Differential Geometry II

Lecture Notes



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Summer term 2024



University of Regensburg

Public Version

Version of May 11, 2024

Preface

These are lecture notes for the lecture "Differential Geometry II" held in Regensburg in the summer term 2024. We assume that the readers of these notes and the audience of the lecture are already familiar with basic notions and results in differential and (semi-)Riemannian gemetry, as taught typically in a one-semester lecture, this includes e.g., the theorems by Hopf–Rinow, Bonnet–Myers and Cartan–Hadamard.

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https://ammann.app.uni-regensburg.de/lehre/2024s_diffgeo2/ Differential_Geometry_II.pdf

I Lie groups and quotients

Tue. 16.4.

The goal of this section is to treat Lie groups, which are defined as manifolds with a compatible group structure. Important examples are O(n), SO(n), U(n), $GL(n, \mathbb{R})$, ...

Lie groups provide many more examples of Riemannian (and more generally semi-Riemannian) manifolds.

1 Lie groups and Lie algebras

Literature for this section: [8], [12], [2], [6], [5]

1.1 Lie groups and their homomorphisms

Definition 1.1. A Lie group consists of a C^{∞} -manifold G together with a smooth map $\mu: G \times G \to G$, $(\sigma, \tau) \mapsto \mu(\sigma, \tau) = \sigma \tau = \sigma \cdot \tau$, called multiplication, such that

- (i) (G,μ) is a group
- (ii) $G \times G \xrightarrow{\tilde{\mu}} G$, $(\sigma, \tau) \mapsto \sigma^{-1}\tau =: \tilde{\mu}(\sigma, \tau)$ is smooth.

As a consequence of (ii) we see that the following maps are smooth

$$\begin{split} \ell_{\sigma} &: G \to G, \quad \tau \mapsto \sigma \tau \quad (\text{left multiplication or left translation} \\ r_{\sigma} &: G \to G, \quad \tau \mapsto \tau \sigma \quad (\text{right multiplication or right translation}) \\ &\text{inv:} G \to G, \quad \tau \mapsto \tau^{-1} \quad (\text{inversion}) \\ &\mu : G \times G \xrightarrow{\mu} G, \quad (\sigma, \tau) \mapsto \sigma \tau \quad (\text{multiplication}) \end{split}$$

Note also that Diff. geom. I, Exercise Sheet 3, Exercise 4 tells us that one can replace (ii) by

(ii') $\mu: G \times G \xrightarrow{\mu} G$, $(\sigma, \tau) \mapsto \sigma \tau$ is smooth

We write 1 for the neutral element of G. Then T_1G is called the Lie algebra of G. It is a vector space that comes with some additional structure discussed below, a "Lie bracket".

Examples 1.2.

- 1.) A finite-dimensional real vector space is a Lie group, if μ is the addition.
- 2.) $\mathbb{C}^*, S^1 \subset \mathbb{C}^*, \mathbb{R}^*$ are Lie groups, if μ is the multiplication.
- 3.) $\operatorname{GL}(n,\mathbb{R})$ is a Lie group, where μ is matrix multiplication. We view $\operatorname{GL}(n,\mathbb{R})$ as an open subset and thus as an n^2 -dimensional submanifold of $\mathbb{R}^{n \times n}$.
- 4.) SL $(n, \mathbb{R}) \coloneqq \{A \in \mathbb{R}^{n \times n} \mid \det A = 1\}.$

In order to show that $SL(n, \mathbb{R})$ is a submanifold of $GL(n, \mathbb{R})$ we show that the determinant det: $GL(n, \mathbb{R}) \to \mathbb{R}^*$ is a submersion, i.e. $d_A \det: T_A GL(n, \mathbb{R}) \to$ $T_{\det A}\mathbb{R}^* \cong \mathbb{R}$ is surjective for all $A \in GL(n, \mathbb{R})$. It follows from this, that $\det^{-1}(t)$ is a submanifold for any $t \in \mathbb{R}^*$. For t = 1, this shows that $SL(n, \mathbb{R}) = \det^{-1}(1)$ is a submanifold.

(a) Let
$$B = (b_{ij})_{ij} \in \operatorname{GL}(n, \mathbb{R}), C(t) := \mathbb{1} + tB = (c_{ij}(t))_{ij} = (\delta_{ij} + tb_{ij})_{ij}.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(\mathbb{1}+tB) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det C(t)$$

$$= \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}}_{t=0} \left(c_{1\sigma(1)}(t)\cdots c_{n\sigma(n)}(t)\right)$$

$$\stackrel{=0 \text{ for } \sigma \neq \mathrm{id}}{\underbrace{=} \frac{\mathrm{d}}{\mathrm{d}t}}\Big|_{t=0} \left(\left(1+tb_{1\sigma(1)}\right)\cdots\left(1+tb_{n\sigma(n)}\right)\right)$$

$$\stackrel{(+)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(1+t(b_{1\sigma(1)}+\cdots+tb_{n\sigma(n)})+P_{\geq 2}(t)\right)$$

$$= b_{1\sigma(1)}+\cdots+tb_{n\sigma(n)}$$

$$= \operatorname{tr} B$$

Here we used at (*) and above that for $\sigma \neq \text{id}$ there are $i \neq j$ with $c_{i\sigma(i)}(0) = c_{j\sigma(j)}(0) = 0$, and after (+) we write $P_{\geq 2}(t)$ for a polyomial in t without constant and without a linear term, i.e., one only with monomials of degree ≥ 2 .

(b) For $A \in GL(n, \mathbb{R})$ we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(A+tB) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(A \cdot (\mathbb{1}+tA^{-1}B))$$
$$= (\det A) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(\mathbb{1}+tA^{-1}B)$$
$$= (\det A) \cdot \operatorname{tr}(A^{-1}B)$$

We conclude

$$d_A \det(B) = \frac{d}{dt} \bigg|_{t=0} \det(A + tB)$$
$$= (\det A) \cdot \operatorname{tr}(A^{-1}B).$$

The linear map $d_A: \mathbb{R}^{n \times n} \to \mathbb{R}$ is surjective as

$$d_A(A) = (\det A) \operatorname{tr} \mathbb{1} = n \cdot \det A \neq 0.$$

Now, we now that $\mathrm{SL}(n,\mathbb{R})$ is a submanifold. Its multiplication is the restriction of the multiplication in $\mathrm{GL}(n,\mathbb{R})$, thus multiplication is smooth as a map $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}: \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$. The image of $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}$ is a subset of the submanifold $\mathrm{SL}(n,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{R})$, and this implies the smoothness of $\mu|_{\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})}: \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$.

Further we have

$$\mathbb{T}_{1} \operatorname{SL}(m.\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \operatorname{tr} A = 0 \}.$$

- 5.) The groups SO(n), O(n), U(n) and SU(n) are Lie groups, see Exercise Sheet 1, Exercise 2
- 6.) If G and H are Lie groups, then $G \times H$ with the product manifold structure and the product group structure

$$(G \times H) \times (G \times H) \to G \times H$$
$$((\sigma, \tau), (\tilde{\sigma}, \tilde{\tau})) \mapsto (\sigma \tilde{\sigma}, \tau \tilde{\tau})$$

is again a Lie group.

7.) Let Γ be a discrete subgroup of \mathbb{R}^n , e.g., $\Gamma = \mathbb{Z}^n$ or another lattice¹ or another discrete subgroup. If we equip \mathbb{R}^n/Γ with the usual addition of equivalence classes, called μ , then $(\mathbb{R}^n/\Gamma, \mu)$ is a Lie group.

Definition 1.3. A homomorphism of Lie groups or a Lie group homomorphism is a smooth map $f: G \to H$, for G and H Lie grous, that is also a group homomorphism. The map f is a Lie group isomorphism if it is additionally a diffeomorphism, it is a Lie group endomorphism if additionally G = H, and it is a Lie group automorphism if G = H and if f is a diffeomorphism. We write Hom(G, H), Iso(G, H), End(G), Aut(G) for the sets/monoid/groups of such homorphisms.

Examples 1.4.

- 1.) The inclusions $SO(n) \hookrightarrow O(n)$, $U(n) \hookrightarrow O(2n)$, etc. are Lie group homomorphisms
- 2.) det_K GL(n, \mathbb{K}) $\rightarrow \mathbb{K}_{\neq 0}$ is a Lie group homomorphism for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.
- 3.) For any $\sigma \in G$, conjugation by σ

$$C_{\sigma}: G \longrightarrow G$$
$$\tau \longmapsto \sigma \tau \sigma^{-1}$$

is a Lie group automorphism, and $C_{\bullet}: G \to \operatorname{Aut}(G), g \mapsto C_g$ is a group homomorphism. We obviously have

$$C_{\sigma} = \ell_{\sigma} \circ r_{\sigma^{-1}} = r_{\sigma^{-1}} \circ \ell_{\sigma} . \tag{1.1}$$

Remarks 1.5.

1.) If G is a Lie group, one might be tempted to define a Lie subgroup as a subgroup H of G such that H is a submanifold as well. However, this is not what one usually does. One says that $H \subset G$ is a Lie subgroup, if there is a Lie group homomorphism $f : H' \to G$, that is injective and an immersion, such that H = image(f). For example consider $G = \mathbb{R}^2/\mathbb{Z}^2$ and $f(t) = [t, \alpha t]$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $f : \mathbb{R} \to G$ is an injective immersion and a Lie group

¹A lattice in \mathbb{R}^n is by definition a discrete subgroup Γ of \mathbb{R}^n , isomorphic to \mathbb{Z}^n . It follows that \mathbb{R}^n/Γ is a compact manifold (without boundary), and that there is an $A \in \mathrm{GL}(n,\mathbb{R})$ with $\Gamma = A \cdot \mathbb{Z}^n$.

homomorphism, but H := image(f) is not a submanifold in the usual sense: a submanifold is always a locally closed subset, but H is not a locally closed subset of G. This leads in books on Lie group, as e.g., in [12, Definition 1.27 (b)] to a slightly generalized definition of a submanifold, however we do not want to elaborate too much on this.

2.) The closed subgroup theorem, see [12, Theorem 3.42], states: Let G be a Lie group, and let H be a subgroup of G (in the sense of group theory) that is closed as a subset, then H is a submanifold of G. It follow any closed subgroup H of G is a Lie group (with induced differentiable structure and induced group structure). Although this result is rather simple to state, the proof is a bit involved. Thus we will not prove it here.

1.2 Lie algebras and their homomorphisms

Let us recall the following exercise from last semester:

Exercise 1.6 (Diff. geom. I, Exercise Sheet 7, Exercise 2). Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. \tilde{X}, \tilde{Y}) be (smooth) vector fields on M (resp. N). We say that X is F-related to \tilde{X} if $dF \circ X = \tilde{X} \circ F$ holds on M.

Show that, if X is F-related to \tilde{X} and Y is F-related to \tilde{Y} , then [X,Y] is F-related to $[\tilde{X}, \tilde{Y}]$.

Definition 1.7. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if for all $\sigma \in G$ we have $d\ell_{\sigma}(X) = X \circ \ell_{\sigma}$, i. e., if the diagram



commutes. Similarly X is called **right-invariant** if for all $\sigma \in G$ we have $dr_{\sigma}(X) = X \circ r_{\sigma}$. If X is left- and right-invariant, we say X is **bi-invariant**.

Using the language of Exercise 1.6, we see that a vector field $X \in \mathfrak{X}(G)$ is leftinvariant (right-invariant, resp.), if, and only if, it is ℓ_{σ} -related (r_{σ} -related, resp.) to itself for any $\sigma \in G$.

Remarks 1.8.

1.) For any $X_0 \in T_{\mathbb{1}}G$ there is a unique left-invariant vector field $X \in \mathfrak{X}(G)$ with $X|_{\mathbb{1}} = X_0$. The uniqueness follows from the calculation

$$X|_{\sigma} = X \circ \ell_{\sigma}(1) = (d\ell_{\sigma} \circ X)(1) = d\ell_{\sigma}(X|_{1}) = d\ell_{\sigma}(X_{0}).$$
(1.2)

On the other hand if we use (1.2) to define X, i.e., if we set $X|_{\sigma} \coloneqq d\ell_{\sigma}(X_0)$, then this vector field is the composition

$$G \xrightarrow{(\mathrm{id}, X_0)} G \times TG \longrightarrow TG$$
$$\sigma \longmapsto (\sigma, X_0) \longmapsto \mathrm{d}\ell_{\sigma}(X_0)$$

which is obviously smooth in σ . In order to show that the vector field X thus obtained is left-invariant we calculate for any fixed $\tau \in G$

$$X \circ \ell_{\tau}(\sigma) = X|_{\tau\sigma} \stackrel{\text{(def)}}{=} d\ell_{\tau\sigma}(X_0) \stackrel{(*)}{=} d\ell_{\tau}(d\ell_{\sigma}(X_0)) \stackrel{\text{(def)}}{=} d\ell_{\tau}(X|_{\sigma})$$

where we used the chain rule $d(f \circ g) = (df) \circ (dg)$ at (*), and thus we have $X \circ \ell_{\tau} = d\ell_{\tau} \circ X$ for all $\tau \in G$.

- 2.) The analogous statement holds as well if we replace left-invariance by rightinvariance.
- 3.) With Exercise 1.6 we see: if $X, Y \in \mathfrak{X}(G)$ are left-invariant (right-invariant, resp.) vector fields, then [X, Y] is also left-invariant (right-invariant, resp.)

Definition 1.9 (Lie bracket on the Lie algebra). Let G be a Lie group with Lie algebra $T_{\mathbb{1}}G$. The vectors $X_0, Y_0 \in T_{\mathbb{1}}G$ are extended to left-invariant vector fields X and Y. We define

$$[X_0, Y_0] \coloneqq [X, Y]|_{\mathbf{1}}$$

This defines a bilinear map $[\bullet, \bullet]$: $T_{\mathbb{1}}G \times T_{\mathbb{1}}G \to T_{\mathbb{1}}G$, called the Lie bracket on the Lie algebra $T_{\mathbb{1}}G$ of G.

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The pair $(T_{\mathbb{1}}G, [\bullet, \bullet])$ satisfies the defining properties of a Lie algebra over \mathbb{R} , which are defined as follows:

Definition 1.10 (Abstract Lie algebra). Let K be a field and \mathfrak{g} a K vector space. A bilinear map $[\bullet, \bullet]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a Lie bracket on \mathfrak{g} if it satisfied

- (i) Alternation: for all $x \in \mathfrak{g}$ we have [x, x] = 0
- (ii) Jacobi identity: for all $x, y, z \in \mathfrak{g}$ we have

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The pair $(\mathfrak{g}, [\bullet, \bullet])$ is then called a Lie algebra (over K).

If the characteristic of K is not 2 – and the field $K = \mathbb{R}$ we are interested in the case that K is of characteristic 0 –, then condition (i) is equivalent to

(i') Antisymmetry: for all $x, y \in \mathfrak{g}$ we have [x, y] = -[y, x].

(In characteristic 2 (i') still implies (i), but the converse is no longer true.)

A Lie subalgebra of \mathfrak{g} is a linear subspace of \mathfrak{g} that is closed under the Liebracket, i. e., then it is itself a Lie algebra.

It is obvious that the Lie bracket on $T_{1}G$ defined in Definition 1.9 satisfies (i') (or equivalently (i)). The Jacobi identity follows immediately in this situation from Exercise 1.6.

Usually for a Lie group the associated Lie algebra, viewed as a vector space with Lie bracket, is denoted by the the associated small fraktur (= gothic) letters, e.g.,

Lie group	G	H	$\operatorname{GL}(n,\mathbb{R})$	O(n)	$\mathrm{SO}(n)$	$\operatorname{GL}(n,\mathbb{C})$	U(n)
Lie algebra	g	h	$\mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{o}(n)$	$\mathfrak{so}(n)$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{u}(n)$

We also will often write Lie(G) for the Lie algebra of G, e.g., $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(H)$, etc.

Examples 1.11.

