,1)$ to the non-open subset $\{0\}$.

3.) The composition of two topological quotients is again a topological quotient. This is part of the following stronger statement: Let $f : X \to Y$ be a topological quotient, Z a topological space, $g : Y \to Z$ a surjective map. Then $h := g \circ f$ is a topological quotient if and only if g is a topological quotient.

"only if": Let h be a topological quotient, then for $U \in Z$:

$$U \text{ open in } Z \iff \underbrace{h^{-1}(U)}_{=f^{-1}\left(g^{-1}(U)\right)} \text{ open in } X \iff g^{-1}(U) \text{ open in } Y,$$

thus g is a topological quotient.

"if": Let g be a topological quotient, then for $U \in Z$:

$$U \text{ open in } Z \iff g^{-1}(U) \text{ open in } Y \iff \underbrace{f^{-1}\left(g^{-1}(U)\right)}_{=h^{-1}(U)} \text{ open in } X,$$

thus h is a topological quotient.

WARNING 3.3. *If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are topological quotients, then in general $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is not a topological quotient. This product map $f_1 \times f_2$ will be continuous, but in some cases the quotient topology on $Y_1 \times Y_2$ has more open subsets than the product of the quotient topologies on Y_1 and Y_2.*

Proposition 3.4. *Assume that a topological group G acts continuously on the topological space X. Then*

$$\pi : X \to G\backslash X$$

$$p \mapsto [p] = G \cdot p$$

is an open map.

Proof: For any $\sigma \in G$ we define $\ell_\sigma := a(\sigma, \bullet) \colon X \to X$, $x \mapsto \sigma x$ which is continuous. As $\ell_{\sigma^{-1}}$ is the inverse to ℓ_σ, we know that ℓ_σ is a homeomorphism. Let $V \subset X$ be open, thus $\ell_\sigma(V)$ is also open. Thus $\bigcup_{\sigma \in G} \ell_\sigma(V) = \pi^{-1}\big(\pi(V)\big)$ is open in X, and finally we get that $\pi(V)$ is open in $G\backslash X$. ■

Tue 7.5.

Corollary 3.5. *Let again $G \curvearrowright X$ (continuously). If X is second countable, then $G\backslash X$ is also second countable.*

Proof: Let $\mathcal{B} = \{U_i \mid i \in I\}$ be a countable basis of the topology of X. Then $\pi(U_i)$ is open in $G\backslash X$ for any $i \in I$. Thus

$$\widetilde{\mathcal{B}} = \left\{\pi(U_i) \mid i \in I\right\}$$

is a countable set of open subsets of $G\backslash X$.

It remains to show that $\widetilde{\mathcal{B}}$ is a basis of the topology of $G\backslash X$.

Let $V \subset G\backslash X$ be open. Then, by definition of the quotient topology, $\pi^{-1}(V)$ is open in X. Thus there is $J \subset I$ such that

$$\pi^{-1}(V) = \bigcup_{j \in J} U_j \,.$$

Using the surjectivity of f, it follows that

$$V = \pi(\pi^{-1}(V)) = \bigcup_{j \in J} \underbrace{\pi(U_j)}_{\in \widetilde{\mathcal{B}}} \,.$$

■

4 Quotient manifolds

4.1 The theorem about smooth structures on quotients

Recall: A smooth map $f \colon M \to N$ is called a

(i) **submersion** iff $\forall x \in M : \mathrm{d}f|_x : T_xM \to T_{f(x)}N$ is surjective

(ii) **immersion** iff $\forall x \in M : \mathrm{d}f|_x : T_xM \to T_{f(x)}N$ is injective

(iii) local diffeomorphism, iff $\forall x \in M$: are open neighborhoods U of x in M and V of $f(x)$ in N such that

$$f|_U : U \to V$$

is a diffeomorphisms. Using the local reversal theorem we obtain:
f is a local diffeomorphism iff $\forall x \in M : \mathrm{d}f|_x : T_xM \to T_{f(x)}N$ is bijective.

Theorem 4.1. *Let a Lie group G act smoothy, freely, and properly on a manifold M. Equip $G\backslash M$ with the quotient topology. Then $G\backslash M$ carries a unique smooth structure such that $\pi: M \to G\backslash M$ is a submersion. Furthermore* $\dim G\backslash M = \dim M - \dim G$.

Examples 4.2.

1.) $\{\pm 1\}$ acts on $S^n \subset \mathbb{R}^{n+1}$, compare Example 2.2 5.). This action is discrete, smooth, free, and proper. We equip $\mathbb{R}\mathrm{P}^n \cong \{\pm 1\}\backslash S^n$ with the quotient topology and the smooth structure given in Diff. geom. I, Exercise Sheet 1, Exercise 3. It is easy to check that the canonical projection map $\pi: S^n \to \mathbb{R}\mathrm{P}^n$ is then smmooth and a submersion.

2.) We consider again the Hopf action of S^1 on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by complex multiplication, compare Example 2.2 6.). This action is non-discrete, smooth, free, and proper. We equip the complex projective space $\mathbb{C}\mathrm{P}^n = S^1\backslash S^{2n+1}$ with the smooth structure given in Exercise 4.3. Then the canonical projection, called the **Hopf fibration**, $\pi: S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a submersion.

3.) Let $\mathbb{Z}^n$ act on $\mathbb{R}^n$ by addition (or equivalently expressed: by translation). This action is discrete, smooth, free, and proper. In Diff. geom. I, Exercise Sheet 3, Exercise 3 we introduced a manifold structure on $\mathrm{T}^n = \mathbb{Z}^n\backslash \mathbb{R}^n$. The topology is the quotient topology, and the smooth structure is the one of the above theorem.

4.) In general, if G is discrete, then $\dim G\backslash M = \dim M$, thus the submersion is in fact a local diffeomorphism. Furthermore, from Exercise Sheet 3, Exercise 3 a) we see that G acts properly discontinuously (and freely). Thus we are in the setting of Diff. geom. I, Exercise Sheet 13, Exercise 4, if we replace the right action by a left action, i. e., by defining in the exercise $R(p,\sigma) := a(\sigma^{-1}, p)$, for $\sigma \in G$, $p \in M$. Thus $M \to G\backslash M$ is a surjective covering.

5.) Conversely, you may ask whether every covering $\pi: M \to N$ that is surjective and a local diffeomorphism arises this way. Here the answer is "No", however, it is yes, M is simply-connected. By choosing $p \in M$, a covering map $\pi: M \to N$ yields a group homomorphism $\pi_*: \pi_1(M,p) \to \pi_1(N,\pi(p))$, and one can show that this is injective. Assuming N and M are connected, the answer the above question is "Yes" if and only if $\pi_*\big(\pi_1(M,p)\big)$ is a normal subgroup of $\pi_1(N,\pi(p))$. Such covers are called **normal coverings** or **Galois coverings**.

Let us formular the complex analogue of Diff. geom. I, Exercise Sheet 1, Exercise 3.

Exercise 4.3 (Potential exercise for Differential Geometry I)**.** *Let $n \in \mathbb{N}$ and $\mathbb{C}\mathrm{P}^n$ be the set of 1-dimensional complex vector subspaces of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$.*

(a) *Identify $\mathbb{C}\mathrm{P}^n$ with the quotient $(\mathbb{C}^{n+1}\backslash\{0\})\backslash\sim$, where*

$$x \sim y \iff \exists\, \lambda \in \mathbb{C}^\times \text{ s.t. } x = \lambda y$$

and endow it with the quotient topology. Show that $\mathbb{C}\mathrm{P}^n$ is a compact Hausdorff space satisfying the second axiom of countability.

(b) *Show that the maps*

$$U_j := \{[x] \in \mathbb{C}\mathrm{P}^n \,|\, x_j \neq 0\} \overset{\varphi_j}{\to} \mathbb{C}^n \cong \mathbb{R}^{2n},\ [z] \mapsto \frac{1}{z_j}(z_1, \dots, \widehat{z_j}, \dots, z_{n+1}),\ 1 \leq j \leq n{+}1,$$

are well-defined homeomorphisms (the "$\widehat{z_j}$" means omitting "z_j", and $z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1}$) .

(c) *Show that $\mathcal{A} = (\varphi_j: U_j \to \mathbb{R}^{2n})_{j \in \{1,2,\dots,n+1\}}$ is an atlas for $\mathbb{C}\mathrm{P}^n$.*

(d) *For $i, j \in \{1, \dots, n+1\}$, $i \neq j$ show that $\varphi_j(U_i \cap U_j)$ is an open subset of $\mathbb{R}^{2n}$ and that*

$$\varphi_i \circ (\varphi_j)^{-1}: \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a C^∞-diffeomorphism.

Lemma 4.4. *Assume that a Lie group acts G smoothly acts on a manifold M, $p \in M$. if $s_p: G \to M$, $\sigma \mapsto \sigma p$ is injective (on a neighborhood of $\mathbb{1}$), then s_p is an immersion.*

In the following, we call this map s_p the **orbit map**[11] of p.

[11] I do not think that this terminology is used in the literature, but it seems a reasonable name

Examples 4.5.

1.) $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is an injective smooth map, but it is not an immersion. Thus it cannot be obtained as a map s_p as above for a suitable smooth action of $G = \mathbb{R}$ on $M = \mathbb{R}$. However, it is the map s_p for $G = M = \mathbb{R}$, $p = 0$ and for the non-smooth, continuous action $a(\sigma, x) = \sqrt[3]{[\sigma + x^3]}$.

2.) For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathrm{T}^2 := \mathbb{Z}^2 \backslash \mathbb{R}^2$ we define $a: \mathbb{R} \times \mathrm{T}^2 \to \mathrm{T}^2$ as

$$a(t, [(x, y)] := [(x + t, y + \alpha t)]$$

where $t, x, y \in \mathbb{R}$, thus $[(x, y)] \in \mathrm{T}^2$.

Figure in the lecture, not yet drawn electronically

A 2-dimensional torus with a line of irrational slope α

For any $p \in \mathrm{T}^2$, the map $s_p: \mathbb{R} \to \mathrm{T}^2$ is an injective immersion. However $\mathbb{R}p = s_p(\mathbb{R})$ is *not a submanifold.*

Proof of Lemma 4.4: We assume that there is an open neighborhood U of $\mathbb{1}$ such that $s_p|_U$ is injective. We write ℓ_σ, $\sigma \in G$, both for left multiplication $\ell_\sigma: G \to G$ and for left multiplication $\ell_\sigma: M \to M$.
"$\mathrm{d}_{\mathbb{1}} s_p: \mathfrak{g} \to T_p M$ is injective": Assume $\mathrm{d}s_p|_{\mathbb{1}}(X) = 0$ for $X \in \mathfrak{g}$ and define $\gamma(t) := \exp(tX)$, i. e., γ is a 1-parameter subgroup and satisfies $\dot{\gamma}(0) = X$ and $\gamma(t_0 + t) = \gamma(t_0)\gamma(t)$. We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \big(\gamma(t) \cdot p\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\gamma(t_0 + t) \cdot p\big) = \mathrm{d}\ell_{\gamma(t_0)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \underbrace{\big(\gamma(t) \cdot p\big)}_{= s_p(\gamma(t))} \right)$$

$$= \mathrm{d}\ell_{\gamma(t_0)} \circ \mathrm{d}s_p\big(\dot{\gamma}(0)\big) = 0\,.$$

to me

As a consequence $\gamma(t) \cdot p = p$. for any $t \in \mathbb{R}$. For t close to 0 we get a contradiction to the injectivity of $s_p|_U$.

"$\mathrm{d}_\sigma s_p \colon \mathfrak{g} \to T_{\sigma p}M$ is injective for all $\sigma \in G$": We calculate

$$s_p\big(\ell_\sigma(\tau)\big) = \ell_\sigma(\tau) \cdot p = (\sigma\tau)p = \sigma(\tau p) = \sigma \cdot s_p(\tau) = \ell_\sigma\big(s_p(\tau)\big).$$

Hence the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ell_\sigma} & G \\ {\scriptstyle s_p}\downarrow & & \downarrow{\scriptstyle s_p} \\ M & \xrightarrow{\ell_\sigma} & M \end{array} \quad \text{and its derivative at } \tau = \mathbb{1} \quad \begin{array}{ccc} T_{\mathbb{1}}G & \xrightarrow{\mathrm{d}_{\mathbb{1}}\ell_\sigma} & T_\sigma G \\ {\scriptstyle \mathrm{d}_{\mathbb{1}}s_p}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_\sigma s_p} \\ T_pM & \xrightarrow{\mathrm{d}_p\ell_\sigma} & T_{\sigma p}M \end{array}$$

commute. As $\mathrm{d}_{\mathbb{1}}\ell_\sigma$ and $\mathrm{d}_p\ell_\sigma$ are isomorphisms and as $\mathrm{d}_{\mathbb{1}}s_p$ is injective, $\mathrm{d}_\sigma s_p = \mathrm{d}_p\ell_\sigma \circ \mathrm{d}_{\mathbb{1}}s_p \circ (\mathrm{d}_{\mathbb{1}}\ell_\sigma)^{-1}$ is injective as well. ∎

Proof of Theorem 4.1:

(a) If M is second countable, then ${}_G\backslash^M$ with the quotient topology is also second countable topological space, see Corollary 3.5. In the literature there two non-equivalent definition of "a manifold" see the section "Conventions and Notations". If you belong to the group of mathematicians for which a manifold is required to be second countable, then you now have seen the proof that ${}_G\backslash^M$ is second countable; and you may proceed with proof item (b). If you belong to the group of mathematicians for which a manifold is only required to be paracompact[12], then
Fr 10.5. you can argue "in each component" in a a similar way.[13]

(b) "${}_G\backslash^M$ is a Hausdorff space.":

[12] A locally Eudlidean Hausdorff space with a C^1-atlas is paracomact, if and only if every connected component is second countable. To understand our lecture, you may use this as a definition of "paracompactness".

[13] More precisely: Consider a connected component M_0 and consider $GM_0 := \{\sigma p \mid \sigma \in G, \quad p \in M_0\}$. One checks that ${}_G\backslash^{GM_0}$ is a connected component of ${}_G\backslash^M$, and any connected component can be obtained this way.

As M is paracompact, M_0 is second countable. The restriction of the canonical map $\pi \colon M \to {}_G\backslash^M$ yields an open and surjective map $M_0 \xrightarrow{\pi} {}_G\backslash^{GM_0}$, see Proposition 3.4, and as in the proof of Corollary 3.5 you see that the second countability for M_0 implies the second countability of ${}_G\backslash^{GM_0}$. Thus we have seen that any connected component ${}_G\backslash^M$ is second countable.

The action is proper, thus by Definition 2.3 the associated shear map

$$G \times M \xrightarrow{\Theta} M \times M$$
$$(\sigma, x) \mapsto (\sigma x, x)$$

is a proper map. Thus by Lemma 2.8

$$\operatorname{image}(\Theta) = \Theta(G \times M) = \left\{(\sigma \cdot x, x) \mid \sigma \in G, \quad x \in M\right\}$$

is closed. Obviously for the equivalence relation ~ defined by being in the same G-orbit, we have

$$\forall x, y \in X : (x, y) \in \operatorname{image}(\Theta) \iff x \sim y \iff [x] = [y] .$$

Suppose $[x] \neq [y]$. Then $(x, y) \notin \operatorname{image}(\Theta)$. As $\operatorname{image}(\Theta)$ is closed, (x, y) is an inner point of $(M \times M) \smallsetminus \operatorname{image}(\Theta)$. This means that x has an open neighborhood U_x in M and y has an open neighborhood U_y in M, such that

$$\left(U_x \times U_y\right) \cap \operatorname{image}(\Theta) = \varnothing .$$

the sets $\pi(U_x)$ and $\pi(U_y)$ are open due to Proposition 3.4, thus they are (open) neighborhoods of $[x]$ and $[y]$. For proving the Hausdorff property, it thus only remains to check that $\pi(U_x) \cap \pi(U_y) = \varnothing$.

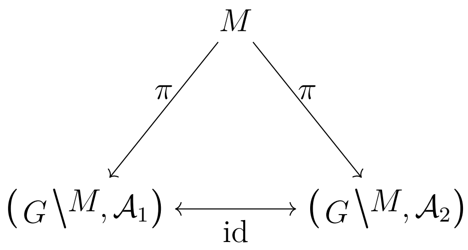
Suppose that $[z] \in \pi(U_x) \cap \pi(U_y)$, and one may choose the representative z of this class such that $z \in U_x$. From $[z] \in \pi(U_y)$ we get the existence of a $w \in U_y$ with $[z] = [w]$. Thus there is a $\sigma \in G$ with $z = \sigma w$. We obtain the contradiction

$$(z, w) = (\sigma w, w) = \Theta(\sigma, w) \in \operatorname{image}(\Theta) \cap \left(U_x \times U_y\right) = \varnothing .$$

(c) "Uniqueness of a smooth structure on $G\backslash M$"
Suppose we have two smooth atlantes $\mathcal{A}_1$ and $\mathcal{A}_2$ on $G\backslash M$ such that $M \xrightarrow{\pi} \left(G\backslash M, \mathcal{A}_i\right)$ is a submersion for $i = 1, 2$. We now apply the following lemma:

Lemma A.1.2. *Let $f : X \to Y$ be a surjective submersion from the C^∞-manifold X to the C^∞-manifold Y, and let Z be a further C^∞-manifold. Let $h : Y \to Z$ be a map. Then h is smooth if and only if $h \circ f$ is smooth.*

We apply the lemma to the diagram



twice:

- Once for $X := M$, $Y := (G\backslash M, \mathcal{A}_1)$, $Z := (G\backslash M, \mathcal{A}_2)$, $f = \pi$ and $h = \mathrm{id}$. Then the smoothness of $h \circ f = \pi\colon M \to (G\backslash M, \mathcal{A}_2)$ gives us the smoothness of $h = \mathrm{id}\colon (G\backslash M, \mathcal{A}_1) \to (G\backslash M, \mathcal{A}_2)$.
- Once for $X := M$, $Y := (G\backslash M, \mathcal{A}_2)$, $Z := (G\backslash M, \mathcal{A}_1)$, $f = \pi$ and $h = \mathrm{id}$. Then the smoothness of $h \circ f = \pi\colon M \to (G\backslash M, \mathcal{A}_1)$ gives us the smoothness of $h = \mathrm{id}\colon (G\backslash M, \mathcal{A}_2) \to (G\backslash M, \mathcal{A}_1)$.

Thus $\mathrm{id}\colon (G\backslash M, \mathcal{A}_1) \to (G\backslash M, \mathcal{A}_2)$ is a diffeomorphism, which says that the two smooth structures are the same.

(d) The construction of a suitable smooth structure on $G\backslash M$ is a bit more involved and will be proven in the next subsection, Subsection 4.2. ∎

4.2 The construction of a suitable smooth structure on the quotient

In this subsection, G will always be a Lie group, acting smoothly and freely on a smooth manifold M.

Definition 4.6. *In the following a submanifold S of M will be called* **transversal** *(to the orbits of G) if we have for all $p \in S$:*

$$T_pS \oplus \mathrm{image}(\mathrm{d}_{\mathbb{1}} s_p) = T_pM\,. \tag{4.1}$$

Recall $\mathrm{d}_{\mathbb{1}} s_p\colon \mathfrak{g} \to T_pM$ is the differential of the orbit map $s_p\colon G \to M$, $\sigma \mapsto \sigma p$.

As s_p is injective, Lemma 4.4 tells us that $\dim \mathrm{image}(\mathrm{d}_{\mathbb{1}} s_p) = \dim G$. Thus for a transversal submanifold we have

$$\dim G = \dim M - \dim S\,.$$

Example 4.7. We continue with Example 4.2 2.). Any non-trivial $\mathbb{R}$-linear function $L: \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \to \mathbb{R}$ defines a hypersurface $N_L := \{x \in S^{2n+1} \mid L(x) = 0\}$ which is a totally geodesic sphere in S^{2n+1} of dimension $2n$.

For which $p \in N_L$ do we have

$$T_pN_L \oplus \mathrm{image}(\mathrm{d}_{\mathbb{1}} s_p) = T_pM\,?$$

Note that $i \in T_{\mathbb{1}}S^1$, and we have the above direct sum decomposition iff $\mathrm{d}_{\mathbb{1}} s_p(i) \notin T_pN_L$. Because $\mathrm{d}_{\mathbb{1}} s_p(i) = ip$ and $T_pN_L = \ker L \cap p^\perp$ we see that the above decomposition is direct precisely on

$$S_L := \{x \in S^{2n+1} \mid L(x) = 0 \text{ and } L(ix) \neq 0\}\,.$$

We see that S_L is an open subset of N_L and

$$N_L \setminus S_L = S^{2n+1} \cap \ker\big(\mathbb{C}^{n+1} \to \mathbb{C}, \quad x \mapsto L(x) - iL(ix)\big)$$

which is a totally geodesic hypersurface in N_L diffeomorphic to S^{2n-1}. Every orbit that passes through $p \in S_L$ is of the form $(\cos t) \cdot p + (\sin t) ip$ which intersects S_L in p and $-p$, i. e., once in the component

$$S_{L,+} := \{x \in S^{2n+1} \mid L(x) = 0 \text{ and } L(ix) > 0\}\,,$$

and once in the component

$$S_{L,-} := \{x \in S^{2n+1} \mid L(x) = 0 \text{ and } L(ix) y 0\}\,.$$

Thus S_L, $S_{L,+}$ and $S_{L,+}$ for any non-trivial $L: \mathbb{C}^{n+1} \to \mathbb{R}$ and all of their open subsets are transversal to the orbits.

Remarks 4.8.

1.) If you try to imagine how a possible manifold S will look like, think of a small manifold. Even when G and M are compact, one can rarely choose compact transveral submanifolds S.

2.) We state that $G \cdot p = \{\sigma p \mid \sigma \in G\} = s_p(G)$ is a submanifold of M, and that $T_{\sigma p}(G \cdot p) = \mathrm{image}(\mathrm{d}_\sigma s_p)$. We will not prove this statement here, and we will not use it in the following. A proof will follow immediately from Theorem 4.1 using the implicit function theorem which then also shows $T_{\sigma p}(G \cdot p) = \ker \mathrm{d}_{\sigma p}\pi$.

Lemma 4.9. *Let $G \curvearrowright M$ freely and smoothly. Then for any $p \in M$, there is a transversal submanifold S through p.*

Proof: Choose a vector space $W \subset T_pM$ with

$$W \oplus \mathrm{image}(\mathrm{d}_{\mathbb{1}} s_p) = T_pM\,.$$

It is easy to construct[14] a smooth submanifold S_0 in M with $p \in S_0$ and and $T_pS_0 = W$. Then

$$S := \left\{ x \in S_0 \;\middle|\; T_xS_0 \cap \mathrm{image}(\mathrm{d}_{\mathbb{1}} s_p) \neq \{0\} \right\}$$

is an open subset S_0 containing p. We see that S is a transversal submanifold with $p \in S$. ∎

Lemma 4.10. *If S is a transversal submanifold, then*

$$\begin{aligned} G \times S &\xrightarrow{\vartheta} M \\ (\sigma, p) &\longmapsto \sigma p \end{aligned}$$

is a local diffeomorphism.

Proof: Because of the local reversal theorem, the statement is equivalent to showing that

$$\mathrm{d}_{(\sigma,p)}\vartheta \colon T_\sigma G \oplus T_pS \longrightarrow T_{\sigma p}M \tag{4.2}$$

is an isomorphism for all $\sigma \in G$ and all $p \in S$.

For $\sigma = \mathbb{1}$, $X \in T_{\mathbb{1}}G$, $Y \in T_pS$ we calculate

$$\mathrm{d}_{(\mathbb{1},p)}\vartheta(X, Y) = \mathrm{d}_{\mathbb{1}} s_p(X) + Y\,.$$

With (4.1) this implies that $\mathrm{image}\left(\mathrm{d}_{(\sigma,p)}\vartheta\right) = T_pM$, i. e., we have (4.2) for $\sigma = \mathbb{1}$.

Now consider arbitrary $\sigma \in G$. From $\sigma(\tau p) = (\sigma\tau)p$ we see that the diagram

[14] The submanifold S_0 can either be constructed in a chart or by taking the Riemannian exponential map $\exp^g$ for some[15] Riemannian metric g and the defining $S_0 := \{\exp X \mid X \in W,\ g(X,X) < \varepsilon^2\}$ for some small $\varepsilon > 0$.

$$\begin{array}{ccc} G \times S & \xrightarrow{\vartheta} & M \\ \Big\downarrow{\scriptstyle \ell_\sigma \times \mathrm{id}_S} & & \Big\downarrow{\scriptstyle \ell_\sigma} \\ G \times S & \xrightarrow[\vartheta]{} & M \end{array}$$

and its derivative at $\tau = \mathbb{1}$

$$\begin{array}{ccc} T_{\mathbb{1}}G \times T_pS & \xrightarrow{d_{(\mathbb{1},p)}\vartheta} & T_pM \\ \Big\downarrow{\scriptstyle d_{\mathbb{1}}\ell_\sigma \times \mathrm{id}_{T_pS}} & & \Big\downarrow{\scriptstyle d_p\ell_\sigma} \\ T_\sigma G \times T_pS & \xrightarrow[d_{(\sigma,p)}\vartheta]{} & T_{\sigma p}M \end{array}$$

commute. Now as three arrows in the last diagram are already known to be isomorphisms, the remaining on, i. e., $d_{(\sigma,p)}\vartheta$, is also an isomorphism. This gives (4.2) in general. ■

End Fr 10.5.
Read following on Fr 17.5.

Definition 4.11. *A transversal submanifold is* **small** *if*

(i) $G \times S \xrightarrow{\vartheta} M$ *is injective and a homeomorphism onto its image (thus an embedding of codimension* 0*).*

(ii) *there is a diffeomorphism* $S \xrightarrow{y} V \Subset \mathbb{R}^{\dim M - \dim G}$

Note that in the literature small transversal submanifolds are often called **slices** of the smooth G-space M.

Lemma 4.12. *Assume that the action* $G \curvearrowright M$ *is smooth, free and* proper*. For each* $p \in M$*, there is a small transversal manifold* S *through* p*.*

Proof: Let S_0 be a transversal submanifold through $p \in M$. We choose a Riemannian metric g on S, which allows us to define the open balls $B_\varepsilon^{(S,g)}(p)$ of radius ε around p in (S,g). We define $S_i := B_{1/i}^{(S,g)}(p)$. For a sufficiently large $i \in \mathbb{N}$, we will prove that $S := S_i$ satisfies Conditions (i) and (ii) in Definition 4.11. Thus we will have proven that S is ia small transversal submanifold through p.

"(i)": Suppose that for all $i \in \mathbb{N}$:

$$G \times S_i \xrightarrow{\vartheta} M$$

is not injective. Thus there are $(\sigma_i, p_i), (\tilde{\sigma}_i, \tilde{p}_i) \in G \times M$, $(\sigma_i, p_i) \neq (\tilde{\sigma}_i, \tilde{p}_i)$ such that $\sigma_i p_i = \tilde{\sigma}_i \tilde{p}_i$. This gives $(\tilde{\sigma}_i)^{-1} \sigma_i p_i = \tilde{p}_i$. Obviously we have

$$\lim_{i \to \infty} p_i = p \text{ and } \lim_{i \to \infty} \tilde{p}_i = p\,.$$

As G acts properly, Proposition 2.7 tells us that a subsequence[16], of $\tau_i := (\tilde{\sigma}_i)^{-1} \sigma_i$ converges to some $\tau_\infty \in G$. In the limit we $\tau_i p_i = \tilde{p}_i$ gives $\tau_\infty p = p$. As G acts freely, this implies $\tau_\infty = \mathbb{1}$.

As ϑ is a local diffeomorphism, there is an open neighborhood U of $(\mathbb{1}, p)$ in $G \times S_0$ such that $\vartheta|_U$ is a diffeomorphism onto its image. There is some $i_0 \in \mathbb{N}$ such that any $i \geq i_0$ satisfies $(\tau_i, p_i) \in U$ and $(\mathbb{1}, \tilde{p}_i) \in U$. Then

$$\vartheta(\tau_i, p_i) = \tau_i p_i = \tilde{p}_i = \vartheta(\mathbb{1}, \tilde{p}_i)\,,$$

and hence $\tau_i = \mathbb{1}$ and $p_i = \tilde{p}_i$, which gives the contradiction $(\sigma_i, p_i) = (\tilde{\sigma}_i, \tilde{p}_i)$.

For any $i \geq i_0$ we thus know that $\vartheta_i := \vartheta|_{G \times S_i} : G \times S_i \to M$ is injective.

Now for $i \geq i_0 + 1$ we will show that ϑ_i is homeomorphism onto its image, i. e., it is also an open map. Note that $S_i \subset \overline{S_i} \subset S_{i-1}$. Let K be a compact neighborhood of $\mathbb{1}$ in G. As ϑ_{i-1} is continuous and injective, this also hold from

$$\vartheta_{i-1}|_{K \times \overline{S_i}} : K \times \overline{S_i} \to \vartheta\big(K \times \overline{S_i}\big) \subset M\,.$$

As any continous bijective map from a compact space to a Hausdorff space is a homeomorphism, the above map is a homeomorphism. Thus by restricting further ontinuous and injective, this also hold for

$$\vartheta_i|_{\mathring{K} \times S_i} : \mathring{K} \times S_i \to \vartheta\big(\mathring{K} \times S_i\big) \subset M\,.$$

We precompose this with the homeomorphism $\ell_{\sigma^{-1}} \times \mathrm{id} : (\sigma \mathring{K}) \times S_i \to \mathring{K} \times S_i$, and thus

$$\vartheta_i|_{(\sigma \mathring{K}) \times S_i} = \vartheta_i|_{\mathring{K} \times S_i} \circ (\ell_{\sigma^{-1}} \times \mathrm{id})$$

defines a homeomorphism $(\sigma \mathring{K}) \times S_i \to \vartheta\big((\sigma \mathring{K}) \times S_i\big)$. Thus the domain of ϑ_i is

[16] We will pass to the subsequence without adapting the notation, for better readability.

covered[17] by a collection of open sets $U_\sigma := \sigma\mathring{K}$, $\sigma \in G$, such that $\vartheta_i|_{U_\sigma}: U_\sigma \to M$ is open. This implies that ϑ_i itself is open and thus a homeomorphism for any $i \geq i_0 + 1$.
"(ii)": Take a chart $y: \hat{U} \to \hat{V}$ of S_0 containing p. Then $S_i \subset \hat{U}$ for sufficiently large $i \geq i_0 + 1$. ■

We are thus may assume that we are in the following setting:

Setting 4.13. *Let $G \curvearrowright M$ be a smooth, proper, free action. Let $p \in M$, $n = \dim M$, and $k = \dim G$. The submanifold S of M is a small G-transversal submanifold through $p \in M$. Then $\vartheta(G \times S)$ is open in M, and $\vartheta: G \times S \to \vartheta(G \times S)$ is a local diffeomorphism and a (global) homeomorphism, thus it is a diffeomorphism. Furthermore ϑ is G-equivariant, i. e., for $\tau, \sigma \in G$ and $q \in S$ we have $\tau \cdot \vartheta(\sigma, q) = \vartheta(\tau\sigma, q)$. In particular, all orbits $G\cdot q$ are submanifolds with $T_{\sigma q}(G\cdot q) = \mathrm{image}(\mathrm{d}_\sigma s_q)$. The Lie group G acts smoothly on $\vartheta(G \times S)$, thus $\vartheta(G \times S)$ is a smooth G-space.*

Furthermore we have a chart $y: S \to V \Subset \mathbb{R}^{n-k}$.

Lemma 4.14. *We assume the setting above. Then the map $\Phi_S := \pi \circ \vartheta \circ (\mathbb{1}, \mathrm{id}) \circ y^{-1}$ obtained by the chain of maps*

$$\begin{array}{l} \mathbb{R}^{n-k} \Supset V \xrightarrow{y^{-1}} S \xrightarrow{(\mathbb{1},\mathrm{id})} G \times S \xrightarrow{\vartheta} \vartheta(G \times S) \xrightarrow{\pi} G\backslash\vartheta(G \times S) \Subset G\backslash M \\ \qquad\qquad\qquad\quad p \longmapsto (\mathbb{1}, p) \end{array}$$

is a homeomorphism.

Corollary 4.15. *$G\backslash M$ is a topological manifold.*

Proof of Lemma 4.14: In the following diagram

$$\begin{array}{ccccc} S & \xrightarrow{(\mathbb{1},\mathrm{id})} & G \times S & \xrightarrow{\vartheta} & \vartheta(G \times S) \\ \Big\downarrow \mathrm{id}_S & \swarrow \mathrm{pr}_S & \Big\downarrow \pi' & & \Big\downarrow \pi \\ S & \underset{\alpha}{\xleftarrow{\cong}} & G\backslash G \times S & \underset{\beta}{\xrightarrow{\cong}} & G\backslash\vartheta(G \times S) \end{array}$$

[17] in German: überdeckt

all maps except α and β are already defined and continuous, and obviously $\mathrm{pr}_S \circ (\mathbb{1}, \mathrm{id}) = \mathrm{id}_S$. The spaces $G \backslash G \times S$ and $G \backslash \vartheta(G \times S)$ carry the quotient topology, thus the maps π and π' are topological quotients. By applying Condition (ii) from Definition and Lemma 3.1, we get a continuous map $\alpha\colon G \backslash G \times S \to S$, making the left square commute. Obviously α is bijective, and $\pi' \circ (\mathbb{1}, \mathrm{id})$ is a continuous right inverse of α, thus α is a homeomorphism. If we apply Condition (ii) from Definition and Lemma 3.1 to $X := G \times S$, $Y := G \backslash G \times S$, $Z := G \backslash \vartheta(G \times S)$, $f := \pi'$, $g := \pi \circ \vartheta$, then we get a well-defined continuous map β making the right square commute. The map β is bijective. If we apply Condition (ii) from Definition and Lemma 3.1 to $X := \vartheta(G \times S)$, $Y := G \backslash \vartheta(G \times S)$, $Z := G \backslash G \times S$, $f := \pi$, $g := \pi' \circ \vartheta^{-1}$, then we see that β^{-1} is also continuous.

So the whole diagram commutes and consists of continuous maps. Thus

$$\pi \circ \vartheta \circ (\mathbb{1}, \mathrm{id}) = \beta \circ \alpha^{-1}$$

is a homeomorphismus. Precomposition with the homomorphism y^{-1} yields the statement. ∎

Lemma 4.16.

$$\mathcal{A} := \left\{ \Phi_S^{-1} \;\middle|\; S \textit{ is a small transversal submanifold} \right\}$$

is a smooth atlas on $G \backslash M$.

Proof: We have already seen that $\mathcal{A}$ is a C^0-atlas for $G \backslash M$. Thus it remains to check that the transition maps are smooth. thus consider two small transversal submanifolds S and $\tilde{S}$, not necessarily running through a common point. The ϑ-map for $\tilde{S}$ will be called $\tilde{\vartheta}$. There are open subsets $U \Subset S$ and $\tilde{U} \Subset \tilde{S}$ such that

$$\vartheta(G \times \tilde{U}) = \vartheta(G \times S) \cap \tilde{\vartheta}(G \times \tilde{S}) = \tilde{\vartheta}(G \times U)\,.$$

Let $y\colon U \to V \Supset \mathbb{R}^{n-k}$ and $\tilde{y}\colon \tilde{U} \to \tilde{V} \Supset \mathbb{R}^{n-k}$ be two charts of U and $\tilde{U}$. We get charts as in Lemma 4.14 for $G \backslash M$

$$\Phi_S^{-1}\colon G \backslash \vartheta(G \times U) \to V, \quad \Phi_{\tilde{S}}^{-1}\colon G \backslash \tilde{\vartheta}(G \times \tilde{U}) \to \tilde{V},$$

We have to show the smoothness of the transition map

$$\Phi_{\tilde S}^{-1}\circ\Phi_S\Big|_{\Phi_S^{-1}(\vartheta(G\times U))}\colon \Phi_S^{-1}\big(\vartheta(G\times U)\big)\longrightarrow \Phi_{\tilde S}^{-1}\big(\tilde\vartheta(G\times\tilde U)\big)$$

which is the composition

$$\begin{array}{ccccccc}
V & \xleftarrow[\cong]{\;y\;} & U & \underset{\mathrm{pr}_U}{\overset{(\mathbb{1},\mathrm{id})}{\rightleftarrows}} & G\times U & \xrightarrow[\cong]{\;\vartheta\;} & \vartheta(G\times U)\\
 & & & & & & \Vert\\
\tilde V & \xleftarrow[\cong]{\;\tilde y\;} & \tilde U & \underset{\mathrm{pr}_{\tilde U}}{\overset{(\mathbb{1},\mathrm{id})}{\rightleftarrows}} & G\times\tilde U & \xrightarrow[\cong]{\;\tilde\vartheta\;} & \vartheta(G\times\tilde U)
\end{array}$$

Thus the smoothness holds, as it is a composition of smooth maps. ■

Lemma 4.17. *We equip $G\backslash M$ with the smooth structure of the preceding lemma. Then $\pi\colon M\to G\backslash M$ is a submersion.*

Proof: We fix a small transversal submanifold S with associated map ϑ. It is sufficient to verify the submersion property of π on the open subset $\vartheta(G\times S)$ as such subsets cover all of M. Now, out of the commuting diagram in the proof of Lemma 4.14, we get the following commuting diagram

$$\begin{array}{ccc}
G\times S & \xrightarrow{\;\vartheta\;} & \vartheta(G\times S)\\
{\scriptstyle \mathrm{pr}_S}\big\downarrow & & \big\downarrow{\scriptstyle \pi}\\
S & \xrightarrow[\;\rho\;]{} & G\backslash\vartheta(G\times S)
\end{array}$$

where the map ϑ is a diffeomorphism by the definition of "small transversal submanifold" and where $\rho := \beta\circ\alpha^{-1}$ is a diffeomorphism by the construction of the smooth structure on $G\backslash M$. As $\mathrm{pr}_S : G\times S\to S$ is obviously a submersion. ■

The formula $\dim G\backslash M = \dim M - \dim G$ is obvious from the construction of the smooth structure on $G\backslash M$. The proof of Theorem 4.1 is thus complete.

5 Further examples

5.1 Frame bundles

For a Riemannian manifold (M, g) and $p \in M$ we define $P_{\mathrm{O}}(M, g)|_p$ as the set of all orthonormal bases of (T_pM, g_p). The group $\mathrm{O}(n)$ acts on the right on this bundle by the usual transformation of basis formula. In fact let $(e_1, \dots, e_n)$ be an orthonormal basis, viewed as a row vector whose coefficients are vectors in T_pM. Let $A = (a_{ij}) \in \mathrm{O}(n)$, then one defines

$$(\tilde{e}_1, \dots, \tilde{e}_n) := (e_1, \dots, e_n) \cdot A$$

by matrix multiplication, i. e., $\tilde{e}_j = \sum_{i=1}^n a_{ij} e_i$. This right action is transitive and free, and there is a unique smooth structure on $P_{\mathrm{O}}(M, g)|_p$ such that $\mathrm{O}(n) \to P_{\mathrm{O}}(M, g)|_p$, $A \mapsto (e_1, \dots, e_n) \cdot A$ is a diffeomorphism. We define $P_{\mathrm{O}}(M, g) := \dot{\bigcup}_{p \in M} P_{\mathrm{O}}(M, g)|_p$. Then $P_{\mathrm{O}}(M, g)$ carries a unique smooth topology, such that for any $U \Subset M$ the following property holds:

> Let $e_i : U \to P_{\mathrm{O}}(M, g)$, $i = 1, 2, \dots, n$ be maps such that
>
> $$\mathcal{E}(p) := (e_1(p), \dots, e_n(p)) \in P_{\mathrm{O}}(M, g)|_p.$$
>
> Then $\mathcal{E}$ is smooth as a map $U \to P_{\mathrm{O}}(M, g)$ if and only if e_i is smooth as a vector field on U for any i.

The group $\mathrm{O}(n)$ acts smoothly, freely, but no longer transitively on $P_{\mathrm{O}}(M, g)$, and the orbits are the subsets $P_{\mathrm{O}}(M, g)|_p$ which are in fact submanifolds. One can check that this action is proper, and we consider the quotient space, which is a smooth manifold by Theorem 4.1. The "canonical"

$$I : M \to P_{\mathrm{O}}(M, g) / \mathrm{O}(n)$$

that maps $p \in M$ to the orbit $P_{\mathrm{O}}(M, g)|_p$ is a diffeomorphism. Usually one identifies M with this quotient.

If M carries other structure than a Riemannian metric, one can often do similar definitions for the structure group and the adapted bases.

Examples 5.1.

1.) If M has no structure at all, we may take all frames. This yields $P_{\mathrm{GL}}(M,g)$, which is a manifold on which $\mathrm{GL}(n,\mathbb{R})$ acts smoothly, freely, and properly, and such that $M \cong P_{\mathrm{GL}}(M,g)/\mathrm{GL}(n,\mathbb{R})$.

2.) If M has an orientation, we may take all positively oriented frames. This yields $P_{\mathrm{GL}_+}(M,g)$, which is a manifold on which $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A > 0\}$ acts smoothly, freely, and properly, and such that $M \cong P_{\mathrm{GL}_+}(M,g)/\mathrm{GL}_+(n,\mathbb{R})$.

5.2 Homogeneous spaces

Read this on May 17, too

We now consider a Lie goup, and we assume that H is a subgroup of G that is closed as a subset of the topological space G. We have already mentioned, that this implies that H is also submanifold. We get an action of H on G as follows $a(\tau,\sigma) := \ell_\tau(\sigma) := \tau\sigma$ for $\tau \in H$ and $\sigma \in G$.

Lemma 5.2. *The action $H \curvearrowright G$ by left multiplication as described above is a proper, free and smooth action. The same holds for other left- and right-actions $H \curvearrowright G$ given in Example 2.2 1.).*

Proof: That the action is free and smooth is obvious. We will check properness using Proposition 2.7.

So let us assume that $(x_i)_{i\in\mathbb{N}}$ is a sequence in G converging to $x_\infty \in G$, and that $(\sigma_i)_{i\in\mathbb{N}}$ is a sequence in H, such that $(\sigma_i \cdot x_i)_{i\in\mathbb{N}}$ converges in G to $y_\infty \in G$. It follows that in G we have the limit

$$\lim_{i\to\infty} \sigma_i = \lim_{i\to\infty}(\sigma_i \cdot x_i \cdot x_i^{-1}) = x_\infty \cdot y_\infty^{-1}.$$

As H is closed, we have $x_\infty \cdot y_\infty^{-1} \in H$. Thus $(\sigma_i)_{i\in\mathbb{N}}$ converges in H to $x_\infty \cdot y_\infty^{-1}$ and the statement follows with Proposition 2.7. ■

We now, let again be H a closed subgroup in a Lie group G. We consider the manifold G/H, whose elements are **left cosets**, i. e., subsets of the form $\sigma \cdot H$ where $\sigma \in G$. Left multiplication turns G/H into a G-space with a transitive G action. We want to argue, that every smooth G-space with a transitive action is of this form, as shown in the following exercise. We will thus additionally assume in this section from now on:

The topology of G is second countable

This assumption is equivalent to the condition, that G has countably many connected components.[18] With this additional condition Sard's theorem[19] implies the following

Proposition 5.3 (Consequence of Sard's theorem)**.** *Let G be a Lie group with countably many connected components acting transitively and smoothly on a smooth manifold M, $p \in M$, then the orbit map*

$$s_p : G \to M, \quad \sigma \mapsto \sigma \cdot p$$

is a submersion.