- 1.) If we consider $G := \mathbb{R}^n$ as a Lie group with $\mu(x, y) = x + y$, then the left-invariant vector fields are the constant ones. As the Lie bracket of constant vector fields vanishes, the Lie bracket on the Lie algebra is the zero map $0: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Thus the Lie algebra is $(\mathbb{R}^n, 0)$.
- 2.) Let V be a finite-dimensional real vector space. We denote the vector space automorphisms of V by GL(V). By choosing a basis of V, and identify $V \cong \mathbb{R}^n$,

 $\operatorname{GL}(V) \cong \operatorname{GL}(n, \mathbb{R})$ we get a Lie group structure on $\operatorname{GL}(V)$, independent of the choice of basis above. Let us write $\operatorname{End}_{\operatorname{lin}}(V)$ for the vector space endomorphisms of V. We have $\operatorname{GL}(V) = \operatorname{det}^{-1}(\mathbb{R} \setminus \{0\})$ for $\operatorname{det}: \operatorname{End}_{\operatorname{lin}}(V) \to \mathbb{R}$, thus $\operatorname{GL}(V)$ is open in $\operatorname{End}_{\operatorname{lin}}(V)$. We obtain $\mathfrak{gl}(V) \coloneqq T_{\operatorname{id}}\operatorname{GL}(V) \cong \operatorname{End}_{\operatorname{lin}}(V)$.

The left-invariant extension of $X_0 \in T_{id} \operatorname{GL}(V) \cong \operatorname{End}_{\operatorname{lin}}(V)$ is $X|_A \coloneqq A \mapsto A \circ X_0 \in T_A \operatorname{GL}(V) \cong \operatorname{End}_{\operatorname{lin}}(V), X \in \mathfrak{X}(\operatorname{GL}(V))$. We proceed similarly for $Y_0 \in T_{id} \operatorname{GL}(V)$ and $Y \in \mathfrak{X}(\operatorname{GL}(V))$. Then

$$\partial_X Y|_A = A \circ \partial_{X_0}|_A (B \mapsto B \circ Y_0) = A \circ X_0 \circ Y_0$$

$$\partial_Y X|_A = A \circ \partial_{Y_0}|_A (B \mapsto B \circ X_0) = A \circ Y_0 \circ X_0$$

$$[X, Y]|_A = \partial_X Y|_A - \partial_Y X|_A = A \circ (X_0 \circ Y_0 - Y_0 \circ X_0)$$

$$[X_0, Y_0] = [X, Y]|_{id} = X_0 \circ Y_0 - Y_0 \circ X_0.$$

Thus the Lie algebra structure on $T_{id} \operatorname{GL}(V) \cong \operatorname{End}_{\operatorname{lin}}(V)$ is given by $(X_0, Y_0) \mapsto_0 \circ Y_0 - Y_0 \circ X_0$, i.e., $[\bullet, \bullet]$ is the usual commutator in $\operatorname{End}_{\operatorname{lin}}(V)$, usually denoted by $[\bullet, \bullet]$ as well.

Definition 1.12 (Lie algebra homomorphism). Let $(\mathfrak{g}, [\bullet, \bullet]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\bullet, \bullet]_{\mathfrak{h}})$ be Lie algebras. A homomorphism of Lie algebras or a Lie algebra homomorphism is a linear map $f: \mathfrak{g} \to \mathfrak{h}$ such that for all $x, y \in \mathfrak{g}$:

$$f([x,y]_{\mathfrak{g}}) = [f(x), f(y)]_{\mathfrak{h}}.$$

Writing \mathfrak{g} for $(\mathfrak{g}, [\bullet, \bullet]_{\mathfrak{g}})$ and \mathfrak{h} for $(\mathfrak{h}, [\bullet, \bullet]_{\mathfrak{h}})$, we denote by $\operatorname{Hom}(\mathfrak{g}, \mathfrak{h})$ the set of all Lie algebra homomorphisms. And similarly to Definition 1.3 we define isomorphisms, endomorphisms, automorphisms and $\operatorname{Iso}(\mathfrak{g}, \mathfrak{h})$, $\operatorname{End}(\mathfrak{g})$ and $\operatorname{Aut}(\mathfrak{g})$.

Proposition 1.13. Let G and H be Lie groups and let $f: G \rightarrow H$ be a Lie group homomorphism. Then

$$\mathrm{d}_{1}f:\mathfrak{g}\to\mathfrak{h}$$

is a Lie algebra homomorphism.

Proof: Assume $X_0, Y_0 \in \mathfrak{g}$. We extend X_0 (resp. Y_0) to a left-invariant vector field $X \in \mathfrak{X}(G)$ (resp. $Y \in \mathfrak{X}(G)$), i. e., $X|_{\sigma} = d_{\mathfrak{l}}\ell_{\sigma}(X_0)$ for all $\sigma \in G$. Also extend $\widehat{X}_0 := d_{\mathfrak{l}}f(X_0) \in \mathfrak{h}$ to a left-invariant vector field $\widehat{X} \in \mathfrak{X}(H)$, and define similarly \widehat{Y}_0 and \widehat{Y} . Thus $\widehat{X}|_{\sigma} = d_{\mathfrak{l}}\ell_{\sigma}(\widehat{X}_0)$ for all $\sigma \in H$. For $\sigma, \tau \in G$ we have $(f \circ \ell_{\sigma})(\tau) = f(\sigma\tau) = f(\sigma)f(\tau) = \ell_{f(\sigma)}(f(\tau))$, thus $f \circ \ell_{\sigma} = \ell_{f(\sigma)} \circ f$. We calculate for $\sigma \in G$.

$$(\mathrm{d}_{\sigma}f)(X|_{\sigma}) = (\mathrm{d}_{\sigma}f \circ \mathrm{d}_{\mathfrak{l}}\ell_{\sigma})(X_{0}) = \mathrm{d}_{\mathfrak{l}}(f \circ \ell_{\sigma})(X_{0})$$
$$= \mathrm{d}_{\mathfrak{l}}(\ell_{f(\sigma)} \circ f)(X_{0}) = \mathrm{d}_{\mathfrak{l}}\ell_{f(\sigma)} \circ \mathrm{d}_{\mathfrak{l}}f(X_{0})$$
$$= \mathrm{d}_{\mathfrak{l}}\ell_{f(\sigma)}\widehat{X}_{0} = \widehat{X}|_{f(\sigma)}.$$

As a result $df \circ X = \widehat{X} \circ f$. And similarly we get $df \circ Y = \widehat{Y} \circ f$. Thus we have just shown that



commutes. This means that X resp. Y is f-related to \widehat{X} resp. \widehat{Y} – in the language of Exercise 1.6. It follows from this exercise that [X, Y] is also f-related to $[\widehat{X}, \widehat{Y}]$. Thus

$$d_{\mathbb{1}}f([X_0, Y_0]) = (df \circ [X, Y])|_{\mathbb{1}}$$
$$= ([\widehat{X}, \widehat{Y}] \circ f)|_{\mathbb{1}} = [\widehat{X}, \widehat{Y}]|_{\mathbb{1}}$$
$$= [\widehat{X}_0, \widehat{Y}_0] = [df_{\mathbb{1}}(X_0), df_{\mathbb{1}}(Y_0)],$$

which is the statement of the proposition.

Corollary of Proposition 1.13. Assume that V is a finite-dimensional real vector space. Let G be a subgroup and submanifold of GL(V). Let \mathfrak{g} be the Lie algebra of G. Then the Lie-bracket on \mathfrak{g} is the commutator bracket on End(V).

Proof: We have seen in Example 1.11 2.) that the Lie bracket on $\mathfrak{gl}(V)$ is the commutator bracket of $\operatorname{End}_{\operatorname{lin}}(V)$. The Lie group homomorphism $i: G \to \operatorname{GL}(V)$ induces an injective Lie algebra homomorphism $di: \mathfrak{g} \to \mathfrak{gl}(V)$, thus \mathfrak{g} the Lie brackt of $\mathfrak{gl}(V)$ restricts to the one on \mathfrak{g} .

1.3 Adjoint representations

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_{\mathfrak{l}}G$. For a given $\sigma \in G$ we differentiate $C_{\sigma}: G \to G$ at $\mathfrak{1}$ and we obtain $\operatorname{Ad}_{\sigma} := d_{\mathfrak{l}}C_{\sigma}: \mathfrak{g} \to \mathfrak{g}$, which is obviously a linear map. For $\sigma, \tau \in G$ differentiating $C_{\sigma\tau} = C_{\sigma} \circ C_{\tau}$ implies $\operatorname{Ad}_{\sigma\tau} = \operatorname{Ad}_{\sigma} \circ \operatorname{Ad}_{\tau}$.

Lemma 1.14. For $\sigma \in G$ the map $\operatorname{Ad}_{\sigma}: \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism.

Proof: Apply Proposition 1.13 to the Lie group homomorphism $C_{\sigma}: G \to G$.

Definition 1.15 (The adjoint representation of a Lie group). The group homomorphism obtained this way

$$\operatorname{Ad}: G \to \operatorname{Aut}(\mathfrak{g})$$

is called the adjoint representation of the Lie group G.

Remarks 1.16.

- 1.) One can show that Aut(g) is itself a Lie-group, in fact a Lie subgoup of the group GL(g) of vector space automorphisms.
- 2.) The Lie algebra of Aut(g) is the Lie algebra Der(g) of derivations of g. A linear map D:g → g, where g is a Lie algebra, is called a derivation of g, if for all x, y ∈ g we have

$$D([x,y]) = [D(x),y] + [x,D(y)].$$

Thus we have $\mathfrak{aut}(\mathfrak{g}) = \operatorname{Der}(\mathfrak{g})$.

We will not prove these statements here, as they will not be used in what follows and they are easier to prove later.

Definition 1.17 (The adjoint representation of a Lie algebra). The differential at 1 of Ad: $G \to GL(\mathfrak{g})$, namely

ad :=
$$d_1 \operatorname{Ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \mapsto \operatorname{ad}_X = d_1(\sigma \mapsto \operatorname{Ad}_\sigma)(X)$$

is called the adjoint representation of the Lie algebra \mathfrak{g} .

Accoding to Remarks 1.16 the adjoint representation of a Lie algebra is in fact a

Lie algebra homomorphism

ad:
$$\mathfrak{g} \to \mathfrak{aut}(\mathfrak{g}) = \operatorname{Der}(\mathfrak{g})$$
.

Lemma 1.18. Let \mathfrak{g} be the Lie algebra of a Lie group. Then the adjoint map ad satisfies. $\operatorname{ad}_X(Y) = [X, Y]$

The proof will be given later.

1.4 The exponential map

In the following $t \in \mathbb{R}$, so $\partial_t \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}$ is the positively oriented vector field on \mathbb{R} of constant length 1. For a smooth map $f:\mathbb{R} \to M$ we also write $\dot{f}(t) = \mathrm{d}f(\partial_r|_t)$. We write $\mathrm{Diff}(M)$ for the group of diffeomorphisms of M.

Definition 1.19. Let M be a manifold and $X \in \mathfrak{X}(M)$. A curve $\gamma: I \to M$ is called integral curve of X or flow line of X, if for all $t \in I$ we have

$$\dot{\gamma}(t) = X|_{\gamma(t)}$$

The theorem of Picard-Lindelöf implies: For any $p \in M$ there is an integral curve γ_p of X with $\gamma_p(0) = p$ and we assume that γ_p is defined on its maximal domain I_p , and this maximal solution is unique. We say that X is complete if $I_p = \mathbb{R}$ for all $p \in M$. We also define $\Phi_t^X(p) \coloneqq \gamma_p(t)$. Thus if X is complete, then we have a group homomorphism $\Phi_{\bullet}^X \colon \mathbb{R} \to \text{Diff}(M), t \mapsto \Phi_t^X$, called the flow of X.

We encourage the reader to check that $t \mapsto \Phi_t^X$ is indeed a group homomorphism.

Lemma 1.20. For a left-invariant vector field X on a Lie group we have:

(1) X is complete,

(2) If γ is an integral curve of X, and $\sigma \in G$, then $\ell_{\sigma} \circ \gamma$ is an integral curve of X as well,

- (3) $\Phi_t^X(\sigma\tau) = \sigma \Phi_t^X(\tau)$ for $t \in \mathbb{R}$, $\sigma, \tau \in G$.
- (4) $\Phi_t^{\lambda X} = \Phi_{\lambda t}^X \text{ for all } \lambda, t \in \mathbb{R}.$

In the proof we use the conventions $\infty + t := \infty$ and $-\infty + t = -\infty$ for all $t \in \mathbb{R}$.

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Proof: Let G be a Lie group and let $X \in \mathfrak{X}(G)$ be a left-invariant vector field. Consider the integral curve $\gamma_{\mathfrak{l}}: I_{\mathfrak{l}} \to G$, with $\gamma_{\mathfrak{l}}(0) = \mathfrak{l}, I_{\mathfrak{l}} = (\alpha, \omega)$. For any $\sigma \in G$ we calculate that the curve $\ell_{\sigma} \circ \gamma_{\mathfrak{l}}$ is also an integral curve of X:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ell_{\sigma}\circ\gamma_{\mathbb{1}}(t)\right) = \mathrm{d}\ell_{\sigma}\left(\dot{\gamma}_{\mathbb{1}}(t)\right) = \mathrm{d}\ell_{\sigma}\left(X\big|_{\gamma_{\mathbb{1}}(t)}\right) = X\big|_{\ell_{\sigma}\circ\gamma_{\mathbb{1}}(t)}$$

Thus $\gamma_{\sigma} := \ell_{\sigma} \circ \gamma_{1}: (\alpha, \omega) \to G$ is the integral curve with $\gamma_{\sigma}(0) = \sigma$. This already shows (2).

Now for $t_0 \in (\alpha, \omega)$ we have

$$\gamma_{\mathbb{1}}(t_0) = \gamma_{\gamma(t_0)}(t_0 - t_0),$$

thus γ_1 and $\gamma_{\gamma_1(t_0)}(\bullet - t_0)$ coincide, including their maximal domains. Hence $(\alpha, \omega) = (\alpha + t_0, \omega + t_0)$, hence $\alpha = -\infty$ and $\omega = \infty$. This proves the completeness, i. e., (1).

The statement (3) follows from the facts that both $t \mapsto \Phi_t^X(\sigma\tau)$ and $t \mapsto \sigma \Phi_t^X(\tau)$ are integral lines for X and that they coincide for t = 0.

In the notation above, and for any $\sigma \in G$ we have $\Phi_{\lambda t}^X(\sigma) = \gamma_{\sigma}(\lambda t)$. We calculate with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{\sigma}(\lambda t) = \lambda \dot{\gamma}_{\sigma}(\lambda t) = \lambda \left(X \Big|_{\gamma_{\sigma}(\lambda t)} \right) = (\lambda X) \Big|_{\gamma_{\sigma}(\lambda t)}$$

Thus $\mapsto \Phi_{\lambda t}^X(\sigma)$ is the integral curve of λX that attains σ for t = 0. Thus, by definition of $\Phi_t^{\lambda X}$, we have (4).

Definition 1.21. A homomorphism $f: \mathbb{R} \to G$ is called a 1-parameter subgroup of G.

Remark. Note that in general $f(\mathbb{R})$ is in general not a submanifold (in the usual sense²) of G, but it is a submanifold in the generalized sense of [12], see Remark 1.5 1.). This explains the usage of the word "subgroup".

Proposition 1.22. Let G be a Lie group. Then the 1-parameter subgroups are the integral curves of some left-invariant vector field through 1. More precisely:

(1) Let $f: \mathbb{R} \to G$ be a 1-parameter subgroup, and take the left-invariant vector field $X \in \mathfrak{X}(G)$ such that $\dot{f}(0) = X|_{\mathfrak{n}}$. Then f is the integral curve of X with $f(0) = \mathfrak{1}$.

²i.e., in the sense of Analysis IV, Differential Geometry I, etc

(2) Let X be a left-invariant vector field and $f: \mathbb{R} \to G$ an integral curve of X with f(0) = 1. Then f is a 1-parameter subgroup.

It follows, that two 1-parameter subgroups $f_1, f_2: \mathbb{R} \to G$ coincide if and only if $\dot{f}_1(0) = \dot{f}_2(0)$.

Proof:

"(1)": Obviously f(0) = 1. As f is a homomorphism $f(t + \bullet) = \ell_{f(t)} \circ f$. Thus

$$\dot{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{0} f(t+\bullet) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{0} \ell_{f(t)} \circ f = \mathrm{d}\ell_{f(t)}(\dot{f}(0)) = \mathrm{d}\ell_{f(t)}(X|_{\mathbb{1}}) = X\Big|_{f(t)}.$$

Thus f is an integral curve of X.

"(2)": Obviously f is smooth. It is defined on \mathbb{R} due to Lemma 1.20 (1). By definition of the flow we have $f(t) = \Phi_t^X(1)$, and thus we calculate, using Lemma 1.20 (1) at (*)

$$f(t+s) = f(s+t) = \Phi_{s+t}^X(\mathbb{1}) = \Phi_s^X(\Phi_t^X(\mathbb{1})) = \Phi_s^X(f(t)) \stackrel{(*)}{=} f(t)\Phi_s^X(\mathbb{1}) = f(t)f(s).$$

Thus f is a Lie group homomorphism.

Definition 1.23. Let G be a Lie group with Lie algebra \mathfrak{g} . We write X for the left-invariant vector field extending $X_0 \in \mathfrak{g}$. The exponential map exp is defined as the map

$$\exp: \mathfrak{g} \to G, \quad X_0 \mapsto \Phi_1^X(\mathfrak{1}).$$

WARNING 1.24. This exponential map is in general not the same as the Riemannian exponential map, even if we know that the metric is left- or right-invariant.³. As a consequence this map is also called the Lie group exponential map in order to distinguish it from the (semi-)Riemannian exponential map. It does however – as will be shown in the exercises – coincide with the Riemannian one for bi-invariant metrics on Lie groups.

Theorem 1.25 (Properties of the exponential map). Let G be a Lie group with Lie algebra fg, and $X \in \mathfrak{g}$, $t, s \in \mathbb{R}$. Then we have

³The notions of left-, right-, and bi-invariant Riemannian metrics are defined in the exercises.

(1) exp is smooth and with our usual identification $T_0 \mathfrak{g} \cong \mathfrak{g}$, its differential $d_0 \exp$ is the identity of \mathfrak{g} . As a consequence there is an open neighborhood U of 0, such that $\exp|_U: U \to \exp(U)$ is a parametrization.

(2) $\exp(tX) = \Phi_t^X(1)$

(3) $t \mapsto \exp(tX) =: f_X(t), \mathbb{R} \to G$ is a 1-parameter subgroup of G and any 1parameter subgroup is of that form for some $X \in \mathfrak{g}$. Furthermore $\dot{f}_X(0) = X$.

(4) The integral curves of the left-invariant vector field associated to X are, the curves $t \mapsto \sigma \exp(tX)$ for $\sigma \in G$

(5) If \overline{X} is the left-invariant vector that extends $X \in \mathfrak{g}$, then for all $t \in \mathbb{R}$ we have

$$\Phi_t^{\overline{X}} = r_{\exp(tX)} \,.$$

Proof:

"(1)": The smoothness of exp follows from the smooth dependence on the initial conditions in the theorem of Picard–Lindelöf. We calculate for the left-invariant vector fields $\overline{X} \in \mathfrak{X}(G)$ extending $X \in \mathfrak{g}$.

$$(\mathrm{d}_{\mathbb{1}} \exp)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp(tX)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t^{\overline{X}}(\mathbb{1}) = \overline{X}\Big|_{\mathbb{1}} = X \,.$$

Thus $d_1 \exp = id_g$.

"(2)": It follows from Lemma 1.20 (4) that $\exp(tX) = \Phi_1^{tX}(1) = \Phi_t^X(1)$.

"(3)": This immediately follows from Proposition 1.22.

"(4)": The integral curves in this item are $t \mapsto \Phi_t^X(\sigma)$ and we have seen in Lemma 1.20 (4) that $\Phi_t^X(\sigma) = \ell_\sigma(\Phi_t^X(1)) = \ell_\sigma \circ \exp(tX)$.

(5) immediately follows from (4) and the definition of $\Phi_t^{\overline{X}}$.

Example 1.26 (Exponential map of matrix groups). We consider again the Lie group GL(V) for a finite-dimensional real vector space V. We have already seen in Example 1.11 2.) that the left-invariant extension of $X_0 \in \mathfrak{g}$ is $X \in \mathfrak{X}(GL(V))$ with $X|_A = A \cdot X_0, A \in GL(V)$.

For $A \in \mathfrak{gl}(V)$ we know from the theory of ordinary differential equations, that

the series

$$\mathrm{EXP}(A) \coloneqq \sum_{i=0}^{\infty} \frac{1}{i!} A^i \tag{1.3}$$

converges (uniformly on compact sets and also all derivatives converge uniformly on compact sets). We obtain a map $\text{EXP}:\mathfrak{gl}(V) \to \text{GL}(V)$ such that for $t, s \in \mathbb{R}$, $A \in \mathfrak{gl}(V)$

$$\operatorname{EXP}((t+s)A) = \operatorname{EXP}(tA) \operatorname{EXP}(sA), \quad \operatorname{EXP}(0) = \mathbb{1}, \quad \operatorname{EXP}(-A) = \operatorname{EXP}(A)^{-1}.$$

Thus $t \mapsto \text{EXP}(tA)$ a 1-parameter subgroup, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{EXP}(tA) = A$$

It follows from Proposition 1.25 (3) that $\text{EXP}(A) = \exp(A)$. So we will write exp instead of EXP from now on. The same holds if G is a submanifold and subgroup of GL(V).

Furthermore from the theory of ordinary differential equations we know that for $t \in \mathbb{R}, A, B \in \mathfrak{gl}(V), M \in \mathrm{GL}(V)$ we have

$$\exp(MAM^{-1}) = M \exp(A)M^{-1}$$
 (1.4)

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A), \text{ if } [A,B] = 0$$
 (1.5)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(tA)\right) = \exp(tA)A = A\exp(tA) \tag{1.6}$$

We would like to have similar properties in adapted form for arbitrary Lie groups. We already have an adapted form of the first equality of (1.6) which is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(tX)\right) = \left(\mathrm{d}\ell_{\exp(tX)}\right)(X)$$

i.e., $t \mapsto \exp(tX)$ is an integral curve of the left-invariant extension of X.

Lemma 1.27. Let G be a Lie group, $\mathfrak{g} = \text{Lie}(G)$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$. Then

$$\operatorname{Ad}_{\exp(tX)}(X) = X$$
.

Proof: One easily checks $C_{\exp(tX)}(\exp(tX)) = \exp(tX)$. If we derive this at t = 0 one gets the equation stated in the lemma.

It immediately follows that $(d\ell_{\exp(tX)})(X) = (dr_{\exp(tX)})(X)$, and we get the following corollary that generalizes (1.6).

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Corollary 1.28.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(tX)\right) = \left(\mathrm{d}\ell_{\exp(tX)}\right)(X) = \left(\mathrm{d}r_{\exp(tX)}\right)(X)$$

Lemma 1.29. If $f: G \to H$ is a homomorphism of Lie groups. Let $\exp^{G}: \mathfrak{g} \to G$ and $\exp^{H}: \mathfrak{h} \to H$ be the exponential maps of G and H. Then the diagram

commutes, i. e., $f \circ \exp^G = \exp^H \circ d_{\mathbb{1}} f$.

Proof: Let $X \in \mathfrak{g}$. Then $t \mapsto \exp^G(tX)$ is a 1-parameter subgroup of G. Thus $t \mapsto f \circ \exp^G(tX)$ is a 1-parameter subgroup of G. We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \exp^G(tX) = \mathrm{d}_{\mathbb{1}}f\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp^G(tX)\right) = \mathrm{d}_{\mathbb{1}}f(X).$$

Thus this is the 1-parameter subgroup $t \mapsto \exp^{H}(t \operatorname{d}_{1}f(X))$, i.e., $f \circ \exp^{G}(tX) = \exp^{H}(t \operatorname{d}_{1}f(X))$ for all $t \in \mathbb{R}$ which implies the statement.

As a corollary we get the Lie group analogon of equation (1.4):

Corollary 1.30. For a Lie G and $\sigma \in G$ we get $C_{\sigma} \circ \exp = \exp \circ \operatorname{Ad}_{\sigma}$, *i. e.*, the diagram



commutes.

Proof: Apply Lemma 1.29 to H = G, $f = C_{\sigma}$ and thus $d_{1}f = Ad_{\sigma}$.

1.5 Proof of Lemma 1.18

We now provide the proof of Lemma 1.18 which is still missing.

Let us recall the following exercise from last semester:

Exercise 1.31 (Diff. geom. I, Exercise Sheet 6, Exercise 4 with changed notation). Let M be a smooth, not necessarily compact, manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on M, i. e., φ is smooth with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s, t \in \mathbb{R}$. Let ξ be the associated tangent vector field on M, defined as

$$\xi|_{x} \coloneqq \frac{d}{dt}\Big|_{t=0} (\varphi_{t}(x)),$$

see also Diff. geom. I, Exercise Sheet 5, Exercise 3. Show that, for any smooth tangent vector field Y on M and point $p \in M$ it is

$$\frac{d}{dt}\Big|_{t=0}\left((\varphi_t)_*\eta\right)\Big|_p = -\left[\xi,\eta\right]\Big|_p,$$

where, for any diffeomorphism $\psi: M \to M$, the term $\psi_*\eta$ denotes the pushforward tangent vector field of η defined by $\psi_*\eta \coloneqq \mathrm{d}\psi \circ \eta \circ \psi^{-1}$.

Proof of Lemma 1.18: ⁴ Let $X, Y \in \mathfrak{g}$ with left-invariant extensions \overline{X} and \overline{Y} . At first, we calculate for $t \in \mathbb{R}$:

$$\begin{aligned} \operatorname{Ad}_{\exp(tX)}(Y) &= \operatorname{d} r_{\exp(-tX)} \circ \operatorname{d} \ell_{\exp(tX)}(\overline{Y}) \big|_{\mathbb{1}} \\ &= \operatorname{d} r_{\exp(-tX)}(\overline{Y}) \big|_{\exp(tX)} \\ &\stackrel{(*)}{=} \operatorname{d} \Phi_{-t}^{\overline{X}}(\overline{Y}) \big|_{\Phi_{-t}^{X}(\mathbb{1})} \\ &\stackrel{(+)}{=} \left(\Phi_{-t}^{\overline{X}} \right)_{*}(\overline{Y}) \big|_{\mathbb{1}} \end{aligned}$$

where we used at (*) Proposition (1) (5), and where we used at (+) the pushforward of vector fields from the preceding exercise. We derive this with respect to t at t = 0, and use the results of the exercise above at (†) for $\xi = -\overline{X}$, $\eta = \overline{Y}$ and $\varphi_t = \Phi_{-t}^{\overline{X}}$. This gives

$$\operatorname{ad}_{X} Y = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \operatorname{Ad}_{\exp(tX)}(Y)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\Phi_{-t}^{\overline{X}}\right)_{*}(\overline{Y})\Big|_{1}$$

⁴We roughly follow [12, 3.46].

$$\stackrel{(\dagger)}{=} \left[\overline{X}, \overline{Y}\right]_{|_{\mathbb{I}}} = \left[X, Y\right]$$

1.6 Commuting elements in Lie groups and Lie algebras

Definition 1.32. Two elements $\sigma, \tau \in G$ in a Lie group commute, if $\sigma \tau = \tau \sigma$. Two elements $X, Y \in \mathfrak{g}$ in a Lie algebra commute, if [x, y].

We want to relate commutativity in a Lie group to commutativity in its Lie algebra.

We start by some considerations on arbitrary manifolds M and N.

Lemma 1.33. Let $f: M \to N$ be a smooth map, and let $X \in \mathfrak{X}(M)$ be f-related to $Y \in \mathfrak{X}(N)$, *i. e.*, $df \circ X = Y \circ f$. Then the flows Φ_t^X and Φ_t^Y of X and Y satisfy

$$\Phi^Y_t \circ f = f \circ \Phi^X_t \,.$$

Proof: For $p \in M$ we will show that $t \mapsto \gamma(t) \coloneqq f \circ \Phi_t^X(p) \in N$ is an integral curve of Y. As one easily checks $\gamma(0) = f(p)$, this proves the statement.

$$\begin{split} \dot{\gamma}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \Big(f \circ \Phi_t^X(p) \Big) \\ &= \mathrm{d}f \circ \Big(\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t^X(p) \Big) \\ &= \mathrm{d}f \circ \Big(X \Big|_{\Phi_t^X(p)} \Big) \\ &= (\mathrm{d}f \circ X) \Big|_{\Phi_t^X(p)} \Big) \\ &= (Y \circ f) \Big|_{\Phi_t^X(p)} \Big) \\ &= Y \Big|_{\gamma(t)}. \end{split}$$

Proposition 1.34. Let X and Y be vector fields on M with flows Φ^X_{\bullet} and Φ^Y_{\bullet} . Then

$$[X,Y] = 0 \Longleftrightarrow \forall s,t \in \mathbb{R}: \ \Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t \,.$$

Proof:

"
—": We apply $\frac{d}{ds}\Big|_{s=0}$ to

$$\Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t$$

and using $\frac{d}{ds}\big|_{s=0}\Phi_s^Y=Y$ we obtain

$$d(\Phi_t^X) \circ Y = Y \circ \Phi_t^X$$

which means $(\Phi_t^X)_* Y = Y$. We apply Exercise 1.31 for $\xi = X$ and thus $\varphi_t = \Phi_t^X$, so we obtain by deriving with respect to t at t = 0:

$$0 = \frac{d}{dt}\Big|_{t=0} Y$$
$$= \frac{d}{dt}\Big|_{t=0} (\Phi_t^X)_* Y$$
$$= -[X, Y]$$

" \Rightarrow ": For $p \in M$ and for $s \in \mathbb{R}$ we define

$$v_p(t) \coloneqq \left(\left(\Phi_t^X \right)_* Y \right) \Big|_p = \mathrm{d} \left(\Phi_t^X \right) \circ Y \circ \Phi_{-t}^X(p) \in T_p M$$

We may differentiate this in the sense of Analysis II, and we write this differential as $v'_p(t)$. Exercise 1.31 tells us that $v'_p(0) = -[X, Y] = 0$ for all $p \in M$. We set $p = \Phi^X_{-s}(q)$ and we get

$$\begin{aligned} v_{\Phi_{-s}^{X}(q)}(t) &= \left(\mathrm{d}(\Phi_{t}^{X}) \circ Y \circ \Phi_{-t-s}^{X} \right) \Big|_{q} \\ &= \left(\mathrm{d}(\Phi_{-s}^{X}) \circ \left(\mathrm{d}(\Phi_{t+s}^{X}) \circ Y \circ \Phi_{-t-s}^{X} \right) \right) \Big|_{q} \\ &= \left(\mathrm{d}(\Phi_{-s}^{X}) \circ \left(\Phi_{t+s}^{X}\right)_{*} Y \right) \Big|_{q} \\ &= \mathrm{d}(\Phi_{-s}^{X}) \left(v_{q}(t+s) \right) \end{aligned}$$

Deriving this with respect to t at t = 0 yields

$$0 = v'_{\Phi^X_{-s}(q)}(0) = \mathrm{d}(\Phi^X_{-s})\left(v'_q(s)\right)$$

and as $d(\Phi_{-s}^X)$ is an isomorphism, this gives $v'_q(s) = 0$ for all $s \in \mathbb{R}$ and all $q \in M$. Thus we have $v_q(t) = v_q(0) = Y$ for all $q \in M$ and $t \in \mathbb{R}$. We have thus proven

$$d(\Phi_t^X) \circ Y = Y \circ \Phi_t^X$$

which means that Y is Φ_t^X -related to itself. Using Lemma 1.33 for M = N, X replaced by Y, $f = \Phi_t^X$ and t replaced by s we get

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X.$$

Corollary 1.35. Let G be a Lie group, $X, Y \in \mathfrak{g} = \text{Lie}(G)$.

- (1) If [X, Y] = 0 then $\exp(X) \exp(Y) = \exp(Y) \exp(X) = \exp(X + Y)$.
- (2) Conversely, if

$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX) \text{ for all } t, s \in \mathbb{R},$$

then [X, Y] = 0.

The first part of the Corollary provides a Lie group analogon of equation (1.5).