With this information the following exercise can be solved:

Exercise 5.4. *Let G be a Lie group acting smoothly and transitively on a manifold M. Let H be the isotropy group of $p \in M$. Show that there is a unique G-equivariant diffeomorphism $F : {}^{G}\!\big/_{H} \to M$ that maps $\mathbb{1} \cdot H$ to p.*

Smooth G-spaces with a transitive G-action are called **homogeneous spaces**. They are essentially given by the pair (G, H) where G is a Lie group and H a closed subgroup. However, a given manifold M can be obtained in several ways as a homogeneous space ${}^{G}\!\big/_{H}$, thus one always has to consider M as a G-space. For example as smooth manifolds we have $S^{2n+1} \cong {}^{\mathrm{O}(2n+2)}\!\big/_{\mathrm{O}(2n+1)} \cong {}^{\mathrm{U}(n+1)}\!\big/_{\mathrm{U}(n)}$, but this does not hold as homogeneous spaces, as it is a G-space for another G. However, when one writes ${}^{G}\!\big/_{H}$, this is usually meant in the sense of G-spaces.

Homogeneous spaces are of tremendous importance, as they provide many examples, and there are man techniques and even computer programs to calculate many properties, e. g., its curvature properties, the spectrum of the Laplace operator on such spaces.

Definition 5.5. *Let G be a Lie group acting smoothly and transitively on a manifold M. Let H be the isotropy group of $p \in M$. Thus for any $h \in H$, $\ell_h : M \to M$ is a diffeomorphism fixing p. We define its* **isotropy representation**

$$I : H \mapsto \mathrm{GL}(T_p M), \quad h \mapsto \mathrm{d}_p \ell_h\,.$$

[18] In many analysis lectures, this assumption is considered anyhow.
[19] we do not want to prove or discuss this here and how to apply this. See [20] for a reference.

Let again $s_p: G \to M$, $\sigma \mapsto \sigma \cdot p$ be the orbit map, and let H the isotropy group at p. If $\mathfrak{p}$ is a complement of $\mathfrak{h}$ in $\mathfrak{g}$, i. e., if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, then the differential $\mathrm{d}s_p: \mathfrak{g} \to T_pM$ of the orbit map s_p is an isomorphism. One thus often identifies M with G/H and T_pM with $\mathfrak{p}$. The isotropy representation is thus a map

$$I: H \to \mathrm{GL}(\mathfrak{p}) .$$

Lemma 5.6. *Let (M, g) be a Riemannian manifold, and assume that $G \curvearrowright M$ is a smooth, transitive and isometric action. If the isotropy representation is irreducible, i. e., if there is no H-invariant linear subspace $W \subset T_pM$ with $\{0\} \neq W \neq T_pM$, then M is an Einstein manifold, i. e., there is a $\lambda \in \mathbb{R}$, such that $\mathrm{ric} = \lambda g$.*

Proof: As the action is isometric, the isotropy representation is a map $H \to \mathrm{O}(T_pM)$. For any $h \in H$ we have $I(h)^* g_p = g_p$ and $I(h)^* ric_p = \mathrm{ric}_p$, thus also $\mathrm{Ric}_p \circ I(h) = I(h) \circ \mathrm{Ric}_p$. As the endomorphism Ric_p is symmetric, it is diagonalizable. Let λ be an eigenvalue of Ric_p and define $W := \ker(\mathrm{Ric}_p - \lambda\,\mathrm{id})$ as the corresponding eigenspace. As $I(h)$ and Ric_p commute, we get $\big(I(h)\big)(W) \subset W$. Thus W is invariant under the action of G given by I. We assumed that $\{0\}$ and T_pM are the only invariant linear subspaces, and as $W \neq \{0\}$ by the choice of λ, we have $W = T_pM$. Thus $\mathrm{ric}_p = \lambda g_p$.

Now consider any point $q \in M$, and we write $q = \sigma^{-1} \cdot p$, $\sigma \in G$. As ℓ_σ acts isometrically we have $\ell_\sigma^* g_p = g_q$ and $\ell_\sigma^* \mathrm{ric}_p = \mathrm{ric}_q$. Thus we have $\mathrm{ric}_q = \lambda g_q$ for all $q \in M$. ∎

Example 5.7. We consider the Lie group $\mathrm{SU}(3)$ with a bi-invariant Riemannian metric g. The adjoint representation $\mathrm{Ad}: \mathrm{SU}(3) \to \mathrm{GL}(\mathfrak{su}(3))$ turns $\mathfrak{su}(3)$ into an $\mathrm{SU}(3)$-space. It has no non-trivial[20] linear subspace W invariant under the $\mathrm{SU}(3)$ action. This implies that the bi-invariant metric is unique up to a constant. We may normalize g such that

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(3)$$

has length 1.

The scalar product on $\mathfrak{su}(3) \subset \mathbb{C}^{3\times 3}$ is then

$$\langle A, B \rangle = \frac{1}{2} \mathrm{tr}(A^* B)$$

[20] Such a subspace is trivial if $W = \{0\}$ or $W = \mathfrak{su}(3)$.

As then SO(3) is a closed subgroup of SU(3), consisting of those matrices in SU(3), where all coefficients are real. The quotient $M := \mathrm{SU}(3)/\mathrm{SO}(3)$ is called the **Wu manifold**[21], and plays an important role in bordism theory. The manifold carries a quotient metric, denoted as $\overline{g}$, and $\dim M = 5$.

We define $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, namely

$$\mathfrak{p} = \mathrm{span}\left\{\begin{pmatrix} i & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{pmatrix}, \begin{pmatrix} 0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{pmatrix}\right\}.$$

One may check[22] that the action of SO(3) on $\mathfrak{p}$ has no non-trivial invariant linear subspaces. Thus $(M, \overline{g})$ is an Einstein manifold. One can check that $\mathrm{ric}^{\overline{g}} = 6\overline{g}$.

5.3 Bi-quotients

We assume that G is a compact connected Lie group, and H a closed subgroup. As discussed above the associated homogeneous space G/H is a G-space. It may happen that a subgroup K of G still acts freely on on G/H. Then the **bi-quotient**

$$K \backslash G/H$$

is a smooth manifold. This gives rise to interesting examples, as e. g., the Gromoll-Mayer sphere, see `https://ncatlab.org/nlab/show/Gromoll-Meyer+sphere`. This is a compact 7-dimensional manifold, homeomorphic, but not diffeomorphic to S^7, and it carries a metric with sectional curvature $K \geq 0$.

6 Riemannian submersions and the O'Neill formula

Tu 14.5.

Notes on literature for this section:

- [10, Chap. 9]: a good and deep, but not easily readable reference
- [23, Chap. 7, Def. 44 and following], textbook by O'Neill, but no proofs
- [25], original article by O'Neill including proofs

[21] In fact it belongs to the family $\mathrm{SU}(n)/\mathrm{SO}(n)$ of so-called Landweber's manifolds
[22] proof omitted here!

Definition 6.1 (Riemannian submersion). *Let (M,g) and $(B.h)$ be Riemannian manifolds, $m = \dim M$, $n = \dim B$, and $f\colon M \to B$ a submersion. For any $p \in M$ we define the* **vertical space** *at p as $\mathcal{V}_p := \ker \mathrm{d}_p f$ which is a vector space of dimension $n - m$. Then $\mathcal{V} := \bigcup_{p\in M} \mathcal{V}_p$ is a submanifold of TM of dimension n. Further we define the* **horizontal space** *at p as*

$$\mathcal{H}_p := (\mathcal{V}_p)^\perp = \{X \in T_pM \mid X \perp \mathcal{V}_p\}\,.$$

The map f is called a **Riemannian submersion** *if*

$$\forall p \in M : \mathrm{d}_p f\big|_{\mathcal{H}_p}\colon \mathcal{H}_p \to T_{f(p)}B \tag{6.1}$$

is an isometry.

We can decompose any $X \in T_pM$ as

$$X = X_{\mathrm{ver}} + X_{\mathrm{hor}}, \quad X_{\mathrm{ver}} \in \mathcal{V}_p,\ X_{\mathrm{hor}} \in \mathcal{H}_p\,.$$

Further we denote the orthogonal projections as $\pi_{\mathrm{hor}}\colon TM \to \mathcal{H}$ and $\pi_{\mathrm{ver}}\colon TM \to \mathcal{V}$ and we define

$$\Gamma(\mathcal{V}) := \{X \in \mathfrak{X}(TM) \in \forall p \in M : X\big|_p \in \mathcal{V}_p\}\,,$$
$$\Gamma(\mathcal{H}) := \{X \in \mathfrak{X}(TM) \in \forall p \in M : X\big|_p \in \mathcal{H}_p\}\,.$$

Remarks 6.2.

1.) Equation 6.1 is "the natural" geometric relation, one may impose to a submersion. More precisely we have for all $p \in M$ and $Y \in T_{f(p)}B$:

$$h(Y,Y) = \inf\{g(X,X) \mid \mathrm{d}_p f(X) = Y\}\,.$$

2.) The notion of a Riemannian submersion is in some sense "dual" to an isometric embedding: let $f\colon (M,g) \to (B,h)$ be a smooth map between Riemannian manifolds, and let $(\mathrm{d}_p f)^*\colon T_{f(p)}B \to T_pM$ be the map adjoint to $\mathrm{d}_p f\colon T_pM \to T_{f(p)}B$. Then

   (a) $\mathrm{d}_p f\colon T_pM \to T_{f(p)}B$ is surjective iff $(\mathrm{d}_p f)^*\colon T_{f(p)}B \to T_pM$ is injective,

   (b) $\mathrm{d}_p f\colon T_pM \to T_{f(p)}B$ is injective iff $(\mathrm{d}_p f)^*\colon T_{f(p)}B \to T_pM$ is surjective,

(c) $\mathrm{d}_p f : T_p M \to T_{f(p)} B$ is surjective and satisfies (6.1) iff $(\mathrm{d}_p f)^* : T_{f(p)} B \to T_p M$ is isometric,

etc.

3.) The Gauß formula relates the curvatures of (M, g) and of (B, h) if f is an isometric immersion. Similarly, the O'Neill formula, discussed below, relates the curvatures of (M, g) and (B, h) if f is an Riemannian submersion.

Examples 6.3.

1.) Assume that G acts smoothly, freely and properly on a Riemannian manifold M with metric g. We additionally assume that the action is **isometric**, i. e., any left multiplication ℓ_σ is an isometry. For $q \in B := G\backslash M$ we want to construct a scalar product on $T_q B$. Choose $p \in f^{-1}(\{q\})$. Let $\mathcal{H}_p$ be the orthogonal complement of $\ker \mathrm{d}_p f$. Then $\mathrm{d}_p f$ defines a vector space isomorphism $\mathcal{H}_p \to T_q B$, and we define the scalar product on $T_q B$ such that this is an isometric isomorphism. One has to check, that this scalar product in independent of the choice of p in $f^{-1}(\{q\})$, which easily follows from the fact that G acts isometrically. The Riemannian metric on B obtained this way, is called the **quotient metric**.

If we equip B with this Riemannian, then $\pi : M \to B = G\backslash M$ is a Riemannian submersion.

2.) We consider the Hopf action $S^1 \curvearrowright S^{2n+1}$. We equip S^{2n+1} with the standard metric g^{sph}. Then the Hopf action is isometric. Following Item 1.) we obtain a quotient metric on $\mathbb{C}\mathrm{P}^n = S^1\backslash S^{2n+1}$. This metric is called the **Fubini–Study metric** g^{FS}.

3.) Let H be a closed subgroup of a Lie group G. Assume that G carries a right-invariant Riemannian metric. Then right multiplication by any $h \in H$ is an isometry. Thus we obtain a Riemannian metric on G/H, such that $\pi : G \to G/H$ is a Riemannian submersion. However, in general the left action G on G/H is not isometric. It is ismetric if the Riemannian metric on G is bi-invariant.

4.) Let $\pi : M \to B$ be an arbitrary submersion. Let $\check{g}$ be a Riemannian metric on B. As always $\mathcal{V}_p := \ker(\mathrm{d}_p \pi)$. We choose an arbitrary Riemannian metric g_0 on M. Define $\mathcal{H}_p := \{X \in T_p M \mid g_0(X, Y) = 0 \quad \forall Y \in \mathcal{V}_p\}$. Then define the Riemannian metric g on M such that

- $\mathrm{d}_p\pi$ defines an isometry of $(\mathcal{H}_p, g)$ to $(T_{\pi(p)}B, \check{g}_p)$,
- $g|_{\mathcal{V}_p\times\mathcal{V}_p} = g_0|_{\mathcal{V}_p\times\mathcal{V}_p}$,
- $g(X,Y) = 0$ for all $X \in \mathcal{H}_p$ and $Y \in \mathcal{V}_p$.

Then $\pi\colon (M,g) \to (B,\check{g})$ is a Riemannian submersion.

In this way one obtains many examples that do not arise from isometric group actions as in Item 1.).

Definition 6.4. *For a Riemannian submersion $f\colon M \to B$ with notation as above we define*

$$A_\Gamma(X,Y) := \pi_{\mathrm{ver}}([X,Y]) \in \Gamma(\mathcal{V})$$

for all $X, Y \in \Gamma(\mathcal{H})$. We thus have a alternating bilinear map $A_\Gamma\colon \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \Gamma(\mathcal{V})$.

We will write $A \in \Gamma(\Lambda^2\mathcal{H}^* \otimes \mathcal{V})$ for a family $(A_p\colon \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{V}_p)_{p\in M}$ of alternating bilinear maps, smoothly depending on p.

Lemma 6.5. *A_Γ is tensorial, i. e., there is a $A \in \Gamma(\Lambda^2\mathcal{H}^* \otimes \mathcal{V})$, such that for all $p \in M$, $X, Y \in \Gamma(\mathcal{H})$ we*[23] *have*

$$(A_\Gamma(X,Y))\big|_p = A_p(X|_p, Y|_p)\,.$$

This tensor A will be called the **A-tensor** of the Riemannian submersion.

Proof: Similarly as for tensors[24], the statement is proven if we show that

$$A_\Gamma\colon \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \Gamma(\mathcal{V})$$

is a bilinear map of $C^\infty(M)$-modules. The additivity of $A_\Gamma(X,Y)$ both in X and in Y is obvious. For some $f \in C^\infty(M)$ we use the identity $[fX,Y] = f[X,Y] - (\partial_Y f)X$

[23] Note that our definition of the A-tensor slightly differs from the one used in [10] and [25]. One the one hand the definitions differ by a factor 2, on the other hand our tensor is defined only on horizontal vectors.

[24] with the same proof

and calculate

$$A_\Gamma(fX,Y) = \pi_{\mathrm{ver}}\big([fX,Y]\big) = \pi_{\mathrm{ver}}\big(f[X,Y]\big) - \underbrace{\pi_{\mathrm{ver}}\big((\partial_Y f)X\big)}_{=0} = fA_\Gamma(X,Y)\,.$$

■

If $X \in \mathfrak{X}(TB)$, then there is a unique vector field $\widehat{X} \in \mathfrak{X}(\mathcal{H})$, such that $\widehat{X}$ is f-related to X, called the **horizontal lift** of X. However, keep in mind, that other non-horizontal vector fields on M may also be f-related to X.

Lemma 6.6. *Let $\widetilde{X} \in \mathfrak{X}(M)$, resp. $\widetilde{Y} \in \mathfrak{X}(M)$ be f-related to $X \in \mathfrak{X}(B)$ resp. $Y \in \mathfrak{X}(B)$, then*

$$A(\widetilde{X},\widetilde{Y}) = [\widetilde{X},\widetilde{Y}] - \widehat{[X,Y]}\,.$$

In other words $\pi_{\mathrm{hor}}\big([\widetilde{X},\widetilde{Y}]\big) = \widehat{[X,Y]}$.

Proof: The f-relations for $\widetilde{X}$ and $\widetilde{Y}$ with X and Y imply, using Exercise 1.6 that $[\widetilde{X},\widetilde{Y}]$ is f-related to $[X,Y]$. Then

$$[\widetilde{X},\widetilde{Y}] - \widehat{[X,Y]}$$

is f-related to $[X,Y] - [X,Y] = 0 \in \mathfrak{X}(TB)$, thus $[\widetilde{X},\widetilde{Y}] - \widehat{[X,Y]}$ is vertical. As $\widehat{[X,Y]}$ is horizontal, it is the horizontal part of $[\widetilde{X},\widetilde{Y}]$, while $A(\widetilde{X},\widetilde{Y})$ is its vertical part. ■

In the following we will use the Koszul formula for the Levi–Civita connection.

Lemma A.1.1. *Let (M,g) be a Riemannian manifold. We write $\langle X,Y\rangle = g(X,Y)$. For $X,Y,Z \in \mathfrak{X}(M)$ we have the* **Koszul formula:**

$$\begin{aligned} &2\langle \nabla_X Y, Z\rangle \\ &= \partial_X \langle Y,Z\rangle + \partial_Y \langle X,Z\rangle - \partial_Z \langle X,Y\rangle \\ &\quad + \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle \end{aligned} \tag{6.2}$$

Lemma 6.7. *For $X,Y \in \mathfrak{X}(B)$ we have*

$$\nabla^M_{\widehat{X}}\widehat{Y} = \widehat{\nabla^B_X Y} + \tfrac{1}{2}A(\widehat{X},\widehat{Y})\,.$$

Proof: Let $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(M)$ be f-related to $X, Y, Z \in \mathfrak{X}(B)$. If $\widetilde{Y}$ or $\widetilde{Z}$ is horizontal, then

$$g(\widetilde{Y}, \widetilde{Z}) = g\big(\pi_{\mathrm{hor}}(\widetilde{Y}), \pi_{\mathrm{hor}}(\widetilde{Z})\big) = h(Y, Z) \circ f\,.$$

Then one checks – again assuming that $\widetilde{Y}$ or $\widetilde{Z}$ is horizontal – that

$$\begin{aligned}\partial_{\widetilde{X}}\Big(g\big(\widetilde{Y}, \widetilde{Z}\big)\Big) &= \partial_{\widetilde{X}}\Big(h(Y,Z) \circ f\Big) = \mathrm{d}\big(h(Y,Z)\big) \circ \mathrm{d}f(\widetilde{X}) \\ &= \mathrm{d}\big(h(Y,Z)\big) \circ X \circ f = \Big(\partial_X\big(h(Y,Z)\big)\Big) \circ f\,.\end{aligned}$$

We apply this formula, and some variant of it where X, Y, Z are permuted, to the Koszul formulas on M and on B. We assume that at least two of the vector fields $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ are horizontal. Then the first lines of the right hand side of (6.2) for M and B cancel and we get – omitting some obvious precompositions with f –

$$\begin{aligned}&2g\big(\nabla^M_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\big) - 2h\big(\nabla^B_X Y, Z\big) \\ &= g\big([\widetilde{X}, \widetilde{Y}], \widetilde{Z}\big) - g\big([\widetilde{Y}, \widetilde{Z}], \widetilde{X}\big) + g\big([\widetilde{Z}, \widetilde{X}], \widetilde{Y}\big) \\ &\quad - h([X,Y],Z) + h([Y,Z],X) - h([Z,X],Y)\,. \end{aligned} \tag{6.3}$$

"vertical part": Consider a vertical vector field $\zeta \in \Gamma(\mathcal{H})$. Because of $\mathrm{d}f(\zeta) = 0$, the field ζ is f-related to $0 \in \mathfrak{X}(B)$. Thus for $X, Y \in \mathfrak{X}(B)$ we apply formula (6.3) for the horizontal lifts $\widetilde{X} := \widehat{X}$ and $\widetilde{Y} := \widehat{Y}$, and for $Z = 0$, $\widetilde{Z} = \zeta$. We get

$$\begin{aligned}2g\big(\nabla^M_{\widehat{X}}\widehat{Y}, \zeta\big) &= g\big([\widehat{X}, \widehat{Y}], \zeta\big) - g\big([\widehat{Y}, \zeta], \widehat{X}\big) + g\big([\zeta, \widehat{X}], \widehat{Y}\big) \\ &\overset{(*)}{=} g\big(A(\widehat{X}, \widehat{Y}), \zeta\big)\end{aligned}$$

where we used at $(*)$ that $\zeta \perp \widehat{[X,Y]}$, that $\pi_{\mathrm{hor}}([\widehat{Y}, \zeta]) = \widehat{[Y,0]} = 0$, and that $\pi_{\mathrm{hor}}([\zeta, \widehat{X}]) = \widehat{[0,X]} = 0$.

As ζ was chosen as an arbitrary smooth vertical vector field, this yields $\pi_{\mathrm{ver}}\big(\nabla^M_{\widehat{X}}\widehat{Y}\big) = \frac{1}{2}A(\widehat{X}, \widehat{Y})$.

"horizontal part": For $X, Y, Z \in \mathfrak{X}(B)$ we consider the horizontal lifts $\widetilde{X} := \widehat{X}$, $\widetilde{Y} := \widehat{Y}$, and $\widetilde{Z} := \widehat{Z}$. Then

$$g\big([\widehat{X}, \widehat{Y}], \widehat{Z}\big) = g\big(\widehat{[X,Y]}, \widehat{Z}\big) = h([X,Y],Z)\,.$$

Applying this, also in permuted forms, formula (6.3) yields

$$g\big(\nabla^M_{\widehat{X}}\widehat{Y}, \widehat{Z}\big) = h\big(\nabla^B_X Y, Z\big)\,,$$

and as $Z \in \mathfrak{X}(B)$ was arbitrary, this implies

$$\pi_{\mathrm{hor}}\big(\nabla^M_{\widehat{X}}\widehat{Y}\big) = \widehat{\nabla^B_X Y}\,.$$

∎

Remark 6.8. We also have the following to extended statements for $X, Y \in \Gamma(\mathcal{H})$, which we will not use and will not prove here.

$$\pi_{\mathrm{ver}}(\nabla^M_X Y) = \tfrac{1}{2}A(X,Y) \text{ and } \mathrm{d}f(\nabla^M_X Y) = \nabla^B_{\mathrm{d}f(X)}\mathrm{d}f(Y)\,.$$

Theorem 6.9 (O'Neill formula)**.** *Let $f\colon M \to B$ be a Riemannian submersion. For all $p \in M$ and all $\xi, \eta, \zeta, \omega \in \mathcal{H}_p$ we have*

$$\begin{aligned}\big\langle R^B\big(\mathrm{d}f(\xi), \mathrm{d}f(\eta)\big)\mathrm{d}f(\zeta), \mathrm{d}f(\omega)\big\rangle &= \langle R^M(\xi,\eta)\zeta,\omega\rangle\\ -\tfrac{1}{2}\langle A(\xi,\eta), A(\zeta,\omega)\rangle + \tfrac{1}{4}&\langle A(\eta,\zeta), A(\xi,\omega)\rangle - \tfrac{1}{4}\langle A(\xi,\zeta), A(\eta,\omega)\rangle\,.\end{aligned}$$

Proof: For any $\xi \in \mathcal{H}_p$, we consider a vector field $X \in \mathfrak{X}(B)$ with $X\big|_{f(p)} = \mathrm{d}f(\xi)$. Then $\widehat{X}\big|_p = \xi$. Similarly, we choose vector fields $Y, Z, W \in \mathfrak{X}(B)$ for $\eta, \zeta, \omega \in \mathcal{H}_p$. As O'Neill's formula is a pointwise relation, it is thus equivalent to proving

$$\begin{aligned}\big\langle R^B\big(X,Y\big)Z,W\big\rangle &= \big\langle R^M(\widehat{X},\widehat{Y})\widehat{Z},\widehat{W}\big\rangle\\ -\tfrac{1}{2}\langle A(\widehat{X},\widehat{Y}), A(\widehat{Z},\widehat{W})\rangle + \tfrac{1}{4}&\langle A(\widehat{Y},\widehat{Z}), A(\widehat{X},\widehat{W})\rangle - \tfrac{1}{4}\langle A(\widehat{X},\widehat{Z}), A(\widehat{Y},\widehat{W})\rangle\end{aligned}$$

for all $X, Y, Z, W \in \mathfrak{X}(B)$. We transform the summands in

$$R^M(\widehat{X},\widehat{Y})\widehat{Z} = \nabla^M_{\widehat{X}}\nabla^M_{\widehat{Y}}\widehat{Z} - \nabla^M_{\widehat{Y}}\nabla^M_{\widehat{X}}\widehat{Z} - \nabla^M_{[\widehat{X},\widehat{Y}]}\widehat{Z}\,,$$

using Lemma 6.7, again suppressing some obvious precompositions with f:

$$\begin{aligned}\big\langle \nabla^M_{\widehat{X}}\nabla^M_{\widehat{Y}}\widehat{Z},\widehat{W}\big\rangle &= \partial_{\widehat{X}}\big\langle \nabla^M_{\widehat{Y}}\widehat{Z},\widehat{W}\big\rangle - \big\langle \nabla^M_{\widehat{Y}}\widehat{Z},\nabla^M_{\widehat{X}}\widehat{W}\big\rangle\\ &= \partial_{\widehat{X}}\big\langle \pi_{\mathrm{hor}}\big(\nabla^M_{\widehat{Y}}\widehat{Z}\big),\widehat{W}\big\rangle - \big\langle \pi_{\mathrm{hor}}\big(\nabla^M_{\widehat{Y}}\widehat{Z}\big),\pi_{\mathrm{hor}}\big(\nabla^M_{\widehat{X}}\widehat{W}\big)\big\rangle\\ &\quad - \big\langle \pi_{\mathrm{ver}}\big(\nabla^M_{\widehat{Y}}\widehat{Z}\big),\pi_{\mathrm{ver}}\big(\nabla^M_{\widehat{X}}\widehat{W}\big)\big\rangle\\ &= \partial_X\big\langle \nabla^B_Y Z, W\big\rangle - \big\langle \nabla^B_Y Z,\nabla^B_X W\big\rangle - \big\langle \tfrac{1}{2}A(\widehat{Y},\widehat{Z}),\tfrac{1}{2}A(\widehat{X},\widehat{W})\big\rangle\\ &= \big\langle \nabla^B_X\nabla^B_Y Z, W\big\rangle - \tfrac{1}{4}\big\langle A(\widehat{Y},\widehat{Z}),A(\widehat{X},\widehat{W})\big\rangle\,,\end{aligned} \tag{6.4}$$

and by exchanging X and Y we also have

$$\left\langle \nabla^M_{\widehat{Y}} \nabla^M_{\widehat{X}} \widehat{Z}, \widehat{W} \right\rangle = \left\langle \nabla^B_Y \nabla^B_X Z, W \right\rangle - \tfrac{1}{4} \left\langle A(\widehat{X}, \widehat{Z}), A(\widehat{Y}, \widehat{W}) \right\rangle . \tag{6.5}$$

Finally, we want to control

$$\nabla^M_{[\widehat{X},\widehat{Y}]} \widehat{Z} = \nabla^M_{\widehat{[X,Y]}} \widehat{Z} + \nabla^M_{A(\widehat{X},\widehat{Y})} \widehat{Z}. \tag{6.6}$$

After multiplication with $\widehat{W}$, Lemma 6.7 turns the first summand into

$$\langle \nabla^M_{\widehat{[X,Y]}} \widehat{Z}, \widehat{W} \rangle = \langle \nabla^B_{[X,Y]} Z, W \rangle . \tag{6.7}$$

However, we cannot apply Lemma 6.7 to the other summand directly, as $\nu := A(\widehat{X}, \widehat{Y})$ is vertical, i. e., $\nu \in \Gamma(\mathcal{V})$. However ν is f-related to 0 and thus $[\nu, \widehat{Z}]$ is f-related to 0 as well, i. e.,

$$\nabla^M_\nu \widehat{Z} - \nabla^M_{\widehat{Z}} \nu = [\nu, \widehat{Y}] \in \Gamma(\mathcal{V}) .$$

Hence, after multiplication with $\widehat{W}$, the last summand of (6.6) turns into

$$\begin{aligned} \langle \nabla^M_{A(\widehat{X},\widehat{Y})} \widehat{Z}, \widehat{W} \rangle = \langle \nabla^M_{\widehat{Z}} \nu, \widehat{W} \rangle &= \partial_{\widehat{Z}} \underbrace{\left\langle \nu, \widehat{W} \right\rangle}_{=0} - \left\langle \nu, \nabla^M_{\widehat{Z}} \widehat{W} \right\rangle \\ &= -\left\langle \nu, \pi_{\mathrm{ver}}\left(\nabla^M_{\widehat{Z}} \widehat{W} \right) \right\rangle = -\tfrac{1}{2} \left\langle \nu, A\left(\widehat{Z}, \widehat{W}\right) \right\rangle \\ &= -\tfrac{1}{2} \left\langle A(\widehat{X}, \widehat{Y}), A\left(\widehat{Z}, \widehat{W}\right) \right\rangle . \end{aligned} \tag{6.8}$$

Finally substracting (6.5), (6.7), and (6.8) from (6.4) we obtain

$$\begin{aligned} \left\langle R^M(\widehat{X}, \widehat{Y}) \widehat{Z}, \widehat{W} \right\rangle = \left\langle R^B\left(X, Y\right) Z, W \right\rangle &- \tfrac{1}{4} \left\langle A(\widehat{Y}, \widehat{Z}), A(\widehat{X}, \widehat{W}) \right\rangle \\ &+ \tfrac{1}{4} \left\langle A(\widehat{X}, \widehat{Z}), A(\widehat{Y}, \widehat{W}) \right\rangle + \tfrac{1}{2} \left\langle A(\widehat{X}, \widehat{Y}), A\left(\widehat{Z}, \widehat{W}\right) \right\rangle , \end{aligned}$$

and this is the stated formula. ∎

In the following K^M (resp. K^B) is the sectional curvature of (M, g), resp. (B, h).

Corollary 6.10. *Let $f: M \to B$ be a Riemannian submersion, $p \in M$, and $X, Y \in \mathcal{H}_p$ two orthonormal horizontal vectors. Then*

$$K^B\Big(\mathrm{span}\big\{\mathrm{d}f(X), \mathrm{d}f(Y)\big\}\Big) = K^M\Big(\mathrm{span}\big\{X, Y\big\}\Big) + \frac{3}{4}\big\|A(X, Y)\big\|^2 .$$

Proof: We use O'Neill's formula for $\zeta := \eta := Y$ and $\xi := \omega = X$. Because $A(\xi,\omega) = 0$, the corresponding term in O'Neill's formula vanishes. Furthermore $\langle A(\xi,\eta), A(\zeta,\omega)\rangle = \langle A(\xi,\zeta), A(\eta,\omega)\rangle = -\|A(X,Y)\|^2$. The corollary thus follows. ∎

Example 6.11 (Hopf fibration and complex projective space)**.** We continue with Example 2.2 6.) resp. Example 4.2 2.) resp. Example 6.3 2.).

One can check as an exercise that for any pair of orthonormal vectors $X, Y \in \mathcal{H}$ we have $0 \le \|A(X,Y)\| \le 2$. Thus the sectional curvature $K^{\mathbb{C}\mathrm{P}^n}$ of $\mathbb{C}\mathrm{P}^n$ with the Fubini-Study metric, i. e., the quotient metric from S^{2n+1} satisfies

$$1 \le K^{\mathbb{C}\mathrm{P}^n} \le 4\,.$$

In the case $n = 1$, $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to S^2, but $g^{\mathrm{FS}} = \frac{1}{4} g^{\mathrm{sph}}$.

Tu 21.5. holiday

Fr 24.5. Overview+discussion

II Some important examples of Lorentzian manifolds

Fr 24.5.

In the chapter we will discuss some special Lorentzian manifolds that are important in general relativity. The theory of Lorentzian manifolds would easily give rise to a one-semester lecture on its own, and important theorems were established in the recent years. For example, let us mention the Penrose singularity theorem for which the Nobel Prize was attributed to Penrose in 2020, see https://www.nobelprize.org/prizes/physics/2020/summary. A good mathematical introduction is [7]. For the related background in general relativity we refer to [8]. We also recommend the more extended reference by O'Neill [23] and further literature given in my previous lectures "Differential Geometry II" (web page https://ammann.app.uni-regensburg.de/lehre/2021s_diffgeo2/ and partial script https://ammann.app.uni-regensburg.de/lehre/2021s_diffgeo2/Diffgeo2.pdf) and "Differential Geometry III" (web page https://ammann.app.uni-regensburg.de/lehre/2021w_diffgeo3/ and partial script https://ammann.app.uni-regensburg.de/lehre/2021w_diffgeo3/Diffgeo3.pdf) where this subject was extended in much more details.

This chapter will be relatively short. In contrast to most other mathematical lectures, we will not prove many theorems. The main goal of this chapter is that you see why semi-Riemannian geometry is relevant in general relativity. After this short chapter, i. e., in the next and last chapter, we will continue studying Riemannian manifolds. Many of these Riemannian theorems have counterparts in the Lorentzian world, and many of the techniques are the same. We will aim for a good understanding of the curvature of Riemannian manifolds, but this also helps for understanding the Lorentzian case. This corresponds to the historical development: the development of Riemannian Geometry started with Riemann's Habilitation in 1854, published in 1876 (Collected works of Riemann) and these techniques where used by Einstein in order to describe the curved spacetime of general relativity in 1915.

1 Special relativity and the Minkowski space

Special relativity goes back to an article titled "On the Electrodynamics of Moving Bodies" by Albert Einstein from 1905.

One of its main ideas is: In special relativity space and time are unified in order to obtain a 3 + 1-dimensional spacetime. Space "is" the Euclidean $\mathbb{R}^3$, time "is" the Euclidean $\mathbb{R}$, and spacetime "is" $\mathbb{R}^{3,1}$.[1]

Definition 1.1. *We equip $\mathbb{R}^{m+k}$, $n = m + k$, $m, k \in \mathbb{N}_0$ with the bilinear form*

$$\langle\!\langle x, y\rangle\!\rangle_{m,k} := \sum_{j=1}^{m} x^j y^j - \sum_{j=m+1}^{m+k} x^j y^j, \quad \text{where } x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \text{ and } y = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix}.$$

If the numbers m and k are obvious from the context, we simply write $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ instead of $\langle\!\langle \bullet, \bullet \rangle\!\rangle_{m,k}$. We write $\mathbb{R}^{m,k}$ for the pair $\big(\mathbb{R}^m, \langle\!\langle \bullet, \bullet \rangle\!\rangle_{m,k}\big)$.

For $k = 1$, we say that $\mathbb{R}^{m,1}$ is the ***$m + 1$-dimensional Minkowski space****.*

The group of isometries $\mathbb{R}^{m,k} \to \mathbb{R}^{m,k}$ is denoted as $\mathrm{O}(m, k)$ and is called the ***generalized orthogonal group****. We have $\mathrm{O}(m) = \mathrm{O}(m, 0) = \mathrm{O}(0, m)$.*

For $k = 1$, we say that $\mathrm{O}(m, 1)$ is the ***Lorentz group*** *in dimension $m + 1$.*

Definition 1.2. *Let V be a finite-dimensional real vector space, with a non-degenerate bilinear map $b : V \times V \to \mathbb{R}$. A* ***generalized orthonormal basis*** *of (V, b) is a basis $(e_1, \ldots, e_n)$ of V, with $\varepsilon_i \in \{-1, +1\}$ such that*

$$b(e_i, e_j) = \varepsilon_i \delta_{ij} .$$

The theorem of Sylvester tells us that for any such (V, b) there are $m, k \in \mathbb{N}_0$, $n = m + k = \dim V$ such that a generalized orthonormal basis exists with $\varepsilon_1 = \ldots = \varepsilon_m = 1 = -\varepsilon_{m+1} = \ldots = -\varepsilon_n$. The choice of such a basis defines an isometric isomorphism $V \to \mathbb{R}^{m,k}$. We have

$$m + k = \dim V, \quad k = \max\left\{ \dim W \;\middle|\; \begin{array}{l} W \text{ is a linear subpace of } V\text{, such} \\ \text{that } b|_{W\times W} \text{ is negative definite} \end{array} \right\},$$

[1] This is only a first approximation, as all these spaces are affine spaces without a natural choice of a basis.

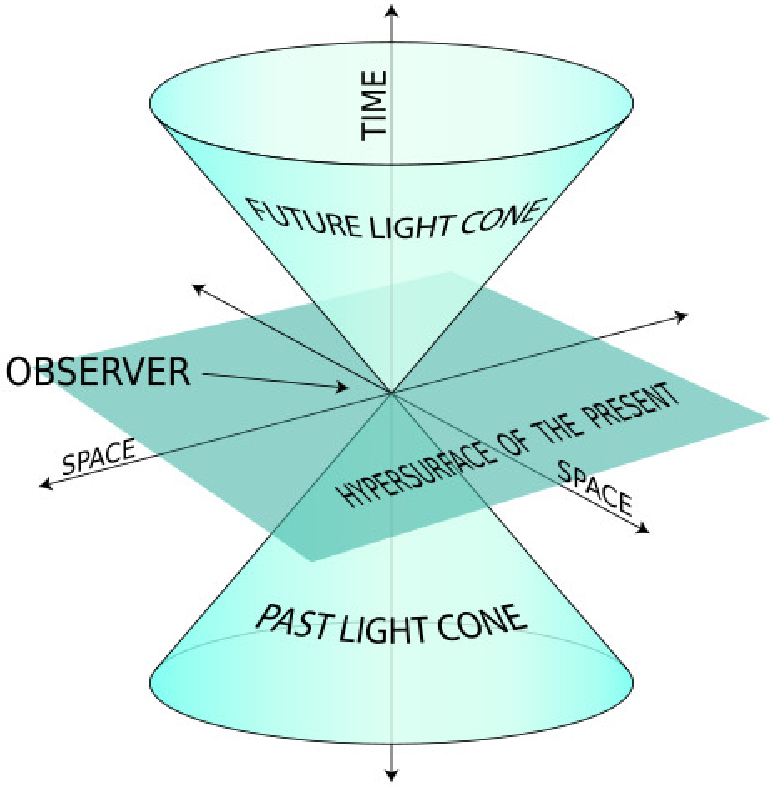


Figure II.1: The lightcone consists of the lightlike vectors and zero. It divides the Minkowski space into three components. Two components contain the timelike vectors and one component contains the spacelike non-zero vectors.©

thus the numbers m and k are already given by (m,k). The unique number k is called the **index** of (V,b). Tu 28.5.

Definition 1.3. *Let* (V,b) *be a finite-dimensional real vector space with non-degenerate symmetric bilinear form* b*. We say that a vector* $X \in V$ *is*

- **timelike** *if* $b(X,X) < 0$,
- **lightlike** *if* $b(X,X) = 0$,
- **causal** *if* $b(X,X) \leq 0$,
- **spacelike** *if* $b(X,X) > 0$.

Note that many sources define $0 \in V$ as spacelike, as an exception. We will not follow this convention.

Example 1.4. We write $X \in \mathbb{R}^{m,1}$ as $(\vec{X}, X_0)$ with $\vec{X} \in \mathbb{R}^m$, $X_0 \in \mathbb{R}$. Then X is timelike if $|X_0| > \|\vec{X}\|$, lightlike if $|X_0| = \|\vec{X}\|$, and spacelike if $|X_0| < \|\vec{X}\|$. The set of timelike vectors $I(0)$ has two components:

- one component, denotes as $I_+(0)$, with $X_0 > 0$
- and one, denoted as $I_-(0)$ with $X_0 < 0$.

We write $E_{m+1} := (0, 0, \dots, 0, 1) \in \mathbb{R}^{m,1}$. Then

$$I_+(0) = \{X \in I(0) \mid b(X, E_{m+1}) < 0\}, \quad I_-(0) = \{X \in I(0) \mid b(X, E_{m+1}) > 0\}.$$

Lemma 1.5. *For $m > 0$, the group $\mathrm{O}(m)$ has two connected components. The surjective continuous group homomorphism*

$$\mathrm{O}(m) \xrightarrow{\det} \{-1, +1\}$$

distinguishes these two components. For $m, k > 0$, the group $\mathrm{O}(m, k)$ has 4 components. There is a group homomorphism

$$\mathrm{O}(m.k) \xrightarrow{\text{t-or}} \{-1, +1\}$$

such that

$$\mathrm{O}(m.k) \xrightarrow{\det \times \text{t-or}} \{-1, +1\} \times \{-1, +1\}$$

is a surjective continuous homomorphism, that distinguishes the four components.

For the group $\mathrm{O}(m) = \mathrm{O}(m, 0) \cong \mathrm{O}(0, m)$ this lemma is classical. We will not prove the lemma if $m \geq 2$ and $k \geq 2$.

Obviously $\mathrm{O}(m, 1)$ and $\mathrm{O}(1, m)$ are isomorphic.

For $A \in \mathrm{O}(m, 1)$ we easily see that $A(I(0)) = I(0)$. We have

$$\text{t-or}(A) = +1 \iff A(I_+(0)) = I_+(0) \iff b(E_{m+1}, AE_{m+1}) < 0.$$

This is a surjective group homomorphism. We define

$$\begin{aligned}
\mathrm{SO}(m, 1) &:= \ker\Big(\det \colon \mathrm{O}(m, 1) \to \{-1, +1\}\Big), \\
\mathrm{O}_\uparrow(m, 1) &:= \ker\Big(\text{t-or} \colon \mathrm{O}(m, 1) \to \{-1, +1\}\Big), \\
\mathrm{SO}_\uparrow(m, 1) &:= \mathrm{SO}(m, 1) \cap \mathrm{O}_\uparrow(m, 1).
\end{aligned}$$

In [2, Chapter I, Lemma 1.2.5] it is shown that $\mathrm{SO}_\uparrow(m,1)$ is connected. This proves the lemma in the case $m \geq 1$ and $k = 1$ and in the case $m = 1$ and $k \geq 1$.

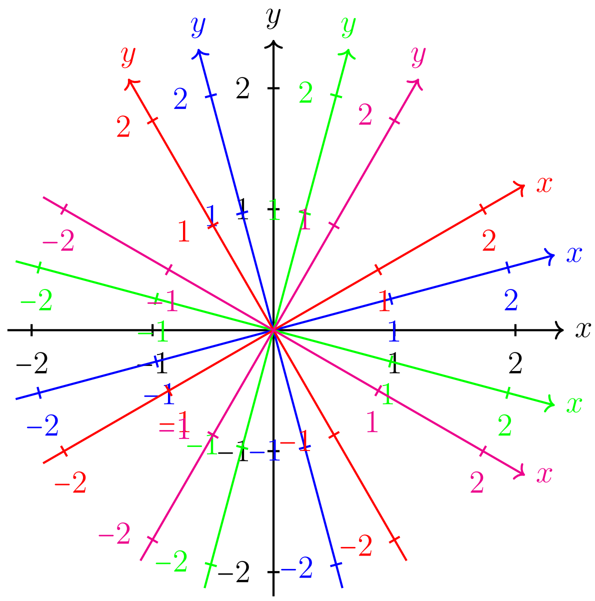


Figure II.2: Euclidean rotations in O(2), acting on the coordinate axes, in comparison to Figure II.3

Example 1.6. Let $m \geq 1$. The map

$$\alpha \mapsto B_\alpha := \begin{pmatrix} \mathbb{1}_{m-1} & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{pmatrix} \in \mathrm{O}(m,1)$$

is a an injective immersive Lie group homomorphism. The isometries defined by B_α are called **Lorentz boosts**, and play a central role in understanding special relativity. See Figure II.2 and Figure II.3 to compare the effects of rotations by elements O(2) with the effect of "rotations" by elements in O(1, 1).