Proof: We extend $X, Y \in \mathfrak{g}$ to left-invariant vector fields, also denoted by X and $Y.^5$

At first, let us assume [X,Y] = 0. As we have $\exp(X) = \Phi_1^X(1)$, the statement $\exp(X)\exp(Y) = \exp(Y)\exp(X)$ follows from " \Rightarrow " in Proposition 1.34. We also have [sX,tY] = st[X,Y] = 0. Thus, we already know $\exp(sX)\exp(tY) = \exp(tY)\exp(sX)$ and this yields

$$\exp((t+s)X)\exp((t+s)Y) = \exp(tX)\exp(sX)\exp(tY)\exp(sY)$$
$$= \left(\exp(tX)\exp(tY)\right)\left(\exp(sX)\exp(sY)\right),$$

thus $t \mapsto \gamma(t) := \exp(tX) \exp(tY)$ is 1-parameter subgroup of G, and $\gamma'(0) = dr_{\exp(0)}X + d\ell_{\exp(0)}Y = X + Y$. Thus implies $\exp(t(X + Y)) = \exp(tX) \exp(tY)$ which gives the remaining statement for t = 1.

The converse statement immediately follows from " \Leftarrow " in Proposition 1.34.

We have seen that $\exp: \mathfrak{g} \to G$ satisfies $d_0 \exp = id$. Thus, the local reversal theorem tells us that there is an open neighborhood U_0 of 0 and and open neighborhood V_0

 $^{^5\}mathrm{We}$ assume it is clear from the context, when a vector is meant, and when we denote a vector field.

of 1 such that $\exp|_{U_0}: U_0 \to V_0$ is a diffeomorphism. Using continuity of multiplication and inversion, we see that there is an open neighborhood U_1 of 0 such that $U_1 \subset U_0$, such that U_1 is starshaped with respect to 0, satisfying $X \in U_1 \iff -X \in U_1$ and $X, Y \in U_1 \Rightarrow X + Y \in U_0$. We put $V_1 \coloneqq \exp(U_1)$ and by shrinking U_1 and V_1 further we can achieve additionally $\mu(V_1 \times V_1) \subset V_0$ and we already have that inversion maps V_1 to itself.

Let $\gamma: [0, b] \to G$ be a continuus path. For any $t \in [0, b]$ we define W_t as the connected component of $\{s \in [0, b] \mid \gamma(t)^{-1}\gamma(s) \in V_1\}$ that contains t. Then $(W_t)_{t \in [0, b]}$ is an open cover⁶ of [0, b]. An elemantary compactness argument for [0, b], treated under the name **Lebesgue number** ε , says: there is an $\varepsilon > 0$ if we have a partition

$$0 = t_0 \le t_1 \le \dots \le t_k = b, \quad \forall i \in \{1, 2, \dots, k\} : \ t_i - t_{i-1} < \varepsilon$$
(1.8)

then

$$\forall i \in \{1, 2, \dots, k\} : t_i \in W_{t_{i-1}}$$

and thus

$$\forall i \in \{1, 2, \dots, k\} : \forall s \in [t_{i-1}, t_i] : \gamma(s)^{-1} \gamma(t_{i-1}) \in V_1 \text{ and } \gamma(t_{i-1})^{-1} \gamma(s) \in V_1.$$
(1.9)

Corollary 1.36. Let G be a connected Lie group with $\mathfrak{g} = \text{Lie}(G)$. Then the following are quivalent:

- (i) G is abelian, i. e., $\sigma \tau = \tau \sigma$ for all $\sigma, \tau \in G$,
- (ii) \mathfrak{g} is abelian, i. e., [X, Y] = 0 for all $X, Y \in \mathfrak{g}$,
- (iii) $\exp:(\mathfrak{g},+) \to (G,\mu)$ is a group homomorphism.

Proof:

"(i) \Rightarrow (ii)": This follows immediately from the second part of Corollary 1.35.

"(ii) \Rightarrow (iii)": This follows immediately from the first part of Corollary 1.35.

"(iii) \Rightarrow (i)": For $\sigma \in G$ we choose a continuous path $\gamma: [0, b] \rightarrow G$ from 1 to σ . We choose a subdivision as (1.8)/(1.9). Then $\sigma_i := \gamma(t_i)^{-1}\gamma(t_{i-1}) \in V_1$ satisfies $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$. We write $\sigma_i = \exp(X_i)$, $X_i \in U_1$. Similarly we decompose $\tau = \tau_1 \tau_2 \cdots \tau_\ell$, $\tau_j = \exp(Y_j)$, $Y_j \in U_1$. Condition (iii) implies that $\sigma_i \tau_j = \tau_j \sigma_i$ for all i, j and thus

⁶in German: "Überdeckung", **nicht** "Überlagerung", the two terms have different meanings, but are denoted with the same words "cover" and "covering" in English, but they are properly distinguished in German

 $\sigma \tau = \tau \sigma$.

1.7 The Baker–Campbell–Hausdorff Formula

We have seen that $\exp: \mathfrak{g} \to G$ satisfies $d_0 \exp = \mathrm{id}$, and in the discussion following Corollary 1.35 we have discussed the diffeomorphism $\exp|_{U_0}: U_0 \to V_0$, and also had the smaller open neighborhoods $U_1 \subset U_0$ of 0 and $V_1 \subset V_0$ of 1. In particular multiplication restricts to a map $V_1 \times V_1 \to V_0$ and inversion maps V_1 to itself. We write $\log: U_0 \to V_0$. In this language, it follows from Corollary 1.35 (1) for all $X, Y \in U_1$:

if
$$[X,Y] = 0$$
, then $\log(\exp(X)\exp(Y)) = X + Y$.

On the other hand it is clear from (the proof of) Corollary 1.35 (2) that this formula no longer holds, if \mathfrak{g} is not abelian. The Baker–Campbell–Hausdorff formula, says that this can be repaired by adding commutator terms.

Exercise 1.37. We define the 3-dimensional Heisenberg group H_3 as

$$H_3 \coloneqq \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

This is a submanifold and subgroup of $GL(3,\mathbb{R})$, thus a Lie group.

(a) Show that its Lie algebra \$\mu_3\$, the 3-dimensional Heisenberg Lie algebra is given by matrices as follows:

$$\mathfrak{h}_3 \coloneqq \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

- (b) Calculate $\exp: \mathfrak{h}_3 \to H_3$, and show that it is a diffeomorphism.
- (c) Show that $\log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A, B]$.
- (d) Show that [X, [Y, Z]] = 0 for all $X, Y, Z \in \mathfrak{h}_3$, i.e. \mathfrak{h}_3 is 2-step nilpotent.

Tu 30.4.

The formula in (c) of this exercise is simple as higher order commutators vanish in the sense of (d). In general one has to work with a power series.

Theorem 1.38 (Baker–Campbell–Hausdorff formula). Let G be a Lie group with Lie algebra \mathfrak{g} . There is a power series BCH whose term of degree $k \in \mathbb{N}_0$ is a homogeneous polynomial of degree k

$$\operatorname{BCH}_k: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that

(1) BCH = $\sum_{k=0}^{\infty}$ BCH_k converges⁷ on a neighborhood U_2 of 0. (We assume $U_2 \subset U_1$ for the U_1 defined above.)

(2) $\log(\exp(X)\exp(Y)) = BCH(X,Y)$ for all $X, Y \in U_2$.

(3) The first terms are $BCH_0(X,Y) = 0$, $BCH_1(X,Y) = X + Y$, $BCH_2(X,Y) = \frac{1}{2}[X,Y]$, $BCH_3(X,Y) = \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]]$.

(4) BCH_k can be expressed by a formula, which only uses the vector space operations of \mathfrak{g} and $[\bullet, \bullet]$.

(5) The formula for BCH_k is the same formula for any Lie group/algebra: obviously the bracket $[\bullet, \bullet]$ is given by \mathfrak{g} , but using this bracket, the formula no longer depends on \mathfrak{g} (or G). This property can also be expressed as follows: if $\varphi: \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then

$$\operatorname{BCH}_k(\varphi(\bullet),\varphi(\bullet)) = \varphi(\operatorname{BCH}_k(\bullet,\bullet)).$$

In other words, we have

 $\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + higher order terms (with at least 3 commutator terms in each summand)$

for X, Y sufficiently close to 0.

We do not prove this theorem here, see [5, Sections 3.1–3.5] for a proof.

⁷uniformly on any compactum in U_2 , and also all derivaties converge uniformly on such a compactum)

1.8 From Lie algebra homomorphisms to Lie group homomorphisms

Theorem 1.39 (Lifting Lie algebra homomorphism to Lie group homomorphisms). Let G and H be Lie groups, $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(H)$. Assume that G is simplyconnected⁸. Then for any Lie algebra homomorphism $f: \mathfrak{g} \to \mathfrak{h}$ there is a unique Lie group homomorphism $F: G \to H$, such that $d_{\mathfrak{l}}F = f$.

A full proof of this theorem is carried out in [12, Theorem 3.27], building on the Frobenius theorem. Another approach, using the Baker–Campbell–Hausdorff formula, is worked out in [5] where the above theorem is Theorem 3.7. We sketch the latter approach.

Sketch of Proof:

(a) On \mathfrak{g} we choose an open neighborhood U_2 of 0 as in Corollary 1.36. On \mathfrak{h} we choose $U_2^{\mathfrak{h}} \ni \mathfrak{1}, U_2^{\mathfrak{h}} \mathfrak{C} \mathfrak{h}$ analogously. We set $U_3 := U_2 \cap f^{-1}(U_2^{\mathfrak{h}})$. We define $V_3 := \exp(U_3)$. We set $F_3: V_3 \to H$ as $F_3 := \exp^H \circ f \circ \log^G$ where exp^H is the exponential map of the Lie group H, and \log^G the local inverse of the exponential map of G. For $\sigma, \tau \in U_3$ we calculate using the Baker–Campbell–Hausdorff formula, more precisely Theorem 1.38 (2) for $X = \log(\sigma)$ and $Y = \log(\tau)$ at (*), Theorem 1.38 (5) at (+) and Theorem 1.38 (2) for $X = f \circ \log(\sigma)$ and $Y = f \circ \log(\tau)$ at (\dagger) :

$$F_{3}(1) = \exp^{H} \circ f \circ \underbrace{\log^{G}(1)}_{=0} = \exp^{H}(0) = 1$$

$$F_{3}(\sigma\tau) = \exp^{H} \circ f \circ \log^{G}(\sigma\tau)$$

$$\stackrel{(*)}{=} \exp^{H} \circ f \left(\operatorname{BCH}(\log^{G}(\sigma), \log^{G}(\tau)) \right)$$

$$\stackrel{(+)}{=} \exp^{H} \left(\operatorname{BCH}(f \circ \log^{G}(\sigma), f \circ \log^{G}(\tau)) \right)$$

$$\stackrel{(\dagger)}{=} \left(\exp^{H} \circ f \circ \log^{G}(\sigma) \right) \cdot \left(\exp^{H} \circ f \circ \log^{G}(\tau) \right)$$

$$= F_{3}(\sigma) F_{3}(\tau). \qquad (1.10)$$

(b) For a given $\sigma \in G$ we choose a path $\gamma: [0,1] \to G$ with $\gamma(0) = 1$ and $\gamma(1) = \sigma$. This is possible, as G is connected and thus path-connected. We restrict the open neighborhood U_3 further to some star-shaped open neighborhood U_4 of 0 that is symmetric with respect to 0, i. e., $X \in U_4 \iff -X \in U_4$, and such that $\mu(U_4 \times U_4) \subset U_3$. We define $V_4 := \exp(U_4)$. We choose a subdvision SUB: $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ as

 $^{^{8}}$ We use the convention that the definition of simply-connectedness includes connectedness

in (1.8) and (1.9) with V_4 instead of V_1 . We define

$$F_{\text{path}}(\gamma, \text{SUB}) \coloneqq \underbrace{F_{3}(\gamma(t_{0})^{-1}\gamma(t_{1}))}_{F_{3}(\gamma(t_{1}))} F_{3}(\gamma(t_{1})^{-1}\gamma(t_{2})) \cdots \underbrace{F_{3}(\gamma(t_{k-1})^{-1}\gamma(t_{k}))}_{F_{3}(\gamma(t_{k-1})^{-1}\sigma)}$$
(1.11)

If SUB' is a subdivision of SUB, then follows from (1.10) that

$$F_{\text{path}}(\gamma, \text{SUB'}) = F_{\text{path}}(\gamma, \text{SUB}).$$

If SUB_1 and SUB_2 are two subdivisions, then we choose SUB' to be a refinement of both of them, and we argue

$$F_{\text{path}}(\gamma, \text{SUB}_1) = F_{\text{path}}(\gamma, \text{SUB'}) = F_{\text{path}}(\gamma, \text{SUB}_2).$$

Thus we now write $F_{\text{path}}(\gamma)$, as this does not depend on the subdivision SUB.

(c) Now, one shows: if γ' is another path as above, and if $\mathcal{H}:[0,1] \times [0,1]$ is a homotopy from γ to γ' with fixed endpoints, then $F_{\text{path}}(\gamma) = F_{\text{path}}(\gamma')$.

For this purpose one chooses a $k \in \mathbb{N}$ (such a number is given again by a "Lebesgue number", whose existence again relies on a compactness argument), such

$$\mathcal{H}\left(\left[i-\frac{1}{k}\right]\times\left[j-\frac{1}{k}\right]\right)\subset V_4$$

for all $i, j \in \{1, ..., k\}$. Now one passes from γ to γ' in k^2 steps by replacing in each step a piece of the curve described by the square $\left[i - \frac{1}{k}\right] \times \left[j - \frac{1}{k}\right]$, see the drawing in the lecture. This proves the claim in this item.

(d) As g is simply-connected, we see that there is a map $F: G \to H$, such that $F(\sigma) = F_{\text{path}}(\gamma)$ if $\gamma(0) = 1$ and $\gamma(1) = \sigma$. The smoothness of F follows from the smoothness of F_3 .

(e) Now let $\sigma, \tau \in G$, we choose paths $\gamma, \rho: [0,1] \to G$ with $\gamma(0) = \rho(0) = 1$, $\gamma(1) = \sigma$, $\rho(1) = \tau$. Then $\ell_{\sigma} \circ \rho$ is a path from σ to $\sigma\tau$. Thus the concatenation $\gamma * (\ell_{\sigma} \circ \rho)$, defined as

$$\gamma * (\ell_{\sigma} \circ \rho)(t) = \begin{cases} \gamma(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ \sigma \cdot \rho(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

is a path from 1 to $\sigma\tau$, and one easily checks⁹ that (1.11) yields

$$F_{\text{path}}(\gamma * (\ell_{\sigma} \circ \rho)) = F_{\text{path}}(\sigma) \cdot F_{\text{path}}(\tau).$$

This gives the homomorphism property $F(\sigma\tau) = F(\sigma) \cdot F(\tau)$.

2 Actions of groups on spaces

2.1 Definitions for groups actions and examples

In this section all topological spaces are assumed to be Hausdorff spaces.

Definitions 2.1.

- 1.) A topological group is a topological space with a map $\mu: G \times G \to G$ such that (G, μ) is a group and such that $(\sigma, \tau) \mapsto \mu(\sigma, \tau^{-1})$ is continuous¹. We write $\sigma\tau := \mu(\sigma, \tau)$. We denote the unit element by 1.
- 2.) A (continuous) left action of a topological group G on a topological space X is a continuous map $a: G \times X \to X$ such that
 - (i) a(1, x) = x for all $x \in X$,
 - (ii) $a(\sigma\tau, x) = a(\sigma, a(\tau, x))$ for all $\sigma, \tau \in G, \forall x \in X$.

One also says that X is a G-space. We often write σx for $a(\sigma, x)$, condition (ii) then reads as $(\sigma \tau)x = \sigma(\tau x)$, so we can omit the parentheses. As a symbol we write $G \curvearrowright X$. Note that a group action induces a group homorphism $G \rightarrow$ Homeo(X), where Homeo(X) denotes the group of homeomorphisms from X to X. However, not every group hommorphism $G \rightarrow$ Homeo(X) defines an action, in general.

3.) In order to get the definition of a right action we replace condition (ii) by

(ii') $a(\sigma\tau, x) = a(\tau, a(\sigma, x))$ for all $\sigma, \tau \in G, \forall x \in X$.

One then writes $x\sigma$ and we have $x(\sigma\tau) = (x\sigma)\tau$.

⁹in fact one has to be careful with the order!

¹This is equivalent to claiming that μ and $\sigma \mapsto \sigma^{-1}$ are continuous.

- 4.) The action of G on X is called discrete if G carries the discrete topology. Then continuity in 1.) is trivially satisfied, and the only conditions on the continuity in 2.) is that for all σ ∈ G, ℓ_σ : X → X is continuous. So a discrete group action of G on X is the same as a group homomorphism G → Homeo(X).
- 5.) An action is smooth, if G is a Lie group, if X is a smooth manifold, and if a is a smooth map. We then say that X is a smooth G-space.
- 6.) An action is free, if

$$\forall \sigma \in G \smallsetminus \{1\} : \forall x \in X : \sigma x \neq x.$$

An action is effective or faithful if

$$\forall \sigma \in G \smallsetminus \{1\} : \exists x \in X : \sigma x \neq x.$$

An action is transitive if

$$\forall x, y \in X : \exists \sigma \in G : \sigma x = y.$$

(For right actions the obvious modification should be done in each definition.)

- 7.) The orbit of x is $Gx := \{\sigma x \mid \sigma \in G\}$. (We then have: G acts transitively \iff Gx = X for all $x \in X \stackrel{X \neq \emptyset}{\iff} Gx = X$ for some $x \in X$.)
- 8.) The stabilizer or isotropy group at $x \in X$ is

$$G_x \coloneqq \{\sigma \in G \mid \sigma x = x\}.$$

This is a closed subgroup of G (obvious). For smooth actions it is a submanifold (more involved, no proof here).

9.) The quotient space is

$$G \setminus X \coloneqq \{Gx \mid x \in X\}.$$

We will clarify its topology and its smooth structure, if it exists.

Examples 2.2.

1.) Let X = G be a topological group (or even a Lie group)

(i)
$$a(\sigma, \tau) = \mu(\sigma, \tau) = \sigma \tau$$

- (ii) $a(\sigma, \tau) = \mu(\tau, \sigma^{-1})$
- (iii) $a(\sigma, \tau) = \mu(\tau, \sigma)$
- (iv) $a(\sigma, \tau) = \mu(\sigma^{-1}, \tau)$

(i) and (ii) are left actions, while (iii) and (iv) are right actions.

- 2.) Let X = G. Conjugation: $a(\sigma, \tau) = \sigma \tau \sigma^{-1} =: C_{\sigma}(\tau)$ is a left action.
- 3.) $\operatorname{Ad}_{\sigma}: \mathfrak{g} \to \mathfrak{g}$ defines a left action $a(\sigma, X) \coloneqq \operatorname{Ad}_{\sigma}(X)$ on \mathfrak{g} .
- 4.) O(n+1) acts on $S^n \subset \mathbb{R}^{n+1}$ transitively and smoothly. Let $e_{n+1} \coloneqq (0, 0, \dots, 0, 1)^{\mathsf{T}}$ Then the stabilizer of O(n+1) at e_{n+1} is

$$O(n+1)\Big|_{e_{n+1}} = \left\{ \begin{pmatrix} A & 0\\ 0 & 1 \end{pmatrix} \middle| A \in O(n) \right\} \cong O(n)$$

- 5.) $\{\pm 1\}$ acts on $S^n \subset \mathbb{R}^{n+1}$ by multiplication. This is a free, discrete smooth action. The quotient $\mathbb{R}P^n = \{\pm 1\} \setminus S^n$ is the **real projective space**.
- 6.) U(1) = $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by multiplication. This is a free, smooth action, called the **Hopf action**. The quotient $\mathbb{C}P^n = S^1 \setminus S^{2n+1}$ is the **complex projective space**.

2.2 Proper maps and proper actions

Definitions 2.3.

- 1.) A continuous map $f: X_1 \to X_2$ is proper if $F^{-1}(K)$ is compact for all compacta $K \subset X_2$.
- 2.) An action of G on X is defined to be proper if

$$G \times X \xrightarrow{\Theta} X \times X$$
$$(\sigma, x) \mapsto (\sigma x, x)$$

is a proper map. The map Θ is called the associated shear map.

Example 2.4. Assume that G acts on X continuously. If G is compact, then the action is proper. In order to prove this, let $K \subset X \times X$ be compact. We write $\operatorname{pr}_i: X \times X \to X$ for the projection to the *i*-th factor. Then $\widehat{K} := \operatorname{pr}_2(K) \subset X$ is also

compact. As a consequence $\Theta^{-1}(K)$ is a closed subset of the compact set $G \times \widehat{K}$, and thus compact as well.

Note: if the action map $a: G \times X \to X$ is a proper map, then the action is proper (i. e., Θ is a proper map). The converse is *not true*. In fact let $a: G \times X \to X$ be a compact map. For some $x_0 \in X$ consider the compact set $A := a^{-1}(\{x_0\}) =$ $\{(g, g^{-1}x_0) \mid g \in G\}$. Then pr_1 defines a continuous surjective map $A \to G$, thus G is compact. Thus it is too restrictive to claim that $a: G \times X \to X$ is a proper map.

As an example, consider the action of $G = (\mathbb{R}, +)$ on $X = \mathbb{R}$ given by $a(\sigma, x)0\sigma + x$. One easily checks, that this action is proper (as an action), but as \mathbb{R} is not compact, the map a is not proper.

For $K \subset X$ we define $\sigma K \coloneqq \{\sigma k \mid k \in K\}$ and

$$G_K \coloneqq \{ \sigma \in G \mid \sigma K \cap K \neq \emptyset \}.$$

In particular, for $x \in X$, $G_{[x]} = G_x$ is the isotropy group of x.

Proposition 2.5. The action of G on X is proper if, and only if, G_K is compact for all compact sets $K \subset X$.

In the special case that G acts smoothly on the manifold X, this is the equivalence of i) and ii) of Exercise Sheet 3, Exercise 2. We will thus currently omit the proof.

Recall the following from the beginners' lectures:

Theorem 2.6 (Bolzano–Weierstrass). Let X be a metrizable topological space (i. e., a space whose topology is induced from a metric). Then

X is compact \iff X is sequentially compact

where a space X is called sequentially compact if X is a Hausdorff space in which any sequence has a convergent subsequence.

All smooth manifolds are metrizable.

Proposition 2.7. Let G and X be metrizable, let X be locally compact, and let G act continuously on X. Then the following are equivalent

(i) the action of G on X is proper.

(ii) Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in M and $(\sigma_i)_{i\in\mathbb{N}}$ a sequence in G such that the sequences $(x_i)_{i\in\mathbb{N}}$ and $(\sigma_i \cdot x_i)_{i\in\mathbb{N}}$ converge. Then we find a convergent subsequence of $(\sigma_i)_{i\in\mathbb{N}}$.

In the special case that G acts smoothly on the manifold X, this is the equivalence of ii) and iii) of Exercise Sheet 3, Exercise 2. We will thus skip the proof.

Lemma 2.8. Let $F: X \to Y$ be a continuous, proper map between (topological) manifolds. Then F is closed.

Note: The Lemma is still correct if one does not require X and Y to be topological manifolds, but to require instead that X and Y are locally compact and metrizable (Hausdorff) spaces. However, an adapted proof is required in this generality.

Proof: Let $A \subset X$ be closed, and take $p \in \overline{F(A)}$. We have to show that $p \in F(A)$.

We choose a chart $U \xrightarrow{y} V$ of Y, containing p, y(p) = 0. We choose an $\varepsilon > 0$ with $\overline{B_{\varepsilon}(0)} \subset V$.

There is a sequence $(q_i)_{i \in \mathbb{N}}$ in A such that $\lim_{i \to \infty} F(q_i) = p$. After removing finitely many exceptions from the sequence, we get $F(q_i) \in y^{-1}(\overline{B_{\varepsilon}(0)}) =: K$ for all i. Obviously, K is compact, and as F is proper $F^{-1}(K)$ is a compact subset of X, hence $K' := F^{-1}(K) \cap A$ is also compact, and we have $q_i \in K'$.

Thus after passing to a subsequence, $q_{\infty} \coloneqq \lim_{i \to \infty} q_i$ exists in $K' \subset A$. Then

$$F(q_{\infty}) = F(\lim_{i \to \infty} q_i) = \lim_{i \to \infty} F(q_i) = p.$$

Hence $p \in \text{image } F$.

3 Topological quotients

Motivation: Let G be a Lie group acting on (smooth) manifold M. We try to find good conditions, such that $G \setminus M$ is a (smooth) manifold.

We now consider the topology on such quotients, admitting a setting that is a bit more general.

Definition and Lemma 3.1. Let X be a topological space. Let $f: X \to Y$ be surjective. Then Y has exactly one topology such that

(i) f is continuous

(ii) for any topological space Z and any continuous map $g: X \to Z$ we have: if there is a map $\overline{g}: Y \to Z$ such that



commutes, then \overline{g} is already continuous.

This topology is called the quotient topology on Y. A surjective map such that Y carries the quotient topology is called a topological quotient. Moreover the quotient topology is characterized by the following:

$$U \subset Y \text{ is open} \iff f^{-1}(U) \subset X \text{ is open.}$$

Proof:

(a) Uniqueness of the topology:

Let \mathcal{O}_1 and \mathcal{O}_2 be two topologies on Y with properties (i) and (ii). Then we get the following commutative diagram



As the maps $id: (Y, \mathcal{O}_1) \to (Y, \mathcal{O}_2)$ and $id: (Y, \mathcal{O}_2) \to (Y, \mathcal{O}_1)$ are both continuous, we have $\mathcal{O}_1 = \mathcal{O}_2$. The topology is thus unique (if it exists).

(b) Existence of such a topology:

We define: a subset $U \subset Y$ is open iff $f^{-1}(U)$ is open. This is a topology:

(i) \emptyset and Y are open in Y, as $\emptyset = f^{-1}(\emptyset)$ and $X = f^{-1}(Y)$ are open in X.

(ii) Assume that U_1, \ldots, U_k are open in Y. Then

$$f^{-1}\left(\bigcap_{i=1}^{k} U_{i}\right) = \bigcap_{i=1}^{k} \left(f^{-1}\left(U_{i}\right)\right)$$

is also open.

(iii) Assume that $U_i \ i \in I$ are open in Y. Then

$$f^{-1}\left(\bigcup_{i\in I}U_i\right) = \bigcup_{i\in I}\left(f^{-1}\left(U_i\right)\right)$$

is also open.

f is continuous: obvious!

Continuity of maps \overline{g} :

Assume Z and \overline{g} as in (ii). Let W be open in Z. Then $g^{-1}(W)$ is open in X. Note that $g^{-1}(W) = f^{-1}(\overline{g}^{-1}(W))$. By the definition of the topology on Y at the beginning of this step, this holds only if $\overline{g}^{-1}(W)$ is open in Y.

Examples 3.2.

- 1.) $f_1: \mathbb{R}^2 \to \mathbb{R}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x$, is a topological quotient. The map f_1 is an open map, i.e., it maps open subsets to open subsets.
- 2.) $f_2: \mathbb{R} \to \mathbb{R},$

$$t \mapsto \begin{cases} t & \text{if } t < 0\\ 0 & \text{if } 0 \le t \le 1\\ t-1 & \text{if } 1 < t \end{cases}$$

is a topological quotient. The map f_2 is not an open map, as it maps to open set (0, 1) to the non-open subset $\{0\}$.

3.) The composition of two topological quotients is again a topological quotient. This is part of the following stronger statement: Let $f: X \to Y$ be a topological quotient, Z a topological space, $g: Y \to Z$ a surjective map. Then $h := g \circ f$ is a topological quotient if and only if g is a topological quotient. "only if": Let h be a topological quotient, then for $U \in Z$:

$$U$$
 open in $Z \iff \underbrace{h^{-1}(U)}_{=f^{-1}\left(g^{-1}(U)\right)}$ open in $X \iff g^{-1}(U)$ open in Y ,

thus g is a topological quotient.

"if": Let g be a topological quotient, then for $U \in Z$:

$$U$$
 open in $Z \iff g^{-1}(U)$ open in $Y \iff \underbrace{f^{-1}(g^{-1}(U))}_{=h^{-1}(U)}$ open in X ,

thus h is a topological quotient.

WARNING 3.3. If $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ are topological quotients, then in general $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is not a topological quotient. This product map $f_1 \times f_2$ will be continuous, but in some cases the quotient topology on $Y_1 \times Y_2$ has more open subsets than the product of the quotient topologies on Y_1 and Y_2 .

Proposition 3.4. Assume that a topological group G acts continuously on the topological space X. Then

$$\pi: X \to G \setminus X$$
$$p \mapsto [p] = G \cdot p$$

is an open map.

Proof: For any $\sigma \in G$ we define $\ell_{\sigma} := a(\sigma, \bullet): X \to X, x \mapsto \sigma x$ which is continuous. As $\ell_{\sigma^{-1}}$ is the inverse to ℓ_{σ} , we know that ℓ_{σ} is a homeomorphism. Let $V \subset X$ be open, thus $\ell_{\sigma}(V)$ is also open. Thus $\bigcup_{\sigma \in G} \ell_{\sigma}(V) = \pi^{-1}(\pi(V))$ is open in X, and finally we get that $\pi(V)$ is open in $G \setminus X$.

Tue 7.5.

Corollary 3.5. Let again $G \curvearrowright X$ (continuously). If X is second countable, then $G \setminus X$ is also second countable.

Proof: Let $\mathcal{B} = \{U_i \mid i \in I\}$ be a countable basis of the topology of X. Then $\pi(U_i)$ is open in $G \setminus X$ for any $i \in I$. Thus

$$\widetilde{\mathcal{B}} = \left\{ \pi(U_i) \mid i \in I \right\}$$

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is a countable set of open subsets of $G \setminus X$.

It remains to show that $\widetilde{\mathcal{B}}$ is a basis of the topology of $G \setminus X$.

Let $V \subset G \setminus X$ be open. Then, by definition of the quotient topology, $\pi^{-1}(V)$ is open in X. Thus there is $J \subset I$ such that

$$\pi^{-1}(V) = \bigcup_{j \in J} U_j \, .$$

Using the surjectivity of f, it follows that

$$V = \pi(\pi^{-1}(V)) = \bigcup_{j \in J} \underbrace{\pi(U_j)}_{\in \widetilde{\mathcal{B}}}.$$

4 Quotient manifolds

4.1 The theorem about smooth structures on quotients

Recall: A smooth map $f: M \to N$ is called a

- (i) submersion iff $\forall x \in M : df|_x : T_x M \to T_{f(x)} N$ is surjective
- (ii) immersion iff $\forall x \in M : df|_x : T_x M \to T_{f(x)}N$ is injective

(iii) local diffeomorphism, iff $\forall x \in M$: are open neighborhoods U of x in M and V of f(x) in N such that

$$f|_U: U \to V$$

is a diffeomorphisms. Using the local reversal theorem we obtain: f is a local diffeomorphism iff $\forall x \in M : df|_x : T_x M \to T_{f(x)}N$ is bijective.

Theorem 4.1. Let a Lie group G act smoothy, freely, and properly on a manifold M. Equip $G \setminus M$ with the quotient topology. Then $G \setminus M$ carries a unique smooth structure such that $\pi: M \to G \setminus M$ is a submersion. Furthermore dim $G \setminus M =$ dim M – dim G.

Examples 4.2.

1.) $\{\pm 1\}$ acts on $S^n \subset \mathbb{R}^{n+1}$, compare Example 2.2 5.). This action is discrete, smooth, free, and proper. We equip $\mathbb{R}P^n \cong \{\pm 1\} \setminus S^n$ with the quotient topology

and the smooth structure given in Diff. geom. I, Exercise Sheet 1, Exercise 3. It is easy to check that the canonical projection map $\pi: S^n \to \mathbb{R}P^n$ is then smmooth and a submersion.