Definition 1.7. *Let (V, b) be a finite-dimensional real vector space with a non-degenerate bilinear form b of index k and dimension $m+k$. We write $\mathcal{B}(V,b)$ for the space of all generalized orthonormal bases with $\epsilon_i = +1 \iff i \geq m$. Then $\mathrm{O}(m,k)$ acts transitively on $\mathcal{B}(V,b)$ by base change. For $m+k \geq 1$, an* **orientation** *of V is the same*[2] *as an element of*

$$\mathcal{B}(V,b) \big/ \mathrm{SO}(m,k) \; .$$

[2]More precisely: for your favorite definition of the word "orientation of a vector space", there is a canonical bijection from the set of orientations to the displayed set

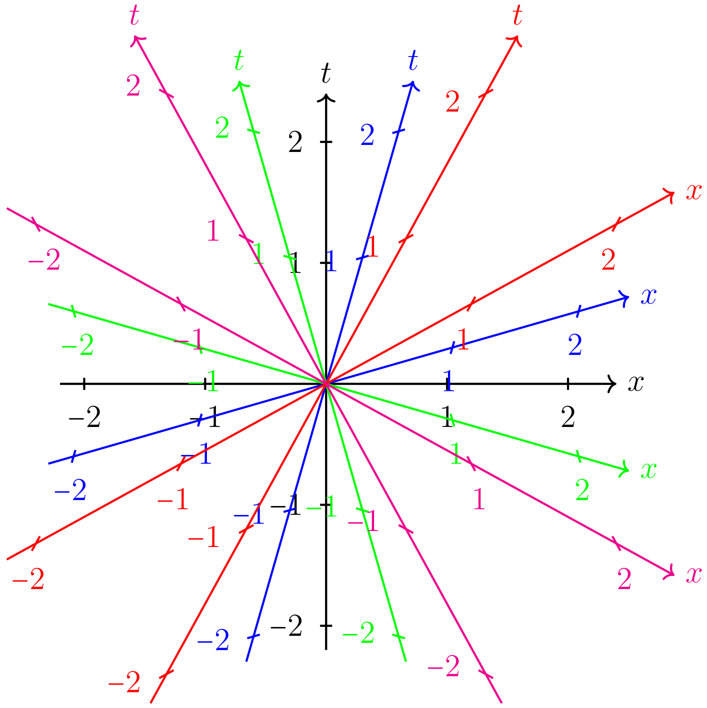


Figure II.3: Hyperbolic “rotations” in $O(1,1)$, more precisely the images of the coordinate axes under the action of the matrices B_α from Examples 1.6 for $\alpha = -0.62$, $\alpha = -0.3$, $\alpha = 0$, $\alpha = 0.3$, $\alpha = 0.62$,

The elements in the chosen orientation are called **positively oriented** *and the elements in the other equivalence class are called* **negatively oriented**. *For $k \geq 1$, a* **time orientation** *of V is the choice of an element of*

$$\mathcal{B}(V,b) / O_\uparrow(m,k) \ .$$

The last k vectors of any base in $\mathcal{B}(V,b)$ are timelike. For $k = 1$ this defines a map $\mathcal{B}(V,b) \to I(0)$ which surjective, continuous, and establishes a bijection from $\mathcal{B}(V,b) / O_\uparrow(m,1)$ to the set of connected components of $I(0) := \{X \in V \mid X \text{ timelike}\}$. Thus, for $k = 1$, a choice of time orientation is the same as the choice of a connected component of $I(0)$. A timelike vector is called **future-directed** *if it is in the chosen component, denoted as $I_+(0)$, and otherwise it is called* **past-directed**, *i. e., an element of $I_-(0)$. A lightlike vector X is called* **future-directed** *(resp.* **past-directed***) if $X \neq 0$ and if $X \in J_+(0) := \overline{I_+(0)}$, (resp. $X \in J_-(0) := \overline{I_-(0)}$).*

Definition 1.8. *The* **Poincaré group** *in dimension $m+1$ is the group of affine maps of the form*

$$\mathbb{R}^{m,1} \to \mathbb{R}^{m,1}, X \mapsto AX + b \,,$$

where $A \in \mathrm{O}(m,1)$ and $b \in \mathbb{R}^{m,1}$. It is thus the group of affine isometries of the Minkowski space. Obviously this is the semi-direct product $\mathbb{R}^{m,1} \rtimes \mathrm{O}(m,1) \ni (b,A)$. Similarly, the **orthochronous Poincaré group** *is $\mathbb{R}^{m,1} \rtimes \mathrm{O}_{\uparrow}(m,1)$.*

Physical setting of special relativity.

In special relativity the **spacetime** is a time-oriented 4-dimensional real affine space whose underlying vector space carries a non-degenerate bilinear form $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ of index 1. After choosing a zero in the affine space and choosing a generalized orthonormal basis of the underlying vector space, we may identify the spacetime with $\mathbb{R}^{3,1}$. Elements in the spacetime $\mathbb{R}^{3,1}$ are called **events**. They consist of a time $t \in \mathbb{R}$ and a (traditional)[3] point $\vec{x}^{\top} = (x^1, x^2, x^3)^{\top}$ in $\mathbb{R}^3$,[4] and this defines $(x^1, x^2, x^3, t)^{\top} = (\vec{x}, t)^{\top} \in \mathbb{R}^{3,1}$. We often write $x^0 := t$. This is the description given by an observer who stays at the point $\vec{p} = 0$ in $\mathbb{R}^3$ with respect to the standard coordinate system.

Physical Postulates 1.9.

(1) *A pointlike body of positive rest mass (or an observer) moves along a (smooth) curve $c\colon I \to \mathbb{R}^{3,1}$, I an interval, such that $\dot{c}(t)$ is timelike and future directed for all $t \in I$. Such curves are called* **timelike, future-directed** *curves.*

$$(\vec{x}, t) \in c(I) \iff \text{the body is at position } \vec{x} \text{ at time } t\,.$$

The parametrization of the curve has no *physical meaning.*

(2) *A timelike curve c is parametrized by* **proper time** *of $\langle\!\langle \dot{c}(t), \dot{c}(t) \rangle\!\rangle \equiv -1$ for all $t \in I$. If c is parametrized by proper time*[5]*, then an observer moving from $c(t_1)$ to $c(t_2)$, $t_1 < t_2$ has the impression that the time $t_2 - t_1$ has passed.*

(3) *Light (in vacuum) travels along a (smooth) curve $c\colon I \to \mathbb{R}^{3,1}$, I an interval, such that $\dot{c}(t)$ is lightlike and future-directed for all $t \in I$.*

An observer travels along a timelike, future-directed curve $\gamma\colon J \to \mathbb{R}^{3,1}$. After reparametrization we assume γ parametrized by proper time. The observer comes with vectors $\tau_1(t), \tau_2(t), \tau_3(t))$, such that $(\tau_1(t), \tau_2(t), \tau_3(t), \dot{\gamma}(t))$ is a generalized orthonormal basis for all $t \in J$, i. e., $(\tau_1(t), \tau_2(t), \tau_3(t), \dot{\gamma}(t)) \in \mathrm{O}(3,1)$.

[3] i. e., a point in our usual 3-dimensional space, that surrounds us.
[4] thus our convention is that $\vec{x}$ is a row vector, i. e., $\vec{x} \in \mathbb{R}^{1\times 3}$
[5] which can be achieved by reparametrization

The standard observer is given by $\gamma(t) = (0,0,0,t)^\top$ and $\tau_1(t) = (1,0,0,0)^\top$, $\tau_2(t) = (0,1,0,0)^\top$, $\tau_3(t) = (0,0,1,0)^\top$.

If the **standard observer** observes an event at $(\vec{x}, x^0)$ and if the observer $(\gamma, \tau_1, \tau_2, \tau_3)$ at proper time t observes this event at $(\vec{y}, y^0)$, then

$$(\vec{x}, x^0)^\top = \big(\tau_1(t), \tau_2(t), \tau_3(t), \dot{\gamma}(t)\big) \cdot (\vec{y}, y^0)^\top + \gamma(t)\,.$$

This implies that abslute time, as known in classical physics, no longer exists:

Abandoned Postulate (Absolute time)**.** *There is an absolute time. More precisely, there is an affine 1-dimensional space T and a map $t : V \to T$ from spacetime V to T which can be measured independently from the observer. (One says $t(\mathcal{E})$ is the time when the event $\mathcal{E}$ occured.)*

So such a t no longer exists, the time of an event always depends on the observer. If γ describes a moving observer, say with $\dot{\gamma}(t)$ constant. Then there are events $\mathcal{E}_1$ and $\mathcal{E}_2$ where the observer γ will say that $\mathcal{E}_1$ happened after $\mathcal{E}_2$, while the standard observer will say that $\mathcal{E}_1$ happened before $\mathcal{E}_2$.

Many classical vector fields together with a function yield a **4-vector field**, i. e., a vector field on $\mathbb{R}^{3,1}$, as you may see in the following table:

classical scalar	name	classical vector	name	relativistic object	name
t	time	$\vec{x}$	position	$x = (\vec{x}, t)$	event
E	energy	$\vec{p}$	momentum	$p = (\vec{p}, E)$	(energy-)momentum 4-vector
ρ	electrical charge	$\vec{j}$	electrical current	$J = (\vec{j}, \rho)$	electrical 4-current
ϕ	electric potential	$\vec{A}$	magnetic potential	$A = (\vec{A}, \phi)$	electromagnetic potential

Physical Postulate 1.10. *Physical laws are invariant under the orthochronous Poincaré group.*

The electro-magnetical field is given by a 2-form $\omega \in \Omega^2(\mathbb{R}^{3,1})$. The famous **Maxwell equations** are then given by

$$d\omega = 0, \quad d^*\omega = \pm J^\flat,$$

where $\mathrm{d}^*\colon \Omega^2(\mathbb{R}^{3,1}) \to \Omega^1(\mathbb{R}^{3,1})$ is the (formally) adjoint differential operator to the exterior differential $\mathrm{d}\colon \Omega^1(\mathbb{R}^{3,1}) \to \Omega^2(\mathbb{R}^{3,1})$, and where $\pm$ depends on several sign conventions. If one considers the electromagnetic field as $\omega \in \Omega^2(V)$, where (V, b) is isometric to $\mathbb{R}^{3,1}$, then ω does not depend on the coordinates, i. e., on the isomorphism to $\mathbb{R}^{3,1}$. All the complicated laws about the transformation of this field in a moving system turn into a mere triviality.

See [2, Section 1.5] for more details on the electromagnetic field.

2 General relativity

General relativity was initiated by work of Albert Einstein published in 1915.

The basic idea of general relativity is that matter curves the space. For physical justification of the precise laws, we refer to physics textbooks. The goal of this section is to explain the role of Lorentzian manifolds and their physical interpretation in this theory.

In general relativity, the **spacetime** is a time-oriented Lorentzian manifold (M, g), mostly of dimension $3+1$. A *time-orientation* is given by a choice of a time-orientation of (T_pM, g_p) for any $p \in M$ which depends continuously on p. Using a partition of unity it is then easy to construction a vector field $X \in \mathfrak{X}(M)$, such that X_p is time-like and future-directed at any $p \in M$. On the other hand any vector field $X \in \mathfrak{X}(M)$ that is everywhere timelike, determines a time-orientation. The time-orientation allows to distinguish past and future, and is important for causality and thermodynamics.

Definition 2.1. *Let (M, g) be an n-dimensional semi-Riemannian manifold. For $h \in \Gamma(\mathcal{T}^{(0,s)}M)$, $s \geq 2$, one defines the* **(metric) contraction** *of the first and second component as*

$$\sum_{j=1}^{n} \epsilon_j h(e_j, e_j, \bullet, \dots, \bullet),$$

where $(e_1, \dots, e_n)$ is a generalized orthonormal basis. One can similarly define contractions in other arguments. In the case $h \in \Gamma(\mathcal{T}^{(0,2)}M)$ we also used the term **metric trace**, *written as $\mathrm{tr}^g h$, for the contraction of the two components. The (usual) trace and the metric trace are related by the formula: if $H \in \Gamma(\mathrm{End}(\mathrm{T}M))$, then*

$$\mathrm{tr}\, H = \mathrm{tr}^g\Big(g\big(H(\bullet), \bullet\big)\Big).$$

For a symmetric $(0,s)$*-tensor* h *on* M*, one defines its* **divergence** $\operatorname{div} h \in \Gamma(\mathcal{T}^{(0,s-1)}M)$ *as the metric trace of* ∇h *in the first two components, i. e.,*

$$\forall p \in M : \forall X_1, \ldots, X_{s-1} \in T_pM : \operatorname{div} h(X_1, \ldots, X_{s-1}) := \sum_{j=1}^{\dim M} \epsilon_j (\nabla_{e_j} h)\big(e_j, X_1, \ldots, X_{s-1}\big).$$

For example $\operatorname{scal} = \operatorname{tr}^g \operatorname{ric} = \operatorname{tr} \operatorname{Ric}$ and

$$\operatorname{ric}(X,Y) = \operatorname{tr}^g \underbrace{g\big(R(X,\bullet)\bullet, Y\big)}_{\in \Gamma(\mathcal{T}^{(0,2)}M)} .$$

Using the second Bianchi identity we have seen in Diff. geom. I, Exercise Sheet 10, Exercise 3 that

$$\operatorname{div} \operatorname{ric} = \tfrac{1}{2} \mathrm{d} \operatorname{scal} . \tag{2.1}$$

Definition 2.2. *For a semi-Riemannian metric* g*, the* **Einstein tensor** *is defined as*

$$G := \operatorname{ric} - \frac{1}{2} \operatorname{scal} \cdot g .$$

A key concept of general relativity is that our universe is modeled by an $n = (m+1)$-dimensional Lorentzian manifold that satisfies the **Einstein equations**:

Physical Postulate 2.3. *The spacetime is a time-oriented Lorentzian manifold that satisfies*

$$G + \Lambda g = 8\pi T, \tag{2.2}$$

where T *is the energy-momentum tensor, and where* Λ *is a real constant. The tensor* T *depends on the fields on* M*.*

The relevant case for traditional general relativity is $n = 3+1$, but in string or super-string theories one also considers other dimensions. This postulate is usually shortly summarized by saying that matter curves spacetime.

When Einstein published the mathematical framework of general relativity in 1915, he was able to use the concepts of Riemannian geometry established by Riemann, Ricci-Curbastro and many others, at the end of the 19th century, by adding signs. It took a while until Einstein got the right type of equations. In fact Einstein got the right equations after he had an extensive exchange with the famous mathematician David Hilbert. At first, Einstein thought the left hand side should be ric, but this led to the contradiction: one always has $\operatorname{div} T = 0$ by construction of T, but equation (2.1) says that $\operatorname{div}\operatorname{ric}$ in general does not vanish. After discussions with Hilbert, he took the Einstein tensor G for the left hand side, i. e., as above with $\Lambda = 0$. Later on he realized that $\Lambda \neq 0$ is also possible. [6] Later again, he regretted this a lot, he said that allowing $\Lambda \neq 0$ was "die größte Eselei meines Lebens". So he came back to G. Nowadays physicists are convinced that $\Lambda \neq 0$, and it is called the **cosmological constant** or the **dark energy**.[7] Fr 31.5.

The tensor T describes the effects of the matter and of the fields. Obviously, as long as we do not have information about T, the statement of the Einstein equations is void, but the explicit form of the T-tensor is involved. Let us mention that one usually describes the energy-momentum tensor by writing down the "action functional" and then by deriving with respect to perturbations of the semi-Riemannian metric. In fact, it is not so simple to write down the Lagrange functional even for simple systems.

Thus, we restrict to summarize some simple properties of T.

(1) If the spacetime describes a vacuum, then $T = 0$.

(2) For all spacetimes we have $\operatorname{div} T = 0$.

(3) It is consensus among physicists[8] that the energy-momentum tensor satisfies several positivity assumptions:

- the **dominant energy condition**: If X, Y are timelike (or causal) tangent vectors with the same basepoint and the same time orientation, then $T(X, Y) \geq 0$.
- the **weak energy condition**: $T(X, X) \geq 0$ for all timelike (or causal) $X \in \mathrm{T}M$.
- the **null energy condition**: $T(X, X) \geq 0$ for all lightlike $X \in \mathrm{T}M$.

Property (2) can be seen from the experimental or the mathematical perspective. It can be seen as the conservation of energy and momentum density, which one can verify experimentally. On the other hand, if we know how to derive the Einstein equations from the action functional, it can be proven mathematically.

[6] It follows from Lovelock's theorem, that – under some suitable assumptions – the equation *has* to be of this form.

[7] Dark enery is not the same as dark matter, although special relativity states that "matter=energy".

[8] as far as I know

The conditions in Property (3) are physically interpreted as non-negative energy density in different senses. The dominant energy condition implies the weak energy condition, and the weak energy condition implies the null energy condition.

Remark 2.4. Another positivity condition found in the literature is the **strong energy condition**: $(T - \frac{1}{n-2}(\operatorname{tr} T)g)(X,X) \geq 0$ for all timelike (or causal) $X \in \mathrm{T}M$. Other than indicated by its name, it does not imply the weak energy condition. Furthermore, there are matter models for which this condition is not satisfied.

Remark 2.5. Assume $n \neq 2$. By taking the metric trace on both sides, (3) implies

$$\operatorname{tr}^g G + n\Lambda = 8\pi \operatorname{tr}^g T.$$

This equation can be used to resolve (3) as a formula for ric

$$\begin{aligned} \mathrm{ric} &= \frac{1}{2}\,\mathrm{scal}\cdot g + 8\pi T \\ &= \frac{n}{n-2}\Lambda \cdot g + 8\pi\left(T - \frac{1}{n-2}(\operatorname{tr}^g T)\cdot g\right). \end{aligned} \tag{2.3}$$

Similarly, we can show that (2.3) implies (3), i. e., (2.3) is an equivalent formulation for (3).

In particular, for $\Lambda = 0$ the strong energy condition is equivalent to *non-positive* Ricci curvature in all timelike directions, i. e., $\mathrm{RIC}(X) \leq 0$ for all timelike X.

Physical Postulate 2.6. *A freely[9] falling body moves along a (timelike) geodesic in the spacetime (M, g).*

3 Black holes and the Schwarzschild solution

The **Schwarzschild solution** is a solution of the Einstein equation for the vaccum, i. e., $T = 0$ and vanishing cosmological constant Λ. It is the model for a non-charged, non-rotating and non-accelerated **black hole** (or the outside of a star) in the vacuum for vanishing dark energy (= cosmological constant).

[9] He experience the effect of the graviational fields, but nothing else

3.1 Historic overview

Amazingly the solution was already derived in 1915 by Karl Schwarzschild, who served as a soldier in World War I at that time. In spite of the effect that this solution was found so immediately after Einstein's publication of general relativity, it took until the 1960's that physicists were convinced that black holes actually exist. It seemed that the Schwarzschild solution was so special (in nature one cannot expect that a heavy body has no angular momentum and no charge at all) that its existence was assumed to be a mathematical construction without physical relevance for many decades.

The appreciation changed drastically in the 60s. On the one hand, the **Kerr solution** was found by Roy Kerr in 1965. It describes a black hole that is stationary (= time-symmetric), axisymmetric, non-charged, with non-zero angular momentum and with vanishing cosmological constant Λ. More precisely this is a whole family of solutions, determined by the size and the direction of the angular momentum. On the other hand Roger Penrose developed a singularity theorem, the Penrose singularity theorem, see [7, Theorem 2.125], which shows that under suitable conditions the spacetimes has to "end" in some sense in some place, an effect that one calls a **singularity in spacetime**, or in more colloquial terms a black hole. For this achievement, Penrose obtaind (half of) the Nobel Prize for Physics 2020, see https://www.nobelprize.org/prizes/physics/2020/popular-information/, the other half went to the astronomers Genzel and Ghez.

These two discoveries were important as since then it is reasonable to assume that any sufficently high concentration of mass converges in some sense for long times towards either a Kerr solution or a Schwarzschild solution, provided that $\Lambda = 0$. With only Schwarzschild solutions and without Kerr metrics such a conjecture was not reasonable, as the preservation of angular momentum would have created problems.

To prove convergence towards the Kerr or Schwarzschild solution in fact turned out to be a very involved task. Important progress was obtained in the recent decades, in particular by the stability of Minkowski space, which says, that a vacuum solution of the Einstein equation that starts as a small (in some suitable sense) perturbation of Minkowski space will converge in some sense for long times to Minkowski space. In physical terms: all gravitational waves will dissipate to infinity. Stability of Minkowski space was proven by Christodoulou and Klainerman in 1992, and their proof was published in the book [11] which contains many estimates[10]. This can be viewed as the extreme case for which we have vanishing mass, charge, angular momentum and cosmological constant. More involved cases are the topic of current reearch, in particular A. Vasy has obtained important progress.

3.2 Axiomatic approach

The Schwarzschild solution has several properties:

(I) *Vacuum solution.* It is a vacuum solution of the Einstein equation (3) with vanishing cosmological constant, i. e., $T \equiv 0$ and $\Lambda = 0$.

(II) *Asymptotic flatness.* The solution is asymptotic (in spatial directions) to the Minkowski space.

(III) *Stationarity.* There is a timelike future-directed Kiling vector field $K \in \Gamma(M)$, i. e., timelike future-directed vector field K with $\mathcal{L}_K g = 0$. After passing to a covering and possibly extending (M, g) this implies the existence of isometric action $A_0\colon \mathbb{R} \times M \to M$, wit $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} A_0(t, \bullet) = K$. At infinity this converges to translation in the time direction of Minkowski space.[11]

(IV) *Spherical symmetry.* There is an effective isometric action $A_1\colon \mathrm{O}(m) \times M \to M$, commuting with A_0. At infinity this converges to the standard $\mathrm{O}(m) \subset \mathrm{O}_\uparrow(m, 1)$-action on $R^{m,1}$. We also assume that the orbits of the A_0 action and the A_1-action are orthogonal to each other.

As these axioms have physical character, we do not want to argue rigorously with these axioms, and to keep the discussion of consequences short. It is thus reasonable to assume that $M = \mathbb{R} \times (\rho_0, \infty) \times S^{m-1} \ni (t, \rho, y)$ and that

$$g = -a(\rho)\,\mathrm{d}t^2 + b(\rho)\,\mathrm{d}\rho^2 + c(\rho) g^{S^{m-1}}$$

where $g^{S^{m-1}}$ is the standard metric on the $(m-1)$-dimensional sphere $\mathbb{S}^{m-1}$. From (II) it is clear that $\rho \to c(\rho)$, is a diffeomorphism onto its image for $\rho > \rho_1$, ρ_1 sufficiently large. Thus assuming that $c\colon (\rho_0, \infty) \to (r_0^2, \infty)$ is a diffeomorphism we

[10] To understand the result, it is advisable to consider secondary literature first, as e.g. recommend in the associated MathSciNet Review by Alan Rendall `https://mathscinet.ams.org/mathscinet/article?mr=1316662`. MR1316662

[11] The property "static" often used in this context, e. g., in [23, Chap. 13], is a little bit stronger, it also includes that $K^\perp$ is integrable. In this situation this immediately follows from the next condition, as the orbits of A_1 yields such integrating submanifolds.

make a change of coordinates $r = \sqrt{c(\rho)}$ and obtain in these coordinates

$$g = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2 g^{S^{m-1}}\,. \tag{3.1}$$

Property (II) yields $\lim_{r\to\infty} A(r) = 1$ and $\lim_{r\to\infty} B(r) = 1$. Property (I) is equivalent to $\mathrm{ric}^g \equiv 0$ and this turns into a system of ODEs where it is a priori unclear whether it has solutions.

Exercise 3.1 (Compare Exercise Sheet 7, Exercise 3)**.** *Let $\mu > 0$ and $n = m+1 \geq 4$. For*

$$A(r) := 1 - \frac{2\mu}{r^{m-2}}, \qquad B(r) := \frac{1}{1-\frac{2\mu}{r^{m-2}}} \tag{3.2}$$

the metric g defined by (3.1) is Ricci-flat. This holds both for $r \in \big((2\mu)^{\frac{1}{m-2}}, \infty\big)$ and for $r \in \big(0, (2\mu)^{\frac{1}{m-2}}\big)$.

One further can show that a Ricci-flat metric g defined by (3.1) with $\lim_{r\to\infty} A(r) = 1$ and $\lim_{r\to\infty} B(r) = 1$ also satisfies (3.2) for some $\mu \in \mathbb{R}$. The constant μ is called the **mass of the Schwarzschild metric**. In the case $n = 4$, thus $m = 3$, which corresponds to the spacetime we experience[12], the mass has an important relation to the mass in classical mechanics, explained below. This leads to the normalization of the variable μ (fo $n = 4$), i. e., a physical reason why we write 2μ in the formula for A and B and not simply μ or even 17μ.

Definition 3.2. *For $\mu \in \mathbb{R}$ we set*

$$r_\mu := \begin{cases} (2\mu)^{\frac{1}{m-2}} & \text{if } \mu \geq 0\,, \\ 0 & \text{if } \mu < 0\,. \end{cases}$$

The Lorentzian manifold $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ with

$$M^{\mathrm{ext}} = \mathbb{R} \times \big(r_\mu, \infty\big) \times S^{m-1}, \quad g^{\mathrm{schw}} = -\left(1 - \frac{2\mu}{r^{m-2}}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{2\mu}{r^{m-2}}}\mathrm{d}r^2 + r^2 g^{S^{m-1}},$$

is called **Schwarzschild exterior spacetime**. *Furthermore, for $\mu > 0$, the manifold*

$$M^{\mathrm{int}} = \mathbb{R} \times \big(0, r_\mu\big) \times S^{m-1}$$

with the Lorentzian metric g^{schw}, defined with the same formula as above, is called **Schwarzschild interior spacetime** *or* **Schwarzschild black hole**. *The Lorentzian*

[12] unless we do string theory

metric g^{schw} *is called the* **Schwarzschild metric***.*

Tu 4.6.

Remarks 3.3.

1.) Only the case $\mu \geq 0$ is physically relevant, and as $\mu = 0$ yields the Minkowski space, the case of interest is $\mu > 0$.

2.) For Schwarzschild interior spacetime the vector field $\frac{\partial}{\partial r}$ is timelike, and $\frac{\partial}{\partial t}$ is spacelike. There are two time-orientations characterized as follows:

- $\frac{\partial}{\partial r}$ is future-directed, we will say this is the **white hole time-orientation**;
- $\frac{\partial}{\partial r}$ is past-directed, we will say this is the **black hole time-orientation**.

The reason for choosing these names for the time-orientations will become apparent in Definition and Lemma 3.9.

3.3 Geodesics

It is helpful to consider a freely falling body in the Schwarzschild exterior spacetime, i. e., we consider timelike geodesics. These are discussed in [23, Chap. 13, pp. 372–380] in detail, we just briefly summarize some results. The geodesic will not be straight line as it is not a geodesic with respect to the Minkowski metric, but with respect to g. We restrict to the case $n = m + 1 = 3 + 1$.

The geodesic has 3 excluding alternatives, depending on the angular momentum of the body and the masses:

(1) It crashes into the center at some time $\tau_1 \in R$, i. e., $\lim_{\tau \nearrow \tau_1} r(\tau) = r_\mu$, and the maximal interval of existence ends at τ_1. Or it escapes from the center at some time $\tau_2 \in R$, i. e., $\lim_{\tau \searrow \tau_2} r(\tau) = r_\mu$, and the maximal interval of existence ends at τ_2. Or both of these sub-alternatives with $\tau_2 < \tau_1$.

(2) The geodesic is defined on $\mathbb{R}$ and $\tau \mapsto r(\tau)$ is periodic.

(3) The geodesic is defined on $\mathbb{R}$ and $\lim_{\tau \to +\infty} r(\tau) = \lim_{\tau \to -\infty} r(\tau) = \infty$.

For $\mu \leq 0$ we only have the last alternative (3).

If we consider an isolated solar system with planets neglecting the forces between planets and without further objects, then the planets move along geodesics in alternative (2). This is a reasonable approximation for our solar system. However, in general the projection of the geodesic to $(r_\mu, \infty) \times S^2$ will not be a closed curve. Sufficiently far away from $r = r_\mu$ this will be almost a closed curve (e. g., Earth, and farther outside), but for orbits as the one of the planet Mercury the non-closedness is well-measurable. This leads to the **perihelion advance of Mercury** which was one of the first predicted results of general relativity that could be experimentally[13] verified.

In the case of alternative (3) and $n = 4$, and for $r_2 := \min_{\tau \in \mathbb{R}} r(\tau)$ sufficiently large, the body will move closely to a freely falling observer in classical mechanics gravitationally attracted by a large body at the center of mass μ. This gives the normalization for μ discussed just before Definition 3.2.

3.4 Extension of the Schwarzschild metric

Definition 3.4. *Let (M, g) be a time-oriented, connected Lorentzian manifold. An* **extension** *of (M, g) is given by a time-oriented, connected Lorentzian manifold (M', g') with $\dim M' = \dim M$ and an isometric time-orientation preserving embedding $i \colon (M, g) \hookrightarrow (M', g')$. It is called* **trivial** *if i is a diffeomorphism. We say that (M, g) is* **extendible** *if a non-trivial extension exists, otherwise it is called* **inextendible**.

Physical Postulate 3.5. *Our spacetime is inextendible in this sense.*

For example Minkowski space is inextendible. More generally geodesically complete space times are inextendible. Furthermore, any compact Lorenzian manifold is inextendible, e. g., the manifold in Exercise Sheet 7, Exercise 2, called the Clifton–Pohl torus.

Obviously, there are several other reasonable notions of extensions and on extendability that one can consider, and many of them are considered in the literature. For example, one could allow extensions by manifolds or metrics of lower regularity, extensions with special causal structure, extensions of vacuum solutions, oriented extensions etc, but we will not go into this discussion.

Question 3.6. *Is Schwarzschild exterior spacetime inextenible (for $n \geq 4$)?*

The answer is no, i. e., non-trivial extensions exist that we will construct now.

[13] in this case: by astronomical observations

We define $(M^{\text{ext}}, g^{\text{schw}})$ as the Schwarzschild exterior spacetime with $\mu > 0$ following Definition 3.2. Recall that we identify

$$M^{\text{ext}} = \mathbb{R} \times (r_\mu, \infty) \times S^{m-1}$$

as smooth manifolds.

Lemma 3.7. *Let* $F\colon (r_\mu, \infty) \to \mathbb{R}$ *be a primitive of* $r \mapsto \left(1 - \frac{2\mu}{r^{m-2}}\right)^{-1}$. *For the diffeomorphism*

$$\begin{aligned} \Phi_-\colon \mathbb{R} \times (r_\mu, \infty) \times S^{m-1} &\longrightarrow \mathbb{R} \times (r_\mu, \infty) \times S^{m-1} \\ (t, r, y) &\longmapsto \big(t - F(r), r, y\big) \end{aligned}$$

we have

$$g^{\text{schw}}_{\text{fut}} := \Phi_-^* g^{\text{schw}} = -\left(1 - \frac{2\mu}{r^{m-2}}\right) \mathrm{d}t \otimes \mathrm{d}t + 2\, \mathrm{d}t \odot \mathrm{d}r + r^2 g^{S^{m-1}}. \tag{3.3}$$

Similarly for $\Phi_+(t, r, y) := \big(t + F(r), r, y\big)$ *we get*

$$g^{\text{schw}}_{\text{past}} := \Phi_+^* g^{\text{schw}} = -\left(1 - \frac{2\mu}{r^{m-2}}\right) \mathrm{d}t \otimes \mathrm{d}t - 2\, \mathrm{d}t \odot \mathrm{d}r + r^2 g^{S^{m-1}}. \tag{3.4}$$

Obviously we also have $\Phi_+ = (\Phi_-)^{-1}$.

Proof:

$$\Phi_-^* \mathrm{d}t = \mathrm{d}\big(t \circ \Phi_+\big) = \mathrm{d}t - F'(r)\mathrm{d}r = \mathrm{d}t - \left(1 - \frac{2\mu}{r^{m-2}}\right)^{-1} \mathrm{d}r$$

As $\Phi_.$ acts as the identity on $(r_\mu, \infty) \times S^{m-1}$ this also yields

$$\Phi_-^* \mathrm{d}r = \mathrm{d}r, \qquad \Phi_-^* g^{S^{m-1}} = g^{S^{m-1}}.$$

This yields using the notation $v \odot w := \frac{1}{2}(v \otimes w + w \otimes v)$ the yields

$$\begin{aligned} \Phi_-^* g^{\text{schw}} &= -\left(1 - \frac{2\mu}{r^{m-2}}\right)\left(\mathrm{d}t - \left(1 - \frac{2\mu}{r^{m-2}}\right)^{-1} \mathrm{d}r\right) \otimes \left(1 - \frac{2\mu}{r^{m-2}}\right)\left(\mathrm{d}t - \left(1 - \frac{2\mu}{r^{m-2}}\right)^{-1} \mathrm{d}r\right) \\ &\quad + \left(1 - \frac{2\mu}{r^{m-2}}\right)^{-1} \mathrm{d}r^2 + r^2 g^{S^{m-1}}, \\ &= -\left(1 - \frac{2\mu}{r^{m-2}}\right) \mathrm{d}t \otimes \mathrm{d}t + 2\, \mathrm{d}t \odot \mathrm{d}r + r^2 g^{S^{m-1}}. \end{aligned}$$

The calculation for Φ_+ is completely analogous. ∎

Extension of $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ to the future.

Thus we can use formula (3.3) to define a Lorentzian metric $g^{\mathrm{schw}}_{\mathrm{fut}}$ defined on all of $\mathbb{R} \times (0, \infty) \times S^{m-1}$. The pullback of the time orientation of $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ extends to a time orientation of this new Lorentzian manifold. We thus have obtained an extension

$$(M^{\mathrm{ext}}, g^{\mathrm{schw}}) \xrightarrow{\Phi_+} \big(\mathbb{R} \times (0, \infty) \times S^{m-1}, g^{\mathrm{schw}}_{\mathrm{fut}}\big).$$

Extension of $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ to the past.

Thus we can use formula (3.4) to define a Lorentzian metric $g^{\mathrm{schw}}_{\mathrm{past}}$ defined on all of $\mathbb{R} \times (0, \infty) \times S^{m-1}$. The pullback of the time orientation of $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ extends to a time orientation of this new Lorentzian manifold. We thus have obtained an extension

$$(M^{\mathrm{ext}}, g^{\mathrm{schw}}) \xrightarrow{\Phi_-} \big(\mathbb{R} \times (0, \infty) \times S^{m-1}, g^{\mathrm{schw}}_{\mathrm{past}}\big).$$

Remark 3.8. Using essentially the same calculation as in Lemma 3.7 we see that the inner set of the new parts of these Lorentz manifolds are isometric to Schwarzschild interior spacetime. More precisely, if we define $F\colon (0, r_\mu) \to (0, r_\mu)$ as a primitive of $r \mapsto \big(1 - \frac{2\mu}{r^{m-2}}\big)^{-1}$, then the functions

$$\Phi^{\mathrm{int}}_{\pm}\colon \mathbb{R} \times (0, r_\mu) \times S^{m-1} \longrightarrow \mathbb{R} \times (0, r_\mu) \times S^{m-1}$$

defined by the same expression as $\Phi_\pm$ in the Lemma yield isometric embeddings

$$(M^{\mathrm{int}}, g^{\mathrm{schw}}) \xrightarrow{\Phi^{\mathrm{int}}_+} \big(\mathbb{R} \times (0, \infty) \times S^{m-1}, g^{\mathrm{schw}}_{\mathrm{fut}}\big)$$

and

$$(M^{\mathrm{int}}, g^{\mathrm{schw}}) \xrightarrow{\Phi^{\mathrm{int}}_-} \big(\mathbb{R} \times (0, \infty) \times S^{m-1}, g^{\mathrm{schw}}_{\mathrm{past}}\big).$$

In both cases the images of M^{ext} and M^{int} in $\mathbb{R} \times (0, \infty) \times S^{m-1}$ are disjoint and the sets

$$\Phi_+(M^{\mathrm{ext}}) \cup \Phi^{\mathrm{int}}_+(M^{\mathrm{int}}) \text{ and } \Phi_-(M^{\mathrm{ext}}) \cup \Phi^{\mathrm{int}}_-(M^{\mathrm{int}})$$

are dense.

Definition and Lemma 3.9. *It is easy to show that we can "glue" $\big(\mathbb{R} \times (0, \infty) \times S^{m-1}, g^{\mathrm{schw}}_{\mathrm{fut}}\big)$ and $\big(\mathbb{R} \times (0, \infty) \times S^{m-1}, g^{\mathrm{schw}}_{\mathrm{past}}\big)$ along the jointly embedded $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ and we obtain a time-oriented Ricci-flat Lorentzian manifold $(M^{\mathrm{schw}}_{\mathrm{fut,past}}, g^{\mathrm{schw}}_{\mathrm{fut,past}})$, called the* **past- and future-extended Schwarzschild solution** *that contains three disjoint open subsets M^{white}, M^{out} and M^{black} such that*

- *There are time-orientation perserving isometries as follows, where the subset of* $M^{\mathrm{schw}}_{\mathrm{fut,past}}$ *carry the induced Lorentzian metrics and time-orientations;*

$$(M^{\mathrm{int}}, g^{\mathrm{schw}}, \textit{white hole time-or.}) \longrightarrow M^{\mathrm{white}}$$
$$(M^{\mathrm{ext}}, g^{\mathrm{schw}}, \textit{standard time-or.}) \longrightarrow M^{\mathrm{out}}$$
$$(M^{\mathrm{int}}, g^{\mathrm{schw}}, \textit{black hole time-or.}) \longrightarrow M^{\mathrm{black}}$$

- $M^{\mathrm{white}} \cup M^{\mathrm{out}} \cup M^{\mathrm{black}}$ *is dense in* $M^{\mathrm{schw}}_{\mathrm{fut,past}}$
- $\partial M^{\mathrm{white}} \cap \partial M^{\mathrm{black}} = \varnothing$ *and* $\partial M^{\mathrm{white}} \cup \partial M^{\mathrm{black}} = \partial M^{\mathrm{out}}$
- *For a smooth curve* $\gamma\colon (a,b) \to M^{\mathrm{schw}}_{\mathrm{fut,past}}$*, such that* $\dot\gamma(t)$ *is causal and future-directed for all* $t \in (a,b)$ *we have:*
  - **Black hole property:** *if* $\gamma(t_0) \in M^{\mathrm{black}}$*, then* $\gamma(t) \in M^{\mathrm{black}}$ *for all* $t \in (t_0, b)$*,*
  - **White hole property:** *if* $\gamma(t_0) \in M^{\mathrm{white}}$*, then* $\gamma(t) \in M^{\mathrm{white}}$ *for all* $t \in (a, t_0)$*,*

The sets have the following names:

- M^{white} *is the* **white hole region of the extended Schwarzschild solution***,*
- M^{out} *is called the* **region of outer communication of the extended Schwarzschild solution***,*
- M^{black} *is the* **black hole region of the extended Schwarzschild solution***.*

This extended spacetime can be extended further, see below. However, this additional extension has no physical consequence as we never can communicate or exchange matter or signals with this additional part of the universe. We obtain the following interpretation:

Physical Interpretation 3.10. *The past- and future-extended Schwarzschild solution* $(M^{\mathrm{schw}}_{\mathrm{fut,past}}, g^{\mathrm{schw}}_{\mathrm{fut,past}})$ *is a good model for a space-time with one single black hole of mass* μ *without angular momentum, without cosmological constant and without charge. We see the following effects: The space around is (almost) vacuum.*

(1) *Assume that an observer, called the* outer oberserver *moves along the curve*

$\tau \mapsto \gamma(\tau) := (\tau, r, y)$ with $r > r_\mu$. For any $x \in M^{\mathrm{schw}}_{\mathrm{fut,past}}$ we have

$$\begin{aligned} & \textit{There is a causal future-directed path from } x \textit{ to } \gamma(\tau) \textit{ for some } \tau \in \mathbb{R}. \\ \Longleftrightarrow\ & \textit{The observer can obtain “a signal” from the event } x. \\ \Longleftrightarrow\ & x \in \overline{M^{\mathrm{white}}} \cup M^{\mathrm{out}}. \end{aligned}$$

We also have

$$\begin{aligned} & \textit{There is a causal future-directed path from } \gamma(\tau) \textit{ to } x \textit{ for some } \tau \in \mathbb{R}. \\ \Longleftrightarrow\ & \textit{The observer can send “a signal” to the event } x. \\ \Longleftrightarrow\ & x \in \overline{M^{\mathrm{black}}} \cup M^{\mathrm{out}}. \end{aligned}$$

(2) *A body that has ever entered the black hole region, will never leave it any more. Because of this $\partial M^{\mathrm{black}}$ is called the* **event horizon of the black hole**. *Once the has entered it, the proper time of its trajectory is bounded from above by $c \times r_\mu$ for some unversal $c > 0$, only depending on m. The outer observer observes that a body never passes the event horizon. In fact this does not happen “in finite outer time”, but in finite proper time of the body.*

(3) *A body that emerges from the white hole, was in the white hole before all the time. Similar effects as abovehold with time-orientation reversed.*

(4) *If M_1 is an inextendible extension of $(M^{\mathrm{ext}}, g^{\mathrm{schw}})$ then the subset of all points of M_1 which can send or receive signals from M^{ext} is isometric to $(M^{\mathrm{schw}}_{\mathrm{fut,past}}, g^{\mathrm{schw}}_{\mathrm{fut,past}})$.*

As already mentioned above, the manifold $(M^{\mathrm{schw}}_{\mathrm{fut,past}}, g^{\mathrm{schw}}_{\mathrm{fut,past}})$ can be extended further. A possible description uses Kruskal-Szekeres coordinates as indeicated in Figure II.4, see `https://en.wikipedia.org/wiki/Kruskal%E2%80%93Szekeres_coordinates`.

Further reading for this part:

- Hawking, Ellis, [16]
- Wald, [27]
- O'Neill, [23]
- Bär, [8] and [7]
- Dan Lee, Geometric Relativity [19]

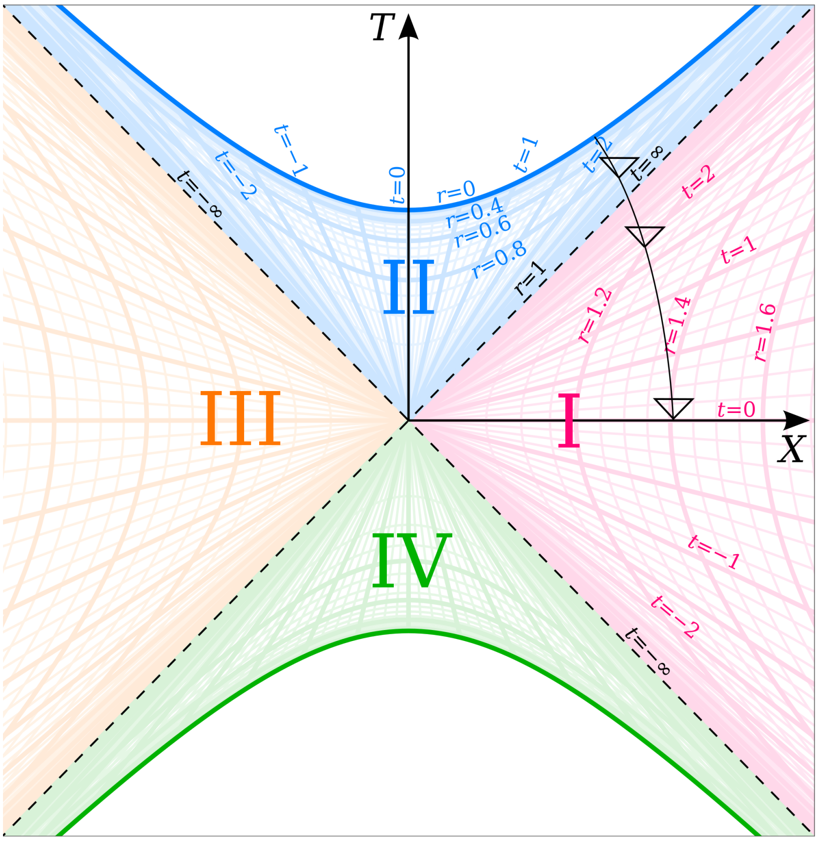


Figure II.4: This picture describes the Kruskal-Szekeres coordinates of an inextendible extensio of the Schwarzschild metric. In this drawing the S^2-component is irgnored. Region I is M^{out}, region II is M^{black} and region IV is M^{white}. Region III is again isometric to $M^{\mathrm{out}} \cong M^{\mathrm{ext}}$ and turns the manifold into an inextendible extension. There is not communication possible between regions I and III. No information can spread from one to another, but each region could send an "embassador" into the black hole where they coould meet and discuss for some time, before they will inevitably hit the singularity at the upper boundary (blue line) on top.
The origin is not part of the manifold. Thus his extension is diffeomorphic to $(a, b) \times S^1 \times S^2$. The universal covering of this space (or any other covering) would also give an inextendible extension. ©

III Comparison theorems in Riemannian geometry

Fr. 7.6.