- 2.) We consider again the Hopf action of S^1 on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by complex multiplication, compare Example 2.2 6.). This action is non-discrete, smooth, free, and proper. We equip the complex projective space $\mathbb{CP}^n = S^1 \setminus S^{2n+1}$ with the smooth structure given in Exercise 4.3. Then the canonical projection, called the Hopf fibration, $\pi: S^{2n+1} \to \mathbb{CP}^n$ is a submersion.
- 3.) Let \mathbb{Z}^n act on \mathbb{R}^n by addition (or equivalently expressed: by translation). This action is discrete, smooth, free, and proper. In Diff. geom. I, Exercise Sheet 3, Exercise 3 we introduced a manifold structure on $T^n = \mathbb{Z}^n \setminus \mathbb{R}^n$. The topology is the quotient topology, and the smooth structure is the one of the above theorem.
- 4.) In general, if G is discrete, then dim $G \setminus M = \dim M$, thus the submersion is in fact a local diffeomorphism. Furthermore, from Exercise Sheet 3, Exercise 3 a) we see that G acts properly discontinuously (and freely). Thus we are in the setting of Diff. geom. I, Exercise Sheet 13, Exercise 4, if we replace the right action by a left action, i.e., by defining in the exercise $R(p, \sigma) \coloneqq a(\sigma^{-1}, p)$, for $\sigma \in G, p \in M$. Thus $M \to G \setminus M$ is a surjective covering.
- 5.) Conversely, you may ask whether every covering $\pi: M \to N$ that is surjective and a local diffeomorphism arises this way. Here the answer is "No", however, it is yes, M is simply-connected. By choosing $p \in M$, a covering map $\pi: M \to N$ yields a group homomorphism $\pi_*: \pi_1(M, p) \to \pi_1(N, \pi(p))$, and one can show that this is injective. Assuming N and M are connected, the answer the above question is "Yes" if and only if $\pi_*(\pi_1(M, p))$ is a normal subgroup of $\pi_1(N, \pi(p))$. Such covers are called **normal coverings** or **Galois coverings**.

Let us formular the complex analogue of Diff. geom. I, Exercise Sheet 1, Exercise 3.

Exercise 4.3 (Potential exercise for Differential Geometry I). Let $n \in \mathbb{N}$ and $\mathbb{C}P^n$ be the set of 1-dimensional complex vector subspaces of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$.

(a) Identify $\mathbb{C}P^n$ with the quotient $(\mathbb{C}^{n+1}\setminus\{0\})\setminus$ ~, where

$$x \sim y \iff \exists \lambda \in \mathbb{C}^{\times} \ s.t. \ x = \lambda y$$

and endow it with the quotient topology. Show that $\mathbb{C}P^n$ is a compact Hausdorff space satisfying the second axiom of countability.

(b) Show that the maps

$$U_j \coloneqq \{ [x] \in \mathbb{C}\mathbb{P}^n \,|\, x_j \neq 0 \} \xrightarrow{\varphi_j} \mathbb{C}^n \cong \mathbb{R}^{2n}, \ [z] \mapsto \frac{1}{z_j} (z_1, \dots, \widehat{z_j}, \dots, z_{n+1}), \ 1 \le j \le n+1,$$

are well-defined homeomorphisms (the " \hat{z}_j " means omitting " z_j ", and $z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$).

- (c) Show that $\mathcal{A} = (\varphi_j : U_j \to \mathbb{R}^{2n})_{j \in \{1,2,\dots,n+1\}}$ is an atlas for $\mathbb{C}P^n$.
- (d) For $i, j \in \{1, ..., n+1\}$, $i \neq j$ show that $\varphi_j(U_i \cap U_j)$ is an open subset of \mathbb{R}^{2n} and that

$$\varphi_i \circ (\varphi_j)^{-1} : \varphi_j (U_i \cap U_j) \to \varphi_i (U_i \cap U_j)$$

is a C^{∞} -diffeomorphism.

Lemma 4.4. Assume that a Lie group acts G smoothly acts on a manifold M, $p \in M$. if $s_p: G \to M$, $\sigma \mapsto \sigma p$ is injective (on a neighborhood of 1), then s_p is an immersion.

In the following, we call this map s_p the orbit map¹ of p.

Examples 4.5.

- 1.) $\mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is an injective smooth map, but it is not an immersion. Thus it cannot be obtained as a map s_p as above for a suitable smooth action of $G = \mathbb{R}$ on $M = \mathbb{R}$. However, it is the map s_p for $G = M = \mathbb{R}, p = 0$ and for the non-smooth, continuous action $a(\sigma, x) = \sqrt[3]{[\sigma + x^3]}$.
- 2.) For $\alpha \in \mathbb{R} \times \mathbb{Q}$, $T^2 := \mathbb{Z}^2 \setminus \mathbb{R}^2$ we define $a: \mathbb{R} \times T^2 \to T^2$ as

$$a(t, [(x, y)] \coloneqq [(x + t, y + \alpha t)]$$

where $t, x, y \in \mathbb{R}$, thus $[(x, y)] \in T^2$.

 $^{^{1}\}mathrm{I}$ do not think that this terminology is used in the literature, but it seems a reasonable name to me

Figure in the lecture, not yet drawn electronically A 2-dimensional torus with a line of irrational slope α

For any $p \in T^2$, the map $s_p: \mathbb{R} \to T^2$ is an injective immersion. However $\mathbb{R}p = s_p(\mathbb{R})$ is not a submanifold.

Proof of Lemma 4.4: We assume that there is an open neighborhood U of 1 such that $s_p|_U$ is injective. We write $\ell_{\sigma}, \sigma \in G$, both for left multiplication $\ell_{\sigma}: G \to G$ and for left multiplication $\ell_{\sigma}: M \to M$.

" $d_{1}s_{p}: \mathfrak{g} \to T_{p}M$ is injective": Assume $ds_{p|_{1}}(X) = 0$ for $X \in \mathfrak{g}$ and define $\gamma(t) := \exp(tX)$, i.e., γ is a 1-parameter subgroup and satisfies $\dot{\gamma}(0) = X$ and $\gamma(t_{0} + t) = \gamma(t_{0})\gamma(t)$. We calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} (\gamma(t) \cdot p) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\gamma(t_0 + t) \cdot p) = \mathrm{d}\ell_{\gamma(t_0)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \underbrace{(\gamma(t) \cdot p)}_{=s_p(\gamma(t))} \right) \\ &= \mathrm{d}\ell_{\gamma(t_0)} \circ \mathrm{d}s_p(\dot{\gamma}(0)) = 0 \,. \end{aligned}$$

As a consequence $\gamma(t) \cdot p = p$. for any $t \in \mathbb{R}$. For t close to 0 we get a contradiction to the injectivity of $s_p|_U$.

"d_{σ}s_p: $\mathfrak{g} \to T_{\sigma p}M$ is injective for all $\sigma \in G$ ": We calculate

$$s_p(\ell_{\sigma}(\tau)) = \ell_{\sigma}(\tau) \cdot p = (\sigma\tau)p = \sigma(\tau p) = \sigma \cdot s_p(\tau) = \ell_{\sigma}(s_p(\tau)).$$

Hence the diagram

commute. As $d_1\ell_{\sigma}$ and $d_p\ell_{\sigma}$ are isomorphisms and as d_1s_p is injective, $d_{\sigma}s_p = d_p\ell_{\sigma} \circ d_1s_p \circ (d_1\ell_{\sigma})^{-1}$ is injective as well.

Proof of Theorem 4.1:

(a) If M is second countable, then $G \setminus M$ with the quotient topology is also second countable topological space, see Corollary 3.5. In the literature there two nonequivalent definition of "a manifold" see the section "Conventions and Notations". If you belong to the group of mathematicians for which a manifold is required to be second countable, then you now have seen the proof that $G \setminus M$ is second countable; and you may proceed with proof item (b). If you belong to the group of mathematicians for which a manifold is only required to be paracompact², then you can argue "in each component" in a a similar way.³

(b) " $_{G} \setminus M$ is a Hausdorff space.":

The action is proper, thus by Definition 2.3 the associated shear map

$$G \times M \xrightarrow{\Theta} M \times M$$
$$(\sigma, x) \mapsto (\sigma x, x)$$

is a proper map. Thus by Lemma 2.8

$$\operatorname{image}(\Theta) = \Theta(G \times M) = \{(\sigma \cdot x, x) \mid \sigma \in G, \quad x \in M\}$$

is closed. Obviously for the equivalence relation ~ defined by being in the same G-orbit, we have

$$\forall x, y \in X : (x, y) \in \operatorname{image}(\Theta) \iff x \sim y \iff [x] = [y].$$

Suppose $[x] \neq [y]$. Then $(x, y) \notin \text{image}(\Theta)$. As $\text{image}(\Theta)$ is closed, (x, y) is an inner point of $(M \times M) \setminus \text{image}(\Theta)$. This means that x has an open neighborhood U_x in

Fr 10.5.

 $^{^{2}}$ A locally Eudlidean Hausdorff space with a C^{1} -atlas is paracomact, if and only if every connected component is second countable. To understand our lecture, you may use this as a definition of "paracompactness".

³More precisely: Consider a connected component M_0 and consider $GM_0 \coloneqq \{\sigma p \mid \sigma \in G, p \in M_0\}$. One checks that $G \setminus GM_0$ is a connected component of $G \setminus M$, and any connected component can be obtained this way.

As M is paracompact, M_0 is second countable. The restriction of the canonical map $\pi: M \to G \setminus M$ yields an open and surjective map $M_0 \xrightarrow{\pi} G \setminus GM_0$, see Proposition 3.4, and as in the proof of Corollary 3.5 you see that the second countability for M_0 implies the second countability of $G \setminus GM_0$. Thus we have seen that any connected component $G \setminus M$ is second countable.

M and y has an open neighborhood U_y in M, such that

$$(U_x \times U_y) \cap \operatorname{image}(\Theta) = \emptyset.$$

the sets $\pi(U_x)$ and $\pi(U_y)$ are open due to Proposition 3.4, thus they are (open) neighborhoods of [x] and [y]. For proving the Hausdorff property, it thus only remains to check that $\pi(U_x) \cap \pi(U_y) = \emptyset$.

Suppose that $[z] \in \pi(U_x) \cap \pi(U_y)$, and one may choose the representative z of this class such that $z \in U_x$. From $[z] \in \pi(U_y)$ we get the existence of a $w \in U_y$ with [z] = [w]. Thus there is a $\sigma \in G$ with $z = \sigma w$. We obtain the contradiction

$$(z,w) = (\sigma w, w) = \Theta(\sigma, w) \in \operatorname{image}(\Theta) \cap (U_x \times U_y) = \emptyset.$$

(c) "Uniqueness of a smooth structure on $G \setminus M$ "

Suppose we have two smooth atlantes \mathcal{A}_1 and \mathcal{A}_2 on $G \setminus M$ such that $M \xrightarrow{\pi} (G \setminus M, \mathcal{A}_i)$ is a submersion for i = 1, 2. We now apply the following lemma:

Lemma A.1.2. Let $f : X \to Y$ be a surjective submersion from the C^{∞} -manifold X to the C^{∞} -manifold Y, and let Z be a further C^{∞} -manifold. Let $h : Y \to Z$ be a map. Then h is smooth if and only if $h \circ f$ is smooth.

We apply the lemma to the diagram



twice:

- Once for X := M, $Y := (G \setminus M, \mathcal{A}_1)$, $Z := (G \setminus M, \mathcal{A}_2)$, $f = \pi$ and h = id. Then the smoothness of $h \circ f = \pi : M \to (G \setminus M, \mathcal{A}_2)$ gives us the smoothness of $h = id: (G \setminus M, \mathcal{A}_1) \to (G \setminus M, \mathcal{A}_2)$.
- Once for X := M, $Y := (G \setminus M, \mathcal{A}_2)$, $Z := (G \setminus M, \mathcal{A}_1)$, $f = \pi$ and $h = \mathrm{id}$. Then the smoothness of $h \circ f = \pi : M \to (G \setminus M, \mathcal{A}_1)$ gives us the smoothness of $h = \mathrm{id} : (G \setminus M, \mathcal{A}_2) \to (G \setminus M, \mathcal{A}_1)$.

Thus $\operatorname{id}: (G \setminus M, \mathcal{A}_1) \to (G \setminus M, \mathcal{A}_2)$ is a diffeomorphism, which says that the two

smooth structures are the same.

(d) The Construction of a suitable smooth structure on $G \setminus M$ is a bit more involved and will be proven in the next subsection, Subsection 4.2.

4.2 The construction of a suitable smooth structure on the quotient

In this subsection, G will always be a Lie group, acting smoothly and freely on a smooth manifold M.

Definition 4.6. In the following a submanifold S of M will be called **transversal** (to the orbits of G) if we have for all $p \in S$:

$$T_p S \oplus \operatorname{image}(d_1 s_p) = T_p M.$$
 (4.1)

Recall $d_1 s_p: \mathfrak{g} \to T_p M$ is the differential of the orbit map $s_p: G \to M, \ \sigma \mapsto \sigma p$.

As s_p is injective, Lemma 4.4 tells us that dim image $(d_{1}s_p) = \dim G$. Thus for a transversal submanifold we have

$$\dim G = \dim M - \dim S \,.$$

Example 4.7. We continue with Example 4.2 2.). Any non-trivial \mathbb{R} -linear function $L: \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \to \mathbb{R}$ defines a hypersurface $N_L := \{x \in S^{2n+1} \mid L(x) = 0\}$ which is a totally geodesic sphere in S^{2n+1} of dimension 2n.

For which $p \in N_L$ do we have

$$T_p N_L \oplus \operatorname{image}(d_1 s_p) = T_p M?$$

Note that $i \in T_{\mathbb{1}}S^1$, and we have the above direct sum decomposition iff $d_{\mathbb{1}}s_p(i) \notin T_pN_L$. Because $d_{\mathbb{1}}s_p(i) = ip$ and $T_pN_L = \ker L \cap p^{\perp}$ we see that the above decomposition is direct precisely on

$$S_L := \{x \in S^{2n+1} \mid L(x) = 0 \text{ and } L(ix) \neq 0\}.$$

We see that S_L is an open subset of N_L and

$$N_L \smallsetminus S_L = S^{2n+1} \cap \ker (\mathbb{C}^{n+1} \to \mathbb{C}, \quad x \mapsto L(x) - iL(ix))$$

which is a totally geodesic hypersurface in N_L diffeomorphic to S^{2n-1} . Every orbit that passes through $p \in S_L$ is of the form $(\cos t) \cdot p + (\sin t)ip$ which intersects S_L in p and -p, i.e., once in the component

$$S_{L,+} \coloneqq \{x \in S^{2n+1} \mid L(x) = 0 \text{ and } L(ix) > 0\},\$$

and once in the component

$$S_{L,-} := \{x \in S^{2n+1} \mid L(x) = 0 \text{ and } L(ix)y0\}.$$

Thus S_L , $S_{L,+}$ and $S_{L,+}$ for any non-trivial $L: \mathbb{C}^{n+1} \to \mathbb{R}$ and all of their open subsets are transversal to the orbits.

Remarks 4.8.

- 1.) If you try to imagine how a possible manifold S will look like, think of a small manifold. Even when G and M are compact, one can rarely choose compact transveral submanifolds S.
- 2.) We state that $G \cdot p = \{\sigma p \mid \sigma \in G\} = s_p(G)$ is a submanifold of M, and that $T_{\sigma p}(G \cdot p) = \text{image}(d_{\sigma}s_p)$. We will not prove this statement here, and we will not use it in the following. A proof will follow immediately from Theorem 4.1 using the implicit function theorem which then also shows $T_{\sigma p}(G \cdot p) = \ker d_{\sigma p}\pi$.

Lemma 4.9. Let $G \curvearrowright M$ freely and smoothly. Then for any $p \in M$, there is a transversal submanifold S through p.

Proof: Choose a vector space $W \subset T_pM$ with

$$W \oplus \operatorname{image}(\operatorname{d}_{\mathbb{1}} s_p) = T_p M$$
.

It is easy to construct⁴ a smooth submanifold S_0 in M with $p \in S_0$ and and $T_pS_0 = W$. Then

$$S \coloneqq \left\{ x \in S_0 \mid T_x S_0 \cap \operatorname{image}(d_1 s_p) \neq \{0\} \right\}$$

⁴The submanifold S_0 can either be constructed in a chart or by taking the Riemannian exponential map \exp^g for some⁵ Riemannian metric g and the defining $S_0 := \{\exp X \mid X \in W, g(X, X) < \}$

 $[\]varepsilon^2$ for some small $\varepsilon > 0$.

is an open subset S_0 containing p. We see that S is a transversal submanifold with $p \in S$.

Lemma 4.10. If S is a transversal submanifold, then

$$G \times S \xrightarrow{\vartheta} M$$
$$(\sigma, p) \longmapsto \sigma p$$

is a local diffeomorphism.

Proof: Because of the local reversal theorem, the statement is equivalent to showing that

$$d_{(\sigma,p)}\vartheta:T_{\sigma}G\oplus T_{p}S\longrightarrow T_{\sigma p}M \tag{4.2}$$

is an isomorphism for all $\sigma \in G$ and all $p \in S$.

For $\sigma = \mathbb{1}, X \in T_{\mathbb{1}}G, Y \in T_pS$ we calculate

$$\mathrm{d}_{(1,p)}\vartheta(X,Y) = \mathrm{d}_{1}s_{p}(X) + Y.$$

With (4.1) this implies that image $(d_{(\sigma,p)}\vartheta) = T_pM$, i.e., we have (4.2) for $\sigma = 1$.

Now consider arbitrary $\sigma \in G$. From $\sigma(\tau p) = (\sigma \tau)p$ we see that the diagram



and its derivative at $\tau=\mathbbm{1}$



commute. Now as three arrows in the last diagram are already known to be isomorphisms, the remaining on, i. e., $d_{(\sigma,p)}\vartheta$, is also an isomorphism. This gives (4.2) in general.

End Fr 10.5. Read following on Fr 17.5.

Definition 4.11. A transversal submanifold is small if

(i) $G \times S \xrightarrow{\vartheta} M$ is injective and a homeomorphism onto its image (thus an embedding of codimension 0).

(ii) there is a diffeomorphism $S \xrightarrow{y} V \mathfrak{C} \mathbb{R}^{\dim M - \dim G}$

Lemma 4.12. Assume that the action $G \curvearrowright M$ is smooth, free and proper. For each $p \in M$, there is a small transversal manifold S through p.

Proof: Let S_0 be a transversal submanifold through $p \in M$. We choose a Riemannian metric g on S, which allows us to define the open balls $B_{\varepsilon}^{(S,g)}(p)$ of radius ε around p in (S,g). We define $S_i := B_{1/i}^{(S,g)}(p)$. For a sufficiently large $i \in \mathbb{N}$, we will prove that $S := S_i$ satisfies Conditions (i) and (ii) in Definition 4.11. Thus we will have proven that S is ia small transversal submanifold through p. "(i)": Suppose that for all $i \in \mathbb{N}$:

$$G \times S_i \xrightarrow{\vartheta} M$$

is not injective. Thus there are $(\sigma_i, p_i), (\tilde{\sigma}_i, \tilde{p}_i) \in G \times M, (\sigma_i, p_i) \neq (\tilde{\sigma}_i, \tilde{p}_i)$ such that $\sigma_i p_i = \tilde{\sigma}_i \tilde{p}_i$. This gives $(\tilde{\sigma}_i)^{-1} \sigma_i p_i = \tilde{p}_i$. Obviously we have

$$\lim_{i \to \infty} p_i = p \text{ and } \lim_{i \to \infty} \tilde{p}_i = p.$$

As G acts properly, Proposition 2.7 tells us that a subsequence⁶, of $\tau_i := (\tilde{\sigma}_i)^{-1} \sigma_i$

 $^{^{6}}$ We will pass to the subsequence without adapting the notation, for better readability.

converges to some $\tau_{\infty} \in G$. In the limit we $\tau_i p_i = \tilde{p}_i$ gives $\tau_{\infty} p = p$. As G acts freely, this implies $\tau_{\infty} = 1$.

As ϑ is a local diffeomorphism, there is an open neighborhood U of (1, p) in $G \times S_0$ such that $\vartheta|_U$ is a diffeomorphism onto its image. There is some $i_0 \in \mathbb{N}$ such that any $i \ge i_0$ satisfies $(\tau_i, p_i) \in U$ and $(1, \tilde{p}_i) \in U$. Then

$$\vartheta(\tau_i, p_i) = \tau_i p_i = \tilde{p}_i = \vartheta(\mathbb{1}, \tilde{p}_i),$$

and hence $\tau_i = 1$ and $p_i = \tilde{p}_i$, which gives the contradiction $(\sigma_i, p_i) = (\tilde{\sigma}_i, \tilde{p}_i)$.

For any $i \ge i_0$ we thus know that $\vartheta_i := \vartheta|_{G \times S_i} : G \times S_i \to M$ is injective.

Now for $i \ge i_0 + 1$ we will show that ϑ_i is homeomorphism onto its image, i.e., it is also an open map. Note that $S_i \subset \overline{S_i} \subset S_{i-1}$. Let K be a compact neighborhood of 1 in G. As ϑ_{i-1} is continuous and injective, this also hold from

$$\vartheta_{i-1}\Big|_{K\times\overline{S_i}}:K\times\overline{S_i}\to\vartheta\left(K\times\overline{S_i}\right)\subset M.$$

As any continuous bijective map from a compact space to a Hausdorff space is a homeomorphism, the above map is a homeomorphism. Thus by restricting further ontinuous and injective, this also hold for

$$\vartheta_i|_{\mathring{K}\times S_i}: \mathring{K}\times S_i \to \vartheta(\mathring{K}\times S_i) \subset M.$$

We precompose this with the homeomorphism $\ell_{\sigma^{-1}} \times \operatorname{id}: (\sigma \mathring{K}) \times S_i \to \mathring{K} \times S_i$, and thus

$$\vartheta_i\big|_{(\sigma \mathring{K}) \times S_i} = \vartheta_i\big|_{\mathring{K} \times S_i} \circ \left(\ell_{\sigma^{-1}} \times \mathrm{id}\right)$$

defines a homeomorphism $(\sigma \mathring{K}) \times S_i \to \vartheta ((\sigma \mathring{K}) \times S_i)$. Thus the domain of ϑ_i is covered⁷ by a collection of open sets $U_{\sigma} \coloneqq \sigma \mathring{K}$, $\sigma \in G$, such that $\vartheta_i|_{U_{\sigma}} \colon U_{\sigma} \to M$ is open. This implies that ϑ_i itself is open and thus a homeomorphism for any $i \ge i_0 + 1$.

"(ii)": Take a chart $y: \hat{U} \to \hat{V}$ of S_0 containing p. Then $S_i \subset \hat{U}$ for sufficiently large $i \ge i_0 + 1$.

We are thus may assume that we are in the following setting:

Setting 4.13. Let $G \curvearrowright M$ be a smooth, proper, free action. Let $p \in M$, $n = \dim M$,

⁷in German: überdeckt

and $k = \dim G$. The submanifold S of M is a small G-transversal submanifold through $p \in M$. Then $\vartheta(G \times S)$ is open in M, and $\vartheta: G \times S \to \vartheta(G \times S)$ is a local diffeomorphism and a (global) homeomorphism, thus it is a diffeomorphism. Furthermore ϑ is G-equivariant, i. e., for $\tau, \sigma \in G$ and $q \in S$ we have $\tau \cdot \vartheta(\sigma, q) =$ $\vartheta(\tau\sigma, q)$. In particular, all orbits $G \cdot q$ are submanifolds with $T_{\sigma q}(G \cdot q) = \operatorname{image}(d_{\sigma}s_q)$. The Lie group G acts smoothly on $\vartheta(G \times S)$, thus $\vartheta(G \times S)$ is a smooth G-space.

Furthermore we have a chart $y: S \to V \ \mathfrak{C} \ \mathbb{R}^{n-k}$.

Lemma 4.14. We assume the setting above. Then the map $\Phi_S := \pi \circ \vartheta \circ (1, id) \circ y^{-1}$ obtained by the chain of maps

$$\mathbb{R}^{n-k} \supseteq V \xrightarrow{y^{-1}} S \xrightarrow{(1,\mathrm{id})} G \times S \xrightarrow{\vartheta} \vartheta(G \times S) \xrightarrow{\pi} G \setminus \vartheta(G \times S) \boxtimes G \setminus M$$
$$p \longmapsto (1,p)$$

is a homeomorphism.

Corollary 4.15. $G \setminus M$ is a topological manifold.

Proof of Lemma 4.14: In the following diagram



all maps except α and β are already defined and continuous, and obviously $\operatorname{pr}_S \circ (\mathbb{1}, \operatorname{id}) = \operatorname{id}_S$. The spaces $G \setminus G \times S$ and $G \setminus \vartheta(G \times S)$ carry the quotient topology, thus the maps π and π' are topological quotients. By applying Condition (ii) from Definition and Lemma 3.1, we get a continuous map $\alpha: G \setminus G \times S \to S$, making the left square commute. Obviously α is bijective, and $\pi' \circ (\mathbb{1}, \operatorname{id})$ is a continuous right inverse of α , thus α is a homeomorphism. If we apply Condition (ii) from Definition and Lemma 3.1 to $X := G \times S$, $Y := G \setminus G \times S$, $Z := G \setminus \vartheta(G \times S)$, $f := \pi'$, $g := \pi \circ \vartheta$, then we get a well-defined continuous map β making the right square commute. The map β is bijective. If we apply Condition (ii) from Definition and Lemma 3.1 to

 $X := \vartheta(G \times S), Y := G \setminus \vartheta(G \times S), Z := G \setminus G \times S, f := \pi, g := \pi' \circ \vartheta^{-1}$, then we see that β^{-1} is also continuous.

So the whole diagram commutes and consists of continuous maps. Thus

$$\pi \circ \vartheta \circ (\mathbb{1}, \mathrm{id}) = \beta \circ \alpha^{-1}$$

is a homeomorphismus. Precomposition with the homomorphism y^{-1} yields the statement. $\hfill\blacksquare$

Lemma 4.16.

$$\mathcal{A} \coloneqq \left\{ \Phi_S^{-1} \mid S \text{ is a small transversal submanifold} \right\}$$

is a smooth atlas on $G \setminus M$.

Proof: We have already seen that \mathcal{A} is a C^0 -atlas for $G \setminus M$. Thus it remains to check that the transition maps are smooth. thus consider two small transversal submanifolds S and \tilde{S} , not necessarily running through a common point. The ϑ -map for \tilde{S} will be called $\tilde{\vartheta}$. There are open subsets $U \Subset S$ and $\tilde{U} \Subset \tilde{S}$ such that

$$\vartheta(G \times \tilde{U}) = \vartheta(G \times S) \cap \tilde{\vartheta}(G \times \tilde{S}) = \tilde{\vartheta}(G \times U).$$

Let $y: U \to V \supseteq \mathbb{R}^{n-k}$ and $\tilde{y}: \tilde{U} \to \tilde{V} \supseteq \mathbb{R}^{n-k}$ be two charts of U and \tilde{U} . We get charts as in Lemma 4.14 for $G \setminus M$

$$\Phi_{S}^{-1}: G \setminus \vartheta(G \times U) \to V, \quad \Phi_{\tilde{S}}^{-1}: G \setminus \tilde{\vartheta}(G \times \tilde{U}) \to \tilde{V},$$

We have to show the smoothness of the transition map

$$\Phi_{\tilde{S}}^{-1} \circ \Phi_{S} \Big|_{\Phi_{s}^{-1}(\vartheta(G \times U))} : \Phi_{S}^{-1}(\vartheta(G \times U)) \longrightarrow \Phi_{\tilde{S}}^{-1}(\tilde{\vartheta}(G \times \tilde{U}))$$

which is the composition

Thus the smoothness holds, as it is a composition of smooth maps.

Lemma 4.17. We equip $G \setminus M$ with the smooth structure of the preceding lemma. Then $\pi: M \to G \setminus M$ is a submersion.

Proof: We fix a small transversal submanifold S with associated map ϑ . It is sufficient to verify the submersion property of π on the open subset $\vartheta(G \times S)$ as such subsets cover all of M. Now, out of the commuting diagram in the proof of Lemma 4.14, we get the following commuting diagram

$$\begin{array}{c|c} G \times S & \xrightarrow{\vartheta} & \vartheta(G \times S) \\ pr_S & & & \\ & & & \\ & & & \\ S & \xrightarrow{\rho} & G \setminus \vartheta(G \times S) \end{array}$$

where the map ϑ is a diffeomorphism by the definition of "small transversal submanifold" and where $\rho := \beta \circ \alpha^{-1}$ is a diffeomorphism by the construction of the smooth structure on $G \setminus M$. As $\operatorname{pr}_S : G \times S \to S$ is obviously a submersion.

The formula dim $G \setminus M = \dim M - \dim G$ is obvious from the construction of the smooth structure on $G \setminus M$. The proof of Theorem 4.1 is thus complete.

5 Further examples

5.1 Frame bundles

For a Riemannian manifold (M,g) and $p \in M$ we define $P_O(M,g)|_p$ as the set of all orthonormal bases of (T_pM,g_p) . The group O(n) acts on the right on this bundle by the usual transformation of basis formula. In fact let (e_1,\ldots,e_n) be an orthonormal basis, viewed as a row vector whose coefficients are vectors in T_pM . Let $A = (a_{ij}) \in O(n)$, then one defines

$$(\tilde{e}_1,\ldots,\tilde{e}_n) \coloneqq (e_1,\ldots,e_n) \cdot A$$

by matrix multiplication, i. e., $\tilde{e}_j = \sum_{i=1}^n a_{ij} e_i$. This right action is transitive and free, and there is a unique smooth structure on $P_O(M,g)|_p$ such that $O(n) \to P_O(M,g)|_p$, $A \mapsto (e_1, \ldots, e_n) \cdot A$ is a diffeomorphism. We define $P_O(M,g) := \bigcup_{p \in M} P_O(M,g)|_p$. Then $P_O(M,g)$ carries a unique smooth topology, such that for any $U \Subset M$ the following property holds:

Let $e_i: U \to P_O(M, g)$, i = 1, 2, ..., n be maps such that

$$\mathcal{E}(p) \coloneqq (e_1(p), \dots, e_n(p)) \in P_{\mathcal{O}}(M, g)|_p$$

Then \mathcal{E} is smooth as a map $U \to P_O(M, g)$ if and only if e_i is smooth as a vector field on U for any i.

The group O(n) acts smoothly, freely, but no longer transitively on $P_O(M,g)$, and the orbits are the subsets $P_O(M,g)|_p$ which are in fact submanifolds. One can check that this action is proper, and we consider the quotient space, which is a smooth manifold by Theorem 4.1. The "canonical"

$$I: M \to P_O(M,g)/O(n)$$

that maps $p \in M$ to the orbit $P_O(M,g)|_p$ is a diffeomorphism. Usually one identifies M with this quotient.

If M carries other structure than a Riemannian metric, one can often do similar definitions for the structure group and the adapted bases.

Examples 5.1.

- 1.) If M has no structure at all, we may take all frames. This yields $P_{\text{GL}}(M,g)$, which is a manifold on which $\text{GL}(n,\mathbb{R})$ acts smoothly, freely, and properly, and such that $M \cong P_{\text{GL}}(M,g)/\text{GL}(n,\mathbb{R})$.
- 2.) If M has an orientation, we may take all positively oriented frames. This yields $P_{\mathrm{GL}_+}(M,g)$, which is a manifold on which $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathbb{R}^{n \times n} \mid \det A > 0\}$ acts smoothly, freely, and properly, and such that $M \cong P_{\mathrm{GL}_+}(M,g)/\mathrm{GL}_+(n,\mathbb{R})$.

5.2 Homogeneous spaces

We now consider a Lie goup, and we assume that H is a subgroup of G that is closed as a subset of the topological space G. We have already mentioned, that this implies that H is also submanifold. We get an action of H on G as follows $a(\tau, \sigma) \coloneqq \ell_{\tau}(\sigma) \coloneqq \tau \sigma$ for $\tau \in H$ and $\sigma \in G$.

Lemma 5.2. The action $H \curvearrowright G$ by left multiplication as described above is a proper, free and smooth action. The same holds for other left- and right-actions $H \curvearrowright G$ given in Example 2.2 1.).