Now we turn back to the Riemannian setting. We aim for a better understanding of the global consquences of special curvature properties such as $K \geq 0$, $K \leq 0$, $\mathrm{ric} \geq 0$, $\mathrm{ric} \leq 0$, etc. The techniques we will learn will often have analogues for statements about Lorentzian manifolds which we will not be able to cover in this lecture for time reasons.

1 Recap: the distance function in Riemannian geometry

Let (M, g) be a connected, non-empty Riemannian manifold. For a piecewise C^1-curve $\gamma\colon [a, b] \to M$ we define the **length** $\mathcal{L}(\gamma)$ as

$$\mathcal{L}(\gamma) := \int_a^b \|\dot{\gamma}(t)\| \,\mathrm{d}t$$

We then obtain the induced **distance function** dist as

$$\mathrm{dist}(x, y) := \inf \left\{ \mathcal{L}(\gamma) \;\middle|\; \gamma \text{ is a curve from } x \text{ to } y \right\}, .$$

Here and in the following, a curve is always assumed to be piecewise C^1, and $\gamma\colon [a, b] \to M$ is called a curve from x to y, iff $\gamma(a) = c$ and $\gamma(b) = y$. It is easy to check, that (M, dist) is a metric space. Thus, in usual topology lectures one would say, that dist is a metric on M. However, we often say that "g is a metric on M" for an abbreviation of "g is a Riemannian metric on M", we replace the word "metric" (in the sense of metric spaces) by the word distance.

We have seen in Differential Geometry I that the topology induced from dist is the original on M. In particular, $\mathrm{dist}\colon M \times M \to \mathbb{R}_{\geq 0}$ is continuous.

In a metric space (M, dist) we define

$$\text{the **open ball** } B_r(p) := \{q \in M \mid \mathrm{dist}(p,q) < r\},$$
$$\text{the **closed ball** } \overline{B}_r(p) := \{q \in M \mid \mathrm{dist}(p,q) \le r\}.$$

In any metric space the open ball is open, the closed ball is closed, and as a consequence $\overline{B_r(p)} \subset \overline{B}_r(p)$. The converse inclusion does not hold for arbitrary metric spaces, e. g.,

$$M = \mathbb{R}, \quad \mathrm{dist}(x,y) := \min\{|x-y|, 1\}, \quad B_1(0) = (-1,1), \quad \overline{B_1(0)} = [-1,1], \quad \overline{B}_1(0) = \mathbb{R}.$$

Lemma 1.1. *On a connected non-empty Riemannian manifold M we have $\overline{B_r(p)} = \overline{B}_r(p)$.*

Proof: We have to show $\overline{B}_r(p) \subset \overline{B_r(p)}$. So, for $q \in \overline{B}_r(p)$ we have to show $q \in \overline{B_r(p)}$. As this is trivial in the case $\mathrm{dist}(p,q) < r$, we assume $\mathrm{dist}(p,q) = r$. For any $\varepsilon > 0$ there is a path $\gamma: [a,b] \to M$ of length $\mathcal{L}(\gamma) \le r + \varepsilon$. By the intermediate value theorem there is a $\tau \in (a,b)$ with $\mathcal{L}(|_{[a,\tau]}) = r - \varepsilon$. Then $q' := \gamma(\tau) \in B_r(p)$ and $\mathrm{dist}(q',p) = 2\varepsilon$. As we may choose ϵ as small as we want, the statement follows. ■

In the last semester we have studied the exponential function in p:

$$\exp_p: \mathcal{D}_p \Subset T_pM \to M$$

where $\mathcal{D}_p \ni 0$ is star-shaped with respect to 0. The curve $t \mapsto \gamma_X(t) := \exp_p(tX)$ is the unique geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}_X(0) = X$. We define(d)[1]

$$\mathrm{injrad}(p) := \sup\left\{ r > 0 \;\middle|\; B_r^{T_pM}(0) \subset \mathcal{D}_p \text{ and } \exp_p|_{B_r^{T_pM}(0)} \text{ injective} \right\}.$$

Exercise 1.2. *Let M, g, p as above and $\rho := \mathrm{injrad}(p)$ the map*

$$\exp_p|_{B_\rho^{T_pM}(0)}$$

is an immersion.

Let us mention a lemma that helps us to understand, but that we will not prove

[1] This definition deviates from the one in Differential Geometry I, but the following lemma implies that they are equivalent.

here and that we will not use currently.

Lemma 1.3. *The function* $\operatorname{injrad}: M \to (0, \infty]$ *is continuous.*

For a proof see [26, Proposition 4.13].

An important ingredient is the Gauss lemma:

Lemma 1.4 (Gauß lemma)**.** *Let* (M, g) *be a Riemannian manifold,* $p \in M$*,* $X, Y \in T_pM$*. Let* $t \in \mathbb{R}$ *such that* $tX \in \mathcal{D}_p$*,* $q := \exp_p(tX)$*. Then*

$$g_q\Big(\mathrm{d}_{tX} \exp_p\big(X\big), \mathrm{d}_{tX} \exp_p\big(Y\big)\Big) = g_p(X, Y)\,.$$

Note that the Gauß lemma does *not* say that $\exp_p$ is an isometry. If $Y \perp X$, then the orthogonality of Y to X is preserved by $\mathrm{d}_{tX} \exp_p$, but not the length of Y. It is a partial isometry. The evolution of the length of $\mathrm{d}_{tX} \exp_p(Y)$ is controlled by the Jacobi equation, more precisely $r \mapsto \mathrm{d}_{rtX} \exp_p(rY)$ is a Jacobi field.

A consequence of the Gauss lemma is

$$B_r(p) = \exp_p\big(B_r(0)\big), \quad \overline{B}_r(p) = \exp_p\big(\overline{B}_r(0)\big)$$

if $r \le \operatorname{injrad}(p)$. One usually applies this in the case $\|X\| = 1$. Let $\rho := \operatorname{injrad}(p)$. We then define a vector field $\frac{\partial}{\partial r}$ on $B_\rho(0) \smallsetminus \{0\}$ by the formula

$$\frac{\partial}{\partial r}\Big|_{\exp_p(rX)} := \mathrm{d}_{rX} \exp_p\big(X\big)\,.$$

In other words, the length of radial vectors is preserved by normal coordinates, and the orthogonality to radial vectors as well. For $Y \perp X$ we define $\tilde{Y} := \exp_p\big(Y\big)$. Then the Gauß lemma say

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1, \quad g\left(\frac{\partial}{\partial r}, \tilde{Y}\right) = 0\,.$$

On this ball one considers $r := \mathrm{d}(\,\bullet\,, p)$ which is smooth at least on $B_\rho(p) \smallsetminus \{p\}$, and $r(\exp_p(X)) = \|X\|$.

If $f: M \to \mathbb{R}$ is a smooth function on a Riemannian manifold (M, g), then we define the **gradient** $\operatorname{grad} f \in \Gamma(TM)$ by the formula

$$g(\operatorname{grad} f, Y) = \mathrm{d}f(Y) = \partial_Y f\,.$$

It thus follows from the Gauss lemma

$$\operatorname{grad} r = \frac{\partial}{\partial r}. \tag{1.1}$$

Furthermore, if $\Phi\colon \Omega \to S^{n-1} \cong S(T_pM) \subset T_pM$ is a parametrization of the unit sphere $S(T_pM)$ in T_pM, then $(0,\rho)\times\Omega \to M$, $(r,y)\mapsto \Psi(r,y) := \exp_p(r\Phi(y))$ is a parametrization of (an open subset of) M, and its inverse Ψ^{-1} is a chart, whose components are called **polar normal coordinates**. For this chart, the coordinate vector field for the coodinate r is $\frac{\partial}{\partial r}$, and this explains the notation.

Let us summarize: for $\rho := \operatorname{injrad}(p)$, the smooth function $r := \operatorname{dist}(\bullet, p)\colon B_\rho(p) \smallsetminus \{p\} \to (0,\rho)$ satisfies

$$\|\operatorname{grad} r\| \equiv 1 \tag{1.2}$$

At this location, we discussed some facts connected to the cut locus, e. g., the smoothness of $\operatorname{dist}(\bullet, p)$ away from p and away from its cut locus. We furthermore discussed that $\|\operatorname{grad}\operatorname{dist}(\bullet,p)\| = 1$. This served as a further motivation to study generalized distance function in the section afterwards. In the script of the lecture these topics will be discussed in Section 6 and will be only sketched here.

Define

$$\mathcal{R}_p := \Big\{x \in M \smallsetminus \{p\} \;\Big|\; \text{there is a (up to parametrization) unique shortest curve } \gamma_x \text{ from } p \text{ to } x\text{, and } p \text{ is not conjugate to } x \text{ along } \gamma_x.\Big\}$$

$$\mathcal{R}_p^{\tan} := \Big\{\dot\gamma_X \mid \gamma_X\colon [0,1] \to M \text{ as above, parametrized proportional to arclength}\Big\}$$

Then $\mathcal{R}_p^{\tan} \to \mathcal{R}_p$, $X \mapsto \exp_p\big(X\big)$ is a diffeomorphism.

We further define the **cut locus** $\mathcal{C}_p$ to p as $\mathcal{C}_p = M \smallsetminus (\{p\}\cup\mathcal{R}_p)$.

Tu 11.6. The function $\operatorname{dist}(\bullet,p)\colon \mathcal{R}_p \to \mathbb{R}$ is smooth and $\|\operatorname{grad}\operatorname{dist}(\bullet,p)\|$.

Theorem 1.5 (Hopf–Rinow, Diff. Geo. I or [26]). *Let (M,g) be a connected, non-empty Riemannian manifold. Then the following are equivalent:*

(i) *(M,dist) is a complete metric space, i. e., all Cauchy sequences converge.*

(ii) *There is a $p \in M$ with $\mathcal{D}_p = T_pM$.*

(iii) *For all* $p \in M$ *we have* $\mathcal{D}_p = T_pM$.

(iv) *There is a* $p \in M$ *such that for all* $r > 0$ *the closed ball* $\overline{B}_r(p)$ *is compact.*

(v) *For all* $p \in M$ *and for all* $r > 0$ *the closure of the open ball* $\overline{B}_r(p)$ *is compact.*

From any of these conditions we may conclude

(vi) *For all* $p, q \in M$ *there is a shortest curve from* p *to* q.

Recall that a curve γ from p to q is called a **shortest curve** if $\mathcal{L}(\gamma) = \mathrm{dist}(p, q)$. After reparametrizing γ proportional to arclength, this is a geodesic.

If any (and thus all) of the conditions (i) to (v) holds, then we say that (M, g) is **complete**.

Fact 1.6 (Unproven in this script)**.** If (M, g) is complete and connected, $p \in M$ and $\dim M \geq 1$. Then $\mathcal{R}_p$ is dense in M.

Summary 1.7. *If* (M, g) *is a complete connected Riemannian manifold,* $p \in M$, *then there is a closed subset* $\mathcal{C}_p \subset M \smallsetminus \{p\}$ *such that the open set* $\mathcal{R}_p = M \smallsetminus (\{p\} \cup \mathcal{C}_p)$ *is dense. We define* $r := \mathrm{dist}(p, \bullet)$. *Then* r *is smooth on* $\mathcal{R}_p$ *and on* $\mathcal{R}_p$ *we have*

$$\|\operatorname{grad} r\| \equiv 1.$$

Furthermore, for $r_0 \in \mathbb{R}_{>0}$ *the level sets* $L_{r_0} = \{x \in \mathcal{R}_p \mid r(x) = r_0\}$ *are smooth hypersurfaces and they are orthogonal to* $\operatorname{grad} r$.

2 Generalized distance functions and the Riccati equation

Definition 2.1. *Let* (M, g) *be a Riemannian manifold. A function* $f \in C^\infty(M)$ *is called* **generalized distance function** *if*

$$\|\operatorname{grad} f\| \equiv 1.$$

Lemma 2.2. *If* f *is a generalized distance function, then*

$$\nabla_{\operatorname{grad} f}(\operatorname{grad} f) = 0.$$

Corollary 2.3. *If γ is an integral curve of* $\operatorname{grad} f$*, i. e., if*

$$\dot\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \operatorname{grad} f\big|_{\gamma(t)}.$$

Then we have $\nabla_{\dot\gamma(t)}(\dot\gamma(t)) = 0$, in other words γ is a geodesic parametrized by arclength.

Proof of the lemma: Recall the definition of the **Hessian** $\operatorname{Hess} f \in \Gamma(T^*M \otimes T^*M)$ of f, defined as

$$\begin{aligned}\operatorname{Hess} f(X,Y) := \nabla \mathrm{d}f(X,Y) := \big(\nabla_X(\mathrm{d}f)\big)(Y) = \partial_X\big(\mathrm{d}f(Y)\big) - \mathrm{d}f\big(\nabla_X Y\big)\\ = \partial_X\langle \operatorname{grad} f, Y\rangle - \langle \operatorname{grad} f, \nabla_X Y\rangle = \langle \nabla_X \operatorname{grad} f, Y\rangle \end{aligned} \qquad (2.1)$$

for $X, Y \in \mathfrak{X}(M)$. With these calculations we also see that

$$\begin{aligned}\operatorname{Hess} f(X,Y) - \operatorname{Hess} f(Y,X) &= \partial_X\big(\mathrm{d}f(Y)\big) - \mathrm{d}f\big(\nabla_X Y\big) - \partial_Y\big(\mathrm{d}f(X)\big) + \mathrm{d}f\big(\nabla_Y X\big)\\ &= \partial_X\partial_Y f - \mathrm{d}f\big(\nabla_X Y\big) - \partial_Y\partial_X f + \mathrm{d}f\big(\nabla_Y X\big)\\ &= \partial_{[X,Y]} f - \mathrm{d}f\big(\nabla_X Y - \nabla_Y X\big)\\ &= \mathrm{d}f\big([X,Y]\big) - \mathrm{d}f\big([X,Y]\big) = 0\,,\end{aligned}$$

i. e., $\operatorname{Hess} f$ is a symmetric tensor. From (2.1) we obtain for $X = \operatorname{grad} f$.

$$\begin{aligned}\langle \nabla_X X, Y\rangle = \langle \nabla_X \operatorname{grad} f, Y\rangle &\overset{(2.1)}{=} \operatorname{Hess} f(X,Y)\\ &= \operatorname{Hess} f(Y,X) = \langle \nabla_Y \operatorname{grad} f, X\rangle = \frac{1}{2}\partial_Y \underbrace{\langle \operatorname{grad} f, \operatorname{grad} f\rangle}_{\equiv 1} = 0\end{aligned}$$

■

Proposition 2.4. *Let (M,g) be a Riemannian manifold, $A \subset M$ a subset. We define*

$$f(x) := \operatorname{dist}(x,A) := \inf_{p\in A} \operatorname{dist}(x,p)\,.$$

Let f be differentiable $q \in M$ and $f(q) > 0$. Then $\|(\operatorname{grad} f)_q\| = 1$.

Proof:

1. Claim: For all $x, y \in M$ we have $|f(x) - f(y)| \le \operatorname{dist}(x,y)$. In particular, f is continuous.

For all $p \in A$ the triangle inequality yields $\operatorname{dist}(x,p) \le \operatorname{dist}(x,y) + \operatorname{dist}(y,p)$. By taking the infimum over all $p \in A$ we obtain $f(x) - f(y) \le \operatorname{dist}(x,y)$. We obtain

$f(y) - f(x) \leq \mathrm{dist}(x,y)$ in an analogous way.

2. Claim: If f is differentiable in q, then $\|(\mathrm{grad}\, f)_q\| \leq 1$.

Assume $X \in T_qM$, $\|X\| = 1$. We have to show $|\langle X, \mathrm{grad}\, f\rangle| \leq 1$. Let $c\colon (-\epsilon,\epsilon) \to M$ be smooth and parametrized by arclength, $c(0) = q$, $\dot{c}(0) = X$. Then

$$|f(c(t)) - f(c(0))| \leq \mathrm{dist}(c(t), c(0)) \leq \mathcal{L}\Big(c\big|_{[0,t]}\Big) = t$$

and as the $t \mapsto f\big(c(t)\big)$ is differentiable in $t = 0$, we get

$$\Big|\Big\langle (\mathrm{grad}\, f)_q, X\Big\rangle\Big| = |(d_qf)(\dot{c}(0))| = \Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(c(t)\big)\Big| \leq 1.$$

3. Claim: $\|(\mathrm{grad}\, f)_q\| \geq 1$.

We define

$$r := \min\left\{\frac{f(q)}{2}, \frac{1}{2}\,\mathrm{injrad}(q)\right\} > 0.$$

Then $\exp_q\big|_{B_r(0_q)}$ is a diffeomorphism onto its image. We define

$$\overline{B} := \overline{B_r(q)} = \exp_q\Big(\overline{B_r(0_q)}\Big),$$

where $\overline{B_r(0_q)}$ is the closed ball of radius r in T_qM. Furthermore, we choose $X \in \overline{B_r(0_q)}$, such that $f(\exp_q(X)) = \min_{y \in \overline{B}} f(y)$. Accordung to the 1. Claim and Lemma 1.4 we have for all $y \in \overline{B}$:

$$f(q) - f(y) \leq f(q) - f(\exp_q(X)) \leq \mathrm{dist}(q, \exp_q(X)) = \|X\| \leq r \leq \frac{f(q)}{2}. \tag{2.2}$$

For $y \in \overline{B}$, we thus have $f(y) \geq \frac{f(q)}{2} > 0$ and it follows $y \notin \overline{A}$. We obtain $\overline{A} \cap \overline{B} = \varnothing$. From the series of inequalities (2.2) we also obtain $f(\exp_q(X)) \geq f(q) - r$.

We now want to prove $f(\exp_q(X)) \leq f(q) - r$. From the definition f it is immediate that for any $\epsilon > 0$ there is a curve $c\colon [0,\ell] \to M$ from q to A, parametrized by arclength, with $f(q) \leq \ell \leq f(q) + \epsilon$. We choose the largest $t_0 \in [0,\ell]$ with $c(t_0) \in \overline{B}$. As t_0 is chosen maximal,we get $c(t_0) \notin B_r(q)$, Thus $\mathrm{dist}\big(c(t_0), q\big) = r$. As c is parametrized by arclength we get $t_0 \geq \mathrm{dist}\big(c(t_0), q\big) = r$, and it follows that

$$\mathrm{dist}\big(c(t_0), c(\ell)\big) \leq \ell - t_0 \leq f(q) + \epsilon - r\,.$$

This implies using the definition of X, the definition of $f\big(c(t_0)\big)$ and because of

$c(\ell) \in A$:

$$f\big(\exp_q(X)\big) \le f\big(c(t_0)\big) \le \operatorname{dist}\big(c(t_0), c(\ell)\big) \le f(q) + \epsilon - r.$$

For $\epsilon \to 0$ we get $f(\exp_q(X)) \le f(q) - r$ as claimed.

We have proven $f(\exp_q(X)) = f(q) - r$ and from (2.2) we see that $\|X\| = r$. We have concluded from the Gauss Lemma 1.4 that $\operatorname{dist}\big(q, \exp_q(Y)\big) = \|Y\|$ for all $Y \in T_qM$ with $\|Y\| < \operatorname{injrad}(q)$. We apply this for $Y = tX$, $t \in [0,1]$ and get

$$f(q) - f(\exp_q(tX)) \overset{(1.\ C)}{\le} \operatorname{dist}(q, \exp_q(tX)) = t\|X\| \le rt \tag{2.3}$$

where we used the 1. Claim at $(1.\ C)$. We continue

$$f\big(\exp_q(tX)\big) - f\big(\exp_q(X)\big) \overset{(1.\ C)}{\le} \operatorname{dist}\big(\exp_q(tX), \exp_q(X)\big) \le (1-t)r. \tag{2.4}$$

Adding up the inequalities (2.3) and (2.4) comparing this to $f(q) - f(\exp_q(X)) = r$ we see that all $\le$-signs in (2.3) and (2.4) can be replaced by $=$. In particular $f\big(\exp_q(tX)\big) = f(q) - rt$ for all $t \in [0,1]$. By differentiation with respect to t at $t = 0$ we obtain

$$\langle (\operatorname{grad} f)_q, X\rangle = df_q(X) = -r = -\|X\|.$$

It follows $\langle (\operatorname{grad} f)_q, -\frac{X}{\|X\|}\rangle = 1$ and thus $\|(\operatorname{grad} f)_q\| \ge 1$. ∎

Corollary 2.5. *Let $A \subset M$ and let U be an open subset of $M \setminus A$ with* $\operatorname{dist}(\bullet, A)$ *smooth on U. Then* $\operatorname{dist}(\bullet, A)|_U$ *is a generalized distance function.*

Let f be a generalized distance function on M and let $s \in \mathbb{R}$. We define the **level set** $N_s := f^{-1}(s)$, which is a (smooth) hypersurface of M, as $\operatorname{grad} f \neq 0$. Further we have

$$(\operatorname{grad} f)_p^\perp = \ker d_p f = T_p N_s, \quad s = f(p)$$

and thus $\nu := \operatorname{grad} f$ is a unit normal field on N_s. We define the **Weingarten map** at $p \in M$, also called **shape operator**[2] at $p \in M$

$$S_p \colon T_pN_s \to T_pN_s, \quad X \mapsto -\nabla_X \nu\,.$$

Note that this map is well-defined as $\|\nu\| \equiv 1$ implies $\nabla_X\nu \perp \nu$, and thus $\nabla_X \nu \in T_pN_s$.

Let $M := \dim M$, thus $\dim N_s = m - 1$. The **mean curvature** H_P of $H_p :=$

[2] In the English community, "shape operator" is more common, in the German one, "Weingarten map" is the usual expression

$\frac{1}{m-1}\operatorname{tr}(S_p)$. The **second fundamental form** II_p in p is defined by

$$\mathrm{II}_p: T_pM \times T_pM \to \mathbb{R}, \quad \mathrm{II}_p(X,Y) = \langle S_p(X), Y\rangle.$$

Remark 2.6. Although we do not need it here, let us recall that for vector fields $X, Y \in \mathfrak{X}(TN_s)$ the normal part of $\nabla_X Y\big|_p$ is $\mathrm{II}_p\left(X\big|_p, Y\big|_p\right)\nu\big|_p$.

We get

$$\mathrm{II}_p(X,Y) = -\langle \nabla_X \nu, Y\rangle = -\langle \nabla_X \operatorname{grad} f, Y\rangle = -(\operatorname{Hess}_p f)(X,Y). \tag{2.5}$$

Fr 14.6.

Theorem 2.7 (Riccati equation). *Let (M,g) be a Riemannian manifold, $f: M \to \mathbb{R}$ a generalized distance function, $N_s := f^{-1}(s)$, $\nu := \operatorname{grad} f$. Then*

$$\nabla_\nu S = R_\nu + S \circ S, \tag{2.6}$$

where S is the Weingarten map of the level set N_s and $R_\nu \in \operatorname{End}(T_pN_{f(p)})$ is defined through $R_\nu(X) := R(X,\nu)\nu$ with Riemannian curvature tensor R.

The set

$$T^fM := \bigcup_{p\in M} \ker d_pf = \bigcup_{p\in M} T_pN_{f(p)}.$$

carries the structure of a vector bundle[3] of rank $m-1$ over M; and we have an isomorphism

$$T^fM \oplus \mathbb{R} \cong TM, \quad (X,t) \mapsto X + t\nu$$

of vector bundles.

Lemma 2.8. *Let again $f: M \to \mathbb{R}$, $\nu = \operatorname{grad} f$ and N_s as above. Let $X \in \Gamma(T^fM)$,*

[3]Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A $\mathbb{K}$-vector bundle over an m-dimensional manifold M is a collection of $\mathbb{K}$-vector spaces V_p, considered as disjoint sets. All vector spaces have a fixed dimension, called the rank $k \in \mathbb{N}_0$. The disjoint union $V = \bigcup_{p\in M} V_p$ carries the structure of an $(m+k)$-dimensional (resp. $(m+2k)$-dimensional for $\mathbb{K} = \mathbb{C}$)manifold, and the map $\pi: V \to M$ defined by the property $\pi^{-1}(\{p\}) = V_p$ is a Riemannian submersion and proper as a map. The vector space operations are smooth maps, and the bundle is locally trivial, i. e., M is covered by open sets U such that $\pi^{-1}(U) \cong U \times \mathbb{K}^k$ where $\cong$ means a diffeomorphism preserving the vector space structure, such that it factors through the identity $U \to U$. There are slightly different ways to formalize this properly, all being equivalent. Examples that you already know are $\pi: TM \to M$, $\pi: T^*M \to M$, $\pi: \Lambda^k T^*M \to M$, $\mathcal{T}^{(r,s)}M, \ldots$. The spaces V_p are called the fibers, M is the base, and the manifold V (smooth) sections of V are denoted as $\Gamma(V)$, vector bundles may carry a connection, a (fiberwise) scalar product etc. We do not want to formalize this properly in the lecture for time reasons. You are encouraged to look this (ask for literature, if you want hints), or to use it intuitively.

$\nu = \operatorname{grad} f$, $\|\nu\| \equiv 1$. *Then we also have* $\nabla_\nu X \in \Gamma(T^f M)$.

Proof: We have

$$\begin{aligned}\langle \nabla_\nu X, \nu \rangle &= \partial_\nu \langle X, \nu \rangle - \langle X, \nabla_\nu \nu \rangle \\ &= 0 + 0 = 0\end{aligned}$$

The first summand vanishes because of $\langle X, \nu \rangle \equiv 0$, the second one, as we have seen $\nabla_\nu \nu =$ in Lemma 2.2. The yields the statement. ■

Definition 2.9. *For* $A \in \operatorname{End}(T^f M)$ *we define* $\nabla_\nu A \in \operatorname{End}(T^f M)$ *by claiming that*

$$(\nabla_\nu A)(X) := \nabla_\nu (A(X)) - A(\nabla_\nu X)$$

for all $X \in \Gamma(T^f M)$.

If $\|\nu\| \equiv 1$, then $(\nabla_\nu A)(X) \in \Gamma(T^f M)$. Furthermore, for any function $f \in C^\infty(M)$ we have:

$$(\nabla_\nu A)(fX) = f(\nabla_\nu A)(X).$$

It follows $(\nabla_\nu A) \in \Gamma\big(\operatorname{End}(T^f M)\big)$.

Properties 2.10.

(1) $\nabla_\nu \operatorname{id}_{T^f M} = 0$

(2) $\operatorname{tr}(\nabla_\nu A) = \partial_\nu (\operatorname{tr} A)$

Proof:
"(1)": obvious.
"(2)": We fix $s_0 \in \mathbb{R}$. For $p \in N_{s_0}$ we choose a local frame $(e_1, \ldots, e_{m-1})$ of TN_{s_0}, defined on an open subset $U \Subset N_{s_0}$ with $p \in U$. We transport $(e_1, \ldots, e_{m-1})$ parallely along the integral lines of ν and we obtain a local orthonormal frame $(e_1, \ldots, e_{m-1})$ on an open subset $\widetilde{U} \Subset M$ with $p \in \widetilde{U}$, i.e., for i we have $e_i \in \mathfrak{X}(\widetilde{U})$ and for any $p \in \widetilde{U} \cap N_s$, we have an orthonormal basis $(e_1(p), \ldots, e_{m-1}(p))$ of $T_p N_s = T_p^f M$.

Because of $\nabla_\nu e_i \equiv 0$ and Lemma 2.2 we have

$$\partial_\nu \langle e_i, \nu \rangle = \langle \nabla_\nu e_i, \nu \rangle + \langle e_i, \nabla_\nu \nu \rangle = 0 + 0 = 0$$

and thus we get $e_i \perp \nu$ on $\widetilde{U}$. Let $a_{ji} \in C^\infty(\widetilde{U})$, such that $Ae_i = \sum_{j=1}^{m-1} a_{ji} e_j$. Then $\partial_\nu(\operatorname{tr} A) = \sum_{i=1}^{m-1} \partial_\nu a_{ii}$ and

$$(\nabla_\nu A)(e_i) + A(\nabla_\nu e_i) = \nabla_\nu(Ae_i) = \sum_{j=1}^{m-1} \Big((\partial_\nu a_{ji}) e_j + a_{ji} \nabla_\nu e_j \Big).$$

Because of $\nabla_\nu e_i \equiv 0$ we obtain $\operatorname{tr}(\nabla_\nu A) = \sum_{i=1}^{m-1} \partial_\nu a_{ii}$. ∎

Remark 2.11. Let $f: M \to \mathbb{R}$ be a generalized distance function, and $N_s := f^{-1}(s)$ for $s \in \mathbb{R}$. Let $\gamma: (-\epsilon, \epsilon) \to M$, $\gamma(0) = p$, $\dot\gamma(t) = \nu|_{\gamma(t)} = (\operatorname{grad} f)|_{\gamma(t)}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t) = \mathrm{d}f(\dot\gamma(t)) = \left\| \operatorname{grad} f \right\|^2 = 1$$

and thus $f(\gamma(t)) = t + f(p)$, hence $\gamma(t) \in N_{f(p)+t}$.

Remarks 2.12.

1.) Let V be a $\mathbb{K}$-vector bundle over M. A connection ∇ is defined as a map $\nabla: \mathfrak{X}(M) \times \Gamma(V) \to \Gamma(V)$ with the following properties

(i) for any $s \in \Gamma(V)$, the map $\nabla V: \mathfrak{X}(M) \to \Gamma(V)$, $X \mapsto \nabla_X s$ is a linear map of $C^\infty(M)$-modules,

(ii) for any $X \in \mathfrak{X}(M)$, the map $\nabla_X: \Gamma(V) \to \Gamma(V)$, $s \mapsto \nabla_X s$ is a linear map of $\mathbb{K}$-vector spaces,

(iii) for any $X \in \mathfrak{X}(M)$, and we have for $f \in C^\infty(M)$, for $X \in \mathfrak{X}(M)$ and for $s \in \Gamma(V)$:

$$\nabla_X(fs) = (\partial_X f)s + f \nabla_X s\,.$$

2.) Now let us assume that the vector bundle V carries a connection ∇. We also assume that M carries a semi-Riemannian metric, thus we have a Levi-Civita connection on M. The **second covariant derivative** of $\xi \in \Gamma(V)$ is then defined as

$$\nabla^2_{X,Y}\xi := \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi$$

for $X, Y \in TM$. Note, that here $\nabla_X Y$ is the Levi-Civita connection.

3.) Now let $V = TM$ and ∇ the Levi-Civita connection. Then for the Riemann curvature tensor

$$R(X,Y)\xi = \nabla^2_{X,Y}\xi - \nabla^2_{Y,X}\xi.$$

Proof of Theorem 2.7: Let $X \in \Gamma(T^f M)$. Then

$$\begin{aligned}(\nabla_\nu S)(X) &= \nabla_\nu(S(X)) - S(\nabla_\nu X) = -\nabla_\nu(\nabla_X \nu) + \nabla_{\nabla_\nu X}\nu \\ &= -\nabla^2_{\nu,X}\nu = R(X,\nu)\nu - \nabla^2_{X,\nu}\nu \\ &= R_\nu(X) - \nabla_X \underbrace{\nabla_\nu \nu}_{=0} + \nabla_{\nabla_X \nu}\nu \\ &= R_\nu(X) - S(\nabla_X \nu) \\ &= R_\nu(X) + S(S(X)),\end{aligned}$$

where we used, once again, Lemma 2.2. ■

Lemma 2.13. *Let V be an n-dimensional Euclidean vector space, $A \in \mathrm{End}(V)$. then*

$$n\,\mathrm{tr}(A^* A) \geq (\mathrm{tr}(A))^2$$

and we have equality if, and only if there is $\lambda \in \mathbb{R}$, such that $A = \lambda\,\mathrm{id}_V$.

Proof: We define a scalar product on $\mathrm{End}(V)$ as follows: for $A, B \in \mathrm{End}(V)$ we set

$$\langle A, B\rangle := \mathrm{tr}(AB^*) = \mathrm{tr}(B^* A) = \mathrm{tr}(BA^*),$$

which turns $(\mathrm{End}(V), \langle .,.\rangle)$ into a Euclidean vector space. Let $e_1, ..., e_n$ by an orthonormal basis of V. Then we have

$$\langle A, B\rangle = \sum_{i=1}^n \langle e_i, B^* A e_i\rangle = \sum_{i=1}^n \langle B e_i, A e_i\rangle,$$

in particular $\langle \mathrm{id}_V, A\rangle = \mathrm{tr}(A)$ and $\langle \mathrm{id}_V, \mathrm{id}_V\rangle = n$.

From the Cauchy–Schwarz inequality we get

$$(\mathrm{tr}\,A)^2 = \langle \mathrm{id}_V, A\rangle^2 \leq \langle \mathrm{id}_V, \mathrm{id}_V\rangle \langle A, A\rangle = n\,\mathrm{tr}(A^* A)$$

and equality hold, iff there is a $\lambda \in \mathbb{R}$ with $A = \lambda\,\mathrm{id}_V$. ■

From Theorem 2.7 we deduce

$$\mathrm{tr}\,\nabla_\nu S = \mathrm{tr}\,R_\nu + \mathrm{tr}(S \circ S)$$

and thus

$$\partial_\nu \operatorname{tr}(S) \geq \operatorname{tr}(R_\nu) + \frac{1}{m-1}(\operatorname{tr} S)^2$$

As we have defined $R_\nu(X) := R(X,\nu)\nu$, we get $\operatorname{tr} R_\nu = \operatorname{ric}(\nu,\nu)$. Now, using $H := \frac{1}{m-1}\operatorname{tr} S$ we obtain:

Theorem 2.14 (Riccati inequality for mean curvature)**.** *Under the conditions of Theorem 2.7 we obtain*

$$\partial_\nu H \geq \frac{1}{m-1}\operatorname{ric}(\nu,\nu) + H^2 \tag{2.7}$$

and equality holds in p, if and only if, $S_p = H_p \operatorname{id}_{T_pN_{f(p)}}$. ∎

Note that a point p in a hypersurface N is called **umbilic**, if there is a $\lambda \in \mathbb{R}$ with $S_p = \lambda \operatorname{id}_{T_pN}$, and in this case $\lambda = H_p$. Thus we have equality in the Riccati inequality at p, if and only if, p is umbilic.

Examples 2.15 (Spaces of constant sectional curvature κ)**.** Here and in the following we write $\mathbb{S}^m$ for the m-dimensional sphere with the standard metric, i. e., the metric induces from $\mathbb{R}^{m+1}$. We consider the following spaces with the standard metrics

$$\begin{aligned} \mathbb{S}^m, \quad \mathbb{R}P^m = \mathbb{S}^m/\{\pm 1\}, &\qquad \kappa = 1, \\ \mathbb{R}^m, \quad \mathbb{R}^m/\mathbb{Z}^m, &\qquad \kappa = 0, \\ \mathbb{H}^m = \left\{x \in \mathbb{R}^{m,1} \mid \langle x,x\rangle = -1, x_{m+1} > 0\right\}, &\qquad \kappa = -1. \end{aligned}$$

By definition we have for all $X,Y,Z \in TM$:

$$R(X,Y)Z = \kappa\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right)$$

and thus

$$R_\nu(X) = R(X,\nu)\nu = \kappa\left(X - \langle X,\nu\rangle\nu\right) = \kappa X,$$

if $X \perp \nu$. It follows $\nabla_\nu S = \kappa \operatorname{id}_{T^fM} + S \circ S$. We additionally assume $S = u \operatorname{id}_{T^fM}$ for a function $u \in C^\infty(M)$.

Case 1: $\kappa = 0$**.**

As in Remark 2.11 we define $\gamma\colon (-\epsilon,\epsilon) \to M$, $\gamma(0) = p$, $\dot\gamma(t) = \nu|_{\gamma(t)}$, and and thus $f(\gamma(t)) = t + f(p) =: s$, hence $\gamma(t) \in N_s$. For $v(s) := u(\gamma(s - f(p)))$ we have $v'(s) = v(s)^2$ following Theorem 2.7. Then $v \equiv 0$ or $v(s) = -\frac{1}{s+C}$, where C is a smooth function on N_s.

Subcase 1a: $M = \mathbb{R}^m$**,** $f(x) = \mathrm{dist}(x,0) = \|x\|$**.**

Then $N_s = S^{m-1}(s) = \left\{x \in \mathbb{R}^m \mid \|x\| = s\right\}$ and $S_p = -\frac{1}{\|p\|}\,\mathrm{id}_{p^\perp}$ for $p \in \mathbb{R}^m$. It follows that $v(s) = -\frac{1}{s}$, $C = 0$ and in Theorem 2.14 we get equality.

Subcase 1b: $M = \mathbb{R}^m$**,** $f(x) = \mathrm{dist}\left(x, \mathbb{R}^{m-1} \times \{0\}\right) = |x_m|$**.**

Then

$$N_s = (\mathbb{R}^{m-1} \times \{s\}) \cup (\mathbb{R}^{m-1} \times \{-s\}).$$

Furthermore we have $\mathrm{II} \equiv 0$, $S \equiv 0$, $u \equiv 0$ and in Theorem 2.14 we get equality.

Subcase 1c: $M = \mathbb{R}^3$**,** $f(x) = \mathrm{dist}(x, \mathbb{R} \times \{(0,0)\}) = \sqrt{x_2^2 + x_3^2}$**.**

Then $N_s = \left\{x \in \mathbb{R}^3 \mid x_2^2 + x_3^2 = s^2\right\}$ and for $p \in N_s$ the shape operator S_p has eigenvalues 0 and $-\frac{1}{s}$. Furthermore $H = -\frac{1}{2s}$, $\partial_\nu H = \frac{1}{2s^2} > H^2 = \frac{1}{4s^2}$.

Case 2: $\kappa = 1$**.**

Diff. geom. I, Exercise Sheet 14, Exercise 2 tells us: If M is complete and connected, then the universal covering of M is isometric to the sphere of radius 1.

Now let $M = \mathbb{S}^m$, $f(x) = \mathrm{dist}(x, e_0)$, $e_0 = (1, 0, ..., 0) \in \mathbb{R}^{m+1}$, where dist is the distance function of S^m. Then $f(x_0, \ldots, x_{m+1}) = \arccos(x_0)$ and $N_s = \{x \in S^m | x_0 = \cos s\}$. For $s \in (0, \pi)$ the level set N_s is a smooth hypersurface.

For $p \in N_s$ we calculate $S_p = -\cot(s)\,\mathrm{id}_{T_pN_s}$ and thus $H(p) = -\cot(s)$. In Theorem 2.14 equality holds, as $\frac{\partial}{\partial s} u = \frac{1}{\sin^2 s} = \kappa + u^2$, because of $\kappa = 1$. For $s \to 0$ we have $u(s) \sim -\frac{1}{s}$, für $s \to \pi$ gilt $u(s) \sim \frac{1}{\pi - s}$.

Case 3: $\kappa = -1$**.**

Diff. geom. I, Exercise Sheet 14, Exercise 2 tells us: If M is complete and connected, then the universal covering is isometric to

$$M = \mathbb{H}^m = \left\{(x_0, ..., x_m) \;\middle|\; -x_{m+1}^2 + \sum_{i=1}^m x_i^2 = -1, x_{m+1} > 0\right\}.$$

Let $f(x) = \mathrm{dist}(x, e_0)$. Then $f(x_1, \ldots, x_{m+1}) = \mathrm{arcosh}(x_{m+1})$ and N_s is a smooth hypersurface for all $s \in (0, \infty)$. For $p \in N_s$ one calculates that $S_p = -\coth(s)\,\mathrm{id}_{T_pN_s}$ and thus $u(p) = -\coth(s)$. In Theorem 2.14 equality holds once again.

3 Riccati equation and Jacobi equation

Tu 18.6.

Let $f\colon M \to \mathbb{R}$ be a generalized distance function, let $p \in N_{t_0} = f^{-1}(t_0)$ and $\xi \in T_p N_{t_0}$. Let $c\colon (-\epsilon, \epsilon) \to N_{t_0}$ be smooth with $c(0) = p$, $\dot{c}(0) = \xi$ and let $\nu := \operatorname{grad} f$.

We consider the geodesic variation

$$\gamma_s(t) := \exp_{c(s)}\big((t - t_0 \nu_{c(s)}\big).$$

Then $t \mapsto \gamma_s(t)$ is an integral line of ν:

$$\dot{\gamma}_s(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma_s(t) = \nu_{\gamma_s(t)}.$$

The calculation in Remark 2.11 implies that $\gamma_s(t) \in N_t$. The vector field

$$J(t) := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\gamma_s(t)$$

is a Jacobi field along γ_s with $J(t_0) = \frac{\partial}{\partial s}\big|_{s=0} c(s) = \xi$. We have

$$\begin{aligned} \frac{\nabla}{dt}J(t) &= \frac{\nabla}{dt}\frac{\partial}{\partial s}\Big|_{s=0}\gamma_s(t) = \frac{\nabla}{ds}\Big|_{s=0}\frac{\partial}{\partial t}\gamma_s(t) = \nabla_{J(t)}\nu = -S\big(J(t)\big), && (3.1)\\ \frac{\nabla^2}{dt^2}J(t) &= -\frac{\nabla}{dt}S\big(J(t)\big) = -(\nabla_\nu S)\big(J(t)\big) - S\left(\frac{\nabla}{dt}J(t)\right) \\ &= -(\nabla_\nu S)\big(J(t)\big) + S^2\big(J(t)\big). && (3.2) \end{aligned}$$

For comparison see that the Jacobi equation is $\frac{\nabla^2}{dt^2}J(t) + R_\nu\big(J(t)\big) = 0$ which yields

$$-(\nabla_\nu S)\big(J(t)\big) + R_\nu\big(J(t)\big) + S^2\big(J(t)\big) = 0. \tag{3.3}$$

Remark 3.1. The following statements are equivalent:

Riccati equation For all level sets of generalized distance functions we have: $-\nabla_\nu S + S^2 = -R_\nu$.	$\Longleftrightarrow$	*Jacobi equation* For all Jacobi fields along any geodesic we have: $-R_\nu(J) = \frac{\nabla^2}{dt^2}J$

In particular, we get a new proof of the Riccati equation.

Proof:

"⇐": Suppose we know that the Jacobi equation holds. Then the arguments in (3.1), (3.2), and (3.3) applied to an arbitrary $\xi = J(t_0)$ yield the Riccati equation (2.6). "⇒": If we want to derive the Jacobi equation from the Riccati equation, we have to find a suitable generalized distance function f and level sets N_s. This is possible, but we do not want to prove this: t requires some tricks, but we do not need this fact later. ■

The Riccati equation as an ODE

In the following, we consider the following setting:

Setting 3.2. *Let (M,g) be a connected non-empty Riemannian manifold, $p \in M$. We assume that we have a geodescis $\gamma\colon [0,b) \to M$, parametrized by arclength, with $\gamma(0) = p$. Let*

$$\mathrm{nc}_{\gamma,0} := \{t \in (0,b) \mid t \textit{ is not conjugate to } 0 \textit{ along } \gamma\}.$$

Exercise Sheet 10, Exercise 2 tells us that $\mathrm{nc}_{\gamma,0}$ is open and dence in $(0,b)$.

We then have $\gamma(t) = \exp_p(tX)$, $t \in [0,b)$ for $\|X\| = 1$. For $\xi \in T_pM$, $\xi \perp X$, $\xi \neq 0$ we define the geodesic variation as

$$\gamma_s(t) = \exp_p\Big(t\big(\cos(s\|\xi\|)X + \sin(s\|\xi\|)\frac{\xi}{\|\xi\|}\big)\Big). \tag{3.4}$$

With

$$J_\xi(t) := \frac{\partial}{\partial s}\Big|_{s=0}\gamma_s(t) = (\mathrm{d}_{tX}\exp_p)(t\xi) = t(\mathrm{d}_{tX}\exp_p)(\xi) \tag{3.5}$$

we obtain $J_\xi(0) = 0$ and

$$\begin{aligned} \frac{\nabla}{dt}\Big|_{t=0} J_\xi &= \frac{\nabla}{dt}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0}\gamma_s(t) = \frac{\nabla}{ds}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\gamma_s(t) \\ &= \frac{d}{ds}\Big|_{s=0}\Big(\cos(s\|\xi\|)X + \sin(s\|\xi\|)\frac{\xi}{\|\xi\|}\Big) = \xi. \end{aligned}$$

For any $t_0 \in \mathrm{nc}_{\gamma,0}$ we have shown in Differential Geometry I, that the map

$$\begin{aligned} \{\xi \in T_pM \mid \xi \perp \dot\gamma(0) = X\} &\to \{\tilde\xi \in T_{\exp_p(t_0X)} \mid \tilde\xi \perp \dot\gamma(t_0)\} \\ \xi &\mapsto J_\xi(t_0) \end{aligned}$$

is a bijection.

Definition 3.3. *We assume Setting 3.2, and* $t_0 \in \mathrm{nc}_{\gamma,0}$, *and we define the bijection* $\xi \mapsto J_\xi(t_0)$ *as above. Then there is a unique endomorphism* $S_{t_0}\colon \dot\gamma(t_0)^\perp \to \dot\gamma(t_0)^\perp$ *such that* $S_{t_0}(J_\xi(t_0)) := -\frac{\nabla}{dt}J_\xi(t_0)$. *Then we have* $S_{t_0} \in \mathrm{End}(\dot\gamma(t_0)^\perp)$.

Lemma 3.4. *Again assume Setting 3.2,* $b_0 \in (0, b]$ *and we define* $\mathrm{dist}_p(x) := \mathrm{dist}(p, x)$.

We assume

(i) $\mathrm{nc}_{\gamma,0} \subset (0, b_0)$, *i. e., there is no point* $(0, b_0)$ *that is conjugate to* 0

(ii) $\gamma\big|_{[0,t]}$ *is a shortest geodesic from* p *to* $\gamma(t)$ *for all* $t \in (0, b_0)$.

(iii) dist_p *is smooth on a neighborhood* U *of* $\gamma\big((0, b_0)\big)$

Then

(1) $S_t(W) = S_{\gamma(t)}(W)$ *for* $W = J_\xi(t)$, *where* $S_{\gamma(t)}$ *is the Weingarten map of* $\mathrm{dist}_p^{-1}(t)$,

(2) $t \mapsto S_t$ *satisfies the Riccati equation.*

Note that in Section 1 we have already briefly discussed arguments that show that (i) and (ii) imply (iii) for sufficiently small U. The conditions ((i)) to ((iii)) are obviously satisfied if $b_0 \le \mathrm{injrad}(p)$.

Proof:
"(1)": For all $t \in (0, b_0)$ and small s we have $\mathrm{grad}\,\mathrm{dist}_p\big|_{\gamma_s(t)} = \dot\gamma_s(t)$. In order to justify this we set

$$Y_s := \cos(s\|\xi\|)X + \sin(s\|\xi\|)\frac{\xi}{\|\xi\|}, \quad \gamma_s(t) := \exp_p(tY_s)$$

and we get with (1.1)

$$\dot\gamma_s(t) = (d_{tY}\exp_p)(Y) = \frac{\partial}{\partial r}\Big|_{\gamma_s(t)} = \mathrm{grad}\,\mathrm{dist}_p\big|_{\gamma_s(t)},$$

if $t \in (0, b_0)$. With $W := J_\xi(t)$ we obtain

$$S_{\gamma(t)}(W) = -\nabla_W \,\mathrm{grad}\,\mathrm{dist}_p = -\frac{\nabla}{ds}\Big|_{s=0}\dot\gamma_s(t) = -\frac{\nabla}{dt}J_\xi(t) = S_t(W)$$

and this yields (1).
"(2)": Let again J_ξ be a Jacobi field with $J_\xi(0) = 0$, $\frac{\nabla}{dt}J_\xi(0) = \xi \perp \dot\gamma(t)$. Then we see

using (3.2).

$$-R_{\dot\gamma(t)}\big(J_\xi(t)\big) = \frac{\nabla^2}{dt^2}J(t) = (-\nabla_{\dot\gamma(t)}S + S^2)(J(t)).$$

For $t \in \mathrm{nc}_{\gamma,0}$ any $W := \dot\gamma(t)^\perp$ is obtaind as $W = J_\xi(t)$ for some $\xi \perp \dot\gamma(0)$. Thus we get (2). ∎

In the following we identify the tangent spaces with parallel transport along γ, provided that we are in the Setting 3.2.

Then equation (2.6) turns into

$$\dot S_t = R_\nu(t) + S_t^2, \tag{3.6}$$

where $R_\nu(t) \in \mathrm{End}(\dot\gamma(t)^\perp) \cong \mathrm{End}(\dot\gamma(0)^\perp)$ is the symmetric endomorphism obtain from the Riemann tensor $R_{\gamma(t)}$ at $\gamma(t)$ as follows

$$R_\nu(t)(X) = R_{\gamma(t)}(X,\nu)\nu\,.$$

Remarks 3.5.

1.) The Riccati equation (2.6) resp. (3.6) along a geodesic as above may still make sense away from conjugate points, although dist_p is no longer smooth, and thus the interpretation of S_t as a Weingarten map fails. For example consider

$$M = \left\{(x,y,z)^\mathsf{T} \in \mathbb{R}^3 \mid x^2 + y^2 = 1\right\}$$

with its metric induced from $\mathbb{R}^3$, which is flat. We consider the geodesic $\gamma\colon[0,\infty) \to M$, $\gamma(t) := (\cos t, \sin t, 0)^\mathsf{T}$. Then $\mathrm{nc}_{\gamma,0} = (0,\infty)$, $p = (1,0,0)^\mathsf{T}$. The universal covering of M is isometric to $\mathbb{R}^2$, so it follows from Example 2.15 Subcase 1a that

$$S_t = -\frac{1}{t}\,\mathrm{id}_{\dot\gamma(t)^\perp}$$

is a solution of the Riccati equation, that arises from dist_p for $t \in (0,\pi)$. However, the function dist_p is not smooth in $\gamma(\pi) = (-1,0,0)^\mathsf{T}$, for example one easily checks that $\mathrm{dist}_p\big(\gamma(\pi)\big) = \pi - |t-\pi|$ for $t \in (0,2\pi)$. Furthermore, $\mathrm{dist}_p^{-1}(\pi)$ is not a smooth hypersurface. Thus S_π is not the Weingarten map of a hypersurface.

2.) The endomorphisms S_t defined above are all self-adjoint. This is true for $t \in (0, \mathrm{injrad}(p))$ from the discussion above, and it is easy to check for a solution

S_t, $t \in (a,b)$ of (3.6): if S_t is self-adjoint for some value of $t = t_0$, then it is self-adjoint for all $t \in (a,b)$.

3.) Any solution S_t, $t \in (a,b)$ of (3.6) arises from a suitable generalized distance function, locally around any $t_0 \in (a,b)$. More precisely for t_0 we construct a (sufficiently small) hypersurface N_{t_0} that intersects γ orthogonally in $p := \gamma(t_0)$, such that its Weingarten map at p is S_{t_0}. We extend $\dot\gamma(t_0)$ to a unit normal field ν along N_{t_0}. For $x \in N_{t_0}$ and $|s|$ sufficiently small, let $\gamma_x(s)$ be the geodesic with intial condition $\gamma_x(0) = x$, $\dot\gamma_x(0) = \nu|_x$. Such geodesics cover a neighborhood U of p in M, such that $(x,s) \mapsto \gamma_x(s)$ defines a diffeomorphism $V \times (-\varepsilon,\varepsilon) \to U$ for some open subset $V \Subset N_{t_0}$, $p \in V$. We then define the generalized distance function $f\colon U \to (t_0-\varepsilon, t_0+\varepsilon) \subset \mathbb{R}$ by requiring that $f\big(\gamma_x(s)\big) = t_0 + s$. This satisfies all the required properties, which is easy to check.

Fr 21.6.

Proposition 3.6. *We assume Setting 3.2 with notations as above, and let S_t be the solution of (3.6) defined in Definition 3.3 for $t \in \mathrm{nc}_{\gamma,0}$. We get*

(1) *S_t diverges for $t \searrow 0$, more precisely $\lim_{t\searrow 0}\big(tS_t\big) = -\mathrm{id}_{\dot\gamma(0)^\perp}$.*

(2) *If $t_0 \in [0,b)$ is conjugate to 0 along γ, then S_t also diverges for $t \nearrow t_0$. In addition, the largest eigenvalue of S_t diverges to ∞.*

Proof:
"(1)": Let $\gamma_s(t)$ be defined as in (3.4), $J_\xi(t) := \frac{\partial}{\partial s}\big|_{s=0}\gamma_s(t)$. Then we have, using (3.5) and $\frac{\nabla}{dt}\big|_{t=0}J_\xi = \xi$:

$$tS_t\big(\underbrace{\mathrm{d}_{tX}\exp_p}_{\to \mathrm{id}}(\xi)\big) \overset{(3.5)}{=} S_t\big(J_\xi(t)\big) \overset{\text{Def. 3.3}}{=} -\frac{\nabla}{dt}J_\xi(t) \to -\xi$$

for $t \to 0$. For all $\xi \in \dot\gamma(0)^\perp$ we thus obtain $\lim_{t\searrow 0}\big(tS_t(\xi)\big) = -\xi$.
"(2)": Let J_ξ be a Jacobi field with $J_\xi(0) = 0$, $J_\xi(t_0) = 0$, $\frac{\nabla}{dt}J_\xi(0) =: \xi \neq 0$. Identifying $T_{\gamma(t)}M$ with $T_{\gamma(t_0)}M$ via parallel transport along γ we obtain via Taylor expansion

$$\mathrm{d}_{tX}\exp_p(t\xi) = J_\xi(t) = \Big(\frac{\nabla}{dt}J_\xi(t_0)\Big)(t-t_0) + O(|t-t_0|^2)$$

and thus

$$-\frac{\nabla}{dt}J_\xi(t_0) = -\lim_{t\nearrow t_0}\frac{\nabla}{dt}J_\xi(t) = \lim_{t\nearrow t_0}S_t(J_\xi(t)) = \lim_{t\nearrow t_0}(t-t_0)S_t\Big(\frac{\nabla}{dt}J_\xi(t_0)\Big).$$

It follows that

$$\lim_{t\nearrow t_0}\Big(\underbrace{(t_0-t)}_{>0}\Big\langle S_t\Big(\frac{\nabla}{dt}J_\xi(t)\Big),\frac{\nabla}{dt}J_\xi(t)\Big\rangle\Big)=\lim_{t\nearrow t_0}\Big\langle\frac{\nabla}{dt}J_\xi(t),\frac{\nabla}{dt}J_\xi(t)\Big\rangle$$

$$=\Big\langle\frac{\nabla}{dt}\Big|_{t=t_0}J_\xi,\frac{\nabla}{dt}\Big|_{t=t_0}J_\xi(t)\Big\rangle>0$$

and thus

$$\lim_{t\nearrow t_0}\Big\langle S_t\Big(\frac{\nabla}{dt}J_\xi(t)\Big),\frac{\nabla}{dt}J_\xi(t)\Big\rangle=\lim_{t\nearrow t_0}\Big(\frac{1}{t_0-t}\Big\langle\frac{\nabla}{dt}J_\xi(t),\frac{\nabla}{dt}J_\xi(t)\Big\rangle\Big)=+\infty$$

and thus the largest eigenvalue of S_t conerges to ∞ for $t\nearrow t_0$. ∎

4 The Rauch comparison theorems

Let $\kappa\in\mathbb{R}$. We defined the **generalized sine function** as $\mathfrak{s}_\kappa:\mathbb{R}\to\mathbb{R}$ durch

$$\mathfrak{s}_\kappa(t):=\begin{cases}\frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t), & \kappa>0\\ t, & \kappa=0\\ \frac{1}{\sqrt{|\kappa|}}\sinh(\sqrt{|\kappa|}t), & \kappa<0\end{cases}$$

and the **generalized cosine function** $\mathfrak{c}_\kappa:\mathbb{R}\to\mathbb{R}$ by $\mathfrak{c}_\kappa:=\mathfrak{s}_\kappa'$. Furthermore we define the **generalized tangent function** $\tan_\kappa:=\frac{\mathfrak{s}_\kappa}{\mathfrak{c}_\kappa}$ and **generalized cotangent function** $\mathfrak{ct}_\kappa:=\frac{\mathfrak{c}_\kappa}{\mathfrak{s}_\kappa}$. The function $f:=-\mathfrak{ct}_\kappa$ solves $f'=\kappa+f^2$, which is the Riccati equation in spaces of constant sectional curvature κ and $S_t=f(t)\,\mathrm{id}_{\dot\gamma(t)^\perp}$.

Figure in the lecture, not yet drawn electronically

Drawing of the function $\mathfrak{ct}_\kappa$ for $\kappa = -1, 0, 1$

Proposition 4.1. *Let $\gamma\colon [0,\ell) \to M$ be a geodesic parametrized by arclength in a Riemannian manifold (M, g). We assume that on $(0,\ell)$ there are no points conjugate to 0 along γ, i. e., $\mathrm{nc}_{\gamma,0} = (0,\ell)$. Further, we assume that S_t is defined by Definition 3.3 for all $t \in (0,\ell)$.*

(1) *Assume that for all $t \in [0,\ell)$ and all planes $E \subset T_{\gamma(t)}M$ we assume $K(E) \geq \kappa_1$. Then $\kappa_1 \ell^2 \leq \pi^2$ and for the eigenvalues $\lambda_1(t) \leq \ldots \leq \lambda_{m-1}(t)$ of S_t, counted with multiplicity, we have:*

$$-\mathfrak{ct}_{\kappa_1}(t) \leq \lambda_i(t) \quad \forall i = 1, \ldots, m-1 \quad \forall t \in (0,\ell)\,.$$

(2) *Assume that for all $t \in [0,\ell)$ and all planes $E \subset T_{\gamma(t)}M$ we assume $K(E) \leq \kappa_2$. Then we have*

$$\lambda_i(t) \leq -\mathfrak{ct}_{\kappa_2}(t) \quad \forall i = 1, \ldots, m-1 \quad \forall t \in (0,\ell^*)\,,$$

where

$$\ell^* := \begin{cases} \ell, & \text{if } \kappa_2 \leq 0 \\ \min\{\ell, \frac{\pi}{\sqrt{\kappa_2}}\}, & \text{if } \kappa_2 > 0. \end{cases}$$

Proof:

(a) For any $\xi \in T_{\gamma(t)}M$, $\xi \perp \dot\gamma(t)$, $\|\xi\| = 1$, we have

$$\langle R_{\dot\gamma(t)}(\xi), \xi\rangle = \big\langle R\big(\xi, \dot\gamma(t)\big)\dot\gamma(t), \xi\big\rangle = K\big(\mathrm{span}\{\xi, \dot\gamma(t)\}\big)\,.$$

For two symmetric tensors A and B we write $A \leq B$ if $B - A$ is positive semi-definite. Thus

$$\begin{aligned} K(E) \leq \kappa\, \forall E \quad &\Rightarrow \quad R_{\dot\gamma(t)} + S_t^2 \leq \kappa + \max_i\{\lambda_i(t)^2\} \\ K(E) \geq \kappa\, \forall E \quad &\Rightarrow \quad R_{\dot\gamma(t)} + S_t^2 \geq \kappa + \min_i\{\lambda_i(t)^2\}\,. \end{aligned}$$

"Proof of (1)":

(b) Let $\xi \in \Gamma\big(\gamma^* TM\big)$ with $\frac{\nabla}{dt}\xi = 0$, $\|\xi\| \equiv 1$, $\xi(t) \perp \dot\gamma(t)$ for all t. We put $F(t) := \langle S_t(\xi(t)), \xi(t)\rangle$. With the Riccati equation (2.6) we get

$$\dot F(t) = \langle (\nabla_t S_t)(\xi), \xi\rangle = \langle S_t^2(\xi), \xi\rangle + \langle R_{\dot\gamma}(\xi), \xi\rangle.$$

From Step (a) we get $\langle R_{\dot\gamma}(\xi),\xi\rangle \geq \kappa_1$, and thus $\dot F(t) \geq \|S_t(\xi)\|^2 + \kappa_1$. From the Cauchy–Schwarz inequality we obtain

$$F(t)^2 = \langle S_t(\xi),\xi\rangle^2 \leq \|S_t(\xi)\|^2\|\xi\|^2 = \|S_t(\xi)\|^2$$

and thus $\dot F(t) \geq F(t)^2 + \kappa_1$.

(c) We define

$$\ell_1 := \begin{cases} \ell, & \text{if } \kappa_1 \leq 0 \\ \min\{\ell, \frac{\pi}{\sqrt{\kappa_1}}\}, & \text{if } \kappa_1 > 0. \end{cases}$$

Thus $-\mathfrak{ct}_{\kappa_1}\colon (0,\ell_1) \to \mathbb{R}$ is a continuous function with $\lim_{t\searrow 0}\big(-\mathfrak{ct}_{\kappa_1}(t)\big) = -\infty$. On the other hand, for any $\varepsilon > 0$, the function $F(\bullet + \varepsilon)\colon [-\varepsilon, \ell-\varepsilon) \to \mathbb{R}$ is continuous. Thus for a (sufficiently small) $t_0(\epsilon) > 0$ we have

$$F(t_0(\epsilon)+\epsilon) \geq -\mathfrak{ct}_{\kappa_1}(t_0(\epsilon)),$$

and obviously we may achieve $\lim_{\epsilon\to 0} t_0(\epsilon) = 0$. Applying Exercise Sheet 10, Exercise 1 for $F_\geq(t) := F(t+\epsilon)$, $F_\leq(t) := \mathfrak{ct}_{\kappa_1}(t)$ and $t_0 := t_0(\epsilon)$ we get $F(t+\epsilon) \geq -\mathfrak{ct}_{\kappa_1}(t)$ for all $t \in [t_0(\epsilon), \ell_1 - \epsilon]$. By taking the limit $\epsilon \to 0$ we obtain $F(t) \geq -\mathfrak{ct}_{\kappa_1}(t)$ for $t \in (0,\ell_1)$.

(d) In the case $\kappa_1 > 0$, $\ell > \frac{\pi}{\sqrt{\kappa_1}}$ we get

$$F(t) \geq -\mathfrak{ct}_{\kappa_1}(t) \nearrow \infty, \quad \text{für } t \to \frac{\pi}{\sqrt{\kappa_1}}$$

which contradicts the continuity of F in $\frac{\pi}{\sqrt{\kappa_1}} < \ell$. As a consequence we get $\kappa_1\ell^2 \leq \pi^2$. This inequality holds trivially for $\kappa_1 \leq 0$.

(e) Let $t_0 \in (0,\ell)$. For any eigenvector v_i of S_{t_0} to the eigenvalue $\lambda_i(t_0)$ with $\|v_i\| = 1$ we may choose a vector field $\xi \in \Gamma\big(\gamma^*TM\big)$ as above with $\xi(t_0) = v_i$. Thus

$$\lambda_i(t_0) = F(t_0) \geq -\mathfrak{ct}_{\kappa_1}(t)\,.$$

Tu 25.6. We have proved (1).
"Proof of (2)": [4]

(f) We define $h(y) := (\mathfrak{ct}_{\kappa_2})^{-1}(-y)$,

$$h\colon \mathbb{R} \to (0, \frac{\pi}{\sqrt{\kappa_2}}), \qquad \text{for } \kappa_2 > 0\,,$$

[4] We omitted this proof in the lecture for time reasons

$$h\colon (-\infty, -\sqrt{|\kappa_2|}) \to (0,\infty) \qquad\qquad \text{for } \kappa_2 \le 0\,.$$

In other words $h(y)$ is well-defined iff $\kappa_2 + y^2 > 0$. One calculates $h'(y) = (\kappa + y^2)^{-1}$. The endomorphismus $A_t := h(S_t) \in \mathrm{End}\big(\dot\gamma(t)^\perp\big)$ is well-defined if $\kappa_2 + \lambda_i(t)^2 > 0$ for all $i \in \{1,\ldots,m-1\}$.

Whenever A_t is well-defined we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t = h'(S_t)\frac{\mathrm{d}}{\mathrm{d}t}S_t = \left(\kappa_2 + S_t^2\right)^{-1}\left(S_t^2 + R_{\dot\gamma(t)}\right) \le \left(\kappa_2 + S_t^2\right)^{-1}\left(S_t^2 + \kappa_2\right) = \mathrm{id}\,. \tag{4.1}$$

(g) We have seen $\lim_{t\searrow 0}(tS_t) = -\,\mathrm{id}_{\dot\gamma(0)^\perp}$, and this implies $\lambda_{m-1}(t) \to -\infty$ for $t \searrow 0$. We also have $-\,\mathfrak{ct}_{\kappa_2}(t) \to -\infty$ for $t \searrow 0$. Similar to above in Step (c), for $\epsilon > 0$ we choose a small $t_0(\epsilon) > 0$ with $\lambda_{m-1}(t_0(\epsilon)) < -\,\mathfrak{ct}_{\kappa_2}(t_0(\epsilon)+\epsilon)$ and $\lim_{\epsilon\to 0} t_0(\epsilon) = 0$.

For a fixed $\varepsilon > 0$ we want to show

$$\lambda_{m-1}(t) < -\,\mathfrak{ct}_{\kappa_2}(t+\epsilon) \text{ for all } t \in (t_0(\epsilon), \ell^* - \epsilon). \tag{4.2}$$

Suppose that the statement is not true. Thus we can define t_1 as the smallest number in $(t_0(\epsilon), \ell^*-\epsilon)$ with $\lambda_{m-1}(t_1) = -\,\mathfrak{ct}_{\kappa_2}(t_1+\epsilon)$. Then we have $\lambda_i(t) < -\,\mathfrak{ct}_{\kappa_2}(t+\epsilon)$ for all $i \in \{1,\ldots,m-1\}$ and all $t \in (t_0(\epsilon), t_1)$. In the case $\kappa_2 \le 0$ we have $-\,\mathfrak{ct}_{\kappa_2}(t) < -\sqrt{|\kappa_2|}$ for all $t \in \mathbb{R}$. Thus for all $i \in \{1,\ldots,m-1\}$ and all $t \in (t_0(\epsilon), t_1)$ we obtain $\kappa_2 + \lambda_i(t)^2 > 0$. As a consequence the endomorphismus $A_t \in \mathrm{End}(\dot\gamma(t)^\perp)$ is well-defined for $t \in (t_0(\epsilon), t_1)$, and A_t depends smoothly on t.

From the choice of $t_o(\varepsilon)$ we get $A_{t_0(\epsilon)} < (t_0(\epsilon)+\epsilon)\,\mathrm{id}_{\dot\gamma(t)^\perp}$. Together with (4.1) we obtain

$$A_{t_1} = A_{t_0} + \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}A_t\,\mathrm{d}t \le A_{t_0} + (t_1 - t_0)\,\mathrm{id} < (t_1+\varepsilon)\,\mathrm{id}\,.$$

However the choice of t_1 implies that $(t_1+\varepsilon)$ is an eigenvalue of A_{t_1}, thus we have obtained a contradiction and (4.2) follows.

(h) Now, taking $\varepsilon \to 0$ in (4.2) yields Statement (2). ∎

Remark 4.2. In the case $\kappa_2 \le 0$ all eigenvalues of S_t remain negative.

Corollary 4.3 (Theorem of Cartan–Hadamard)**.** *If (M,g) is a connected Riemannian manifold with $K \le 0$, then there are no conjugate points. If addtionally (M,g) is complete and $p \in M$, then $\exp_p\colon T_pM \to M$ is a local diffeomorphism and a cover-*

ing.

Proof: The preceding proposition immediately yields that there are no conjugate points, thus $d_X \exp_p$ is invertible for any X in the domain of $\exp_p$. Thus a local diffeomorphism. It follows that $\tilde{g} := \exp_p^* g$ is a Riemannian metric on T_pM, which is in general different from g_p. However one easily checks for a smooth curve γ in T_pM with $\gamma(0) = 0$: γ is a geodesic with respect to g_p iff it is a straight line parametrized proportional to arclength, iff it is a geodesic with respect to $\tilde{g}$. Thus, the Hopf-Rinow theorem implies that $(T_pM, \tilde{g})$ is complete. An isometric local diffeomorphism from a complete connected Riemannian manifold to another Riemannian manifold is a covering due to Diff. geom. I, Exercise Sheet 14, Exercise 1b. ∎

The proof is also an exercise, see Exercise Sheet 11, Exercise 1.

Theorem 4.4 (Rauch comparison theorems)**.** *Let (M, g) be a Riemannian manifold, $\gamma: [0, \ell) \to M$ a geodesic parametrized by arclength, let J be a Jacobi field along γ with $J(0) = 0$, $\frac{\nabla}{dt}\big|_{t=0} J \perp \dot\gamma(0)$ (and thus $J(t) \perp \dot\gamma(t)$ for all $t \in [0, \ell)$).*

(1) *If we have $K \geq \kappa_1$ and if 0 is not conjugate to any point on $(0, \ell)$, then*

$$t \mapsto \frac{\|J(t)\|}{\mathfrak{s}_{\kappa_1}(t)}$$

is monotonically decreasing on $(0, \ell)$, and for all $t \in (0, \ell)$ we have

$$\|J(t)\| \leq \left\| \frac{\nabla}{dt}\Big|_{t=0} J \right\| \mathfrak{s}_{\kappa_1}(t) .$$

(2) *We we have $K \leq \kappa_2$ and if we define*

$$\ell^* := \begin{cases} \ell, & \text{if } \kappa_2 \leq 0 \\ \min\{\ell, \frac{\pi}{\sqrt{\kappa_2}}\}, & \text{if } \kappa_2 > 0, \end{cases}$$

then

$$t \mapsto \frac{\|J(t)\|}{\mathfrak{s}_{\kappa_2}(t)}$$

is monotonically increasing on $(0, \ell^*)$, and for $t \in (0, \ell^*)$ we have

$$\|J(t)\| \geq \left\| \frac{\nabla}{dt}\Big|_{t=0} J \right\| \mathfrak{s}_{\kappa_2}(t).$$

Note that there are other versions of this theorem, e. g., versions comparing a given

Riemannian manifold with another Riemannian manifold that is not necessarily of constant sectional curvature, see [12, Chap. 10, Theorem 2.3].

Remark 4.5. It follows that in the case $K \equiv \kappa$ constant we have $\|J(t)\| = \|\frac{\nabla}{dt}J(0)\|\,\mathfrak{s}_\kappa(t)$ for all $t \geq 0$ with $t^2\kappa < \pi^2$.

Proof of the theorem: We define S_t as in Definition 3.3, and thus we have $S_t(J(t)) := -\frac{\nabla}{dt}J(t)$. The function $t \mapsto \|J(t)\|$ is differentiable, whenever $J(t) \neq 0$. In this case we get

$$\|J(t)\|' = \sqrt{\langle J(t), J(t)\rangle}' = \frac{1}{\|J(t)\|}\langle J(t), \frac{\nabla}{dt}J(t)\rangle$$

and thus

$$\begin{aligned}\left(\frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}\right)' &= \frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}\left(\frac{1}{\|J(t)\|^2}\langle J(t), \frac{\nabla}{dt}J(t)\rangle - \frac{\mathfrak{s}_\kappa'(t)}{\mathfrak{s}_\kappa(t)}\right)\\ &= \frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}\left(-\frac{\langle S_t(J(t)), J(t)\rangle}{\|J(t)\|^2} - \mathfrak{ct}_\kappa(t)\right). \qquad (4.3)\end{aligned}$$

We use the rule of de l'Hôpital and $\mathfrak{s}_\kappa'(0) = \mathfrak{c}_\kappa(0) = 1$, and we calculate

$$\lim_{t\to 0}\frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)} = \lim_{t\to 0}\frac{\|J\|'(t)}{\mathfrak{s}_\kappa'(t)} = \lim_{t\to 0}\frac{1}{\|J(t)\|}\left\langle \frac{\nabla}{dt}J(t), J(t)\right\rangle.$$

In this formular we use the Taylor expansion $J(t) = t\cdot\left(\frac{\nabla}{dt}J\right)(0) + O(t^2)$ in order to substitute two of the terms $J(t)$:

$$\lim_{t\to 0}\frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)} = \lim_{t\to 0}\left(\frac{1}{t\|\frac{\nabla}{dt}J(0)\|}t\|\frac{\nabla}{dt}J(0)\|^2 + O(t^2)\right) = \left\|\left(\frac{\nabla}{dt}J\right)(0)\right\|. \qquad (4.4)$$

"(1)": Let $\kappa = \kappa_1$ and $K \geq \kappa_1$. From Proposition 4.1 (1) we get

$$\frac{\langle S_t(J(t)), J(t)\rangle}{\|J(t)\|^2} \geq -\mathfrak{ct}_\kappa$$

and thus we obtain with (4.3)

$$\left(\frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}\right)' \leq \frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}(\mathfrak{ct}_\kappa(t) - \mathfrak{ct}_\kappa(t)) = 0\,.$$

Thus the function $t \mapsto \frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}$ is monotonically decreasing. Then for any $t \in (0,\ell)$ we get using (4.4)

$$\frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)} \le \lim_{\tau\to 0} \frac{\|J(\tau)\|}{\mathfrak{s}_\kappa(\tau)} \le \left\| \left(\frac{\nabla}{dt} J\right)(0) \right\|.$$

"(2)": Let $\kappa = \kappa_2$ and $K \le \kappa_2$. From Proposition 4.1 ((2)) we get

$$\frac{\langle S_t(J(t)), J(t)\rangle}{\|J(t)\|^2} \le -\mathfrak{ct}_\kappa$$

and thus we obtain with (4.3)

$$\left(\frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}\right)' \ge \frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}(\mathfrak{ct}_\kappa(t) - \mathfrak{ct}_\kappa(t)) = 0$$

and thus the function $t \mapsto \frac{\|J(t)\|}{\mathfrak{s}_\kappa(t)}$ is monotonically increasing. The inequality then follows from (4.4), analogously to the proof of the inequality in (1). ∎

5 Growth of $\det(d\exp_p)$

We again assume Setting 3.2, i. e., let $\gamma\colon [0,\ell) \to M$ be a geodesic in M^m, $\|\dot\gamma(t)\| = 1$ for all t. Let $\mathrm{ric} \ge (m-1)\kappa g$ for some $\kappa \in \mathbb{R}$ and S_t a solution[5] the Riccati equation (3.6) for $t \in (0,\ell)$. We define $H_t := \frac{1}{m-1}\operatorname{tr}(S_t)$. The Riccati inequality 2.7 for H_t implies $\frac{\partial}{\partial t} H_t \ge \kappa + H_t^2$. In the following we use the convention

$$\frac{\pi}{\sqrt{\kappa}} := \begin{cases} \infty, & \text{if } \kappa \le 0 \\ \frac{\pi}{\sqrt{\kappa}}, & \text{if } \kappa > 0. \end{cases}$$

Then $\mathfrak{ct}_\kappa\colon (0, \frac{\pi}{\sqrt{\kappa}}) \to \mathbb{R}$ is well-defined Again we identify $T_{\gamma(t)}M$ with $T_{\gamma(0)}M$ via parallel transport along γ. With $\exp_p(t\dot\gamma(0)) = \gamma(t)$ we obtain

$$\mathrm{d}_{t\dot\gamma(0)} \exp_p\colon T_{\gamma(0)}M \to T_{\gamma(t)}M \cong T_{\gamma(0)}M$$

and thus $\mathrm{d}_{t\dot\gamma(0)} \exp_p \in \mathrm{End}(T_{\gamma(0)}M)$.

From now on we assume

- Setting 3.2, i. e., let $\gamma\colon [0,\ell) \to M$ be a geodesic in M^m, $\|\dot\gamma(t)\| = 1$ for all t,
- $\mathrm{nc}_{\gamma,0} = (0,\ell)$, i. e., on $[0,\ell)$ there are no points conjugate to 0 along γ,

[5] we do not assume so far that S_t is given by of Definition 3.3, but that it is defined on $(0,\ell)$.

- S_t is given by Definition 3.3.

From Proposition 3.6 (1) we conclude that

$$\lim_{t\searrow 0}\left(tH_t\right) = -1\,.$$

With similar arguments as in Proposition 4.1 (1) we get

$$H_t \geq -\mathfrak{ct}_\kappa(t) \tag{5.1}$$

for all $t \in (0,\ell)$. We have equality in(5.1), iff S_t is of the form $u(t)\,\mathrm{id}$ and if $\mathrm{ric}(\dot\gamma(t),\dot\gamma(t)) = (m1)\kappa$ and this in turn equivalent to $K(E) = \kappa$ for all planes $E \subset T_{\gamma(t)}$ with $\dot\gamma(t) \in E$ and $t \in (0,\ell)$.

In the following we now usually write r instead of t as the role of that variable will be typically the distance to a point p, thus the radial function in normal coordinates.

Proposition 5.1. *We assume the three assumptions above and additionally* $\mathrm{ric} \geq (m-1)\kappa g$, $p := \gamma(0)$. *Then the function* $h\colon (0,\ell) \to \mathbb{R}$

$$t \mapsto \frac{t^{m-1}\,\det(\mathrm{d}_{t\dot\gamma(0)}\exp_p)}{(\mathfrak{s}_\kappa(t))^{m-1}} =: h(t)$$

is monotonically decreasing and satisfies $\lim_{t\searrow 0} h(t) = 1$ *(and thus* $0 < h(t) \leq 1$ *for all* $t \in (0,\ell)$*).*

Corollary 5.2. *Again we assume the three assumptions above and additionally* $\mathrm{ric} \geq (m-1)\kappa g$, $p := \gamma(0)$. *Then* $\ell \leq \frac{\pi}{\sqrt{\kappa}}$.

Proof of the Corollary: As the statement is void for $\kappa \leq 0$, we assume $\kappa > 0$ and $r_0 := \frac{\pi}{\sqrt{\kappa}} < \ell$. For $t \to \frac{\pi}{\sqrt{\kappa}}$ we have $\mathfrak{s}_\kappa(t) \to 0$. As a consequence

$$\det(\mathrm{d}_{r_0\dot\gamma(0)}\exp_p) = \lim_{t\nearrow r_0}\det(\mathrm{d}_{t\dot\gamma(0)}\exp_p) \to 0\,.$$

Hence $\mathrm{d}_{r_0\dot\gamma(0)}\exp_p$ is not invertible, thus $r_0 = \frac{\pi}{\sqrt{\kappa}}$ is conjugate to 0. ■

Proof:

(a) Obviously $h\colon (0,\ell) \to \mathbb{R}$ is smooth, and with similar arguments as in the proof of the Corollary, we see that $h > 0$ as there are no conjugate points in $(0, \ell)$.

$$\lim_{t \searrow 0} \det(\mathrm{d}_{t\dot\gamma(0)} \exp_p) = \det(\mathrm{d}_0 \exp_p) = \det(\mathrm{id}_{T_pM}) = 1\,.$$

For $t \to 0$ we have $\frac{\mathfrak{s}_\kappa(t)}{t} \to 1$ and we obtain

$$\lim_{t\to 0} h(t) = \lim_{t\to 0} \det(\mathrm{d}_{t\dot\gamma(0)} \exp_p) \cdot \lim_{t\to 0} \frac{t}{\mathfrak{s}_\kappa(t)} = 1\,.$$

Fr 28.6.

(b) After identifying tangent vectors via parallel transport along γ we obtain

$$(\mathrm{d}_{t\dot\gamma(0)} \exp_p)(\dot\gamma(0)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \exp_p(s\dot\gamma(0)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \gamma(s) = \dot\gamma(t) \cong \dot\gamma(0)$$

and similarly

$$(\mathrm{d}_{t\dot\gamma(0)} \exp_p)(\dot\gamma(0)^\perp) = (\dot\gamma(t))^\perp \cong (\dot\gamma(0))^\perp.$$

If follows that

$$\mathrm{d}_{t\dot\gamma(0)} \exp_p \cong \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

where $B \in \mathbb{R}^{(m-1)\times(m-1)}$. We write $W := \dot\gamma(0)^\perp$. We obtain

$$\det(\mathrm{d}_{t\dot\gamma(0)} \exp_p) = \det(\mathrm{d}_{t\dot\gamma(0)} \exp_p|_W).$$

Let $\xi \in W$ and let J_ξ be the Jacobi field with $J_\xi(0) = 0$ and $\frac{\nabla}{dt} J_\xi(0) = \xi$. We have seen

$$(\mathrm{d}_{t\dot\gamma(0)} \exp_p)(t\xi) = J_\xi(t) =: A_t(\xi)$$

with

$$A_t = t\mathrm{d}_{t\dot\gamma(0)} \exp_p|_W\colon \quad W \to \dot\gamma(t)^\perp \cong W.$$

The endomorphism $A_t \in \mathrm{End}(W)$ is invertible as t is not conjugate to 0. It follows that

$$S_t\big(A_t(\xi)\big) = S_t\big(J_\xi(t)\big) = -\frac{\nabla}{dt} J_\xi(t) = -\Big(\frac{d}{dt} A_t\Big)(\xi)$$

and thus $S_t = -(\frac{\mathrm{d}}{\mathrm{d}t} A_t) \circ A_t^{-1}$. Because of the formula for the derivative of the determinant, see Lemma 5.3 below, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(A_t) = \mathrm{tr}(\dot A_t A_t^{-1}) \det(A_t) = -(m-1) H_t \det(A_t).$$

Because of $\det(A_t) > 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\big(\det(A_t)\big) = -(m-1)H_t \overset{(5.1)}{\leq} (m-1)\,\mathfrak{ct}_\kappa(t) = (m-1)\frac{\mathrm{d}}{\mathrm{d}t}\log(\mathfrak{s}_\kappa(t))$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\Big(\frac{\det(A_t)}{\mathfrak{s}_\kappa(t)^{m-1}}\Big) \leq 0.$$

It follows that $t \mapsto \frac{\det(A_t)}{\mathfrak{s}_\kappa(t)^{m-1}}$ is monotonically decreasing, and because of

$$\det(A_t) = \det(t d_{t\dot\gamma(0)}\exp_p|_W) = t^{m-1}\det(d_{t\dot\gamma(0)}\exp_p)$$

the statement of the Proposition follows. ■

Lemma 5.3. *Let W be a vector space over $\mathbb{R}$, $\dim W < \infty$ and let $A_t \in \mathrm{GL}(W)$ be smooth in t. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(A_t) = \mathrm{tr}(\dot A_t A_t^{-1})\det(A_t).$$

This lemma was already proven in Examples 1.2 4.). **Proof:** We identify W with $\mathbb{R}^n$, thus $A_t \in \mathrm{GL}(n,\mathbb{R})$. We fix t_0 and define $B_t := A_{t+t_0}A_{t_0}^{-1}$. Then $B_0 = \mathrm{id}_W$. We write $B_t = \big(b_{ij}(t)\big)$, $b_{ij}(0) = \delta_{ij}$. Let $\mathcal{S}_n$ be the symmetric group in n elements, i. e., the permutations of $\in \{1,\dots,n\}$, and let $\mathrm{sgn}: \mathcal{S}_n \to \{\pm 1\}$ be the parity function, also called signum.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det(B_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)b_{1\sigma(1)}(t)\cdots b_{n\sigma(n)}(t)$$

If $\sigma \neq \mathrm{id}$, then the product $b_{1\sigma(1)}(t)\cdots b_{n\sigma(n)}(t)$ has at least two factors that vanish at $t = 0$, thus by the product formular

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(b_{1\sigma(1)}(t)\cdots b_{n\sigma(n)}(t)\big) = 0\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det(B_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(b_{11}(t)\cdots b_{nn}(t)\big) = \dot b_{11}(0) + \ldots + \dot b_{nn}(0) = \mathrm{tr}\,\dot B_0\,.$$

Because of $\dot B_0 = \dot A_{t_0}A_{t_0}^{-1}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0}\det(A_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det\big(A_{t+t_0}\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\det\big(B_tA_{t_0}\big)$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(B_t) \right) \det\left(A_{t_0}\right) = \operatorname{tr}(\dot{A}_{t_0} A_{t_0}^{-1}) \det\left(A_{t_0}\right)$$

and the statement follows. ∎

Remark 5.4. In the case $K \equiv \kappa$, we have equality in the Riccati inequality and we obtain that $t \mapsto h(t)$ is constant 1.

Examples 5.5. We follow the numbering of Example 2.15.

Case 1: $\kappa = 0$**, Subcase 1a:** $M = \mathbb{R}^m$**,** $f(x) = \operatorname{dist}(x, 0) = \operatorname{dist}_0(x) = \|x\|$**.**

Let $\gamma(t) = tX$ with $\|X\| = 1$ and let $S_t = -t^{-1}\,\mathrm{id}$. We have $\mathrm{d}_{\gamma(t)} \exp_p = \mathrm{d}_{\gamma(t)}\,\mathrm{id} = \mathrm{id}$, thus $\det(\mathrm{d}_{\gamma(t)} \exp_p) = 1$. Furthermore $\mathfrak{s}_\kappa(t) = t$. Thus

$$\frac{t^{m-1} \det(\mathrm{d}_{\gamma(t)} \exp_p)}{\mathfrak{s}_\kappa(t)^{m-1}} = 1$$

for all t, as predicted in the remark above.

$$\operatorname{vol}_{m-1}(f^{-1}(t)) = \operatorname{vol}_{m-1}(\mathbb{S}^{m-1})\, t^{m-1}.$$

Case 2: $K \equiv \kappa = 1$**,** $M = \mathbb{S}^m$**.**

We have the generalized distance function $f = \operatorname{dist}_{e_0}$ with $p = e_0 = (1, 0, \ldots, 0)^\top \in \mathbb{S}^m$. Let $X \in T_{e_0} S^m$, $\|X\| = 1$, and let $\gamma(t) = \cos(t) e_0 + \sin(t) X$. We have $S_t = -\cot(t)\,\mathrm{id}$. Again the prediction by the remark above is $h \equiv 1$.

$$\det(\mathrm{d}_{t\dot{\gamma}(0)} \exp_p) = \left(\frac{\sin(t)}{t} \right)^{m-1}$$

it follows

$$\operatorname{vol}_{m-1}(f^{-1}(t)) = \operatorname{vol}_{m-1}(\mathbb{S}^{m-1}) \sin(t)^{m-1}.$$

which may also be obtained by elementary methods.

Case 3: $\kappa = -1$**.**

$f = \operatorname{dist}_{e_0}$, $e_0 = p$. Analogously $h \equiv 1$. It follows

$$\operatorname{vol}_{m-1}(f^{-1}(t)) = \operatorname{vol}_{m-1}(\mathbb{S}^{m-1}) \sinh(t)^{m-1}.$$

We now reprove the theorem by Bonnet and Myers from last semester.