Proof: That the action is free and smooth is obvious. We will check properness using Proposition 2.7.

So let us assume that $(x_i)_{i\in\mathbb{N}}$ is a sequence in G converging to $x_{\infty} \in G$, and that $(\sigma_i)_{i\in\mathbb{N}}$ is a sequence in H, such that $(\sigma_i \cdot x_i)_{i\in\mathbb{N}}$ converges in G to $y_{\infty} \in G$. It follows that in G we have the limit

$$\lim_{i \to \infty} \sigma_i = \lim_{i \to \infty} (\sigma_i \cdot x_i \cdot x_i^{-1}) = x_\infty \cdot y_\infty^{-1}.$$

As *H* is closed, we have $x_{\infty} \cdot y_{\infty}^{-1} \in H$. Thus $(\sigma_i)_{i \in \mathbb{N}}$ converges in *H* to $x_{\infty} \cdot y_{\infty}^{-1}$ and the statement follows with Proposition 2.7.

We now, let again be H a closed subgroup in a Lie group G. We consider the manifold $G/_H$, whose elements are **left cosets**, i.e., subsets of the form $\sigma \cdot H$ where $\sigma \in G$. Left multiplication turns $G/_H$ into a G-space with a transitive G action. We want to argue, that every smooth G-space with a transitive action is of this form, as shown in the following exercise. We will thus additionally assume in this section from now on:

The topology of G is second countable

Summer term 2024

Read this on May 17,

This assumption is equivalent to the condition, that G has countably many connected components.¹ With this additional condition Sard's theorem² implies the following

Proposition 5.3 (Consequence of Sard's theorem). Let G be a Lie group with countably many connected components acting transitively and smoothly on a smooth manifold $M, p \in M$, then the orbit map

$$s_p: G \to M, \quad \sigma \mapsto \sigma \cdot p$$

is a submersion.

With this information the following exercise can be solved:

Exercise 5.4. Let G be a Lie group acting smoothly and transitively on a manifold M. Let H be the isotropy group of $p \in M$. Show that there is a unique G-equivariant diffeomorphism $F: G/H \to M$ that maps $1 \cdot H$ to p.

Smooth G-spaces with a transitive G-action are called homogeneous spaces. They are essentially given by the pair (G, H) where G is a Lie group and H a closed subgroup. However, a given manifold M can be obtained in several ways as a homogeneous space $G/_H$, thus one always has to consider M as a G-space. For example as smooth manifolds we have $S^{2n+1} \cong O(2n+2)/O(2n+1) \cong U(n+1)/U(n)$, but this does not hold as homogeneous spaces, as it is a G-space for another G. However, when one writes $G/_H$, this is usually meant in the sense of G-spaces.

Homogeneous spaces are of tremendous importance, as they provide many examples, and there are man techniques and even computer programs to calculate many properties, e.g., its curvature properties, the spectrum of the Laplace operator on such spaces.

Definition 5.5. Let G be a Lie group acting smoothly and transitively on a manifold M. Let H be the isotropy group of $p \in M$. Thus for any $h \in H$, $\ell_h: M \to M$ is a diffeomorphism fixing p. We define its isotropy representation

 $I: H \mapsto \operatorname{GL}(T_p M), \quad h \mapsto \mathrm{d}_p \ell_h.$

¹In many analysis lectures, this assumption is considered anyhow.

²we do not want to prove or discuss this here and how to apply this. See [8] for a reference.

Let again $s_p: G \to M$, $\sigma \mapsto \sigma \cdot p$ be the orbit map, and let H the isotropy group at p. If \mathfrak{p} is a complement of \mathfrak{h} in \mathfrak{g} , i. e., if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, then the differential $\mathrm{d} s_p: \mathfrak{g} \to T_p M$ of the orbit map s_p is an isomorphism. One thus often identifies M with G/H and $T_p M$ with \mathfrak{p} . The isotropy representation is thus a map

$$I: H \to \mathrm{GL}(\mathfrak{p})$$
.

Lemma 5.6. Let (M,g) be a Riemannian manifold, and assume that $G \sim M$ is a smooth, transitive and isometric action. If the isotropy representation is irreducible, i. e., if there is no H-invariant linear subspace $W \subset T_pM$ with $\{0\} \neq W \neq T_pM$, then M is an Einstein manifold, i. e., there is a $\lambda \in \mathbb{R}$, such that ric = λg .

Proof: As the action is isometric, the isotropy representation is a map $H \to O(T_pM)$. For any $h \in H$ we have $I(h)^*g_p = g_p$ and $I(h)^*ric_p = \operatorname{ric}_p$, thus also $\operatorname{Ric}_p \circ I(h) = I(h) \circ \operatorname{Ric}_p$. As the endomorphism Ric_p is symmetric, it is diagonalizable. Let λ be an eigenvalue of Ric_p and define $W := \ker(\operatorname{Ric}_p - \lambda \operatorname{id})$ as the corresponding eigenspace. As I(h) and Ric_p commute, we get $(I(h))(W) \subset W$. Thus W is invariant under the action of G given by I. We assumed that $\{0\}$ and T_pM are the only invariant linear subspaces, and as $W \neq \{0\}$ by the choice of λ , we have $W = T_pM$. Thus $\operatorname{ric}_p = \lambda g_p$.

Now consider any point $q \in M$, and we write $q = \sigma^{-1} \cdot p$, $\sigma \in G$. As ℓ_{σ} acts isometrically we have $\ell_{\sigma}^* g_p = g_q$ and $\ell_{\sigma}^* \operatorname{ric}_p = \operatorname{ric}_q$. Thus we have $\operatorname{ric}_q = \lambda g_q$ for all $q \in M$.

Example 5.7. We consider the Lie group SU(3) with a bi-invariant Riemannian metric g. The adjoint representation $Ad:SU(3) \rightarrow GL(\mathfrak{su}(3))$ turns $\mathfrak{su}(3)$ into an SU(3)-space. It has no non-trivial³ linear subspace W invariant under the SU83) action. This implies that the bi-invariant metric is unique up to a constant. We may normalize g such that

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{su}(3)$$

has length 1.

The scalar product on $\mathfrak{su}(3) \subset \mathbb{C}^{3 \times 3}$ is then

$$\langle A,B\rangle = \frac{1}{2}\operatorname{tr}(A^*B)$$

³Such a subspace is trivial if $W = \{0\}$ or $W = \mathfrak{su}(3)$.

As then SO(3) is a closed subgroup of SU(3), consisting of those matrices in SU(3), where all coefficients are real. The quotient $M := \frac{SU(3)}{SO(3)}$ is called the **Wu manifold**⁴, and plays an important role in bordism theory. The manifold carries a quotient metric, denoted as \overline{g} , and dim M = 5.

We define \mathfrak{p} as the orthogonal complement of \mathfrak{h} in \mathfrak{g} , namely

$$\mathfrak{p} = \operatorname{span} \left\{ \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \right\}$$

One may check⁵ that the action of SO(3) on \mathfrak{p} has no non-trivial invariant linear subspaces. Thus (M, \overline{g}) is an Einstein manifold. One can check that $\operatorname{ric}^{\overline{g}} = 6\overline{g}$.

5.3 Bi-quotients

We assume that G is a compact connected Lie group, and H a closed subgroup. As discussed above the associated homogeneous space G/H is a G-space. It may happen that a subgroup K of G still acts freely on on G/H. Then the **bi-quotient**

$$K \setminus G/H$$

is a smooth manifold. This gives rise to interesting examples, as e.g., the Gromoll-Mayer sphere, see https://ncatlab.org/nlab/show/Gromoll-Meyer+sphere. This is a compact 7-dimensional manifold, homeomorphic, but not diffeomorphic to S^7 , and it carries a metric with sectional curvature $K \ge 0$.

6 Riemannian submersions and the O'Neill formula

Tu 14.5.

Notes on literature for this section:

- [3, Chap. 9]: a good and deep, but not easily readable reference
- [9, Chap. 7, Def. 44 and following], textbook by O'Neill, but no proofs
- [10], original article by O'Neill including proofs

⁴In fact it belongs to the family SU(n)/SO(n) of so-called Landweber's manifolds ⁵proof omitted here!

Definition 6.1 (Riemannian submersion). Let (M,g) and (B.h) be Riemannian manifolds, $m = \dim M$, $n = \dim B$, and $f: M \to B$ a submersion. For any $p \in M$ we define the **vertical space** at p as $\mathcal{V}_p := \ker d_p f$ which is a vector space of dimension n - m. Then $\mathcal{V} := \bigcup_{p \in M} \mathcal{V}_p$ is a submanifold of TM of dimension n. Further we define the **horizontal space** at p as

$$\mathcal{H}_p \coloneqq (\mathcal{V}_p)^{\perp} = \{ X \in T_p M \mid X \perp \mathcal{V}_p \} \,.$$

The map f is called a Riemannian submersion if

$$\forall p \in M : d_p f|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{f(p)} B \tag{6.1}$$

is an isometry.

We can decompose any $X \in T_pM$ as

$$X = X_{\text{ver}} + X_{\text{hor}}, \quad X_{\text{ver}} \in \mathcal{V}_p, \ X_{\text{hor}} \in \mathcal{H}_p.$$

Further we denote the orthogonal projections as $\pi_{\text{hor}}: TM \to \mathcal{H}$ and $\pi_{\text{ver}}: TM \to \mathcal{V}$.

$$\Gamma(\mathcal{V}) \coloneqq \{ X \in \mathfrak{X}(TM) \in \forall p \in M : X|_p \in \mathcal{V}_p \}$$

$$\Gamma(\mathcal{H}) \coloneqq \{ X \in \mathfrak{X}(TM) \in \forall p \in M : X|_p \in \mathcal{H}_p \}$$

A Mathematical Appendices

A.1 Supplements from the theory of smooth manifolds

A.1.1 The Koszul formula

Let (M, g) be a semi-Riemannian manifold. We write $\langle X, Y \rangle$ for g(X, Y). In the lecture "Differential Geometry I" we have shown that there is a unique connection ∇ on TM, called the Levi-Civita connection such that it is metric and torsionfree.

We give here a version of the Koszul identity that differs slightly from the one given in that lecture. It gives a formula for the Levi–Civita connection.

Lemma A.1.1 (Koszul formula). For $X, Y, Z \in \mathfrak{X}(M)$ we have

$$2 \langle \nabla_X Y, Z \rangle$$

= $\partial_X \langle Y, Z \rangle + \partial_Y \langle X, Z \rangle - \partial_Z \langle X, Y \rangle$
+ $\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle$

In order to prove this lemma, one verifies that the right hand side defines a connection that is torsionfree and compatible with the metric.

The proof of this Lemma is given, e.g., in [4, Eq. (9) in Proof of Theorem 3.6]. It is also worked out in [1, Def. 2.7.2] for submanifolds of \mathbb{R}^n , but the same proofs also works for arbitrary semi-Riemannian manifolds.

A.1.2 A lemma on surjective submersions

Lemma A.1.2. Let $f : X \to Y$ be a surjective submersion from the C^{∞} -manifold X to the C^{∞} -manifold Y, and let Z be a further C^{∞} -manifold. Let $h: Y \to Z$ be a

map. Then h is smooth if and only if $h \circ f$ is smooth.

Proof: It is obvious that $h \circ f$ is smooth if h is smooth, as every submersion is by definition a smooth map.

Now assume that $h \circ f$ is smooth. For a given $y \in Y$ we want to show that h is smooth on a neighborhood of y. As y may be arbitrarily chosen, this then implies that h is smooth.

Let $n \coloneqq \dim X$ and $k \coloneqq \dim Y$.

At first we choose a preimage $x \in X$ of y, i.e. f(x) = y. (Here we use the surjectivity of f.) We choose a chart $\tilde{\varphi}_0 : \tilde{U}_0 \to \tilde{V}_0$ of Y with $y \in \tilde{U}_0$, then we choose a chart $\varphi_0 : U_0 \to V_0$ of X with $x \in U_0$

We obtain a smooth map $F: V_1 \to \widetilde{V}_0$, $F \coloneqq \widetilde{\varphi}_0 \circ f \circ \varphi_0^{-1}$, $V_1 \coloneqq V_0 \cap \varphi_0(f^{-1}(\widetilde{U}_0))$. As $df|_x: T_x X \to T_y Y$ is surjective, we see that $d(\widetilde{\varphi}_0 \circ \varphi_o^{-1})|_{\varphi_0(x)}$ is surjective. The implicit function theorem thus says that there is a small neighborhood V_2 of $\varphi_0(x)$ in V_1 , a diffeomorphism $\psi: V_2 \to W_1 \times W_2$, W_1 open in \mathbb{R}^k , W_2 open in \mathbb{R}^{n-k} , that there is an open neighborhood \widetilde{V}_2 of $\widetilde{\varphi}_0(y)$ in \widetilde{V}_0 and a diffeomorphism $\psi: \widetilde{V}_2 \to W_1$, such that $\widetilde{\psi} \circ F \circ \psi^{-1}: W_1 \times W_2 \to W_1$ is the projection to W_1 , i.e. $\widetilde{\psi} \circ F \circ \psi^{-1}(x_1, x_2) = x_1$ where $x_i \in W_i$.

In the following diagram all symbols α denote open subsets.



We set $U \coloneqq \varphi_0^{-1}(V_2)$, $\widetilde{U} \coloneqq \widetilde{\varphi}_0(\widetilde{V}_2)$, $\varphi \coloneqq \psi \circ \varphi_0 \colon U \to W_1 \times W_2$, $\widetilde{\varphi} \coloneqq \widetilde{\psi} \circ \widetilde{\varphi}_0 \colon \widetilde{U} \to W_1$. Then $\varphi \colon U \to W_1 \times W_2$ and $\widetilde{\varphi} \colon \widetilde{U} \to W_1$ are charts with $x \in U$ and $y \in \widetilde{U}$. Furthermore $\widetilde{\varphi} \circ f \circ \varphi^{-1} \colon W_1 \times W_2 \to W_1$ is the projection pr_{W_1} to W_1 .



Now as $h \circ f$ is smooth, $h \circ f \circ \varphi^{-1} : W^1 \times W_2 \to Z$ is smooth as well. As the map

$$h \circ f \circ \varphi^{-1} = (h \circ \tilde{\varphi}^{-1}) \circ \operatorname{pr}_{W_1} : W^1 \times W_2 \to Z$$

is smooth, it is in particular smooth in the W_1 direction (for fixed element in W_2), but this is just the map $h \circ \tilde{\varphi}^{-1} \to W_1$, which is thus smooth. This implies that $h|_{\tilde{U}}$ is smooth.

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ad adjoint representation of a Lie algebra	p. 10
Aut(G) group of automorphisms of the Lie group G	p. 4
$\mathbb{C}\mathbf{P}^n$ complex projective space	p. 28
Diff(M) group of diffeomorphisms of M	p. 11
ℓ_{σ} left multiplication	p. 1
End(G) monoid of endomorphisms of the Lie group G	p. 4
$\operatorname{End}_{\operatorname{lin}}(V)$ vector space endomorphisms of V	p. 8
exp exponential map	p. 13
\mathfrak{h}_3 3-dimen. Heisenberg Lie algebra	p. 22
GL(V) automorphism groups of the vector space V	p. 7
$\operatorname{Hom}(\mathfrak{g},\mathfrak{h})$ Lie algebra homomorphisms from \mathfrak{g} to \mathfrak{h}	p. 8
$\operatorname{Hom}(G, H)$ set of homomorphisms of Lie groups from G to $H \dots$	p. 4
\mathcal{H}_p horizontal space at p	p. 53
Iso(G, H) set of isomorphisms of Lie groups from G to H	p. 4
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$G \setminus X$ quotient by a group (as a set)	p. 27
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Ric_p Ricci endomorphism at p	p. 51
ric_p Ricci form at p	p. 51
$\mathbb{R}\mathbf{P}^n$ real projective space	p. 28
\mathcal{V}_p vertical space at p	p. 53
$G \curvearrowright X$ G acts on X from the left	p. 26
H_3 3-dimen. Heisenberg group	p. 22
r_{σ} right multiplication	p. 1

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topological group	topologische Gruppe

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freie Operation	free operation
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Geradenbündel	line bundleii
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innerer Punkt	. inner point
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