Theorem 5.6 (Bonnet-Myers). *Let (M^m, g) be a complete connected Riemannian manifold with* $\mathrm{ric} \geq (m-1)\kappa g$ *for some* $\kappa > 0$*.*

(1) *Then* $\mathrm{diam}(M, g) \leq \frac{\pi}{\sqrt{\kappa}}$. *In particular M is compact.*

(2) $\pi_1(M)$ *is finite.*

Proof:
"(1)": If $\mathrm{diam}(M, g) > \frac{\pi}{\sqrt{\kappa}}$, then there are $p, q \in M$ with $\ell := \mathrm{dist}(p, q) > \frac{\pi}{\sqrt{\kappa}}$. By the Hopf–Rinow Theorem 1.5, there is a shortest geodesic $\gamma\colon [0, \ell] \to M$ from p to q, parametrized by arclength.

Similarly to Exercise Sheet 9, Exercise 3, we may prove: If $c\colon [0, \ell] \to M$ is a geodesic and if $t \in (0, \ell)$ is conjugate to 0, then for $s \in (t, \ell]$ the curve $c|_{[0,s]}$ is not a shortest curve from $c(0)$ to $c(s)$. Applying this to $c := \gamma$, we obtain, that the are no points $t \in (0, \ell)$ conjugate to 0 along γ, i. e., $\mathrm{nc}_{\gamma,0} = (0, \ell)$. We define S_t via Definition 3.3. Then the conditions of Proposition 5.1 and Corllary 5.2 are satisfied. This yields the contradiction $\ell \leq \frac{\pi}{\sqrt{\kappa}}$. Thus we obtain the claimed diameter bound. As the diameter is finite, it follows from Hopf–Rinow Theorem 1.5 that M is compact.
"(2)": We apply Part (1) to the universal covering $\widetilde{M}$ of M, and obtain the compactness of $\widetilde{M}$. For $\pi\colon \widetilde{M} \to M$, and $p \in M$ $\pi^{-1}(\{p\})$ is compact and discrete, thus finite. However, as $\widetilde{M}$ is connected, we get a bijection

$$\pi_1(M) \to \pi^{-1}(\{p\}), \quad [\alpha] \mapsto [\alpha \cdot \bullet].$$

■

6 The cut locus

Sorry, this part is not yet fully translated. This part was partially treated in the lecture earlier. Because of this and for time reasons, we will only summarize the results of this chapter in the lecture. If you do not understand the German part, please have a look at Sakai's book [26].

Let (M^m, g) be a complete connected Riemannian manifold, $p \in M$. We $S_pM :=$

$\{X \in T_pM | \|X\| = 1\}$. The set

$$SM := \bigcup_{p \in M} S_pM$$

is a submanifold of TM of dimension $2m-1$ (recall: $\dim TM = 2m$). SM is called the **unit tangent bundle** over M.

Definition 6.1. *Let $X \in S_pM$. We define*

$$\mathrm{c}(X) := \sup\{t \geq 0 | \operatorname{dist}(p, \exp_p(tX)) = t\} \in (0, \infty]$$

and we obtain a map $\mathrm{c}: SM \to (0, \infty]$.

Remarks 6.2.

1.) If we have $\operatorname{dist}(p, \exp_p(tX)) = t$ for all $t \in (0, r)$, then $\operatorname{dist}(p, \exp_p(rX)) = r$. This can be seen as follows:

   a) $\operatorname{dist}(p, \exp_p(rX)) \leq r$ holds as $\tau \mapsto \exp_p(\tau X)$ is a curve of length r from p to $\exp_p(rX)$.

   b) $\operatorname{dist}(p, \exp_p(rX)) \geq r$ holds, as for $t \in (0, r)$ we have

   $$t = \operatorname{dist}(p, \exp_p(tX)) \leq \operatorname{dist}(p, \exp_p(rX)) + \operatorname{dist}(\exp_p(rX), \exp_p(tX))$$

   and as we have $\operatorname{dist}(\exp_p(rX), \exp_p(tX)) \leq t - r$.

2.) For $t \in (0, \operatorname{injrad}(p))$ we have $\operatorname{dist}(p, \exp_p(tX)) = t$ according to the Gauß lemma. It follows $\mathrm{c}(X) \geq \operatorname{injrad}(p)$.

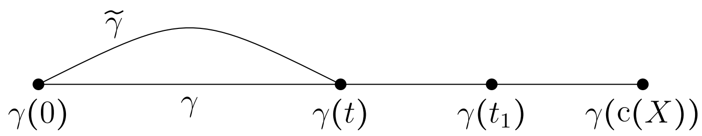
Lemma 6.3. *In the case $\mathrm{c}(X) < \infty$ the supremum in Definition 6.1 is attained.*

Proof: Suppose $\mathrm{c}(X) < \infty$. We write $t_\infty = \mathrm{c}(X)$. then there is a sequence $(t_i)_{i \in \mathbb{N}}$ in $\mathbb{R}$ with $t_i \nearrow t_\infty$. It follows $\operatorname{dist}(p, \exp_p(t_iX)) = t_i \to t_\infty$. On the other hand we have $\exp_p(t_iX) \to \exp_p(t_\infty X)$ und thus $\operatorname{dist}(p, \exp_p(t_iX)) \to \operatorname{dist}(p, \exp_p(t_\infty X))$. It follows $\operatorname{dist}(p, \exp_p(t_\infty X)) = t_\infty$. ■

Lemma 6.4. *Let $t \in (0, \mathrm{c}(X))$. Then – up to reparametrization – there is a unique shortest curve from p to $\exp_p(tX)$.*

Proof:

"Existence": $r \mapsto \exp_p(rX)$ is a shortest curve.
"Uniqueness": Let $\gamma(t) = \exp_p(tX)$.



Suppose that $\widetilde{\gamma}\colon [0,t] \to M$ is a shortest curve from $\gamma(0)$ to $\gamma(t)$. Choose $t_1 \in (t, \mathrm{c}(X))$. Then $\gamma|_{[0,t_1]}$ is a shortest curve from $\gamma(0)$ to $\gamma(t_1)$. Furthermore $\widetilde{\gamma} * \gamma|_{[t,t_1]}$, defined by

$$\widetilde{\gamma} * \gamma|_{[t,t_1]}(s) := \begin{cases} \widetilde{\gamma}(s) & \text{if } s \in [0,t] \\ \gamma(s) & \text{if } s \in [t, t-1] \end{cases}$$

is a shortest curve from $\gamma(0)$ to $\gamma(t_1)$. Thus this composed curve is also geodesic, and it coincides of $[t, t_1]$ with γ. Thus it is equal to γ on all of $[, t_1]$. We have proven $\widetilde{\gamma} = \gamma|_{[0,t]}$. ∎

Examples 6.5.

1.) $M = \mathbb{R}^m$ with Euclidean metric: $\mathrm{c}(X) = \infty$ for all $X \in S\mathbb{R}^m = S^{m-1} \times \mathbb{R}^m$.

2.) $M = \mathbb{S}^m$ with the standard round metric g^{sph}. Then $\mathrm{c}(X) = \pi$ for all $x \in S\mathbb{S}^m$.

3.) $M = \mathbb{R}P^m = \mathbb{S}^m/\{\pm 1\}$ with the metric induced from g^{sph}. Then $\mathrm{c}(X) = \frac{\pi}{2}$ for all $x \in S\mathbb{R}P^m$.

We recapitulate the following formula:

Proposition 6.6 (Second variation formula for the energy). *Let $\gamma_s(t)$ be a smooth variation of a geodesic $\gamma_0 = \gamma\colon [a,b] \to M$. Let*

$$\xi(t) := \frac{d}{ds}\Big|_{s=0} \gamma_s(t), \quad \zeta(t) := \frac{\nabla}{ds}\Big|_{s=0} \frac{d}{ds}\gamma_s(t)$$

and let

$$E(\gamma_s) := \frac{1}{2}\int_a^b \left\| \frac{d\gamma_s}{dt}(t) \right\|^2 dt.$$

Then

$$\frac{d^2}{ds^2}\Big|_{s=0} E(\gamma_s) \quad = \quad \int_a^b \left(\left\| \frac{\nabla}{dt}\xi(t) \right\|^2 + \langle R(\dot\gamma(t), \xi(t))\dot\gamma(t), \xi(t)\rangle \right) dt$$

$$+\langle\zeta(b),\dot\gamma(b)\rangle-\langle\zeta(a),\dot\gamma(a)\rangle. \tag{6.1}$$

Addendum: *Gibt es eine Unterteilung $a=t_0<t_1<\ldots<t_k=b$ und ist $\gamma_s(t)$ auf $[a,b]\times(-\epsilon,\epsilon)$ stetig und auf allen $[t_i,t_{i+1}]\times(-\epsilon,\epsilon)$ glatt, dann gilt die 2. Variationsformel (6.1) ebenso, da sich die Randterme im Inneren wegheben.*

Lemma 6.7. *Let (M,g) be a Riemannian manifold, $\gamma:[0,\ell)\to M$ a geodesic, $\|\dot\gamma(t)\|=1$ for all t. Let $\ell_0\in(0,\ell)$, such that 0 and ℓ_0 are conjugated. then for $t\in(\ell_0,\ell)$ the curve $\gamma|_{[0,t]}$ is not a shortest curve.*

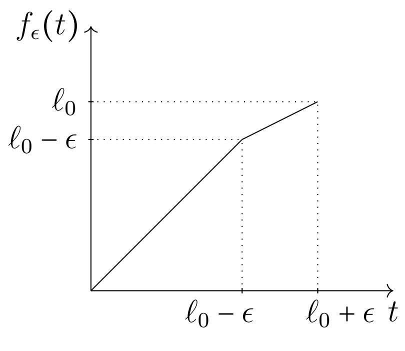
For a proof, see Exercise Sheet 9, Exercise 3 or the following.

Proof: Wähle ein Jacobi-Feld $J\not\equiv 0$ mit $J(0)=0$, $J(\ell_0)=0$. Dann ist $W:=\frac{\nabla}{dt}J(\ell_0)\neq 0$ und $J(t)\perp\dot\gamma(t)$ für alle t. Wähle einen parallelen Orthonormal-Rahmen $E_1,\ldots,E_m$ längs γ, so dass $E_1(t)=\dot\gamma(t)$ und schreibe

$$J(t)=\sum_{i=2}^m a_i(t)E_i(t)$$

mit glatten Funktionen a_i. Zu $\epsilon>0$, $\epsilon<\ell-\ell_0$ definiere

$$f_\epsilon(t):=\begin{cases} t, & \text{für } 0\le t\le\ell_0-\epsilon\\ t-\frac{t-\ell_0+\epsilon}{2}, & \text{für } |t-\ell_0|\le\epsilon.\end{cases}$$



Setze

$$\xi(t):=\sum_{i=2}^m a_i(f_\epsilon(t))E_i(t).$$

f_ϵ und ξ seien auf $[0,\ell_1]$ mit $\ell_1:=\ell_0+\epsilon$ definiert. Für $t\in[0,\ell_1]$ definiere weiter $\gamma_s(t):=\exp_{\gamma(t)}(s\xi(t))$. Dann gilt

$$\frac{d}{ds}\gamma_s(t)=d_{s\xi(t)}\exp_{\gamma(t)}(\xi(t))$$

und nach Definition von J gilt $\xi(0) = 0$, $\xi(\ell_1) = 0$. Für $\zeta(t) = \frac{\nabla}{ds}\big|_{s=0}\frac{d}{ds}\gamma_s(t)$ folgt $\zeta(0) = 0$ und $\zeta(\ell_1) = 0$. Mit der 2. Variationsformel (6.1) folgt

$$\frac{d^2}{ds^2}\Big|_{s=0} E[\gamma_s] = \int_0^{\ell_1} \underbrace{\left(\left\|\frac{\nabla}{dt}\xi(t)\right\|^2 + \langle R(\dot\gamma(t),\xi(t))\dot\gamma(t),\xi(t)\rangle\right)}_{=:I(t)} dt.$$

1. Teil:

$$\begin{aligned}
\int_0^{\ell_0-\epsilon} \left\|\frac{\nabla}{dt}\xi(t)\right\|^2 dt &= \int_0^{\ell_0-\epsilon} \left\langle \frac{\nabla}{dt}J(t), \frac{\nabla}{dt}J(t)\right\rangle dt \\
&= \left[\left\langle \frac{\nabla}{dt}J(t), J(t)\right\rangle\right]_0^{\ell_0-\epsilon} - \int_0^{\ell_0-\epsilon} \left\langle \frac{\nabla^2}{dt^2}J(t), J(t)\right\rangle dt \\
&= \left\langle \frac{\nabla}{dt}J(\ell_0-\epsilon), J(\ell_0-\epsilon)\right\rangle \\
&\quad - \int_0^{\ell_0-\epsilon} \langle R(\dot\gamma(t),J(t))\dot\gamma(t),J(t)\rangle\, dt.
\end{aligned}$$

Es folgt

$$\int_0^{\ell_0} I(t)\, dt = \left\langle \frac{\nabla}{dt}J(\ell_0-\epsilon), J(\ell_0-\epsilon)\right\rangle.$$

Es gilt

$$J(\ell_0-\epsilon) = J(\ell_0) - \epsilon W + O(\epsilon^2) = -\epsilon W + O(\epsilon^2)$$

und wegen $\frac{\nabla}{dt}J(\ell_0-\epsilon) = -W + O(\epsilon)$ folgt

$$\int_0^{\ell_0} I(t)\, dt = -\epsilon\|W\|^2 + O(\epsilon).$$

Weiter gilt

$$\begin{aligned}
\int_{\ell_0-\epsilon}^{\ell_0+\epsilon} \left\|\frac{\nabla}{dt}\xi(t)\right\|^2 dt &= \sum_{i=2}^m \int_{\ell_0-\epsilon}^{\ell_0+\epsilon} \left|\frac{\mathrm{d}}{\mathrm{d}t}(a_i\circ f_\epsilon)(t)\right|^2 dt \\
&= \frac{1}{4}\sum_{i=2}^m \int_{\ell_0-\epsilon}^{\ell_0+\epsilon} |a_i'(f_\epsilon(t))|^2\, dt \\
&= \frac{1}{2}\sum_{i=2}^m \int_{\ell_0-\epsilon}^{\ell_0} |a_i'(\tau)|^2\, d\tau \\
&= \frac{1}{2}\int_{\ell_0-\epsilon}^{\ell_0} \left\|\frac{\nabla}{dt}J(t)\right\|^2 dt \\
&= \frac{1}{2}\epsilon\|W\|^2 + O(\epsilon^2)
\end{aligned}$$

sowie

$$\left|\int_{\ell_0-\epsilon}^{\ell_0+\epsilon} \langle R(\dot\gamma(t),\xi(t))\dot\gamma(t),\xi(t)\rangle\, dt\right|$$

$$
\begin{aligned}
&\le 2\epsilon \sup_{t\in[\ell_0-\epsilon,\ell_0+\epsilon]} |\langle R(\dot\gamma(t),\xi(t))\dot\gamma(t),\xi(t)\rangle| \\
&\le 2\epsilon \sup_{\substack{E\subset T_{\gamma(t)}M \text{ Ebene},\\ t\in[\ell_0-\epsilon,\ell_0+\epsilon]}} K(E)\|\xi(t)\|^2 \\
&\le O(\epsilon^3).
\end{aligned}
$$

Insgesamt erhalten wir

$$\int_0^{\ell_0+\epsilon} I(t)\,dt = -\frac{1}{2}\epsilon\|W\|^2 + O(\epsilon^2) < 0$$

für kleine positive ϵ. Fixiere ein solches $\epsilon > 0$. Dann gilt

$$\frac{d}{ds}\Big|_{s=0} E[\gamma_s] = 0, \quad \frac{d^2}{ds^2}\Big|_{s=0} E[\gamma_s] < 0$$

und daher $E[\gamma_s] < E[\gamma]$ für kleine $s > 0$. Mit der Cauchy-Schwarz-Ungleichung folgt

$$\frac{1}{2\ell_1}L[\gamma_s]^2 \le E[\gamma_s] < E[\gamma] = \frac{1}{2}\ell_1 = \frac{1}{2\ell_1}L[\gamma]^2$$

und daher $L[\gamma_s] < L[\gamma]$ für kleine $s > 0$. Weiter ist $\gamma_s(0) = \gamma(0)$ und $\gamma_s(\ell_1) = \gamma(\ell_1)$ für alle s. Für kleine $s > 0$ ist daher γ_s ein kürzerer Weg von $\gamma(0)$ nach $\gamma(\ell_1)$ als γ. Da $\ell_1 - \ell_0 = \epsilon$ beliebig klein gewählt werden kann, ist für jedes $t \in (\ell_0, \ell)$ die Kurve $\gamma|_{[0,t]}$ keine Kürzeste. ■

Proposition 6.8. *Sei (M, g) eine vollständige zusammenhängende riemannsche Mannigfaltigkeit, $p \in M$, $X \in S_pM$ und $\mathrm{c}(X)$ wie oben. Dann gilt*

$$\mathrm{c}(X) = \min\left\{ \begin{array}{l} \inf\{t>0 \mid 0 \text{ und } t \text{ sind konjugiert längs } \tau\mapsto \exp_p(\tau X)\}, \\ \inf\left\{ t>0 \,\middle|\, \begin{array}{l} \text{Es gibt eine von } \tau\mapsto\exp_p(\tau X) \\ \text{verschiedene Kürzeste} \\ \text{von } p \text{ nach } \exp_p(tX) \end{array} \right\} \end{array} \right\}.$$

Proof: "≤" gilt nach Lemma 6.4 und Lemma 6.7.

zu "≥": Sei $\gamma(t) = \exp_p(tX)$. Zu zeigen ist

(∗) Angenommen für ein $\ell > 0$ sei $\gamma|_{[0,\ell]}$ die eindeutige Kürzeste von $\gamma(0)$ nach $\gamma(\ell)$ und ℓ sei nicht konjugiert zu 0. Dann gilt $\mathrm{c}(X) > \ell$.

Aus $(*)$ folgt "$\geq$", denn: Wähle ein $\ell > 0$, das kleiner als die beiden Infima in der Behauptung ist. Dann erfüllt $\gamma|_{[0,\ell]}$ die Voraussetzung von $(*)$. Es folgt $\mathrm{c}(X) > \ell$ und daher folgt "$\geq$".

Zeige nun $(*)$: Angenommen $\mathrm{c}(X) \leq \ell$. Dann ist $\mathrm{c}(X) = \ell$ und für alle $t > \ell$ gilt $\mathrm{dist}(p, \gamma(t)) < t$. Sei $i \in \mathbb{N}$. Wähle nach Hopf-Rinow eine nach Bogenlänge parametrisierte Kürzeste γ_i von p nach $\gamma(\ell + \frac{1}{i})$. Dann ist $L[\gamma_i] < L[\gamma|_{[0,\ell+\frac{1}{i}]}]$. Da ℓ nicht konjugiert ist zu 0, ist $d_{\ell X}\exp_p$ invertierbar. Daher existieren offene Umgebungen U von ℓX in T_pM und V von $\exp_p(\ell X)$ in M, so dass $\exp_p|_U : U \to V$ ein Diffeomorphismus ist. Für $i \in \mathbb{N}$ definiere

$$\ell_i := \mathcal{L}(\gamma_i) = \mathrm{dist}(p, \gamma(\ell + \frac{1}{i})).$$

Dann konvergiert $\gamma_i(\ell_i) = \gamma(\ell + \frac{1}{i})$ gegen $\gamma(\ell)$ für $i \to \infty$, und daher ist $\gamma_i(\ell_i) \in V$ für große i. Wir definieren $X_i := \dot\gamma_i(0)$. Dann ist $X_i \neq X$ für alle i. Nun gilt

$$\gamma_i(\ell_i) = \exp_p(\ell_i X_i) = \exp_p((\ell + \frac{1}{i})X)$$

aber $\ell_i X_i \neq (\ell + \frac{1}{i})X$. Da $\exp_p|_U$ ein Diffeomorphismus ist, folgt, dass $\ell_i X_i \notin U$ für große i. Weiter gilt

$$\ell_i = \mathrm{dist}(p, \gamma(\ell + \frac{1}{i})) \to \mathrm{dist}(p, \gamma(\ell)) = \ell$$

für $i \to \infty$. Nach Übergang zu einer Teilfolge gilt daher $X_i \to Y$ und $\ell_i X_i \to \ell Y$ für ein $Y \in S_pM$. Da $\ell Y \notin U$ ist, folgt $Y \neq X$. Es ist aber

$$\exp_p(\ell Y) = \exp_p(\ell X) = \gamma(\ell),$$

und daher ist die Kurve $\tau \mapsto \exp_p(\tau Y)$ eine weitere Kürzeste von $\gamma(0)$ nach $\gamma(\ell)$. Dies ist ein Widerspruch zur Voraussetzung in $(*)$. ∎

Corollary 6.9. *Let (M, g) be a complete connected Riemannian manifold. Let γ be a geodesic parametrized by arclength with $\dot\gamma(0) = X$, and let $s_0 := \mathrm{c}(X)$. Then we have $\mathrm{c}(-\dot\gamma(s_0)) = \mathrm{c}(X)$.*

Proof: Both infima in Proposition 6.8 are symmetric under permutation of the endpoints. ∎

Tu 2.7.

Definition 6.10. *Let (M,g) be a complete Riemannian manifold, $p \in M$. For $X \in S_pM$ we define $\gamma(t) := \exp_p(tX)$ and*

$$\mathrm{conj}(X) := \inf\{t > 0 | 0 \text{ and } t \text{ are conjugate along } \tau \mapsto \exp_p(\tau X)\}.$$

and we get a map $\mathrm{conj}\colon S_pM \to (0,\infty]$. The set

$$\mathrm{Conj}_p^{\tan} := \{\mathrm{conj}(X)X \mid X \in S_pM, \mathrm{conj}(X) < \infty\}$$

is called **tangential conjugation locus***. The set*

$$\mathcal{C}_p^{\tan} := \{\mathrm{c}(X)X \mod X \in S_pM, \mathrm{c}(X) < \infty\}$$

is called **tangential cut locus***. The set*

$$\mathrm{Conj}_p := \exp_p(\mathrm{Conj}_p^{\tan})$$

is called **conjugation locus***. The set*

$$\mathcal{C}_p := \exp_p(\mathcal{C}_p^{\tan})$$

is called **cut locus***. Furthermore we define*

$$\mathcal{C}^{\tan} := \bigcup_{p \in M} \mathcal{C}_p^{\tan} \subset TM.$$

Lemma 6.11. *The map $\mathrm{c}\colon SM \to (0,\infty]$ is continuous.*

Proof: Let $(p_i)_{i\in\mathbb{N}}$ be a sequence in M with $p_i \to p$ for $i \to \infty$ and let $(X_i)_{i\in\mathbb{N}}$ be a sequence in SM with $X_i \in T_{p_i}M$ for all i and $X_i \to X$ for $i \to \infty$.

Für alle i und für alle $t < \mathrm{c}(X_i)$ gilt $\mathrm{dist}(p_i, \exp_{p_i}(tX_i)) = t$. Sei nun

$$t < \limsup_{k\to\infty} \mathrm{c}(X_k)$$

fixiert. Dann gibt es unendlich viele i mit $t < \mathrm{c}(X_i)$, und für diese i gilt

$$\mathrm{dist}(p_i, \exp_{p_i}(tX_i)) = t.$$

Läßt man $i \to \infty$ laufen, so erhält man $\mathrm{dist}(p, \exp_p(tX)) = t$. Es folgt

$$\mathrm{c}(X) \geq \limsup_{k\to\infty} \mathrm{c}(X_k).$$

Wir definieren nun $\sigma := \liminf_{k\to\infty} \mathrm{c}(X_k)$. Zu zeigen bleibt $\mathrm{c}(X) \leq \sigma$.

O.B.d.A. sei $\sigma < \infty$. Nach Übergang zu einer Teilfolge gilt $\lim_{i\to\infty} \mathrm{c}(X_i) = \sigma$ und $\mathrm{c}(X_i) < \infty$ für alle i. Die Geodätische $[0, \mathrm{c}(X_i)] \to M$, $\tau \mapsto \exp_p(\tau X_i)$ hat

a) konjugiert Endpunkte

oder b) es existiert eine weitere Kürzeste zwischen den beiden Endpunkten $[0, \mathrm{c}(X_i)] \to M$, $\tau \mapsto \exp_p(\tau Y_i)$ mit $Y_i \in S_{p_i}M$ und $Y_i \neq X_i$. Nach Übergang zu einer Teilfolge gilt

[a] für alle i gilt Eigenschaft a)

oder [b] für alle i gilt Eigenschaft b).

Fall [a]: Für alle i ist $\mathrm{d}_{\mathrm{c}(X_i)X_i} \exp_{p_i}$ nicht invertierbar. Für $i \to \infty$ folgt, dass $d_{\sigma X} \exp_p$ nicht invertierbar ist. Es folgt $\mathrm{c}(X) \leq \sigma$.

Fall [b]: Angenommen $\sigma < s(X)$. Dann ist $\mathrm{d}_{\sigma X} \exp_p$ invertierbar. Es bezeichne $\pi\colon TM \to M$ die Projektion und

$$(\pi, \exp): \quad TM \to M \times M, \quad Y \mapsto (\pi(Y), \exp_{\pi(Y)}(Y)).$$

Es folgt, dass $d_{\sigma X}(\pi, \exp)$ invertierbar ist. Daher gibt es ein $\epsilon > 0$, so dass die Einschränkung $(\pi, \exp)|_{B_\epsilon(\sigma X)}$ ein Diffeomorphismus auf eine offene Umgebung von $(p, \exp_p(\sigma X))$ ist. Daher gibt es ein $\epsilon > 0$ und ein $\delta > 0$, so dass für alle $\widetilde{p} \in B_\delta(p)$ die Einschränkung $\exp_{\widetilde{p}}|_{B_\epsilon(\sigma X)\cap T_{\widetilde{p}}M}$ ein Diffeomorphismus auf eine offene Umgebung $U \subset M$ von $\exp_p(\sigma X)$ ist. Für große i gilt

$$\exp_{p_i}\big(\mathrm{c}(X_i)X_i\big) = \exp_{p_i}(\mathrm{c}(X_i)Y_i) \in U$$

und $\mathrm{c}(X_i)X_i \in B_\epsilon(\sigma X)$. Falls $\mathrm{c}(Y_i)X_i \in B_\epsilon(\sigma X)$ gilt, so folgt $\mathrm{c}(X_i)X_i = \mathrm{c}(X_i)Y_i$ und daher $X_i = Y_i$, ein Widerspruch. Nehme daher an, dass $\mathrm{c}(X_i)Y_i \notin B_\epsilon(\sigma X)$. Nach Übergang zu einer Teilfolge gilt für $i \to \infty$, dass $Y_i \to Y \in S_pM$. Wegen $\sigma Y \notin B_\epsilon(\sigma X)$ ist $Y \neq X$. Jedoch gilt $\exp_p(\sigma Y) = \exp_p(\sigma X)$. Daher ist $\tau \mapsto \exp_p(\tau Y)$ eine weitere Kürzeste von p nach $\exp_p(\sigma X)$. Nach Proposition 6.8 folgt $\mathrm{c}(X) \leq \sigma$. ∎

Folgerung 6.12. *Die Menge $\mathcal{C}^{\tan}$ ist abgeschlossen in TM.*

Proof: Sei tX ein Häufungspunkt von $\mathcal{C}^{\tan}$, wobei $X \in S_pM$, $t \geq 0$. Dann gibt es eine Folge $(p_i)_{i\in\mathbb{N}}$ in M und eine Folge $(X_i)_{i\in\mathbb{N}}$ in SM, so dass $X_i \in S_{p_i}M$ und $\mathrm{c}(X_i)X_i \to tX$ für $i \to \infty$. Nach Übergang zu einer Teilfolge gilt $X_i \to Y$ und $\mathrm{c}(X_i)X_i \to \mathrm{c}(Y)Y$ für $i \to \infty$. Wegen $\mathrm{c}(Y) > 0$ folgt $\mathrm{c}(Y) = t$ und $X = Y$. Daher gilt $\mathrm{c}(X) = t$ und $tX \in \mathcal{C}^{\tan}$. ∎

Folgerung 6.13. $\mathcal{C}_p^{\tan} \subset T_pM$ *is a zero set.*

Proof: Graphs of continuous functions are zero sets. ∎

Corollary 6.14. $\exp_p(\mathcal{C}_p^{\tan}) = \mathcal{C}_p \subset M$ *is a zero set as well.*

The set

$$\mathcal{R}_p^{\tan} := \{tX \mid X \in S_pM, 0 \leq t < \mathrm{c}(X)\}$$

called the **tangential interior set** at p is starshaped with respect to 0.

Lemma 6.15. $\exp_p|_{\mathcal{R}_p^{\tan}} : \mathcal{R}_p^{\tan} \to M \setminus \mathcal{C}_p =: \mathcal{R}_p$ *is a diffeomorphism.*

The set $\mathcal{R}_p$ is called the **interior set** at p.

Proof:

(a) Let $q \in M$. We choose a shortest curveγ from p to q, $\gamma: [0,\ell] \to M$, $\|\dot{\gamma}(0)\| = 1$. Dann ist $X := \dot{\gamma}(0) \in S_pM$, $\exp_p(\ell X) = q$ und $\ell \leq \mathrm{c}(X)$. Es folgt $\ell X \in \mathcal{C}_p^{\tan}$ (falls $\ell = \mathrm{c}(X)$) oder $\ell X \in \mathcal{R}_p^{\tan}$ (falls $\ell < \mathrm{c}(X)$). Daher ist $\exp_p|_{\mathcal{R}_p^{\tan}}$ surjektiv und

$$M = \exp_p(\mathcal{R}_p^{\tan}) \cup \exp_p(\mathcal{C}_p^{\tan}).$$

(b) Es gilt $\exp_p(\mathcal{R}_p^{\tan}) \cap \exp_p(\mathcal{C}_p^{\tan}) = \varnothing$. Es gilt nämlich $q \in \exp_p(\mathcal{R}_p^{\tan})$ genau dann, wenn es genau eine Kürzeste von p nach q gibt, so dass die Endpunkte nicht konjugiert sind. Andererseits gilt $q \in \exp_p(\mathcal{C}_p^{\tan})$ genau dann, wenn es mehrere Kürzeste von p nach q gibt oder mindestens eine Kürzeste mit konjugierten Endpunkten.

(c) $\exp_p|_{\mathcal{R}_p^{\tan}}$ ist injektiv: Angenommen $\exp_p(X) = \exp_p(Y)$ für $X, Y \in \mathcal{C}_p^{\tan}$, $X \neq Y$. Dann gäbe es zwei Kürzeste von p nach $\exp_p(X)$, ein Widerspruch.

(d) Es ist $d_X \exp_p$ invertierbar für alle $X \in \mathcal{R}_p^{\tan}$. Daher ist $\exp_p|_{\mathcal{R}_p^{\tan}}$ eine bijektive Immersion und wegen $\dim \mathcal{R}_p^{\tan} = \dim M$ ein Diffeomorphismus.

■

Corollary 6.16. *Sei (M, g) eine vollständige riemannsche Mannigfaltigkeit, $p \in M$. Dann ist $\mathrm{dist}_p := \mathrm{dist}(p, \bullet)$ glatt auf $M \smallsetminus (\mathcal{C}_p \cup \{p\})$. Insbesondere ist $\mathrm{dist}_p\big|_{M \smallsetminus (\mathcal{C}_p \cup \{p\})}$ eine verallgemeinerte Abstandsfunktion.*

Proof: Das Diagramm

$$\begin{array}{ccc} T_pM \smallsetminus \{0\} & \xrightarrow{\|.\|} & (0, \infty) \\ {\scriptstyle incl} \uparrow & & \uparrow {\scriptstyle \mathrm{dist}_p} \\ \mathcal{R}_p^{\tan} \smallsetminus \{0\} & \xrightarrow{\exp_p} & M \smallsetminus (\mathcal{C}_p \cup \{p\}) \end{array}$$

kommutiert, denn für alle X mit $\|X\| \le \mathrm{c}(\frac{X}{\|X\|})$ gilt $\mathrm{dist}_p(X) = \|X\|$. Insbesondere gilt dies für alle $X \in \mathcal{R}_p^{\tan} \smallsetminus \{0\}$. Da die Abbildung $\|.\|$ auf $T_pM \smallsetminus \{0\}$ glatt ist, folgt die Behauptung. ■

Examples 6.17.

1. $M = \mathbb{R}^m$ mit euklidischer Metrik, $p \in M$. Dann gilt $\mathrm{Conj}_p = \varnothing$, $\mathcal{C}_p = \varnothing$, $\mathrm{conj} \equiv \infty$, $\mathrm{c} \equiv \infty$.
2. $M = \mathbb{H}^m$ mit Standardmetrik, $p \in M$. Dann gilt $\mathrm{Conj}_p = \varnothing$, $\mathcal{C}_p = \varnothing$, $\mathrm{conj} \equiv \infty$, $\mathrm{c} \equiv \infty$.
3. Sei $\pi \colon \widetilde{M} \to M$ eine Überlagerung und g eine riemannsche Metrik auf M. Dann ist $\widetilde{g} := \pi^* g$ eine riemannsche Metrik auf $\widetilde{M}$. Sei $\pi(\widetilde{p}) = p$. Dann gilt $\mathrm{Conj}_{\widetilde{p}}^{\tan} \cong \mathrm{Conj}_p^{\tan}$, aber im allgemeinen nicht $\mathcal{C}_{\widetilde{p}}^{\tan} \cong \mathcal{C}_p^{\tan}$.

Sei M kompakt. Dann ist $\mathrm{c}(X) \le \mathrm{diam}\, M < \infty$ für alle $X \in SM$. Weiter ist M homöomorph zu $\mathcal{C}_p \cup \overline{B_1(0_p)}/\sim$ mit der Quotiententopologie, wobei $X \in S_pM = S^{m-1}(0_p)$ identifiziert wird mit $\exp_p(\mathrm{c}(X)X) \in \mathcal{C}_p$.

Example 6.18. $M = \mathbb{R}^2/\mathbb{Z}^2$, $p = [(0,0)] \in M$. ETC

Corollary 6.19. *Sei M^m kompakt. Die Abbildung $i \colon \mathcal{C}_p \to M^m$ induziert*

1. *Isomorphismen $i_* \colon \pi_j(\mathcal{C}_p) \to \pi_j(M)$, $1 \le j \le m-2$,*
2. *einen Epimorphismus $i_* \colon \pi_{m-1}(\mathcal{C}_p) \to \pi_{m-1}(M)$,*

3. *Isomorphismen* $i_*: H_j(\mathcal{C}_p, R) \to H_j(M, R)$ *für* $j \in \mathbb{N} \setminus \{m-1, m\}$ *sowie* i^**:* $H^j(\mathcal{C}_p, R) \to H^j(M, R)$ *für* $j \in \mathbb{N} \setminus \{m-1, m\}$.

Proof: [26], Kap. III, Abschnitt 4. ■

Lemma 6.20. *Let* (M, g) *be a complete Riemannian manifold,* $p \in M$*. Then* $\mathrm{injrad}(p) = \min_{X \in S_pM} \mathrm{c}(X)$.

Proof:

(a) Das Minimum wird angenommen: Die Funktion

$$S_pM \to [0, \infty), \quad X \mapsto \frac{1}{\mathrm{c}(X)},$$

wobei $\frac{1}{\infty} = 0$, ist stetig auf einem Kompaktum und nimmt daher ihr Maximum an.

(b) "$\leq$": this is easy to check

(c) "$\geq$": Sei $\ell \leq \mathrm{c}(X)$ für alle $X \in S_pM$. Dann ist $d_Y \exp_p$ invertierbar für alle Y mit $\|Y\| < \ell$, d.h. $\exp_p|_{B_\ell(0_p)}$ ist eine Immersion.

(d) Claim: $\exp_p|_{B_\ell(0_p)}$ is injektive.

(Daraus folgt dann $\ell \leq \mathrm{injrad}(p)$, und daraus die Behauptung des Lemmas, indem man $\ell = \min_{X \in S_pM} \mathrm{c}(X)$ setzt).

Proof of the claim: Angenommen $\exp_p(X) = \exp_p(Y)$ für $X, Y \in B_\ell(0_p)$, $X \neq Y$, o.B.d.A. sei $\|X\| \leq \|Y\|$. Es folgt $\mathrm{c}(\frac{Y}{\|Y\|}) \leq \|Y\|$.

Begründung: Case (1): Es gilt $\|X\| < \|Y\|$. Dann ist $\tau \mapsto \exp_p(\tau X)$ eine kürzere Verbindung von p nach $\exp_p(X)$ als $\tau \mapsto \exp_p(\tau Y)$.

Case (2): Es gilt $\|X\| = \|Y\|$. Dann gilt $\frac{X}{\|X\|} \neq \frac{Y}{\|Y\|}$, und daher gibt es mindestens zwei verschiedene Kürzeste von p nach $\exp_p(X)$.

It follows that

$$\mathrm{c}\Big(\frac{Y}{\|Y\|}\Big) \leq \|Y\| < \ell \leq \mathrm{c}\Big(\frac{Y}{\|Y\|}\Big),$$

which is a contradiction. The Claim follows. ■

Corollary 6.21. *The map* $p \mapsto \mathrm{injrad}(p)$ *is continuous.*

Proof: Let $(p_i)_{i\in\mathbb{N}}$ be a sequence in M with $p_i \to p$.

(a) We show $\limsup_{i\to\infty} \operatorname{injrad}(p_i) \le \operatorname{injrad}(p)$: We choose $X \in S_pM$ with $\mathrm{c}(X) = \operatorname{injrad}(p)$ and we choose $X_i \in S_{p_i}M$ with $X_i \to X$. It follows that

$$\limsup_{i\to\infty} \operatorname{injrad}(p_i) \le \lim_{i\to\infty} \mathrm{c}(X_i) = \mathrm{c}(X).$$

(b) We show $\lambda := \liminf_{i\to\infty} \operatorname{injrad}(p_i) \ge \operatorname{injrad}(p)$: We choose a subsequence[6] with $\operatorname{injrad}(p_i) \to \lambda$ for $i \to \infty$. For each i we choose $X_i \in S_{p_i}M$ with $\mathrm{c}(X_i) = \operatorname{injrad}(p_i)$. After passage to a subsequence we get $X_i \to Y$. It follows

$$\operatorname{injrad}(p) \le \mathrm{c}(Y) = \lim_{i\to\infty} \mathrm{c}(X_i) = \lambda.$$

From (a) and (b) we get the statement. ∎

7 Volume Growth of balls

Definition 7.1. *Let V and W be m-dimensional oriented real vector spaces and $F\colon V \to W$ be a linear map. We define $\det F := \pm\sqrt{\det(F^* \circ F)}$. This expression is 0 iff F is not an isomorphism. we choose the $+$-sign, if F preserves orientation, otherwise the $-$-sign.*

For positively oriented orthonormal bases $(e_1, \dots, e_m)$ and $(b_1, \dots, b_m)$ of V and W, and $A = (\langle b_i, F(e_j)\rangle)_{i,j=1}^m \in \mathbb{R}^{m\times m}$ we get

$$\det F = \pm\sqrt{\det A^\mathsf{T} A}.$$

We can integrate over Riemannian with the tools from the Analysis IV-lecture, e. g., [1]. In particular, if $F\colon N \to M$ is a diffeomorphism, h resp. g a Riemannian metric on N resp. M, and f is an integrable function on (M, g), i. e., if f is

[6] we suppress the passage to the subsequence in notation ot avoid interated indices

measurable and if $\int_M |f| \,\mathrm{dvol}^g < \infty$, then we have the **transformation formula**[7]

$$\int_N |\det F| \cdot (f \circ F) \,\mathrm{dvol}^h = \int_M f \,\mathrm{dvol}^g \,.$$

If $\Psi\colon \Omega \to M$ is a global parametrization of M, $\Omega \Subset \mathbb{R}^m$, and if we define $g_{ij} := (\Psi^* g)(E_i, E_j)$, then this yields for the Lebesgues measure μ

$$\int_\Omega (f \circ \Psi) \cdot \sqrt{\det(g_{ij})} \,\mathrm{d}\mu = \int_M f \,\mathrm{dvol}^g \,.$$

In particular, for the volume we get

$$\mathrm{vol}(M, g) := \int_M 1 \,\mathrm{dvol}^g = \int_\Omega |\det \mathrm{d}\Psi| \,\mathrm{d}\mu = \int_\Omega \sqrt{\det(g_{ij})} \,\mathrm{d}\mu \,.$$

Examples 7.2. In the following let M_κ^m be a simply-connected complete space of constant sectional curvature $\kappa \in \mathbb{R}$. In particular, up to resclaing M_κ^m is isometric $\mathbb{R}^m$, $\mathbb{S}^m$, or $\mathcal{H}^m$. As a parametrization Ψ (up to a zero set) we use a parametrization by normal coordinates

$$\begin{array}{ccccc} B_{\pi/\sqrt{\kappa}}(0) & \xrightarrow{\cong} & \mathcal{R}_p^{\tan} \cup \{0\} & \xrightarrow{\exp_p} & \mathcal{R}_p \cup \{p\} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{R}^m & \xrightarrow{\cong} & T_p M_\kappa^m & \xrightarrow{\exp_p} & M_\kappa^m \end{array}$$

where $\hookrightarrow$ means inclusion and where the first map in the second line is a linear isometry. Furthermore in $B_{\pi/\sqrt{\kappa}}(0)$ we used again the conventions that for $\kappa \leq 0$ we define $\pi/\sqrt{\kappa} = \infty$ and thus $B_{\pi/\sqrt{\kappa}}(0) = \mathbb{R}^m$. Furthermore the parametrization Ψ by normal coordinates is the diagonal map $\Psi\colon \Omega := B_{\pi/\sqrt{\kappa}}(0) \to M_\kappa^m$. Define $\omega_n := \mathrm{vol}(S^n) = \frac{\pi^{(n+1)/2}}{\Gamma\left((n+1)/2\right)}$ and $\mathcal{V}_\kappa(r) := \mathrm{vol}(B_r(p))$ for some (and thus any) ball $B_r(p) \subset M_\kappa^m$. From Proposition 5.1 and Remark **??** we know that for $X \in T_p M_\kappa^m$, $\|X\| = 1$, with $X \cong \hat{X} \in \Omega$ we have:

$$|\mathrm{d}_{r\hat{X}} \Psi| = |\det \mathrm{d}_{rX} \exp_p| = (\mathfrak{s}_\kappa(r)/r)^{m-1} \,.$$

[7] We assume that the reader is already familiar with integrations on Riemannian manifold. In case this is unclear, please consult [5]. We have a different approach here. We take the transformation formula as known and explain how to remember out if the formula for integration on a manifold.

We get

$$\mathcal{V}_\kappa(r) = \int_0^r \int_{\mathbb{S}^{m-1}\subset T_pM} |\det \mathrm{d}_{\rho X} \exp_p| \, \rho^{m-1} \, \mathrm{d}\rho$$
$$= \omega_{m-1} \int_0^r \mathfrak{s}_\kappa(\rho)^{m-1} \, \mathrm{d}\rho \tag{7.1}$$

for $0 \le r \le \pi/\sqrt{\kappa}$, which again has to be read as $0 \le r < \infty$ for $\kappa \le 0$.

1.) $\kappa = 0$, i. e., $M^m_\kappa = \mathbb{R}^m$. Then

$$\mathcal{V}_0(r) = \frac{\omega_{m-1}}{m} r^m.$$

2.) $\kappa > 0$, i. e., $M^m_\kappa = \mathbb{S}^m\left(\frac{1}{\sqrt{\kappa}}\right) = \frac{1}{\sqrt{\kappa}}\mathbb{S}^m$

$$\mathcal{V}_\kappa(r) = \omega_{m-1} \int_0^r \mathfrak{s}_\kappa(\rho)^{m-1} \, \mathrm{d}\rho = \omega_{m-1} \int_0^r \frac{\sin(\sqrt{\kappa}\,\rho)^{m-1}}{\sqrt{\kappa}^{m-1}} \, \mathrm{d}\rho$$

for $0 \le r \le \pi/\sqrt{\kappa}$ and by partial integration we obtain $\mathcal{V}_\kappa(\pi/\sqrt{\kappa}) = \omega_m \kappa^{-m/2}$.

3.) $\kappa < 0$, i. e., $M^m_\kappa = \frac{1}{\sqrt{|\kappa|}}\mathbb{H}^m$

$$\mathcal{V}_\kappa(r) = \omega_{m-1} \int_0^r \mathfrak{s}_\kappa(\rho)^{m-1} \, \mathrm{d}\rho \quad = \quad \omega_{m-1} \int_0^r \frac{\sinh(\sqrt{|\kappa|}\,\rho)^{m-1}}{\sqrt{|\kappa|}^{m-1}} \, \mathrm{d}\rho$$

Literature: [5, Satz 6.7.1 (Bishop–Gromov) and Satz 6.7.2]

Theorem 7.3 (Bishop–Gromov)**.** *Let (M, g) be a complete connected m-dimensional Riemannian manifold with* $\mathrm{ric} \ge \kappa(m-1)g$. *Then the function*

$$(0, \infty) \to \mathbb{R}, \quad r \mapsto \frac{\mathrm{vol}(B_r(p))}{\mathcal{V}_\kappa(r)}$$

is monotonically decreasing, and we have $\mathrm{vol}(B_r(p)) \le \mathcal{V}_\kappa(r)$.

Proof: We view $\mathbb{S}^{m-1}$ as the unit sphere in $\mathbb{R}^m$.

$$\mathrm{vol}(B_r(p)) = \int_{\mathbb{S}^{m-1}} \int_0^{\min\{r, c(X)\}} |\det \, \mathrm{d}_{\rho X} \exp_p| \cdot \rho^{m-1} \, \mathrm{d}\rho \, \mathrm{d}X \,.$$

This yields

$$\frac{\operatorname{vol}(B_r(p))}{\mathcal{V}_\kappa(r)} = \frac{\int_{\mathbb{S}^{m-1}} \int_0^{\min\{r,c(X)\}} |\det \, \mathrm{d}_{\rho X} \exp_p| \cdot \rho^{m-1} \, \mathrm{d}\rho \, \mathrm{d}X}{\int_{\mathbb{S}^{m-1}} \int_0^r (\mathfrak{s}_\kappa(\rho))^{m-1} \, \mathrm{d}\rho \, \mathrm{d}X}$$
$$= \frac{\int_{\mathbb{S}^{m-1}} \int_0^r f(X,\rho)(\mathfrak{s}_\kappa(\rho))^{m-1} \, \mathrm{d}\rho \, \mathrm{d}X}{\int_{\mathbb{S}^{m-1}} \int_0^r (\mathfrak{s}_\kappa(\rho))^{m-1} \, \mathrm{d}\rho \, \mathrm{d}X}, \tag{7.2}$$

where

$$f(X,\rho) = \begin{cases} \frac{|\det \, \mathrm{d}_{\rho X} \exp_p| \cdot \rho^{m-1}}{\mathfrak{s}_\kappa(\rho)^{m-1}} & \text{for } 0 \le \rho \le \min\{r, c(X)\}\,, \\ 0 & \text{for } \rho \ge \min\{r, c(X)\}\,. \end{cases}$$

From Proposition 5.1 we know that $f(X,\rho)$ is decreasing in ρ. As (7.3) describes an average of f with respect to the measure $(\mathfrak{s}_\kappa(\rho))^{m-1} \, \mathrm{d}\rho$, then integrated over $\mathbb{S}^{m-1}$, we get the claimed monotonicity. For $\rho \to 0$, the function $f(X,\rho)$ converges uniformly in X to 1. It follows

$$\lim_{r \searrow 0} \frac{\operatorname{vol}(B_r(p))}{\mathcal{V}_\kappa(r)} = 1\,,$$

and combining this with the monotonicity implies $\operatorname{vol}(B_r(p)) \le \mathcal{V}_\kappa(r)$. ∎

As a corollary we obtain a new proof of Bonnet–Myers and a regidity discussion going back to Cheng.

Corollary 7.4 (Bonnet–Myers–Cheng)**.** *Let (M,g) be a complete connected m-dimensional Riemannian manifold with* $\operatorname{ric} \ge \kappa(m-1)g$, $\kappa > 0$ *Then* $\operatorname{diam}(M,g) \le \pi/\sqrt{\kappa}$*. If we have equality, then (M,g) is isometric to $\frac{1}{\sqrt{\kappa}}\mathbb{S}^m$.*

Proof: For $r > \frac{1}{\sqrt{\kappa}}\mathbb{S}^m$, $\mathcal{V}_\kappa(r)$ is constant, thus may not increase either $\operatorname{vol}(B_r(p))$, i. e., $\operatorname{vol}(B_{1/\sqrt{\kappa}}(p)) = \operatorname{vol}(M,g)$. As $\mathbb{R}_p$ is open, this implies $B_{1/\sqrt{\kappa}}(p) \cup \mathcal{R}_p = M$, which implies $\overline{B}_{1/\sqrt{\kappa}}(p) = M$. Thus $\operatorname{diam}(M,g) \le \pi/\sqrt{\kappa}$. The equality discussion for the Riccati equation shows that $\operatorname{diam}(M,g) = \pi/\sqrt{\kappa}$ leads to a round sphere (= rescaled sphere). ∎

Fr 5.7.

Corollary 7.5. *Let (M,g) be a complete, connected Riemannian manifold with* $\operatorname{ric} \ge 0$*, . Then $r \mapsto \mathcal{V}_\kappa(r)$ grows "at most polynomially of degree $m = \dim M$". More precisely, we have*

$$\mathcal{V}_\kappa(r) \le \omega_m r^m$$

for all $r \ge 1$.

Proof: Follows from the Theorem 7.3 of Bishop–Gromov. ∎

Theorem 7.6. *Let (M, g) be a complete connected m-dimensional Riemannian manifold, $K \le \kappa$, $\kappa \in \mathbb{R}$, $p \in M$. Then for $0 < r < \min\{\mathrm{injrad}(p), \frac{\pi}{\sqrt{\kappa}}\} \ge r_0$:*

$$(0, r_0) \to \mathbb{R}, \quad r \mapsto \frac{\mathrm{vol}(B_r(p))}{\mathcal{V}_\kappa(r)}$$

is monotonically increasing and

$$\lim_{r \searrow 0} \frac{\mathrm{vol}(B_r(p))}{\mathcal{V}_\kappa(r)} = 1\,.$$

Thus we also have x$\mathrm{vol}(B_r(p)) \ge \mathcal{V}_\kappa(r)$.

Proof: Similar to the previous proof we calculate for r as above, (which implies $r < c(X)$)

$$\begin{aligned}\frac{\mathrm{vol}(B_r(p))}{\mathcal{V}_\kappa(r)} &= \frac{\int_{\mathbb{S}^{m-1}} \int_0^r |\det\, \mathrm{d}_{\rho X} \exp_p| \cdot \rho^{m-1}\, \mathrm{d}\rho\, \mathrm{d}X}{\int_{\mathbb{S}^{m-1}} \int_0^r (\mathfrak{s}_\kappa(\rho))^{m-1}\, \mathrm{d}\rho\, \mathrm{d}X} \\ &= \frac{\int_{\mathbb{S}^{m-1}} \int_0^r f(X,\rho)(\mathfrak{s}_\kappa(\rho))^{m-1}\, \mathrm{d}\rho\, \mathrm{d}X}{\int_{\mathbb{S}^{m-1}} \int_0^r (\mathfrak{s}_\kappa(\rho))^{m-1}\, \mathrm{d}\rho\, \mathrm{d}X}, \end{aligned} \tag{7.3}$$

where

$$f(X,\rho) = \frac{|\det\, \mathrm{d}_{\rho X} \exp_p| \cdot \rho^{m-1}}{\mathfrak{s}_\kappa(\rho)^{m-1}}$$

for $0 \le \rho \le r$.

From the Rauch comparison Theorem 4.4, Part (2) we know that $f(X,\rho)$ is increasing in ρ. As (7.3) describes an average of f with respect to the measure $(\mathfrak{s}_\kappa(\rho))^{m-1}\, \mathrm{d}\rho$ and integrating over $\mathbb{S}^{m-1}$, we get the claimed monotonicity. For $\rho \to 0$, the function $f(X,\rho)$ converges uniformly in X to 1. It follows

$$\lim_{r \searrow 0} \frac{\mathrm{vol}(B_r(p))}{\mathcal{V}_\kappa(r)} = 1\,,$$

and combining this with the monotonicity implies $\mathrm{vol}(B_r(p)) \ge \mathcal{V}_\kappa(r)$. ∎

Corollary 7.7 (Milnor)**.** *Let (M, g) be a complete, connected Riemannian manifold with $K \le \kappa < 0$, pin$M$. Then $r \mapsto \mathrm{vol}(B_r^M(p))$ grows "at most exponentially". More* *precisely, there is a constant $c \in \mathbb{R}_{>0}$ such that*

$$\mathrm{vol}(B_r^M(p)) \geq c e^{\sqrt{|\kappa|}(m-1)r}$$

for all $r \geq 1$.

Proof: With the Theorem 7.3 of Bishop–Gromov we calculate

$$\begin{aligned} \mathrm{vol}(B_r^M(p)) &\geq \mathcal{V}_\kappa(r) \\ &= \omega_{m-1} \int_0^r \frac{\sinh(\sqrt{|\kappa|}\rho)^{m-1}}{\sqrt{|\kappa|}^{m-1}} \, d\rho \\ &\geq C_1 e^{\sqrt{|\kappa|}(m-1)r} - C_2 \\ &\geq C_3 e^{\sqrt{|\kappa|}(m-1)r}, \end{aligned}$$

where the last inequality holds for $r \geq 1$. ∎

Remark 7.8. If we also have $\mathrm{ric} \geq -(m-1)\hat{\kappa}$, then by the Theorem 7.3 of Bishop–Gromov we also have an upper exponential bound. We then say that $r \mapsto \mathcal{V}_\kappa(r)$ **grows exponentially**. For example if $\widetilde{M}$ is the universal covering of a closed manifold with negative sectional curvature, the the volume of balls in $\widetilde{M}$ grows exponentially.

8 Growth of balls in the fundamental group

Literature: [5, Abschnitt 6.8] [24] Some of the content is also often treated in lectures by Clara Löh on geometric group theory.

Let G be a group with set of generators Γ, let $\Gamma^{-1} := \{\gamma^{-1} | \gamma \in \Gamma\}$. The **Cayley graph** $\mathrm{Cay}^{G,\Gamma}$ to (G, Γ) is the graph defined as follows:

The vertices of $\mathrm{Cay}^{G,\Gamma}$ are the elements of G. For $g_1, g_2 \in G$ there is an edge between g_1 and g_2 iff $g_1^{-1} g_2 \in \Gamma \cup \Gamma^{-1}$. As G is generated by Γ, the graph $\mathrm{Cay}^{G,\Gamma}$ is connected.

Examples 8.1.

1.) $G = (\mathbb{Z}, +)$, $\Gamma = \{1\}$

2.) $G = (\mathbb{Z}, +)$, $\Gamma = \{3, 5\}$

3.) Let G_k be an arbitrary group of order $k \in \mathbb{N}$, $\Gamma = G_k \setminus \{1\}$. Then $\mathrm{Cay}^{G_k, G_k \setminus \{1\}}$ is the 1-skeleton of a $(k-1)$-simplex.

4.) $G = \mathbb{Z}$, $\Gamma = \mathbb{Z} \setminus \{0\}$

5.) $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\Gamma = \{(0,1), (1,0)\}$

Figure in the lecture, not yet drawn electronically

The Cayley graphs for the above groups

Now take $h, g_1, g_2 \in G$. We then have:

$$\begin{aligned} & \text{there is an edge from } g_1 \text{ to } g_2 \\ \iff\ & g_1^{-1} g_2 \in \Gamma \cup \Gamma^{-1} \\ \iff\ & (hg_1)^{-1} h g_2 \in \Gamma \cup \Gamma^{-1} \\ \iff\ & \text{there is an edge from } hg_1 \text{ to } hg_2 \end{aligned}$$

Thus G acts on $\mathrm{Cay}^{G,\Gamma}$ by left-multiplication.

On $\mathrm{Cay}^{G,\Gamma}$ we consider the largest metric (in the sense of metric spaces) such that all edges are isometric (as metric spaces) to an interval of length 1. As a consequence, any path $\widetilde{\gamma}$ in $\mathrm{Cay}^{G,\Gamma}$ has a well-defined length $L(\widetilde{\gamma})$. Furthermore for $g_1, g_2 \in G$ we define;

$$\begin{aligned} \mathrm{dist}^{G,\Gamma}(g_1, g_2) &:= \inf\{L(\widetilde{\gamma}) | \widetilde{\gamma} \text{ path from } g_1 \text{ to } g_2\} \\ &= \min\{\ell | \exists \gamma_1, \ldots, \gamma_\ell \in \Gamma \cup \Gamma^{-1}, \text{ such that } g_2 = g_1 \gamma_1 \ldots \gamma_\ell\}. \end{aligned}$$

This metric resp. distance function is called the **word metric**.

Remark 8.2. $\mathrm{dist}^{G,\Gamma}$ is a left-invariant metric, i. e., for all $h, g_1, g_2 \in G$ we have

$$\mathrm{dist}^{G,\Gamma}(hg_1, hg_2) = \mathrm{dist}^{G,\Gamma}(g_1, g_2).$$

For $R \in \mathbb{R}$ we define

$$\overline{B}_R^\Gamma := \left\{h \in G \mid \mathrm{dist}^{G,\Gamma}(h, 1) \le R\right\}$$

the closed ball of radius R. Furthermore, let

$$N_\Gamma : \mathbb{N}_0 \to \mathbb{N} \cup \{\infty\}, \quad N_\Gamma(R) := \#\overline{B}_R^\Gamma$$

be the **growth function** of G **with respect to** Γ.

We ask: Does $\mathrm{dist}^{G,\Gamma}$ depend on Γ? Does N_Γ depend on Γ? The answer to both questions is “Yes”, as you may see from the Example 8.1 above:

Example 8.1, con’td	1.)	2.)	4.)	
$\mathrm{dist}(0,1)$	1	3	1	
$\mathrm{dist}(0,5k)$	$\lvert 5k\rvert$	$\lvert k\rvert$	1	$k \in \mathbb{Z},\ k \neq 0$
$\mathrm{dist}(0,k)$	$\lvert k\rvert$	$\approx \frac{\lvert k\rvert}{5}$	1	
$N_\Gamma(k)$	$2k+1$	$\approx 10k$	∞	$k \in \mathbb{N},\ k \neq 0$

Furthermore we obtain the following statements:

Lemma 8.3. *Assume that* Γ *is a finite generating set of* G *and let* dist' *be a left-invariant metric on* G*. Then there is* $C \in \mathbb{R}$*,* $C > 0$*, such that*

(1) *For all* $g_1, g_2 \in G$*:* $\mathrm{dist}'(g_1, g_2) \le C\, \mathrm{dist}^{G,\Gamma}(g_1, g_2)$.

(2) *For all* $R \in \mathbb{N}$*:* $N_{\mathrm{dist}'}(CR) := \#\left\{h \in G \mid \mathrm{dist}'(h,1) \le CR\right\} \ge N_\Gamma(R)$.

See Exercise Sheet 13, Exercise 2 or:

Proof:
“(1)”: Define

$$C := \max\{\mathrm{dist}'(\gamma, 1) \mid \gamma \in \Gamma \cup \Gamma^{-1}\}.$$

Let $g_1, g_2 \in G$. Define $\ell := \mathrm{dist}^{G,\Gamma}(g_1, g_2)$. Then there are $\gamma_1, \dots, \gamma_\ell \in \Gamma \cup \Gamma^{-1}$ with

$g_2 = g_1\gamma_1 \ldots \gamma_\ell$. It follows

$$\begin{aligned} \operatorname{dist}'(g_1, g_2) &\le \underbrace{\operatorname{dist}'(g_1, g_1\gamma_1)}_{=\operatorname{dist}'(1,\gamma_1)} + \underbrace{\operatorname{dist}'(g_1\gamma_1, g_1\gamma_1\gamma_2)}_{=\operatorname{dist}'(1,\gamma_2)} + \ldots + \underbrace{\operatorname{dist}'(g_1\gamma_1\ldots\gamma_{\ell-1}, g_2)}_{=\operatorname{dist}'(1,\gamma_\ell)} \\ &\le C\ell \end{aligned}$$

and we get Statement (1).
"(2)": According to Statement (1) we have

$$\overline{B}\Gamma^R = \left\{g \in G \mid \operatorname{dist}^{G,\Gamma}(g, 1) \le R\right\} \subset \{g \in G \mid \operatorname{dist}'(g, 1) \le CR\} =: \overline{B}^R_{\operatorname{dist}'}.$$

It follows that $N_\Gamma(R) \le N_{\operatorname{dist}'}(CR)$. ■

Application 8.4. *If Γ and Γ' are finite generating sets of G, then there is $C \in \mathbb{R}_{>0}$, such that for all $g_1, g_2 \in G$ and all $R \in \mathbb{N}$ we have*

$$\begin{aligned} &\frac{1}{C}\operatorname{dist}^{G,\Gamma}(g_1, g_2) \le \operatorname{dist}^{G,\Gamma'}(g_1, g_2) \le C\operatorname{dist}^{G,\Gamma}(g_1, g_2), \\ &N_\Gamma\left(\frac{R}{C}\right) \le N_{\Gamma'}(R) \le N_\Gamma(CR) \end{aligned}$$

Definition 8.5. *Let Γ be a finite generating set of G.*

(i) *G has* **(at least) exponential growth**, *if there are $C_1, C_2 > 0$ such that for all $R \ge 1$ we have*

$$N_\Gamma(R) \ge C_1 e^{C_2 R}.$$

This condition is equivalent to claiming the existence of some $C_1, C_2 > 0$, $C_3 \in \mathbb{R}$, such that we have for all $R \ge 1$:

$$N_\Gamma(R) \ge C_1 e^{C_2 R} - C_3.$$

(ii) *G has* **polynomial growth** *of* **degree** *$\le n$, if there is a polynomial $P \in \mathbb{R}[X]$ of degree $\le n$ such that for all $R \ge 0$:*

$$N_\Gamma(R) \le P(R).$$

Remarks 8.6. We have:

1.) These properties do not depend on the choice of Γ, as long as Γ is finite.

2.) The Properties (i) and (ii) are mutally exclusive. However there are groups - not so easy to construct that grow slower than exponentially, but faster than exponential.

3.) If the growth is polynomial, then the degree of the growth does not depend on the choice of Γ.

4.) The constant C_2, in general, does depend on the choice of Γ.

Example 8.7. Suppose that G is abelian and finitely generated. Then we have $G \cong \mathbb{Z}^n \times F$ with a finite abelian group F, the torsion subgroup. We choose the generating set

$$\Gamma := \{e_1, ..., e_n\} \cup (F \setminus \{0\}).$$

Any $g \in G$ may be written uniquely as

$$g = \sum_{i=1}^{n} a_i e_i + t$$

where $a_i \in \mathbb{Z}$ and $t \in F$. Thus

$$\mathrm{dist}^{G,\Gamma}(g,0) = \begin{cases} \sum_{i=1}^n |a_i| + 1, & t \neq 0 \\ \sum_{i=1}^n |a_i|, & t = 0 \end{cases}$$

and as a consequence

$$\sup_{1\le i\le n} |a_i| \le \mathrm{dist}^{G,\Gamma}(g,0) \le n \sup_{1\le i\le n} |a_i| + 1.$$

By counting we find that

$$\#\{g \in G | \sup_{1\le i\le n} |a_i| \le R\} \le (\#F)(2R+1)^n.$$

We conclude

$$N_\Gamma(R) \le (\#F)(2R+1)^n.$$

Furthermore, we have

$$\left\{ \sum_{i=1}^{n} a_i e_i \middle| \forall i : |a_i| \le \frac{R}{n} \right\} \subset \overline{B}_R^\Gamma$$

and thus $N_\Gamma(R) \ge (2\frac{R}{n} + 1)^n$. We have seen that G grows polynomially of degree n.

Examples 8.8. If $\#\Gamma = m$, then $\#\big(\Gamma \cup \Gamma^{-1} \cup \{1\}\big) \le 2m + 1$. For all $R \in \mathbb{N}$ we have

$$\overline{B}_R^\Gamma \quad = \quad \left\{\gamma_1 \cdots \gamma_R \mid \gamma_1, \ldots, \gamma_R \in \Gamma \cup \Gamma^{-1}\right\} \cup \overline{B}_\Gamma^{R-1}$$

$$= \left\{\gamma_1 \cdots \gamma_R \mid \gamma_1, \ldots, \gamma_R \in \Gamma \cup \Gamma^{-1} \cup \{1\}\right\}.$$

It follows that $N_\Gamma(R) \le (2m+1)^R = e^{R\log(2m+1)}$. As a consequence $N_\Gamma(R)$ grows at most exponentially and $\#G = \lim_{R\to\infty} N_\Gamma(R)$ is at most countably infinite.

Example 8.9 (Heisenberg group)**.** The 3-dimensional discrete Heisenberg group, i. e.,

$$\mathcal{H}_3(\mathbb{Z}) := \left\{ M_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

grows polynomially of degree 4. See Exercise Sheet 13, Exercise 4 for details.

Tu 9.7.

Let (M,g) be a compact Riemannian manifold, and let $\pi\colon \widetilde{M} \to M$ be the universal covering. We fix $p \in M$ and some $\widetilde{p} \in \pi^{-1}(p) \subset \widetilde{M}$. Then $\pi_1(M,p)$ acts on $P := \pi^{-1}(p)$ as follows:

Let $g := [\gamma] \in \pi_1(M,p)$ with $\gamma\colon [0,1] \to M$, $\gamma(0) = \gamma(1) = p$. A lift of γ is a curve $\widetilde{\gamma}\colon [0,1] \to \widetilde{M}$ with $\pi \circ \widetilde{\gamma} = \gamma$. As π is a covering map, there is a unique lift $\widetilde{\gamma}$ of γ with $\widetilde{\gamma}(1) = \widetilde{p}$. We define $g \cdot \widetilde{p} := \widetilde{\gamma}(0)$, which yields a map $\pi_1(M,p) \times \{\widetilde{p}\} \to \widetilde{M}$.

Now, for some $\widetilde{q} \in \widetilde{M}$ we want to define $g \cdot \widetilde{q}$. For this we choose path $\tau_1\colon [0,1] \to \widetilde{M}$ from $\widetilde{p}$ to $\widetilde{q}$. Define $\tau_2\colon [0,1] \to \widetilde{M}$ as the unique lift of $\pi \circ \tau_1$ starting in $\widetilde{p}$. Then we define $g \cdot \widetilde{q} := \tau_2(1)$.

One easily checks that this defines a well-defined smooth group action

$$\pi_1(M,p) \times \widetilde{M} \to \widetilde{M},$$

and $\pi\colon \widetilde{M} \to M$ induces a well-defined bijective map $\pi_1(M,p) \backslash \widetilde{M} \to M$.

Keep in mind that the action $\pi_1(M,p) \times \widetilde{M} \to \widetilde{M}$, in general depends on the choice of $\widetilde{p}$.

We equip $\widetilde{M}$ with the pull-back metric $\widetilde{g} := \pi^* g$. Let $\widetilde{\operatorname{dist}}_g$ be the associated distance function. According to Diff. geom. I, Exercise Sheet 14, Exercise 1 the space $(\widetilde{M}, \widetilde{\operatorname{dist}}_g)$ is a complete metric space. Now take $\widetilde{p} \in \widetilde{M}$ from above, $p = \pi(\widetilde{p})$. For $h_1, h_2 \in \pi_1(M,p)$ we define

$$\operatorname{dist}_g(h_1, h_2) := \widetilde{\operatorname{dist}}_g(h_1 \cdot \widetilde{p}, h_2 \cdot \widetilde{p}).$$

Then dist_g is a metric on $\pi_1(M,p)$ and dist_g is left-invariant under the action of $\pi_1(M,p)$ on $(\widetilde{M},\widetilde{\mathrm{dist}}_g)$.

The set

$$\overline{B}_{3\,\mathrm{diam}(M)}(\widetilde{p}) = \exp_{\widetilde{p}}(\overline{B}_{3\,\mathrm{diam}(M)}(0_{\widetilde{p}}))$$

is compact, and thus

$$\Gamma := \overline{B}_{3\,\mathrm{diam}(M)}(\widetilde{p}_0) \cap \pi^{-1}(\{p\})$$

is finite. An element $h \in \pi_1(M,p)$ lies in Γ if, and only if a loop γ based in p exists with $[\gamma] = h$ and $\mathcal{L}(\gamma) \le 3\,\mathrm{diam}(M)$.

In the following we define

$$G := \pi_1(M,p)\,.$$

Lemma 8.10. *The set Γ defined above generates G, and there is a $C > 0$, such that for all $h_1, h_2 \in G$:*

$$\mathrm{dist}^{G,\Gamma}(h_1,h_2) \le C\,\mathrm{dist}_g(h_1,h_2).$$

Proof: For $h_1, h_2 \in G$ we choose a geodesic $\widetilde{\gamma}\colon [0,\ell] \to \widetilde{M}$, parametrized by arclength, from $h_1 \cdot \widetilde{p}$ to $h_2 \cdot \widetilde{p}$ with

$$\mathcal{L}(\widetilde{\gamma}) = \widetilde{\mathrm{dist}}_g(h_1 \cdot \widetilde{p}, h_2 \cdot \widetilde{p}) = \ell.$$

Determine $N \in \mathbb{N}$ with

$$\frac{\ell}{N-1} > \mathrm{diam}(M) \ge \frac{\ell}{N}.$$

In the case $\mathrm{diam}(M) \ge \ell$, we put $N = 1$. For $j = 0, \dots, N$ we define

$$t_j := \frac{j\ell}{N}, \quad \widetilde{q}_j := \widetilde{\gamma}(t_j),$$

and we choose a path τ_j from p to $\pi(\widetilde{q}_j)$ with $\mathcal{L}(\tau_j) \le \mathrm{diam}(M)$. Let $\widetilde{\tau}_0$ be the constant path $\widetilde{q}_0$ and $\widetilde{\tau}_N$ the constant path $\widetilde{q}_N$. For $j = 1, \dots, N-1$ let $\widetilde{\tau}_j$ be the lift of τ_j in $\widetilde{M}$ with endpoint $\widetilde{q}_j$. For $j = 1, \dots, N$ we define

$$\widetilde{\gamma}_j := \widetilde{\tau}_{j-1} * \widetilde{\gamma}\big|_{[t_{j-1},t_j]} * \widetilde{\tau}_j^{-1}.$$

Then for any j the curve $\pi \circ \widetilde{\gamma}_j$ is a loop in M with basepoint p and

$$\mathcal{L}(\pi \circ \widetilde{\gamma}_j) = \mathcal{L}(\widetilde{\gamma}_j) = \mathcal{L}(\widetilde{\tau}_{j-1}) + \underbrace{\mathcal{L}\Big(\widetilde{\gamma}\big|_{[t_{j-1},t_j]}\Big)}_{=\frac{\ell}{N} \le \mathrm{diam}(M)} + \mathcal{L}(\widetilde{\tau}_j^{-1}) \le 3\,\mathrm{diam}(M),$$

thus $[\pi\circ\widetilde{\gamma}_j]\in\Gamma$. Furthermore, $\widetilde{\gamma}$ is homotopic to $\widetilde{\gamma}_1 * \ldots * \widetilde{\gamma}_N$.

Consider the special case $h_2 = 1$. According to the definition of the action of G on P we have

$$h_1 = [\pi\circ\widetilde{\gamma}] = [\pi\circ\widetilde{\gamma}_1] * \cdots * [\pi\circ\widetilde{\gamma}_N] \in \overline{B}_N^{\Gamma},$$

in particular Γ generates G.

For abitrary, but different $h_1, h_2 \in G$ we claim that $\mathrm{dist}_g(h_1,h_2) \geq 2\,\mathrm{injrad}(M)$. If this estimate did not hold, then for $\widetilde{\gamma}$ defined as above, the curve $\pi\circ\widetilde{\gamma}$ were a geodesic loop at p, that runs entirely in $B_{\mathrm{injrad}(M)}(p)$. The only such geodesic loop is the constant curve $t\mapsto p$, and we get the contradiction $h_1 = h_2$.

According to the choice of N we have

$$\begin{aligned}\mathrm{dist}^{G,\Gamma}(h_1,h_2) &\leq N < 1 + \frac{\ell}{\mathrm{diam}(M)} = 1 + \frac{\mathrm{dist}_g(h_1,h_2)}{\mathrm{diam}(M)}\\ &\leq \Big(\frac{1}{2\,\mathrm{injrad}(M)} + \frac{1}{\mathrm{diam}(M)}\Big)\mathrm{dist}_g(h_1,h_2).\end{aligned}$$

The statement of the lemma is proven. ∎

We combine the previous results in order to get:

Theorem 8.11 (Švarc–Milnor). *Let (M,g) be a compact Riemannian manifold, let dist_g be the distance function on $G=\pi_1(M,p)$, induced by the pull back of the Riemannian metric g to $\widetilde{g}$ on $\widetilde{M}$. Let Γ be a finite generating set of G. Then there is a constant $C>0$, such that for all $h_1,h_2\in G$ we have*

$$\frac{1}{C}\,\mathrm{dist}^{G,\Gamma}(h_1,h_2) \leq \mathrm{dist}(h_1,h_2) \leq C\,\mathrm{dist}^{G,\Gamma}(h_1,h_2)$$

and for the associated growth functions and all $R>0$ we have

$$N_\Gamma\Big(\frac{R}{C}\Big) \leq N_{\mathrm{dist}_g}(R) \leq N_\Gamma(CR).$$

The balls in the Cayley graph grow exponentially resp. polynomially resp. polynomially of degree k if and only if the balls in (G,dist_g) grow exponentially resp. polynomially resp. polynomially of degree k.

Proof: This statement follows by combining Application 8.4 with Lemma 8.3 and Lemma 8.10. ∎

We now compare the growth of balls in (G, dist_g) with the growth of balls in $\widetilde{M}$. Let $\delta := \mathrm{injrad}(M)$. Then according to the triangle inequality we have for $r > 0$

$$\bigsqcup_{\widehat{p}\in P\cap B_r(\widetilde{p})} B_\delta(\widehat{p}) \subset B_{r+\delta}(\widetilde{p}).$$

As G acts isometrically on $\widetilde{M}$ we have for all $\widehat{p} \in P$

$$\mathrm{vol}\big(B_\delta(\widehat{p})\big) = \mathrm{vol}\big(B_\delta(\widetilde{p})\big).$$

We conclude

$$\mathrm{vol}\big(B_{r+\delta}(\widetilde{p})\big) \geq \mathrm{vol}\big(B_\delta(\widetilde{p})\big)\ \#\big(P \cap \overline{B}_r(\widetilde{p})\big)$$

As $h \in P \cap \overline{B}_r(\widetilde{p})$ holds if, and only if $\mathrm{dist}_g(h, 1) \leq r$, we get for all $r > 0$

$$N_{\mathrm{dist}_g}(r) \leq \frac{1}{\mathrm{vol}\big(B_\delta(\widetilde{p})\big)}\,\mathrm{vol}(B_{r+\delta}\big(\widetilde{p})\big). \tag{8.1}$$

Now we claim that for $D := \mathrm{diam}(M)$ and for all $r > 0$ we have

$$B_r(\widetilde{p}) \subset \bigcup_{\widehat{p}\in P\cap\overline{B}_{D+r}(\widetilde{p})} \overline{B}_D(\widehat{p}).$$

To prove this, let $\widetilde{s} \in B_r(\widetilde{p})$. We connect $\pi(\widetilde{s})$ with p in M by a curve γ with $\mathcal{L}(\gamma) \leq D$. Let $\widetilde{\gamma}$ be the lift of γ with start point $\widetilde{s}$. Then $\widetilde{\gamma}$ ends in some $\widehat{p} \in P \cap \overline{B}_{D+r}(\widetilde{p})$. It follows that $\widetilde{s} \in \overline{B}_D(\widehat{p})$. It follows that

$$\mathrm{vol}\big(B_r(\widetilde{p})\big) \leq \mathrm{vol}\big(\bigcup_{\widehat{p}\in P\cap\overline{B}_{D+r}(\widetilde{p})} \overline{B}_D(\widehat{p})\big) \leq \mathrm{vol}\big(\overline{B}_D(\widetilde{p})\big)\cdot\#\big(P\cap\overline{B}_{D+r}(\widetilde{p})\big)$$

and thus

$$\mathrm{vol}\big(B_r(\widetilde{p})\big) \leq \mathrm{vol}\big(\overline{B}_D(\widetilde{p})\big)\cdot N_{\mathrm{dist}_g}(D + r). \tag{8.2}$$

Theorem 8.12. *We use the notation from above. Then $\pi_1(M, p)$ grows exponentially resp. polynomially resp. polynomially of degree k, if, and only if the volume of balls in $\widetilde{M}$ of radius ≥ 1 grow exponentially resp. polynomially resp. polynomially of degree k.*

Proof: This follows from (8.1) and (8.2). We then use the Theorem 8.11 by Švarc–Milnor, in order to replace $N_{\mathrm{dist}_g}(D + \bullet)$ by N_Γ. ■

Theorem 8.13 (Milnor)**.** *Let (M^m, g) be compact with $K \leq \kappa$ for some $\kappa < 0$, and* *let $p \in M$. Then $\pi_1(M,p)$ has exponential growth.*

Proof: Corollary 7.7 tells us that there is some $c > 0$ such that for balls in the universal covering $\widetilde{M}$ of M:

$$\mathrm{vol}(B_r^{\widetilde{M}}(p)) \geq c e^{\sqrt{|\kappa|}(m-1)r}$$

for all $r \geq 1$. Then the statement follows from Theorem 8.12. ∎

Theorem 8.14 (Milnor). *Let (M,g) be a compact, connected m-dimensional Riemannian manifold with* $\mathrm{ric} \geq 0$*, let $p \in M$. Then $\pi_1(M,p)$ has polynomial growth of degree $\leq m$.*

Proof: Accroding to the Theorem 7.3 by Bishop–Gromov we have for all $r > 0$:

$$\mathrm{vol}\big(B_r(\widetilde{p}_0)\big) \leq \mathcal{V}_0(r) = \omega_{m-1} \int_0^r \rho^{m-1}\, \mathrm{d}\rho = \frac{\omega_{m-1}}{m} r^m.$$

The statement then follows from Theorem 8.12. ∎

Examples 8.15.

1.) Let $(M,g) = (\mathbb{R}^m/\mathbb{Z}^m, g_0)$ with the Riemannian metric g_0 induced from the Euclidean metric on $\mathbb{R}^m$. We have $\mathrm{ric} \equiv 0$. Then $\pi_1(M) = \mathbb{Z}^m$ grows polynmially of degree $m = \dim M$.

2.) $(M,g) = (\mathbb{R}^m/\mathbb{Z}^m \times S^k, g_0 \oplus g^{\mathrm{sph}})$, where g_0 is as above, and where g^{sph} is the standard metric on S^k, $k \geq 1$. We have $\mathrm{ric} \geq 0$. Then $\pi_1(M) = \mathbb{Z}^m$ grows polynomially of degree $m < \dim M$.

3.) For a ring R, recall from Exercise I.1.37

$$\mathcal{H}_3(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in R \right\}.$$

We consider $G := \mathcal{H}_3(\mathbb{Z})$. Then $M := {}_G\backslash^{\mathcal{H}_3(\mathbb{R})}$ is a compact manifold with $\pi_1(M) = G$, $\widetilde{M} = \mathcal{H}_3(\mathbb{R})$.

The volume growth in $\widetilde{M}$ resp. in G is polynomially of degree 4, see Exercise Sheet 13, Exercise 4 and also Example 8.9.

Using Theorem 8.14 there is no Riemannian metric on M with $\mathrm{ric} \geq 0$.

Fr. 12.7.

9 Laplace Operator and harmonic maps

Sorry, this part is not yet fully translated. The text below is also in some other aspects preliminary.

Let (M^m, g) be a Riemannian manifold.

Definition 9.1. *The co-differential is is defined by*

$$\delta \colon \Gamma(T^*M) \to C^\infty(M), \quad (\delta\alpha)_p := -\sum_{i=1}^m (\nabla\alpha)(e_i, e_i),$$

where $e_1, \ldots, e_m$ is an orthonormal basis of T_pM.

Lemma 9.2. *$(\delta\alpha)_p$ does not depend on the choice of orthonormal basis.*

Proof: The metric g induces the **musical isomorphism**[8]

$$\flat \colon TM \to T^*M, \quad X \mapsto X^\flat := g(X, \bullet).$$

The inverse is denoted as $\sharp \colon T^*M \to TM$. For any $X \in TM$, we define $A(X) := (\nabla_X \alpha)^\sharp \in TM$, thus $A \in \mathrm{End}(TM)$. We have for $X, Y \in TM$:

$$g(AX, Y) = (\nabla\alpha)(X, Y).$$

As a consequence

$$-\operatorname{tr}(A) = -\sum_{i=1}^m g(A(e_i), e_i) = -\sum_{i=1}^m (\nabla\alpha)(e_i, e_i),$$

and thus $(\delta\alpha)_p$ is independent on the choice of basis $(e_i)_i$. ■

Remark 9.3. With this notation we have

$$\mathrm{grad}(f) = (\mathrm{d}f)^\sharp, \quad \mathrm{div}(X) = -\delta(X^\flat).$$

[8] In index notation this isomorphism lowers an index, while its inverse raises an index. This is why one often uses the symbols $\flat$ and $\sharp$, as in musics s $\flat$ lowers a note, while $\sharp$ raises it. However in music $\sharp$ and $\flat$ are not inverses of each other.

According to the divergence theorem we have for all $\alpha \in \Gamma(T^*M)$ with compact support:

$$0 = \int_M \operatorname{div}(\alpha^\sharp)\, d\mu^g = -\int_M \delta\alpha\, d\mu^g.$$

According to the following lemma δ is the adjoint of d.

Lemma 9.4. *Let $f \in C^\infty(M)$ and $\alpha \in \Gamma(T^*M)$ with $\operatorname{supp}(f) \cap \operatorname{supp}(\alpha)$ compact. Then*

$$\int_M \langle \mathrm{d}f, \alpha\rangle\, \mathrm{d}\mu^g = \int_M f\,\delta\alpha\, \mathrm{d}\mu^g.$$

Proof: We have for an orthonormal basis $(e_1, \ldots, e_m)$:

$$\delta(f\alpha) = -\sum_{i=1}^m \Big((\partial_{e_i} f) \cdot \alpha(e_i) + f \cdot (\nabla\alpha)(e_i, e_i) \Big) = f\delta\alpha - \langle \mathrm{d}f, \alpha\rangle.$$

Because of $\operatorname{supp}(\delta(f\alpha)) \subset \operatorname{supp}(f) \cap \operatorname{supp}(\alpha)$ we know that $\operatorname{supp}(\delta(f\alpha))$ is compact. It follows that

$$0 = \int_M \delta(f\alpha)\, d\mu^g = \int_M f\,\delta\alpha\, d\mu^g - \int_M \langle df, \alpha\rangle\, d\mu^g$$

and thus the statement. ■

We now define the **Laplace operator on smooth functions**, also called **Laplacian (on smooth functions)** or **Laplace–Beltrami operator**.

$$\Delta := \delta \circ \mathrm{d} \colon C^\infty(M) \to C^\infty(M). \tag{9.1}$$

For $f_1, f_2 \in C^\infty(M)$ with compact $\operatorname{supp}(f_1) \cap \operatorname{supp}(f_2)$ we have

$$\int_M (\Delta f_1) f_2\, \mathrm{d}\mu^g = \int_M \langle \mathrm{d}f_1, \mathrm{d}f_2\rangle\, \mathrm{d}\mu^g = \int_M f_1 \Delta f_2\, \mathrm{d}\mu^g.$$

Examples 9.5.

1.) $M = \mathbb{R}^m$. Then the definition spells out to

$$\Delta = -\sum_{i=1}^m \frac{\partial^2}{(\partial x^i)^2}. \tag{9.2}$$

2.) In local coordinates with $g_{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $(g^{ij}) = (g_{ij})^{-1}$ we obtain the formula

$$\Delta f = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{k,\ell=1}^{m} \partial_\ell\Big(g^{\ell k}\sqrt{\det(g_{ij})}\partial_k f\Big). \tag{9.3}$$

Note that there two sign conventions in the literature: the "geometer's Laplacian" has no minus sign in formula (9.1), but a minus sign in (9.3). In contrast to this the "analyst's Laplacian" has no minus sign in (9.2) and in case (s)he considers curved spaces, no minus sign in (9.3). This involves a sign change in (9.1), usually by replacing δ by div.[9]

Definition 9.6. *A function $f \in C^\infty(M)$ is called* **harmonic**, *if we have $\Delta f = 0$.*

An important tool in analysis is the maximum principle for harmonic functions.

Proposition 9.7 (Maximum principle for smooth functions)**.** *Let $f \in C^\infty(M)$, $p \in M$, M connected. We assume $\Delta f \le 0$. Then f is constant.*

We will proof a more general statement in Theorem 9.10 below, from which the above proposition will follow.

Definition 9.8. *Let (M, g) be a connected Riemannian manifold. A continuous function $f: M \to \mathbb{R}$ is called* **subharmonic**, *if for all $p \in M$ and for all $\epsilon > 0$ there is a neighborhood W of p and a smooth function $\varphi: W \to \mathbb{R}$ (called* **support function** *in p, such that $\varphi(p) = f(p)$, $\varphi \le f|_W$, and $\Delta\varphi \le \epsilon$ on W.*

Lemma 9.9. *Let $f \in C^\infty(M)$. Then f is subharmonic, iff $\Delta f \le 0$.*

Proof:
"$\Leftarrow$": Put $\varphi = f$.
"$\Rightarrow$": As $f - \varphi$ has a local minimum in p, we have $d_p(f - \varphi) = 0$ and $\nabla^2(f - \varphi)|_p \ge 0$. Because of

$$\Delta f = -\operatorname{tr}(\nabla^2 f) = -\operatorname{tr}(\operatorname{Hess} f)$$

we also obtain $\Delta(f - \varphi)|_p \le 0$, and together with $\Delta\varphi|_p \le \epsilon$ we conclude $\Delta f|_p \le \epsilon$. In the limit$\epsilon \to 0$ we get $\Delta f|_p \le 0$ for all $p \in M$. ■

[9] This is lecture on differential geometry, and not on geometrically motivated analysis, which determines the sign convention.

Theorem 9.10 (Maximum principle for subhharmonic functions)**.** *Let (M,g) be a connected Riemannian manifold, let $f: M \to \mathbb{R}$ be continuous and subharmonic. If f attains its maximum, then f is constant.*

Proof: After possibly restricting M we may assume that there is a $p \in M$ in which the maximum is attained, i. e., $f(p) = \max_{x\in M} f(x)$. We show that an open neighborhood of p exists, on which f is constant. Daraus folgt dann, dass $f^{-1}(f(p))$ offen und abgeschlossen in M und daher gleich M ist. Angenommen, die Behauptung wäre falsch. Dann wähle ein $r > 0$, so dass $f|_{\partial B_r(p)}$ nicht konstant gleich $f(p)$ ist und $r < \frac{1}{2}\operatorname{injrad}(p)$ gilt. Let

$$A := \left\{x \in \partial B_r(p) \mid f(x) = f(p)\right\}$$

and choose $q \in \partial B_r(p) \setminus A$, thus $f(q) < f(p)$.

Wir benutzen ohne Beweis, dass es eine Karte

$$\rho: B_{2r}(p) \to V \Subset \mathbb{R}^m$$

mit $\rho(A) \subset (-\infty, 0) \times \mathbb{R}^{m-1}$, $\rho(q) \in (0,\infty) \times \mathbb{R}^{m-1}$ und $\rho(p) = 0$ gibt.

Der Beweis der Behauptung verläuft dann wie folgt. Let $\pi_1: \mathbb{R}^m \to \mathbb{R}$ be the projection to the first component. We define

$$x^1 := \pi_1 \circ \rho: B_{2r}(p) \to \mathbb{R},$$

i. e., x^1 is the first coordinate function of the chart ρ. We obtain $x^1\big|_A < 0$, $x^1(q) > 0$, $x^1(p) = 0$ and

$$\mathrm{d}x^1 = \mathrm{d}\pi_1 \circ \mathrm{d}\rho = \pi_1 \circ \mathrm{d}\rho$$

and thus $\left\|\mathrm{d}x^1|_{\overline{B_r(p)}}\right\| > 0$. Für ein α, das später noch gewählt wird, definiere

$$h := e^{\alpha x^1} - 1: \quad B_{2r}(p) \to \mathbb{R}.$$

Dann gilt $h\big|_A < 0$, $h(q) > 0$, $h(p) = 0$ und

$$\begin{aligned}\Delta h &= \delta(d(e^{\alpha x^1} - 1)) = \delta(e^{\alpha x^1}\alpha dx^1) = \alpha e^{\alpha x^1}\delta x^1 - \alpha^2\langle dx^1, dx^1\rangle e^{\alpha x^1}\\ &= e^{\alpha x^1}(-\alpha^2\|\mathrm{d}x^1\|^2 + \alpha\Delta x^1).\end{aligned}$$

Choose

$$\alpha \geq 2\frac{\max_{\overline{B_r(p)}} \Delta x^1}{\min_{\overline{B_r(p)}} \|\mathrm{d}x^1\|^2}.$$

Dann ist $\alpha < \infty$ und $\Delta h < 0$. Bestimme nun eine offene Umgebung U von A, so dass $h|_U < 0$ gilt. Dann gilt für alle $x \in U$ und für alle $\beta > 0$ die Ungleichung $(f + \beta h)(x) < f(p)$. Nach Konstruktion gilt

$$\min_{x \in \partial B_r(p) \setminus U} (f(p) - f(x)) > 0.$$

Daher kann man $\beta > 0$ so wählen, dass für alle $x \in \partial B_r(p) \setminus U$ die Ungleichung $f(p) - f(x) > \beta h(x)$ gilt. Für alle $x \in \partial B_r(p)$ gilt dann

$$(f + \beta h)(x) < f(p) = (f + \beta h)(p).$$

Daher kann man $p_1 \in B_r(p)$ wählen $(p_1 \notin \partial B_r(p))$, so dass gilt

$$(f + \beta h)(p_1) = \max_{x \in \overline{B_r(p)}} (f + \beta h)(x).$$

Wähle eine Stützfunktion φ zu f in p_1:

$$\varphi : \quad B_{\overline{r}}(p_1) \to \mathbb{R}, \quad \varphi(p_1) = f(p_1), \quad \varphi \leq f, \quad \Delta\varphi \leq \epsilon.$$

Dann ist $\Phi := \varphi + \beta h$ eine Stützfunktion zu $f + \beta h$ in p_1 und

$$\Delta\Phi\big|_{p_1} = \Delta\varphi\big|_{p_1} + \beta\Delta h\big|_{p_1} \leq \epsilon + \beta\Delta h\big|_{p_1}$$

wird negativ für $\epsilon \to 0$ und β fest. Es gilt nun

$$\max_{x \in B_{\overline{r}}(p)} \Phi(x) = \Phi(p_1)$$

und $\mathrm{tr}(\mathrm{Hess}\big|_{p_1} \Phi) = -\Delta\Phi\big|_{p_1} > 0$, ein Widerspruch. ∎

Tu, 16.7.

Corollary 9.11. *If M is a compact connected Riemannian manifold with non-empty boundary, and let $f: M \to \mathbb{R}$ be continuous and subharmonic. Then f attains its maximum on the boundary,*

Proof: As M is compact, there is a point $p_0 \in M$ in which the maximum of f is attained. If $p_0 \in \mathring{M}$, then we may apply the maximum principle Theorem 9.10 for $\mathring{M}$ instead of M and then f is constant on $\mathring{M}$, and thus on M, thus the statement follows. If $p_0 \in \partial M$ then the statement is immediate. ∎

Lemma 9.12. *Let $f: M \to \mathbb{R}$ be a continuouand subharmonic function, and let $-f$ be subharmonic as well. Then f is smooth and $\Delta f = 0$.*

We will not give a complete proof as we will use (without proof) that there is a solution h to the boundary value problem below.

Sketch of proof: Let $p \in M$, $\epsilon > 0$. Let us use (without proof) that there is a unique harmonic function $h: \overline{B_\epsilon(p)} \to \mathbb{R}$,continous on $\overline{B}_\epsilon(p)$ and smooth on $B_\epsilon(p)$ such that $h|_{\partial B_\epsilon(p)} = f|_{\partial B_\epsilon(p)}$. Define

$$F := f - h: \overline{B}_\epsilon(p) \to \mathbb{R}.$$

We thus have $F|_{\partial B_\epsilon(p)} = 0$. Our conditions imply that both F and $-F$ are subharmonic. According to the corollary of the maximum principle Corollary 9.11 the maximum and the minimum of F are attained on $\partial B_\epsilon(p)$. It follows that $F \equiv 0$. ∎

Proposition 9.13. *Let (M^m, g) be a complete connected Riemannian manifold witht* ric ≥ 0 *and let $\gamma: [0, \infty) \to M$ be a geodesic ray,*[10] *parametrized by arclength. Then the* **Busemann function** *defined as*

$$b_\gamma: M \to \mathbb{R}, \quad b_\gamma(q) := \lim_{t \to \infty} (t - \operatorname{dist}(\gamma(t), q))$$

is a subharmonic function.

Note that up to a sign, this Busemann function coincides with the Busemann function defined in Exercise Sheet 9, Exercise 2.

Proof: We fix a point $p \in M$ and for $t \geq$ define $\ell_t := \operatorname{dist}(p, \gamma(t))$. Let $c_t: [0, \ell_t] \to M$ be a shortest curve from p to $\gamma(t)$, parametrized by arclength, and thus a geodesic. For any $t \geq 0$ we have $\dot{c}_t(0) \in S_pM$, and as S_pM is compact, there is a sequence $(t_i)_{i \in \mathbb{N}}$ with $t_i \to \infty$, such that $X := \lim_{i \to \infty} \dot{c}_{t_i}(0)$ exists. Define

$$\gamma_p: [0, \infty) \to M, \quad \gamma_p(s) := \exp_p(sX).$$

[10] Recall that a **geodesic ray parametrized by arclength**, is a curve $\gamma: [0, \infty) \to M$ with $\operatorname{dist}(\gamma(t), \gamma(s)) = |t - s|$ for all $s, t \in [0, \infty)$.

For each $s \in [0,\infty)$ we have

$$\mathrm{dist}\big(\gamma_p(0),\gamma_p(s)\big) = \mathrm{dist}\Big(p, \lim_{i\to\infty} \underbrace{\exp_p\big(s\dot c_{t_i}(0)\big)}_{=c_{t_i}(s)}\Big) = \lim_{i\to\infty}\mathrm{dist}\big(p, c_{t_i}(s)\big) = s.$$

Thus γ_p is a ray, parametrized by arclength. Now we fix $s_0 \in (0,\infty)$ as well. There is a neighborhood $W_p^{s_0}$ of p, such that

$$f\colon W_p^{s_0} \to \mathbb{R}, \quad q \mapsto \mathrm{dist}\big(q,\gamma_p(s_0)\big)$$

is smooth. Otherwise p would be in the cut locus of $\gamma_p(s_0)$, and thus for any $\epsilon > 0$, the curve $\gamma_p|_{[0,s_0+\epsilon]}$ would not be a shortest curve, which is a contradiction. Now define

$$\varphi := \varphi_p^{s_0}\colon W_p^{s_0} \to \mathbb{R}, \quad \varphi(q) := b_\gamma(p) + s_0 - \mathrm{dist}(q,\gamma_p(s_0))$$

which is obviously a smooth function.

We now show that φ is a support function for b_γ in p. At first one checks that $\varphi(p) = b_\gamma(p)$. Further we have for $q \in W_p^{s_0}$:

$$\begin{aligned}
\varphi(q) - b_\gamma(q) &= b_\gamma(p) + s_0 - \mathrm{dist}(q,\gamma_p(s_0)) + \lim_{t\to\infty}(-t + \mathrm{dist}(\gamma(t),q))\\
&= \lim_{t\to\infty}\Big(t - \mathrm{dist}\big(\gamma(t),p\big) + s_0 - \mathrm{dist}\big(q,\gamma_p(s_0)\big) - t + \mathrm{dist}\big(\gamma(t),q\big)\Big)\\
&= \lim_{i\to\infty}\Big(-\mathrm{dist}\big(\gamma(t_i),p\big) + s_0 - \mathrm{dist}\big(q,c_{t_i}(s_0)\big) + \mathrm{dist}\big(\gamma(t_i),q\big)\Big).
\end{aligned}$$

We have for all sufficiently large i

$$\mathrm{dist}(\gamma(t_i),p) - s_0 = \ell_{t_i} - s_0 = \mathrm{dist}\big(c_{t_i}(s_0), c_{t_i}(\ell_{t_i})\big) = \mathrm{dist}\big(c_{t_i}(s_0),\gamma(t_i)\big).$$

Thus

$$\varphi(q) - b_\gamma(q) = \lim_{i\to\infty}\Big(-\mathrm{dist}\big(c_{t_i}(s_0),\gamma(t_i)\big) - \mathrm{dist}\big(q,c_{t_i}(s_0)\big) + \mathrm{dist}\big(\gamma(t_i),q\big)\Big).$$

From the triangle inequality we get $\varphi(q) - b_\gamma(q) \le 0$ for all $q \in W_p^{s_0}$.

It remains to show that for any $\epsilon > 0$ there is an $s_0 > 0$ with $\Delta\varphi_p^{s_0} \le \epsilon$. The function $f\colon W_p^{s_0} \to \mathbb{R}$, defined as above, is a generalized distance function. Consider its level set

$$N_\sigma := \{x \in W_p^{s_0} | f(x) = \sigma\} \quad \sigma \in \mathbb{R}\,.$$

Let II_x be the second fundamental form of $N_{f(x)}$ in x and $H_x := \frac{1}{m-1}\,\mathrm{tr}(\mathrm{II}_x)$ the mean curvature of $N_{f(x)}$ in x. Following (2.5) we have for all $Y, Z \in T_x N_{f(x)}$

$$\mathrm{II}_x(Y,Z) = -(\mathrm{Hess}_x f)(Y,Z)$$

and thus $H_x = \frac{1}{m-1}\Delta f\big|_x$. Let

$$\tilde{c}\colon [0, \mathrm{dist}(\gamma_p(s_0), q)] \to M$$

the shortest curve from $\gamma_p(s_0)$ to q, parametrized by arclength. According to the Riccati inequality from Theorem 2.14 we have

$$\frac{\partial}{\partial\tau} H\big|_{\tilde{c}(\tau)} \geq \frac{1}{m-1}\,\mathrm{ric}\Big(\dot{\tilde{c}}(\tau), \dot{\tilde{c}}(\tau)\Big) + \Big(H\big|_{\tilde{c}(\tau)}\Big)^2 \geq \Big(H\big|_{\tilde{c}(\tau)}\Big)^2$$

and we further have $H\big|_{\tilde{c}(\tau)} \sim -\frac{1}{\tau}$ for $\tau \to 0$. It follows that $H\big|_{\tilde{c}(\tau)} \geq -\frac{1}{\tau}$ for all $\tau \geq 0$, and thus we have for all q in $W_p^{s_0}$

$$\frac{1}{m-1}\Delta f\big|_q = H_q \geq -\frac{1}{\mathrm{dist}\big(\gamma_p(s_0), q\big)} = -\frac{1}{f(q)} \to 0$$

for $s_0 \to \infty$. We thus obtain

$$\Delta\varphi\big|_q = \Delta(\mathrm{const} - f)\big|_q = -\Delta f\big|_q \leq \epsilon(s_0, p),$$

with $\epsilon(s_0, p) \to 0$ for $s_0 \to \infty$. In this argument p is fixed and the above estimate is uniform in $q \in W_p^{s_0}$ for $s_0 \geq s_1$ for some $s_1 > 0$, as long as $\bigcup_{s_0 \geq s_1} W_p^{s_0}$ is uniformly bounded. ■

10 The Cheeger splitting theorem

Theorem 10.1 (Cheeger splitting theorem). *Let (M, g) be a complete connected Riemannian manifold with $\mathrm{ric} \geq 0$ and let $\gamma\colon \mathbb{R} \to M$ be a line*[11]*. Then M is isometric to a Riemannian product $\mathbb{R} \times N$, where (N, g^N) is a complete connected Riemannian manifold of dimension $\dim M - 1$ with $\mathrm{ric}^N \geq 0$.*

Proof: We define

$$\gamma_+\colon [0, \infty) \to M, \quad \gamma_+(t) := \gamma(t),$$

[11] Recall that a **line**, is a curve $\gamma\colon \mathbb{R} \to M$ with $\mathrm{dist}(\gamma(t), \gamma(s)) = |t - s|$ for all $s, t \in \mathbb{R}$.

$$\gamma_-:[0,\infty)\to M,\quad \gamma_-(t):=\gamma(-t)$$

and the Busemann function

$$b_+:=b_{\gamma_+},\quad b_-:=b_{\gamma_-}$$

for the rays γ_+ and γ_- as in Proposition 9.13. Then b_+ and b_- are subharmonic. Furthermore we have for all $q\in M$:

$$\begin{aligned} b_+(q)+b_-(q) &= \lim_{t\to\infty}\Big(t-\operatorname{dist}\big(\gamma(t),q\big)+t-\operatorname{dist}\big(\gamma(-t),q\big)\Big)\\ &= \lim_{t\to\infty}\Big(\operatorname{dist}\big(\gamma(t),\gamma(-t)\big)-\operatorname{dist}\big(\gamma(t),q\big)-\operatorname{dist}\big(\gamma(-t),q\big)\Big)\\ &\le 0 \end{aligned}$$

and $b_+(q)+b_-(q)=0$, if $q\in\gamma(\mathbb{R})$. Thus b_++b_- is a subharmonic function, that attains its maximum on $\gamma(\mathbb{R})$. The maximum principle Theorem 9.10 implies $b_++b_-\equiv 0$. Hence $-b_+=b_-$ is subharmonic as well. With Lemma 9.12 it follows that b_+ is smooth and harmonic. According to Exercise Sheet 9, Exercise 2 b) we have $\|\operatorname{grad} b_+\|\equiv 1$
Fr 19.7. and thus b_+ is a generalized distance function. The level sets of b_+ are denoted as

$$N_s:=\big\{x\in M\mid b_+(x)=s\big\}.$$

Then $\nu:=\operatorname{grad} b_+$ is a unit normal field for N_s. The mean curvature of N_s is $H=\frac{1}{m-1}\Delta b_+=0$. With Corollary 2.3 we see that the integral lines of $\operatorname{grad} b_+$ are geodesics. The Riccati inequality for mean curvature, i. e., Theorem 2.14, tells us

$$\underbrace{\partial_\nu H}_{=0}\ge\frac{1}{m-1}\operatorname{ric}(\nu,\nu)+\underbrace{H^2}_{=0}.$$

Thus we have equality in this inequality and by the equality discussion in this Theorem 2.14 we conclude that for any $x\in M$ we have $S_x=H_x\operatorname{id}_{T_xN_{b_+(x)}}=0$. We have obtained $S\equiv 0$, i. e., all N_s are totally geodesic.

Let $\varphi_t:M\to M$ be the flow of $\operatorname{grad} b_+$, i. e., for all t and all $x\in M$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)=\operatorname{grad} b_+\big|_{\varphi_t(x)}\text{ and }\varphi_0(x)=x\,.$$

The flow exists for all $t\in\mathbb{R}$ and of all of M, as the boundedness of $\operatorname{grad} b_+$ and the completeness of M imply that the flow lines cannot leave compacta in finite time, and the global existence follows from standard ODE results. Define

$$\psi\colon N_0 \times \mathbb{R} \to M, \quad (x,t) \mapsto \varphi_t(x).$$

We now show that ψ is an isometry. For all $x \in N_0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} b_+(\varphi_\tau(x)) = \mathrm{d}b_+\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_\tau(x)\Big) = \mathrm{d}b_+(\operatorname{grad} b_+|_{\varphi_\tau(x)}) = \langle \operatorname{grad} b_+|_{\varphi_\tau(x)}, \operatorname{grad} b_+|_{\varphi_\tau(x)}\rangle = 1$$

and thus

$$b_+\big(\psi(x,t)\big) = b_+\big(\varphi_t(x)\big) = \underbrace{b_+\big(\varphi_0(x)\big)}_{=0} + \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} b_+\big(\varphi_\tau(x)\big)\,\mathrm{d}\tau = t.$$

Define

$$\Phi\colon M \to N_0 \times \mathbb{R}, \quad y \mapsto \big(\varphi_{-b_+(y)}(y), b_+(y)\big).$$

Then Φ is smooth and $\Phi \circ \psi = \mathrm{id}$, $\psi \circ \Phi = \mathrm{id}$. Thus ψ is a diffeomorphism. It remains ot show that

$$\mathrm{d}_{(x,t)}\psi\colon T_{(x,t)}(N_0 \times \mathbb{R}) \to T_{\varphi_t(x)}M$$

is a linear isometry. For all $x \in N_0$ the vector

$$\mathrm{d}_{(x,t)}\psi\Big(\frac{\partial}{\partial t}\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x) = \operatorname{grad} b_+|_{\varphi_t(x)}$$

is orthogonal to $T_{\varphi_t(x)}N_t$ and has length 1. For any $Y \in T_x N_0$ the vector

$$\mathrm{d}_{(x,t)}\psi(Y) \in T_{\varphi_t(x)}N_t$$

is thus orthogonal to $\mathrm{d}_{(x,t)}\psi(\frac{\partial}{\partial t})$. Hence, in order to show that $\mathrm{d}_{(x,t)}\psi$ is a linear isometry, it suffices to show that for any $Y, \widetilde{Y} \in T_{\varphi_t(x)}N_t$ we have

$$\big\langle \mathrm{d}_{(x,t)}\psi(Y), \mathrm{d}_{(x,t)}\psi(\widetilde{Y})\big\rangle = \big\langle Y, \widetilde{Y}\big\rangle. \tag{10.1}$$

We extend Y, $\widetilde{Y}$ to smooth vector fields $Y, \widetilde{Y} \in \Gamma(TM)$, such that $Y|_{N_0} \in \Gamma(TN_0)$, $\widetilde{Y}|_{N_0} \in \Gamma(TN_0)$ and $\nabla_\nu Y \equiv 0$, $\nabla_\nu \widetilde{Y} \equiv 0$. Then the vector fields Y and $\widetilde{Y}$ are tangential to all level sets as

$$\partial_\nu\langle \nu, Y\rangle = \langle \underbrace{\nabla_\nu \nu}_{=0}, Y\rangle + \langle \nu, \underbrace{\nabla_\nu Y}_{=0}\rangle = 0,$$

Where we used Lemma 2.2. We also calculate

$$[\nu, Y] = \nabla_\nu Y - \nabla_Y \nu = S(Y) = 0.$$

We conclude

$$0 = [\nu, Y]\big|_p = (\mathcal{L}_\nu Y)\big|_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}\varphi_{-t}\Big(Y\big|_{\varphi_t(p)}\Big).$$

With $p = \varphi_{t_0}(q)$ and $\widetilde{t} = t + t_0$ it follows that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}\varphi_{-t}\Big(Y\big|_{\varphi_t(\varphi_{t_0}(q))}\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}\varphi_{t_0} \mathrm{d}\varphi_{-(t+t_0)}\Big(Y\big|_{\varphi_{t+t_0}(q)}\Big) \\ &= \mathrm{d}\varphi_{t_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\widetilde{t}=t_0} \mathrm{d}\varphi_{-\widetilde{t}}\Big(Y\big|_{\varphi_{\widetilde{t}}(q)}\Big)\right). \end{aligned}$$

Replacing $\tilde{t}$ by t and q by p we have obtained

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}\varphi_{-t}\Big(Y\big|_{\varphi_t(p)}\Big) = 0$$

for all t and all p. Using $\varphi_0 = \mathrm{id}$ this then implies that

$$\mathrm{d}\varphi_{-t}\Big(Y\big|_{\varphi_t(p)}\Big) = \mathrm{d}\varphi_0\Big(Y\big|_{\varphi_0(p)}\Big) = Y\big|_p$$

for all t and all p and thus

$$Y\big|_{\varphi_t(p)} = \mathrm{d}\varphi_t\Big(Y\big|_p\Big) = \Big(\mathrm{d}\big|_{(p,t)}\psi\Big)(Y)$$

for all $Y \in T_p N_0$. In order to show (10.1) we thus should prove

$$\Big\langle Y\big|_{\varphi_t(p)}, \widetilde{Y}\big|_{\varphi_t(p)}\Big\rangle = \Big\langle Y\big|_p, \widetilde{Y}\big|_p\Big\rangle.$$

This equation follows immediately as we may express $Y\big|_{\varphi_t(p)}$ using parallel transport as follows amd writing $\rho_p(t) := \varphi_t(p)$

$$Y\big|_{\varphi_t(p)} = P_{\rho_p|_{[0,t]}}\Big(Y\big|_p\Big)$$

and then using the fact that parallel transport is an isometry. ∎

Lemma 10.2. *Let (M, g) be a compact, connected Riemannian manifold, $p \in M$.*

Then its fundamental group $\pi_1(M,p)$ is infinite, if and only if the universal covering $\widetilde{M}$ contains a line.

Proof: Let $\widetilde{M}$ be the universal covering of M with diameter $\operatorname{diam}\widetilde{M}$.
$\operatorname{diam}\widetilde{M} < \infty \iff \widetilde{M}$ is compact
$\iff \widetilde{M} \to M$ is a finite covering
$\iff \pi_1(M)$ is finite.
"$\Leftarrow$": If there exists a line on $\widetilde{M}$, then clearly, $\operatorname{diam}\widetilde{M} = \infty$.
"$\Rightarrow$": Let $\widetilde{\operatorname{dist}}$ be the distance funtion on $\widetilde{M}$ with respect to the pullback metric $\tilde{g}$. If $\operatorname{diam}\widetilde{M} = \infty$, then there are points $p_i, q_i \in \widetilde{M}$ with $\ell_i := \widetilde{\operatorname{dist}}(p_i, q_i)/2 \to \infty$. We choose a shortest curve $\sigma_i\colon [-\ell_i, \ell_i] \to \widetilde{M}$, parametrized by arclength with $\sigma_i(-\ell_i) = p_i$ and $\sigma_i(\ell_i) = q_i$, parametrized by arclength. Obviously we have for $k \geq i$:

$$\widetilde{\operatorname{dist}}\big(\sigma_k(-\ell_i), \sigma_k(\ell_i)\big) = 2\ell_i\,.$$

As the deck transformation group acts cocompactly on $\widetilde{M}$ and on the unit tangent bundle of $\widetilde{M}$, we can assume – without loss of generality – that there is a compact set K containing $\dot{\sigma}_i(0)$ for all $i \in \mathbb{N}$. This allows us to pass to a subsequence such that $v_\infty := \lim_{k\to\infty} \dot{\sigma}_k(0)$ exists. We define the geodesic $\sigma_\infty\colon \mathbb{R} \to \widetilde{M},\ t \mapsto \exp(tv_\infty)$. We calculate

$$\begin{aligned}
\widetilde{\operatorname{dist}}\big(\sigma_\infty(-\ell_i), \sigma_\infty(\ell_i)\big) &= \widetilde{\operatorname{dist}}\big(\exp_p(-\ell_i v_\infty), \exp_p(\ell_i v_\infty)\big)\\
&= \widetilde{\operatorname{dist}}\Big(\exp_p\big(-\ell_i \lim_{k\to\infty} \dot{\sigma}_k(0)\big), \exp_p\big(\ell_i \lim_{k\to\infty} \dot{\sigma}_k(0)\big)\Big)\\
&= \widetilde{\operatorname{dist}}\Big(\lim_{k\to\infty} \exp_p\big(-\ell_i\dot{\sigma}_k(0)\big), \lim_{k\to\infty} \exp_p\big(\ell_i\dot{\sigma}_k(0)\big)\Big)\\
&= \lim_{k\to\infty} \widetilde{\operatorname{dist}}\big(\sigma_k(-\ell_i), \sigma_k(\ell_i)\big) = 2\ell_i\,.
\end{aligned}$$

Thus σ_∞ is a line. ∎

As a consequence we get

Corollary 10.3. *Let (M,g) be a connected compact Riemannian manifold with* $\operatorname{ric} \geq 0$ *and infinite fundamental group, then the universal covering is isometric to a Riemannian product $N \times \mathbb{R}$.*

This can be strengthened to the following:

Theorem 10.4. *Let (M,g) be a connected compact Riemannian manifold with* $\operatorname{ric} \geq 0$ *and infinite fundamental group, then the universal covering is isometric to a*

Riemannian product $Q_k \times \mathbb{R}^k$*, where* Q_k *is a compact simply-connected Riemannian manifold of dimension* $\dim M - k$ *with* $\mathrm{ric} \geq 0$*.*

Proof: As a first step we apply the corollary and obtain $\widetilde{M} = Q_1 \times \mathbb{R}$. Now the theorem follows from the following claim:

> If $\widetilde{M} = Q_j \times \mathbb{R}^j$ and if Q_j is non-compact, then $Q_j = Q_{j+1} \times \mathbb{R}$ for some Riemannian manifold Q_{j+1}.

In view of the Cheeger splitting Theorem10.1 we have to show that Q_j contains a line. Let $\widetilde{\mathrm{dist}}_j$ be $\widetilde{\mathrm{dist}}|_{Q_j \times Q_j}$ which coincides with the distance function of the Riemannian manifold Q_j. If $R := \mathrm{diam}\, Q_j < \infty$, then $Q_j = \exp_p\big(\overline{B}_R(0)\big)$ for some $p \in Q_j$ and for the ball $\overline{B}_R(0) \subset T_pQ_j$ and thus Q_j is compact. If $\mathrm{diam}\, Q_j = \infty$, then we may as in the proof of Lemma 10.2. We obtain a family $\dot{\sigma}_i(0) \in SQ_j \times \mathbb{R}^j$, where again SQ_j is the unit tangent bundle of Q_j. Now we do *not* know whether the deck transformation groups acts cocompactly on Q_j, but it acts cocompactly and isometrically on $Q_j \times \mathbb{R}^j$. Thus after passing to a subsequence and possibly applying some deck transformationswe can assume $\dot{\sigma}_i(0) \to (v_\infty, q_\infty)$ for some $v_\infty \in SQ_j$ and $q_\infty \in \mathbb{R}^j$. The geodesic $\sigma_\infty(t) := \exp(tv_\infty)$ is a line in Q_j. ■

Literature in this part.

- Eschenburg-Heintze [13], [14],
- Karcher MAA-Stud [18]
- Ballmann [9]
- Bär Skript Differentialgeometrie 2006/2010 [5]
- Sakai [26]
- Peter Li: “Geometric analysis” [22] und “Lecture Notes on Geometric Analysis” [21]

A Mathematical Appendices

A.1 Supplements from the theory of smooth manifolds

A.1.1 The Koszul formula

Let (M, g) be a semi-Riemannian manifold. We write $\langle X, Y \rangle$ for $g(X, Y)$. In the lecture "Differential Geometry I" we have shown that there is a unique connection ∇ on TM, called the **Levi–Civita connection** such that it is metric and torsionfree.

We give here a version of the Koszul identity that differs slightly from the one given in that lecture. It gives a formula for the Levi–Civita connection.

Lemma A.1.1 (Koszul formula). *For $X, Y, Z \in \mathfrak{X}(M)$ we have*

$$\begin{aligned} & 2 \langle \nabla_X Y, Z \rangle \\ = & \partial_X \langle Y, Z \rangle + \partial_Y \langle X, Z \rangle - \partial_Z \langle X, Y \rangle \\ & + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \end{aligned}$$

In order to prove this lemma, one verifies that the right hand side defines a connection that is torsionfree and compatible with the metric.

The proof of this Lemma is given, e.g., in [12, Eq. (9) in Proof of Theorem 3.6]. It is also worked out in [1, Def. 2.7.2] for submanifolds of $\mathbb{R}^n$, but the same proofs also works for arbitrary semi-Riemannian manifolds.

A.1.2 A lemma on surjective submersions

Lemma A.1.2. *Let $f : X \to Y$ be a surjective submersion from the C^∞-manifold X to the C^∞-manifold Y, and let Z be a further C^∞-manifold. Let $h : Y \to Z$ be a* *map. Then h is smooth if and only if $h \circ f$ is smooth.*

Proof: It is obvious that $h \circ f$ is smooth if h is smooth, as every submersion is by definition a smooth map.

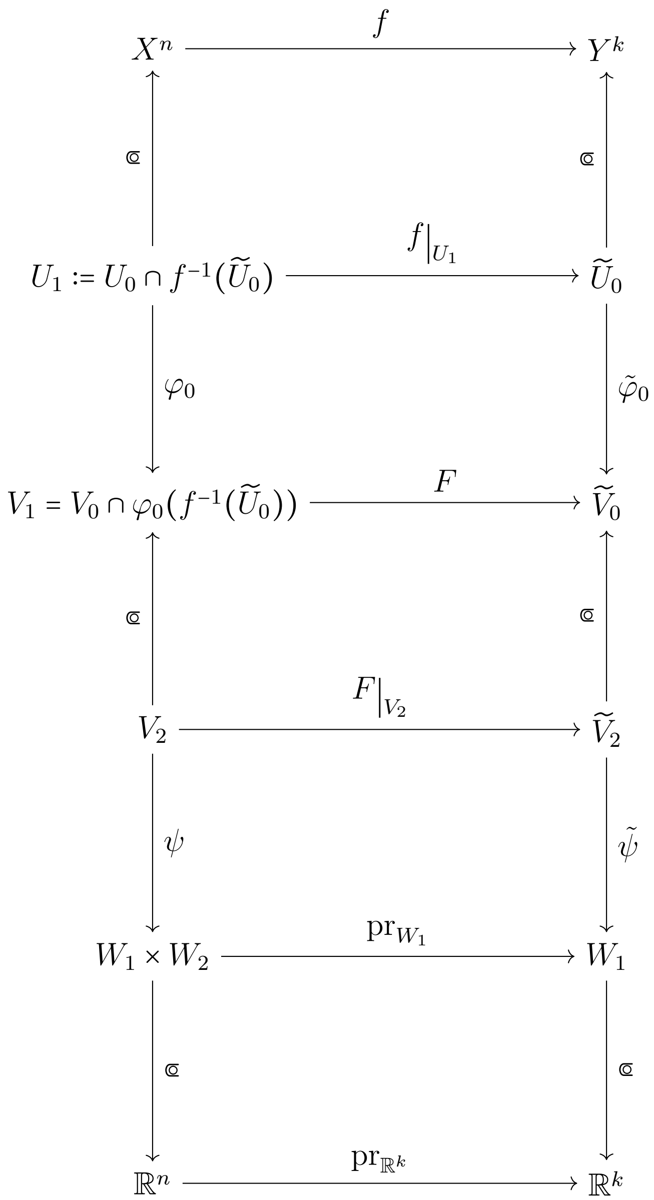
Now assume that $h \circ f$ is smooth. For a given $y \in Y$ we want to show that h is smooth on a neighborhood of y. As y may be arbitrarily chosen, this then implies that h is smooth.

Let $n \coloneqq \dim X$ and $k \coloneqq \dim Y$.

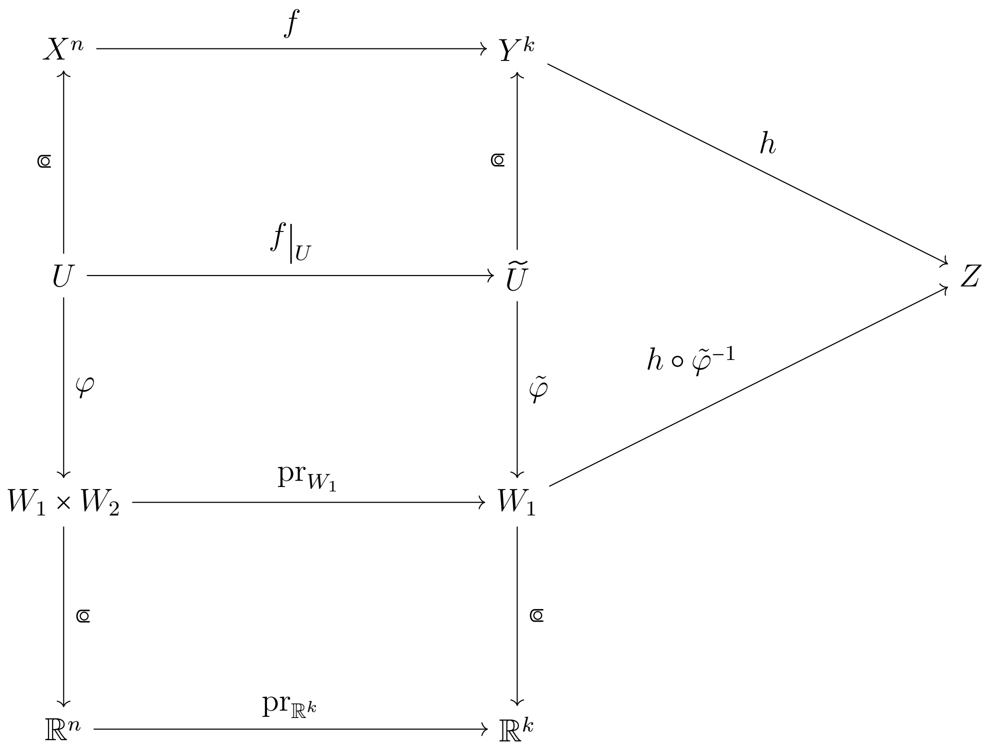
At first we choose a preimage $x \in X$ of y, i. e., $f(x) = y$. (Here we use the surjectivity of f.) We choose a chart $\tilde{\varphi}_0 : \widetilde{U}_0 \to \widetilde{V}_0$ of Y with $y \in \widetilde{U}_0$, then we choose a chart $\varphi_0 : U_0 \to V_0$ of X with $x \in U_0$

We obtain a smooth map $F\colon V_1 \to \widetilde{V}_0$, $F \coloneqq \tilde{\varphi}_0 \circ f \circ \varphi_0^{-1}$, $V_1 \coloneqq V_0 \cap \varphi_0(f^{-1}(\widetilde{U}_0))$. As $df|_x : T_xX \to T_yY$ is surjective, we see that $d(\tilde{\varphi}_0 \circ \varphi_o^{-1})\big|_{\varphi_0(x)}$ is surjective. The implicit function theorem thus says that there is a small neighborhood V_2 of $\varphi_0(x)$ in V_1, a diffeomorphism $\psi\colon V_2 \to W_1 \times W_2$, W_1 open in $\mathbb{R}^k$, W_2 open in $\mathbb{R}^{n-k}$, that there is an open neighborhood $\widetilde{V}_2$ of $\tilde{\varphi}_0(y)$ in $\widetilde{V}_0$ and a diffeomorphism $\tilde{\psi} : \widetilde{V}_2 \to W_1$, such that $\tilde{\psi} \circ F \circ \psi^{-1} : W_1 \times W_2 \to W_1$ is the projection to W_1, i. e., $\tilde{\psi} \circ F \circ \psi^{-1}(x_1, x_2) = x_1$ where $x_i \in W_i$.

In the following diagram all symbols $\Subset$ denote open subsets.



We set $U := \varphi_0^{-1}(V_2)$, $\widetilde{U} := \tilde{\varphi}_0(\widetilde{V}_2)$, $\varphi := \psi \circ \varphi_0 : U \to W_1 \times W_2$, $\tilde{\varphi} := \tilde{\psi} \circ \tilde{\varphi}_0 : \widetilde{U} \to W_1$. Then $\varphi : U \to W_1 \times W_2$ and $\tilde{\varphi} : \widetilde{U} \to W_1$ are charts with $x \in U$ and $y \in \widetilde{U}$. Furthermore $\widetilde{\varphi} \circ f \circ \varphi^{-1} : W_1 \times W_2 \to W_1$ is the projection pr_{W_1} to W_1.



Now as $h \circ f$ is smooth, $h \circ f \circ \varphi^{-1} : W^1 \times W_2 \to Z$ is smooth as well. As the map

$$h \circ f \circ \varphi^{-1} = (h \circ \tilde{\varphi}^{-1}) \circ \mathrm{pr}_{W_1} : W^1 \times W_2 \to Z$$

is smooth, it is in particular smooth in the W_1 direction (for fixed element in W_2), but this is just the map $h \circ \tilde{\varphi}^{-1}) \to W_1$, which is thus smooth. This implies that $h|_{\widetilde{U}}$ is smooth. ∎

B Indices

Keyword Index

J

K

L

M

N

O

P

U

V

W

Nomenclature

English-German translations

X is second countable	X erfüllt das 2. Abzählbarkeitsaxiom	v
f-related	f-verwandt	6
abandon	aufgeben	68
absolute time	absolute Zeit	68
adjoint representation	adjungierte Darstellung	10
bi-invariant	bi-invariant	5
Busemann function	Busemann-Funktion	147
charge	Ladung	68
Cheeger splitting theorem	Cheegerscher Spaltungssatz	149
co-differential	Kodifferential	142
conjugation locus	Konjugationsort	122
covering or cover	Überdeckung	v
covering or cover	Überlagerung	v
cut locus	Schnitt-Ort	88
cutl locus	Schnittort	122
derivation	Derivation	10
distance function	Abstandfunktion	85
edge	Kante (eines Graphs)	132
effective	effektiv	27
event	Ereignis	67
exponential growth	exponentielles Wachstum	135
faithful	treu	27
free operation	freie Operation	27
generalized cosine function	verallgemeinerte Cosinusfunktion	104
generalized cotangent function	verallgemeinerte Cotangensfunktion	104
generalized distance function	verallgemeinerte Abstandsfunktion	89
generalized orthogonal group	verallgemeinerte orthogonale Gruppe	62
generalized orthonormal basis	verallgemeinerte orthonormale Basis	62
generalized sine function	verallgemeinerte Sinusfunktion	104
generalized tangent function	verallgemeinerte Tangensfunktion	104
geodesic	Geodätische	v
harmonic	harmonisch	144

German-English translations

Theorem list

P

R

S

Copyright Notes

Bibliography

[1] Ammann, B.: *Analysis IV*, 2020. Lecture Notes, avalailable online, https://ammann.app.uni-regensburg.de/lehre/2020s_analysisIV/AnalysisIV.pdf. ↑7, A

[2] ———: *Differential Geometry II*, 2021. Partial Lecture Notes, avalailable online, https://ammann.app.uni-regensburg.de/lehre/2021s_diffgeo2/Diffgeo2.pdf. ↑1, 1

[3] ———: *Differential Geometry III*, 2021/22. Partial Lecture Notes, avalailable online, https://ammann.app.uni-regensburg.de/lehre/2021w_diffgeo3/Diffgeo3.pdf. ↑

[4] Baues, O.: *Lie-Gruppen und Lie-Algebren*, 2006. Lecture Notes, avalailable online, https://www.math.kit.edu/iag2/~baues/media/lie.pdf. ↑1

[5] Bär, C.: *Differentialgeometrie*, 2006/2010. Lecture Notes, avalailable online, https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-DiffGeoErw.pdf. ↑7, 7, 8, 10

[6] ———: *Differential Geometry*, 2023. Lecture Notes, avalailable online, https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/DiffGeo.pdf. ↑

[7] ———: *Lorentzian Geometry*, 2004/2020. Lecture Notes, avalailable online, https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Veranstaltungen/WS0405-SS08/LorentzianGeometryEnglish13Jan2020.pdf. ↑II, 3.1, 3.4

[8] ———: *General Relativity*, 2023. Lecture Notes, avalailable online, https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/GR.pdf. ↑II, 3.4

[9] Ballmann, W.: *Riccati equation and volume estimates*, 2016. ↑10

[10] Besse, A. L.: *Einstein manifolds*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10. Springer-Verlag, Berlin, 1987. ↑6, 23

[11] Christodoulou, D.; Klainerman, S.: *The global nonlinear stability of the Minkowski space*. Princeton Mathematical Series, vol. 41. Princeton University Press, Princeton, NJ, 1993. ↑3.1

[12] do Carmo, M. P.: *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, Portuguese edition, 1992. ↑4, A

[13] Eschenburg, J.-H.; Heintze, E.: Comparison theory for Riccati equations. *Manuscripta Math.* **68** (1990), 209–214. ? MR1063226 ↑10

[14] Eschenburg, J.; Heintze, E.: An elementary proof of the Cheeger-Gromoll splitting theorem. *Ann. Global Anal. Geom.* **2** (1984), 141–151. ? MR0777905 ↑10

[15] Hall, B.: *Lie groups, Lie algebras, and representations.* Graduate Texts in Mathematics, vol. 222. Springer, Cham, 2nd ed., 2015. An elementary introduction. ↑1, 1.7, 1.8

[16] Hawking, S. W.; Ellis, G. F. R.: *The large scale structure of space-time.* Cambridge Monographs on Mathematical Physics, vol. No. 1. Cambridge University Press, London-New York, 1973. ↑3.4

[17] Humphreys, J. E.: *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics, vol. 9. Springer-Verlag, New York-Berlin, 1978. Second printing, revised. ↑1

[18] Karcher, H.: Riemannian comparison constructions., Global differential geometry, MAA Stud. Math., vol. 27, Math. Assoc. America, Washington, DC, 170–222, 1989. ↑10

[19] Lee, D. A.: *Geometric relativity.* Graduate Studies in Mathematics, vol. 201. American Mathematical Society, Providence, RI, 2019. ↑3.4

[20] Lee, J. M.: *Introduction to smooth manifolds.* Graduate Texts in Mathematics, vol. 218. Springer, New York, 2nd ed., 2013. ↑1, 19

[21] Li, P.: *lecture Notes on Geometric Analysis*, 1992/96. ↑10

[22] ______: *Geometric analysis.* Cambridge Studies in Advanced Mathematics, vol. 134. Cambridge University Press, Cambridge, 2012. ↑10

[23] O'Neill, B.: *Semi-Riemannian geometry.* Pure and Applied Mathematics, vol. 103. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity. ↑6, II, 11, 3.3, 3.4

[24] Milnor, J.: A note on curvature and fundamental group. *J. Differential Geometry* **2** (1968), 1–7. ? MR0232311 ↑8

[25] O'Neill, B.: The fundamental equations of a submersion. *Michigan Math. J.* **13** (1966), 459–469. ↑6, 23

[26] Sakai, T.: *Riemannian geometry.* Translations of Mathematical Monographs, vol. 149. American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author. ↑(document), 1, 1.5, 6, 6, 10

[27] Wald, R. M.: *General relativity.* University of Chicago Press, Chicago, IL, 1984. ↑3.4

[28] Warner, F. W.: *Foundations of differentiable manifolds and Lie groups.* Graduate Texts in Mathematics, vol. 94. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition. ↑1, 1.), 2.), 1.4, 4, 1.8